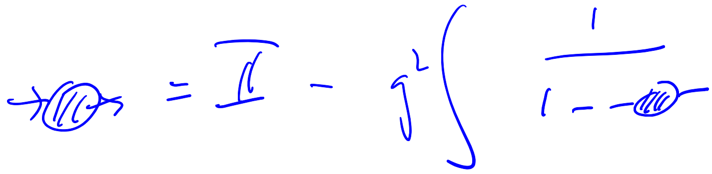
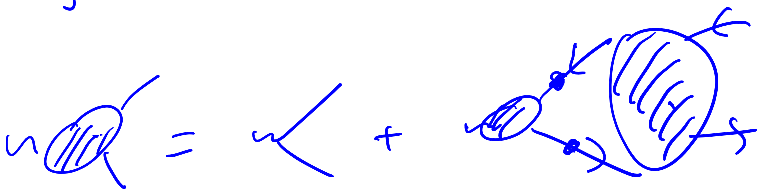
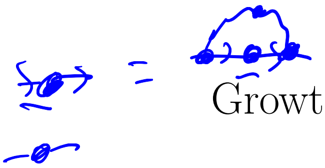
Nordita Lecture notes on Field Th by Cvitanovic



Bjorken Drell QFT



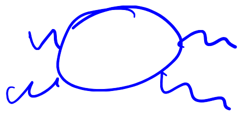
Ch. 19



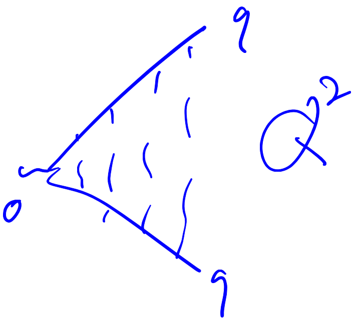
Growth estimates for Dyson-Schwinger equations

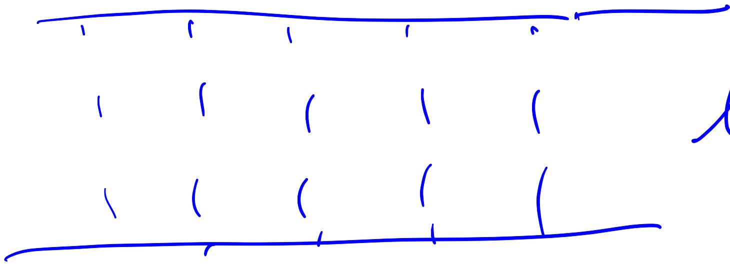
Karen Amanda Yeats

October 22, 2018

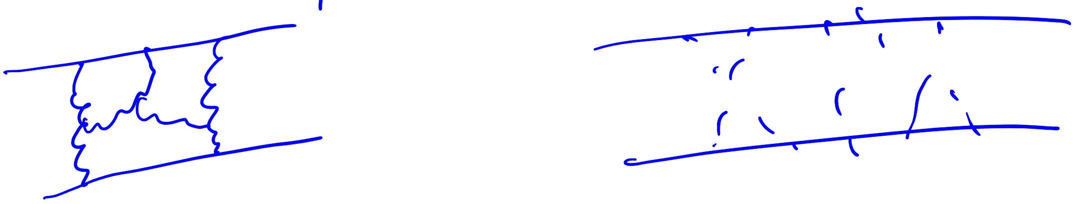


Rivers





big



small

Acknowledgments

I would like to thank Dirk Kreimer for his wisdom, insight, and never-ending store of ideas. Second, I would like to thank David Fried for a very detailed reading. Third, I would like to thank the remainder of my committee, Maciej Szczesny, David Rohrlich, and Takashi Kimura. I would also like to thank everyone who got excited about the differential equation including Paul Krapivsky, Cameron Morland, David Uminsky, and Guillaume Van Baalen.

Cameron has further been invaluable professionally for his great skill with plots and personally for hugs, geekiness, and love. Finally, Russell Morland has prevented me from getting work done in the best possible way.

