

Talk 1: graphical functions.

Framework: φ^4 -theory (massless)

The period map

φ^4 -theory $\leadsto P \subseteq \mathbb{R}_+$

- Take a 4-regular, internally 6-connected graph (A 4-point graph with ∞). 

- Delete a vertex ($\text{label } \infty$) 

- Label vertices $0, 1, x_1, \dots, x_{v-3}$ 

- Write for every edge the propagator

$$\overline{x_i x_j} = \frac{1}{\|x_i - x_j\|^2} \quad 0: \text{origin}, 1: \text{en unit-vector}$$

- Integrate over x_1, \dots, x_{v-3} the product of all propagators with $\frac{1}{\pi^2} \int d^4 x_i$

$$\frac{1}{\pi^2} \int d^4 x_1 \frac{1}{\pi^2} \int d^4 x_2 \frac{1}{\|x_1 - x_2\|^2 \|x_1 - x_3\|^2 \|x_2 - x_3\|^2 \|x_2 - x_4\|^2} = 6 \zeta(3)$$

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \text{. Riemann } \zeta\text{-function.}$$

First systematic study due to Broadhurst, Kreimer 1995
The image of the map is always a period
in the sense of Kontsevich-Zagier.

Let $P_F \subseteq P$ the \mathbb{Q} -vector space of Feynman periods

final philosophy: A Hopf-algebra Galois-coach on P . $\Delta: P \rightarrow P \otimes_{\mathbb{Q}} X$

Main conjecture:

There exists a Hopf-algebra \mathcal{H}_{44} that coacts

$$\text{on } \mathcal{P}_{\text{44}}. \quad \Delta_{\mathcal{P}_{\text{44}}} : \mathcal{P}_{\text{44}} \rightarrow \mathcal{P}_{\text{44}} \otimes \mathcal{H}_{\text{44}}$$

- \mathcal{P}_{44} is a co-module. The \mathbb{Q} -vector-space of 4^{th} -periods is a co-module.
- \mathcal{P}_{44} is extremely sparse in \mathcal{P} .

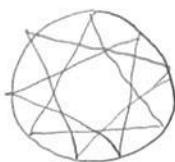
This makes it a very strong conjecture.

Example $S(3)S(2) \notin \mathcal{P}_{\text{44}}$ because

$S(2) \in \mathcal{P}_{\text{44}}$. The conjecture becomes stronger with the weight. One can have new objects in 4^{th} -periods only in a very restricted way.

- The Hopf algebra of iterated integrals in the letters $0, 1, S_6 = \frac{1+\sqrt{-3}}{2}$ modulo $2\pi i$ is a sub-Hopf-algebra of \mathcal{H}_{44} .

(First example



$\mathcal{P}_{7,11}$ at wt 11.

another example at weight 13).

We want to understand Feynman periods.

How can we calculate them? easy
Problem: Find an algorithm to numerically evaluate Feynman periods
 This was unclear for ≈ 19 years (starting from

1995). Now we can calculate most Feynman periods with ≤ 8 loops (wt 13) and many more up to ≤ 10 loops.

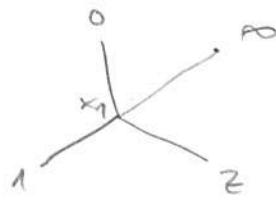
Tools:

- parametric integration (Feynman/Schwinger parameters ≤ 7 loops)
- graphical functions
- generalized single-valued hyperlogs ($?.\text{ta}(?)$)

graphical functions

- Leave one variable z un-integrated in the calculation of a period

example

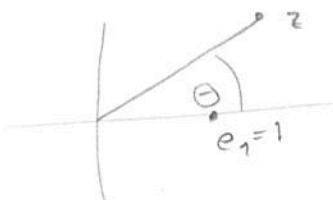


$$f_k(z) = \frac{1}{\pi^2} \int d^4 x_1 \frac{1}{\|x_1\|^2 \|x_1 - e_1\|^2 \|x_1 - z\|^2}$$

- This gives a function of a single variable z .
- graphical-functions may be considered as convergent n-point-functions in ϕ^4 -theory with $n \geq 4$ where certain external vertices are identified so that one obtains an effective 4-point-function.
- They are SYM correlators or amplitudes
- James Drummond, Claude Duhr, Burkhard Eden
Paul Heslop, Jeffrey Pennington, Vladimir A. Smirnov.