The following software was used in the research behind and the presentation of this work. This document is typeset in LaTeX using a thesis style file originally written by Stephen Gildea and modified by Paolo Gaudiano, Jonathan Polimeni, Janusz Konrad, and Cameron Morland. Symbolic and numerical computation was done using GiNaC and Maple. Plots were prepared with gnuplot both directly and via octave.

Abstract

Dyson-Schwinger equations are integral equations in quantum field theory that describe the Green functions of a theory and mirror the recursive decomposition of Feynman diagrams into subdiagrams. Taken as recursive equations, the Dyson-Schwinger equations describe perturbative quantum field theory. However, they also contain non-perturbative information.

Using the Hopf algebra of Feynman graphs we will follow a sequence of reductions to convert the Dyson-Schwinger equations to the following system of differential equations,

$$\gamma_1^r(x) = P_r(x) - \mathrm{sign}(s_r)\gamma_1^r(x)^2 + \left(\sum_{j \in \mathcal{R}} |s_j|\gamma_1^j(x)\right) x\partial_x \gamma_1^r(x)$$

where $r \in \mathcal{R}$, $\mathcal{R}$ is the set of amplitudes of the theory which need renormalization, γ_1^r is the anomalous dimension associated to r, $P_r(x)$ is a modified version of the function for the primitive skeletons contributing to r, and x is the coupling constant.

Next, we approach the new system of differential equations as a system of recursive equations by expanding $\gamma_1^r(x) = \sum_{n \geq 1} \gamma_{1,n}^r x^n$. We obtain the radius of convergence of $\sum \gamma_{1,n}^r x^n/n!$ in terms of that of $\sum P_r(n)x^n/n!$. In particular we show that a Lipatov bound for the growth of the primitives leads to a Lipatov bound for the whole theory.

Finally, we make a few observations on the new system considered as differential equations.

Contents

List of Tables

List of Figures

List of Symbols

1PI	1-particle irreducible, that is, 2-connected
A	the gauge field in QED
$\mathbf{A}^r(x)$	generating function for a_n^r
$\mathbf{A}(x)$	generating function for a_n
a_n^r	$\gamma_{1,n}^r/n!$
a_n	$\gamma_{1,n}/n!$
a_n^1, a_n^2	coefficients for an example system
β	the physicists' β-function describing the nonlinearity of a Green function
B_+	insertion into a Hopf algebra primitive taken generically
B_+^γ	insertion into the primitive γ
$B_+^{k,i;r}$	insertion into the k-loop primitive with residue r indexed by i
$B_+^{k,i}$	insertion into a primitive at k loops, with i an index running over primitives; that is, $B_+^{k,i;r}$ in the case with only one r
$\mathbf{B}^r(x)$	generating function for b_n^r
$\mathbf{B}(x)$	generating function for b_n
b_n^r	a particular lower bound for a_n^r
b_n	a particular lower bound for a_n
$\mathbf{bij}(\gamma, X, \Gamma)$	the number of bijections of the external edges of X with an insertion place of γ such that the resulting insertion gives Γ
$\mathbf{C}^r(x)$	generating function for c_n^r implicitly depending on an $\epsilon > 0$
$\mathbf{C}(x)$	generating function for c_n implicitly depending on an $\epsilon > 0$
c_n^r	a particular upper bound for a_n^r implicitly depending on an $\epsilon > 0$
c_n	a particular upper bound for a_n implicitly depending on an $\epsilon > 0$
Δ	the coproduct of $\mathcal{H}$
d^4	integration over $\mathbb{R}^4$
D	dimension of space-time
η	the counit of $\mathcal{H}$
e	the unit map of $\mathcal{H}$
E	an edge type, viewed as a pair of half edge types
F_p	the Mellin transform associated to the Hopf algebra primitive p
$F_{k,i}^r$	the Mellin transform associated to the k-loop primitive with residue r indexed by i
$F_{k,i}(\rho)$	$F_{k,i}^r$ in the case with only one r
$f^r(x)$	$\sum_{k\geq 1} x^k p^r(k)/k!$ when $\sum_{k\geq 1} x^k p^r(k)$ is Gevrey-1
$f(x)$	$\sum_{k\geq 1} x^k p(k)/k!$ when $\sum_{k\geq 1} x^k p(k)$ is Gevrey-1

γ_1^r	the anomalous dimension of the Green function indexed by the amplitude r
γ_k^r	k-th leading log term of the Green function indexed by the amplitude r
γ_k	γ_k^r in the case with only one r
$\gamma_{1,n}^r$	coefficient of x^n in γ_1^r
$\gamma_{1,n}$	coefficient of x^n in γ_1
$(\gamma\|X)$	the number of insertion places for X in γ
$\gamma \cdot U$	$\sum \gamma_k U^k$
G, Γ, γ	graphs
$\Gamma(x)$	the Γ function extending the factorial function to the complex numbers
G/γ	the graph G with the subgraph γ contracted
$G^r(x, L)$ ...	Green function indexed by the amplitude r
$\mathcal{H}$	the Hopf algebra of Feynman graphs
$\mathcal{H}_{\text{lin}}$	the linear piece of $\mathcal{H}$
H	set of half edge types
$\mathbb{I}$	the empty graph as the unit element of $\mathcal{H}$
id	the identity map on $\mathcal{H}$
k	an internal momentum appearing as an integration variable
$\mathcal{L}$	a Lagrangian
L	$\log(q^2/\mu^2)$, the second variable on which the Green functions depend, where q^2 is a kinematical variable and μ^2 is a subtraction point
m	multiplication on $\mathcal{H}$ or a mass
$\text{maxf}(\Gamma)$ ...	the number of insertion trees corresponding to Γ
$d\Omega_k$	angular integration over the $D-1$ sphere in $\mathbb{R}^D$ where $k \in \mathbb{R}^D$
ϕ	a scalar field or the (unrenormalized) Feynman rules
ϕ^3	scalar field theory with a 3 valent vertex
ϕ^4	scalar field theory with a 4 valent vertex
ϕ_R	the renormalized Feynman rules
ψ	the fermion field in QED
P_ϵ, P_ϵ^r	polynomials depending on ϵ
P_r	a modified version of the function of the primitive skeletons with residue r
P_{lin}	projection onto the linear piece of $\mathcal{H}$
$p_i^r(k)$	coefficient giving the contribution of primitive i at k loops with external leg structure r
$p_i(k)$	$p_i^r(k)$ in the case with only one r
$p^r(k)$	$-\sum_i r_{k,i;r} p_i^r(k)$, the overall contribution of all primitives at k loops
$p(k)$	$-\sum_i r_{k,i} p_i(k)$, the overall contribution of all primitives at k loops in the case with only one r
q	an external momentum
Q	(combinatorial) invariant charge
QCD	quantum chromodynamics
QED	quantum electrodynamics
ρ	the argument of Mellin transforms with 1 insertion place or the radius of convergence of $f(x)$
ρ_a	radius of convergence of $\mathbf{A}(x)$