Universal properties of graphical functions

- They depend only on $\|z\|$ and the angle between z and e_1 . They may hence be considered as functions on the complex plane



- ② $f_B(z) = f_B(\bar{z})$ by symmetry.
- ③ f_B is real analytic on $\mathbb{C} \setminus \{0, 1\}$
(parametric representation, E. Panzica, H. Götze)
- ④ f_B is single-valued
(when we vary γ the integral changes continuously.)

- ⑤ If B_γ has no edges between $0, 1, z_1, \infty$ and ∂M
least one edges to z then at $0, 1, \infty$ we have expansions

$$f_B(z) = \sum_{k=0}^K \sum_{l,m=0}^k c_{klm}^B \ln^k(z-a)(\bar{z}-a) \cdot (z-a)^l (\bar{z}-a)^m \quad a \in \{0, 1\}$$

$$f_B(z) = \sum_{k=0}^K \sum_{\substack{l,m \geq 0 \\ l+m \leq 2K}} c_{klm}^B \ln^k z \bar{z} \quad z^l \bar{z}^m \quad \text{at infinity.}$$

example  : $f_+(z) = \frac{4iD(z)}{z-\bar{z}}$

$$D(z) = \Im \left(\operatorname{Li}_2(z) + \ln(1-z) \ln(z) \right) \quad \text{Bloch-Wigner}$$

def. $\operatorname{Li}_2(z) = \sum_{k=0}^{\infty} \frac{z^k}{k^2}$ for $|z| < 1$.

Check: ① or, ② by $D(\bar{z}) = -D(z)$, ③ : Singularity
at $z = \bar{z}$ cancels except for $z = 0$ or $z = 1$

④ from $D(z)$ ⑤ explicit calculation.

Construction of graphical functions

- Start with a small graph with known graphical function. e.g. $y = \dots$ $f_B = 1$.
→ parametric integration, Feigenbaums - technique, ...
- Add external edges

$$\overline{-z} = \frac{1}{z\bar{z}} \quad , \quad \overline{1-z} = \frac{1}{(z-1)(\bar{z}-1)} \quad .$$

- Add an edge to the external vertex z



Theorem: f_Γ is the unique solution with
① - ⑤ of the differential equation

$$-\frac{1}{z - \bar{z}} \partial_z \partial_{\bar{z}} (z - \bar{z}) f_{\Gamma}(z) = f_{\Gamma}(z).$$

Proof: The differential equation is the 2-dimensional analogue of $\square \frac{1}{(z-x)^2} = -4\pi^2 \delta^{(4)}(z-x)$

Proof of uniqueness:

Assume f is a solution of the dgl.

Then $f_\Gamma(z) = f + g$, $g \in \ker(\text{dgl})$

$$\Rightarrow g = \frac{g_1(z) + g_2(\bar{z})}{z - \bar{z}}$$

$$\textcircled{2} \Rightarrow f_\Gamma(z) = f + \frac{g_1(z) - g_1(\bar{z})}{z - \bar{z}}$$

$$\Rightarrow \partial_z (z - \bar{z})(f_\Gamma - f) = g_1'(z)$$

\textcircled{4} $\Rightarrow g_1'$ is single-valued

\textcircled{5} \Rightarrow l.h.s. is of the form

$$\sum_{k=0}^{\infty} \sum_{\substack{l+m \geq 0 \\ l+m \geq 1}} \text{d}_{k,l,m} (k \ln^{k-1}(z) + l \ln^l(z)) (z-a)^{l-1} (\bar{z}-a)^m$$

Comparison: $m=0, l \geq 1, k=0$

$\Rightarrow g_1'(z)$ is analytic at 0, 1

\textcircled{5} $\Rightarrow g_1'(z)$ vanishes at ∞

$$\Rightarrow g_1'(z) = 0 \Rightarrow g_1(z) = c \Rightarrow f_\Gamma = f. \quad \square$$

exercise Use the Theorem to prove that

$$f_+(z) = \frac{4iD(z)}{z-\bar{z}}.$$

Remark • Repeated application of the theorem leads to the proof of the zigzag theorem.

$$P(\text{graphical function}) = 4 \frac{(2n-2)!}{n!(n-1)!} \left(1 - \frac{1 - (-1)^n}{2^{2n-3}}\right) S(2n-3)$$

Proof: substitute $z=0$ in a graphical function (w. F. Brown)

- In general graphical functions can be very complicated (first examples of complicated gf's have 8 vertices).

- There exist more tools to calculate graphical functions. For ≤ 3 vertices the most general tool is parametric integration (F. Brown, E. Panzer)

- We want to understand gf's which can be expressed in terms of iterated integrals (with a certain alphabet). These functions lead to generalizations of single-valued -multiple polylogarithms (like $D(z)$) \rightarrow generalized single-valued hyperlogarithms \rightarrow next talk.