ρ_ϵ	the radius of convergence of $\mathbf{C}(x)$
ρ_i	the argument of the Mellin transform which marks the ith insertion place
ρ_r	the radius of convergence of $f^r(x)$
$\mathcal{R}$	amplitudes which need renormalization, used as an index set
$\mathbb{R}$	the real numbers
R	the map from Feynman graphs to regularized Feynman integrals
$r_{k,i;r}$	residue of $\rho F^r_{k,i}(\rho)$, especially after reducing to geometric series
$r_{k,i}$	residue of $\rho F_{k,i}(\rho)$, especially after reducing to geometric series
$\star$	the convolution product of functions on $\mathcal{H}$
S	the antipode of $\mathcal{H}$
s_r	the power of X^r in Q^{-1}
s	the power of X in Q^{-1} in the case with only one r
$\mathrm{sign}(s)$	the sign of the real number s
T	a combinatorial physical theory
t^r_k	upper bound for the index over primitives at k loops with external leg structure r
t_k	upper bound for the index over primitives at k loops in the case with only one r
V	a vertex type viewed as a set of half edge types
ξ	a gauge variable
$[x^n]$	the coefficient of x^n operator
x	the coupling constant used as an indeterminate in series with coefficients in $\mathcal{H}$ and used as one of the variables on which the Green functions depend
$\|X\|_\vee$	the number of distinct graphs obtainable by permuting the external edges of X
$X^r(x)$	sum of all graphs with external leg structure r, as a series in the coupling constant x
X	X^r in the case with only one r

Chapter 1

Introduction

Dyson-Schwinger equations are integral equations in quantum field theory that describe the Green functions of a theory and mirror the recursive decomposition of Feynman diagrams into subdiagrams. Taken as recursive equations, the Dyson-Schwinger equations describe perturbative quantum field theory, while as integral equations they also contain non-perturbative information.

Dyson-Schwinger equations have a number of nice features. Their recursive nature gives them a strong combinatorial flavor, they tie Feynman diagrams and the rest of perturbation theory to non-perturbative quantum field theory, and on occasion they can be solved, for example [5]. However, in general they are complicated and difficult to extract information from.

The goal of the present work is to show how the Dyson-Schwinger equations for a physical theory can be transformed into the more manageable system of equations

$$\gamma_1^r(x) = P_r(x) - \mathrm{sign}(s_r)\gamma_1^r(x)^2 + \left(\sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)\right) x\partial_x \gamma_1^r(x) \tag{1.1}$$

where r runs over $\mathcal{R}$, the amplitudes which need renormalization in the theory, x is the coupling constant, $\gamma_1^r(x)$ is the anomalous dimension for r, and $P_r(x)$ is a modified version of the function of the primitive skeletons contributing to r, see Chapter 7 for details.

Chapter 2 discusses the general background with a focus on definitions and examples rather than proofs. The approach taken is that Feynman graphs are the primary objects. In an attempt to make matters immediately accessible to a wide range of mathematicians and to accentuate the combinatorial flavor, the physics itself is mostly glossed over. Readers with a physics background may prefer to skip this chapter and refer to existing surveys, such as [15], for the Hopf algebra of Feynman graphs.

Chapter 3 discusses the more specific background and setup for Dyson-Schwinger equations and the insertion operators B_+ on Feynman graphs. Proofs are again primarily left to other sources. [1] covers combinatorially similar material for rooted trees. Some important subtleties concerning B_+ for Feynman diagrams are discussed in more detail in [21] with important results proved in [32]. The approach to disentangling the analytic and combinatorial information comes from [24]. This chapter leaves us with the following input to the upcoming analysis: combinatorial Dyson-Schwinger equations and a Mellin transform for each connected, divergent, primitive graph. The former consists of recursive equations at the level of Feynman graphs with the same structure as the original analytic Dyson-Schwinger equations. The latter contains all the analytic information.

The next four chapters derive (1.1) expanding upon the discussion in [25]. Chapter 4 derives a preliminary recursive equation in two different ways, first from the renormalization group equation, and second from the Connes-Kreimer scattering-type formula [9]. Chapter 5 reduces to the case of single variable Mellin transforms and a single external scale. The Mellin transform variables correspond to the different insertion places in the graph, so we refer to this as the single insertion place case, though this is only literally true for simple examples. The cost of this reduction is that we are forced to consider non-connected primitive elements in the Hopf algebra. Chapter 6 reduces to the case where all Mellin transforms are geometric series to first order in the scale parameters by exchanging unwanted powers of the Mellin transform variable for a given primitive with lower powers of the variable for a primitive with a larger loop number, that is, with a larger number of independent cycles. The cost of this reduction is that we lose some control over the residues of the primitive graphs. Chapter 7 applies the previous chapters to derive (1.1).