Decorated intervals

Let $S = \{s_1, s_2, \dots\} \subseteq \mathbb{C}$ a finite set of singular points and $w_i = \frac{dz}{z-a_i}$, $a_i \in S$

Fix a path $\gamma: [0,1] \rightarrow \mathbb{C} \setminus S$ with $\gamma(0) = a_0$ and $\gamma(1) = a_{n+1}$. Then

$$\mathcal{I}_\gamma(a_0, a_1, a_2, \dots, a_n; a_{n+1}) = \int \dots \int_{\text{order-ctd}} \gamma^* w_1(t_1) \wedge \dots \wedge \gamma^* w_n(t_n).$$

- If $a_i \in \mathbb{Q}$ then \mathcal{I}_γ is a period with known coaction (Gorchakov).

- \mathcal{I}_γ depends only on the homotopy class of γ .
- \mathcal{I}_γ may be regularized for the case $a_0, a_{n+1} \in S$.
- \mathcal{I}_γ has path concatenation.
- We may hence set $a_0 = 0$ and take $\gamma =$ straight path (if possible) to obtain a function of the end point $\mathcal{I}(0, a_n, z)$: hyperlogarithm. If $a_i \in \mathbb{Q}, \mathbb{R}$ multiple polylogarithm.
- Hyperlogarithms span a shuffle algebra over \mathbb{C} .

Examples: $\mathcal{I}(0, \underbrace{0 \dots 0}_{n}, z) = \frac{1}{n!} \ln^z z$ because

$$\mathcal{I}(0, 0, z) = \ln z, \quad \mathcal{I}(0, a, z) = \ln(1 - \frac{z}{a}), \quad a \neq 0.$$

$$\mathcal{I}(0, \underbrace{0 \dots 0}_{n}, z) = -\ln(z) = -\sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

$$\text{Differentiation } \partial_z \mathcal{I}(0, w, z) = \frac{1}{z-w} \mathcal{I}(0, w, z).$$

• Thatched integrals have monodromy around $S \in S$.

For graphical functions we need single-valued objects. Idea: combine hyperlogs with their complex conjugates to obtain s.v. hyperlogs.

Example $\ln z + \ln \bar{z} = \ln z\bar{z}$ s.v.

$D(z) = \Im(\ln(z) + \ln(1-z)\ln|z|)$ is s.v.

• In general we obtain a shuffle algebra of single-valued hyperlogarithms (F. Brown).

• For graphical functions we need more general objects.

Example $f(z) = \int_{S^1} \frac{\ln z\bar{z}}{z - \frac{1}{2}} dz$ does it exist?

Does it have only pointlike singularities?

• Answer in terms of iterated integrals:

$$f(z) = I(0, 0, \frac{1}{2}, z) + I(0, \frac{1}{2}, z) I(0, 0, \bar{z})$$

fulfills the diff. eq. $\partial_z f = \frac{\ln z\bar{z}}{z - \frac{1}{2}}$ and is single valued.

General construction method

$$f = \int_{S^1} g d\bar{z}$$

If f is given as $\int_{S^1} g d\bar{z}$ and as $\int_{S^1} g_z d\bar{z}$ then we can obtain f by ordinary integration.

Def : $\mathcal{G}_n, \partial_z \mathcal{G}_n$ are defined recursively by

① $\mathcal{G}_0 = \mathbb{C}, \partial_z \mathcal{G}_0 = \{0\}$.

② \mathcal{G}_n is spanned (over \mathbb{C}) by hyperlogarithms in z and \bar{z} over letters $\gamma(\bar{z}) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$

③ $f \in \mathcal{G}_n \Rightarrow \partial_z f \in \partial_z \mathcal{G}_n, f(0) = 0$.

④ Let $f \in \mathcal{G}_n$ or $f \in \partial_z \mathcal{G}_n$. Then

$$f: \bar{\mathbb{C}}^2 \setminus \bigcup_i (s_i \times \bar{c} \cup \bar{c} \times s_i^*) , s_i \in S.$$

$f(z, \bar{z})$ is single-valued for $\bar{z} = z^k$

$$f(z, \bar{z}) = \sum_{k=0}^K \sum_{l=-1(0)}^{\infty} \sum_{m=0}^{\infty} c_k^a \ln^k (z-a)(\bar{z}-a^k) \cdot (z-a)^l (\bar{z}-a^k)^m$$

$$|z-a| < R_a, a \in \mathbb{C} \text{ and}$$

$$f(z, \bar{z}) = \sum_{k=0}^K \sum_{l=-1(0)}^{\infty} \sum_{m=0}^{\infty} c_k^a \ln^k z \bar{z} \cdot z^l \bar{z}^m \quad |z| > R_a.$$