Chapter 8 considers (1.1) as a system of recursions. It is devoted to the result of [25] where we bound the radii of convergence of the Borel transforms of the γ_1^r in terms of those of P_r. For systems with nonnegative coefficients we determine the radius exactly as $\min\{\rho_r, 1/b_1\}$, where ρ_r is the radius of the Borel transform of P_r, the instanton radius, and b_1 the first coefficient of the β-function[1]. In particular this means that a Lipatov bound[2] for the superficially convergent Green functions leads to a Lipatov bound for the superficially divergent Green functions. This generalizes and mathematizes similar results obtained in particular cases, such as ϕ^4, through quite different means by constructive field theory [13]. Both approaches require estimates on the convergent Green functions which can also be obtained in some cases from constructive field theory, for example [26].

Chapter 9 considers (1.1) as a system of differential equations. We are not able to prove any non-trivial results, and so simply discuss some tantalizing features of vector field plots of some important examples. More substantial results will appear in [31].

[1]This is the physicists' β-function, see Section 4.1, not the Euler β function.

[2]A Lipatov bound for $\sum d_n n^k$ means that $|d_n| \leq c^n n!$ for some c.

Chapter 2

Background

2.1 Series

Definition 2.1. If $\{a_n\}_{n\geq 0}$ is a sequence then $\mathbf{A}(x) = \sum_{n\geq 0} a_n x^n$ is its (ordinary) generating function and $\sum_{n\geq 0} a_n x^n/n!$ is its exponential generating function.

Bold capital letters are used for the ordinary generating function for the sequence denoted by the corresponding lower case letters. ρ will often denote a radius of convergence.

We will make use of the standard combinatorial notation for extracting coefficients.

Definition 2.2. If $\mathbf{A}(x) = \sum_{n\geq 0} a_n x^n$ then $[x^n]\mathbf{A}(x) = a_n$.

Definition 2.3. Call a power series $\sum_{k\geq 0} a(k)x^k$ *Gevrey-n* if $\sum_{k\geq 0} x^k a(k)/(k!)^n$ has nonzero radius of convergence.

For example, a convergent power series is Gevrey-0 and $\sum_{k\geq 0}(xk)^k$ is Gevrey-1 due to Stirling's formula. Trivially, a series which is Gevrey-n is also Gevrey-m for all $m \geq n$.

Gevrey-1 series are important in perturbative quantum field theory since being Gevrey-1 is necessary (but not sufficient) for Borel resummation. Resummation and resurgence are an enormous topic which will not be touched further herein; one entry point is [29]. Generally very little is known about the growth rates of the series appearing in perturbation theory. They are usually thought to be divergent, though this is questioned by some [11], and hoped to be Borel resummable.

2.2 Feynman graphs as combinatorial objects

Feynman graphs are graphs, with multiple edges and self loops permitted, made from a specified set of edge types, which may include both directed and undirected edges, with a specified set of permissible edge types which can meet at any given vertex. Additionally there are so-called external edges, weights for calculating the degree of divergence, and there may be additional colorings or orderings as necessary.

There are many possible ways to set up the foundational definitions, each with sufficient power to fully capture all aspects of the combinatorial side of Feynman graphs. However it is worth picking a setup which is as clean and natural as possible.

For the purposes of this thesis graphs are formed out of half edges. This naturally accounts for external edges and symmetry factors and permits oriented and unoriented edges to be put on the same footing.

Definition 2.4. A *graph* consists of a set H of half edges, a set V of vertices, a set of vertex - half edge adjacency relations ($\subseteq V \times H$), and a set of half edge - half edge adjacency relations ($\subseteq H \times H$), with the requirements that each half edge is adjacent to at most one other half edge and to exactly one vertex.

Graphs are considered up to isomorphism.