⑤ $\partial_z \mathcal{G}_n$ is spanned by functions

$$\frac{f_{n-1}(z)}{z - \gamma(\bar{z})}, f_{n-1} \in \mathcal{G}_{n-1}.$$

For $f \in \partial_z \mathcal{G}_n$ $\underset{z=a}{\text{res}} f = c_{0,-1,0}^a, \underset{\bar{z}=a}{\text{res}} f = c_{0,0,-1}^a$

$\mathcal{G} = \bigoplus_{n \geq 0} \mathcal{G}_n$

\mathcal{G}_1 is spanned by $\ln(z-s_i)(\bar{z}-s_i^*)$, $s_i \in S$.

Similarly $\partial_{\bar{z}} \mathcal{G}_n$.

Lemma:

① Let $f \in \partial_z \mathcal{G}_n + \partial_{\bar{z}} \mathcal{G}_1$ residue-free, then there exists an anti-residue-free $g \in \partial_{\bar{z}} \mathcal{G}_n + \partial_z \mathcal{G}_1$ such that $\partial_{\bar{z}} f = \partial_z g$.

② $\int_{S^1} f dz = \int_{S^1} g d\bar{z} \in \mathcal{G}_n + \mathcal{G}_2$

③ Every $f \in \partial_z \mathcal{G}_n$ has a single-valued primitive in \mathcal{G}_n

Proof (sketch):

Induction on the weight. Weight $n=0$ trivial.

Let $f_n = \frac{f_{n-1}(z)}{z - \gamma(z)}$ (linear-combinations by linearity).

$$f_{n-1} \in \mathcal{G}_{n-1} + \mathcal{G}_0.$$

$$\partial_{\bar{z}} f = \partial_{\bar{z}} \left(\frac{f_{n-1}(z)}{z - \gamma(z)} \right) + \underbrace{\frac{\partial_{\bar{z}} f_{n-1}}{z - \gamma(z)}}_{(*)} - \underbrace{\frac{\partial_{\bar{z}} f_{n-1}}{z - \gamma(z)}}_{(*)}$$

(if γ is not invertible then the terms with γ^{-1} are absent.)

$$\bullet \quad \frac{f_{n-1}(z)}{z - \gamma(z)} \in \partial_{\bar{z}} \mathcal{G}_n + \partial_{\bar{z}} \mathcal{G}_1$$

• In (*) we have terms $\frac{1}{z - \gamma_1(z)} \frac{1}{z - \gamma_2(z)}$ f_{n-2} with $f_{n-2} \in \mathcal{G}_{n-2}$.

• A partial fraction decomposition of the rational pre-factor numerically gives a sum of terms of the form

$$\frac{1}{A\bar{z} + B} \quad \frac{1}{z - \gamma_i(z)} \quad i=1,2. \quad [\text{possibly } -\frac{B}{A} \notin S]$$

• Non simple terms ($\gamma_1 = \gamma_2$ or $(A\bar{z} + B)^{-k}, k > 1$) are not possible (?)

• $\frac{f_{n-2}}{z - \gamma_i(z)} \in \partial_{\bar{z}} \mathcal{G}_{n-2}$ by (3) and induction

• Then exists a $g_{n-1} \in \mathcal{G}_{n-1}$ with $\partial_{\bar{z}} g_{n-1} = \frac{f_{n-2}}{z - \gamma_i(z)}$

• $\frac{1}{A\bar{z} + B} g_{n-1} \in \partial_{\bar{z}} \mathcal{G}_n$. Hence there exists a $g_n \in \partial_{\bar{z}} \mathcal{G}_n$ with $\partial_{\bar{z}} g_n = (*)$

• Set $g' = \frac{f_{n-1}}{z - \gamma(z)} + g_n \in \partial_{\bar{z}} \mathcal{G}_n + \partial_{\bar{z}} \mathcal{G}_1$

then $\partial_{\bar{z}} g' = \partial_{\bar{z}} f$.

• Set $g = g' - \sum_a \frac{\text{Res}(g')}{z-a} \frac{1}{z-a}$ anti-residue with $\partial_{\bar{z}} g = \partial_{\bar{z}} f$. $\Rightarrow \textcircled{1}$

Define F such that $\partial_z F = f$ and $\partial_{\bar{z}} F = g$. - 5 -

To do so set

$$F_1 = \int f dz \text{ (ordinary integral)} . \quad \partial_{\bar{z}} F_1 = h$$

$$\partial_z h = \partial_z \partial_{\bar{z}} F_1 = \partial_{\bar{z}} f = \partial_z g \Rightarrow h = g + h_1(\bar{z})$$

$$\text{Set } F = F_1 - \int h_1(\bar{z}) d\bar{z} . \text{ Then } \partial_z F = f \text{ and } \partial_{\bar{z}} F = h - h_1 = g.$$

- Because $\partial_z F$ and $\partial_{\bar{z}} F$ are sv., F can only have constant monodromies.

$$\Rightarrow F_0 = F - \sum_{i=1}^N c_i \ln(z - s_i) , a_i \in S \text{ is sv.}$$

$$\partial_{\bar{z}} F_0 = \partial_{\bar{z}} F = g \text{ anti-residue-free}$$

$$\Rightarrow \partial_z F_0 \text{ has no terms } \frac{c_{00-i}}{z - a^k} \text{ in the expansion at } a$$

$$\Rightarrow F_0 \text{ has no term } \ln(z-a)(\bar{z}-a^k) \text{ in the expansion at } a . \Rightarrow c_{00} = 0$$

$$\Rightarrow \partial_z F_0 \text{ has no residues.}$$

$$\Rightarrow \underbrace{\partial_z F_0 - \partial_z F}_{f} = - \sum_{i=1}^N \frac{c_i}{z - a_i} \text{ has no residues}$$

$$\Rightarrow c_i = 0 \Rightarrow F \text{ is sv.} \Rightarrow F = \int f dz = \int g d\bar{z} \Rightarrow ②.$$

To show ③ we take $f \in \partial_z \mathcal{G}_n$

$$f - \sum_a \sum_{z=a} \frac{1}{z-a} \in \partial_z \mathcal{G}_n + \partial_z \mathcal{G}_1 \text{ residue-free}$$

$$② \Rightarrow F^1 = \int_{sv} \left(f - \sum_a \sum_{z=a} \frac{1}{z-a} \right) dz \text{ exists}$$

$$\text{Set } F = F^1 + \sum_a \sum_{z=a} \ln(z-a)(\bar{z}-a^k) \in \mathcal{G}_n + \mathcal{G}_1$$

$$\text{and } \partial_z F = f \Rightarrow F \in \mathcal{G}_n \Rightarrow ③ \quad \square$$

- Remark: generalized single-valued hyperlogarithms generate a bi-differential algebraation in which single-valued primitives always exist.

Conjecture

If $f(z) = \frac{a_i z + b_i}{c_i z + d_i}$, $a_i, b_i, c_i, d_i \in \mathbb{C}$
 exist such that a graphical function, f_g is
 expressible in terms of hyperlogarithms in z and \bar{z}
 in the alphabet $\{f_k\}$ with coefficients in $\mathbb{C}(z, \bar{z})$, $\frac{1}{z-f_1(\bar{z})}, \frac{1}{\bar{z}-f_1(z)}$
 then $f_g \in \mathcal{A}$ and the differential equation

$$-\frac{1}{z-\bar{z}} \partial_z \partial_{\bar{z}} (z-\bar{z}) f_g = f_g$$

has a solution in \mathcal{A} .

Problem: Terms like $z\bar{z}+1$ which spoil regularity
 but do not affect real-analyticity.

Remark: You can permute external labels of a graph. Then

Corollary: Let G be a graph with $f_G \in \mathcal{A}$ then
 one can add edges between external
 vertices and append an edge to an
 external vertex without leaving \mathcal{A} .

period is obtained by $-\frac{1}{2\pi i} \int (z-\bar{z})^2 f_G(z) dz$

Lemma: Let $f \in \mathcal{A}$ such that $\frac{1}{\pi} \int f(z) dz$ exists.

Let $F = \int_S f dz \in \mathcal{A}$. Then

$$\frac{1}{\pi} \int f(z) dz = \lim_{z \rightarrow \infty} F - \sum_{z=s_i} F, \quad s_i \in S$$

Proof: l.h.s. $= -\frac{1}{2\pi i} \int_{\mathbb{C} \setminus S} dz \wedge d\bar{z} f(z) = \frac{1}{2\pi i} \int_{\mathbb{C} \setminus S} d(F dz)$

$$= \frac{1}{2\pi i} \int_{\partial(\mathbb{C} \setminus S)} F dz + \text{explicit calculation.} \quad \square$$