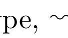
Definition 2.5. Half edges which are not adjacent to another half edge are called *external edges*. Pairs of adjacent half edges are called *internal edges*.

Definition 2.6. A *half edge labelling* of a graph with half edge set H is a bijection $H \to \{1, 2, \ldots, |H|\}$. A graph with a half edge labelling is called a *half edge labelled graph*.

2.2.1 Combinatorial physical theories

Feynman graphs will be graphs with extra information and requirement. In order to define this extra structure we need to isolate the combinatorial information that the physical theory, such as quantum electrodynamics (QED), scalar ϕ^4, or quantum chromodynamics (QCD), requires of the graph.

Each edge in the graph corresponds to a particle and a given physical theory describes only certain classes of particles, hence the physical theory determines a finite set of permissible *edge types*. For our half edge based setup, an edge type E consists of two, not necessarily distinct, *half edge types*, with the restriction that each half edge type appears in exactly one edge type. An edge composed of two adjacent half edges, one of each half edge type in E, is then an edge of type E. An edge type made up of the same half edge type twice is called an *unoriented edge type*. An edge type made up of two distinct half edge types is called an *oriented edge type*. The half edge types themselves contain no further structure and thus can be identified with $\{1, \ldots, n\}$ for appropriate n.

For example in QED there are two edge types, an unoriented edge type,  , representing a photon, and an oriented edge type, , representing an electron or positron[1]. At the level of half edge types we thus have a half photon, a front half electron, and a back half electron.

Each vertex in the graph corresponds to an interaction of particles and only certain interactions are permitted in a given physical theory, hence the physical theory also determines a set of permissible *vertex types*. A vertex type V consists of a multiset of half edge types with $3 \leq |V| < \infty$. A vertex in a graph which is adjacent to half edges one of each half edge type in V is then a vertex of type V. For example in QED there is one type of vertex,  .

The physical theory determines a formal integral expression for each graph by associating a factor in the integrand to each edge and vertex according to their type. This map is called the Feynman rules, see subsection 2.3.1. On the combinatorial side the only part of the Feynman rules we need is the net degree of the integration variables appearing in the factor of the integrand

[1]If we chose a way for time to flow through the graph then the edge would represent an electron or positron depending on whether it was oriented in the direction of time or not. However part of the beauty of Feynman graphs is that both combinatorially and analytically they do not depend on a flow of time.

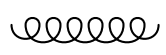
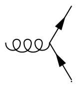
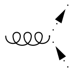
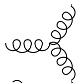
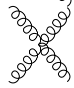
name	graph	weight
gluon		2
fermion		1
ghost		1
		0
		0
		-1
		0

Table 2.1: Edge and vertex types in QCD with power counting weights

associated to each type. Traditionally this degree is taken with a negative sign; specifically for a factor N/D this net degree is $\deg(D) - \deg(N)$, which we call the *power counting weight* of this vertex type or edge type.

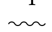
The other thing needed in order to determine the divergence or convergence of these integrals at large values of the integration variables, which will be discussed further in subsection 2.2.4, is the *dimension of space time*. We are not doing anything sophisticated here and this value will be a nonnegative integer, 4 for most theories of interest.

Thus we define,

Definition 2.7. A *combinatorial physical theory T* consists of a set of half edge types, a set of edge types with associated power counting weights, a set of vertex types with associated power counting weights, and a nonnegative integer dimension of space-time.

More typically the dimension of space-time is not included in the definition of the theory, and so one would say a theory T in dimension D to specify what we have called a physical theory.

Our examples will come from five theories

Example 2.8. QED describes photons and electrons interacting electromagnetically. As a combinatorial physical theory it has 3 half-edge types, a half-photon, a front half-electron, and a back half-electron. This leads to two edge types a photon,  , with weight 2, and an electron, , with weight 1. There is only one vertex consisting of one of each half-edge type and with weight 0. The dimension of space-time is 4.

Example 2.9. Quantum chromodynamics (QCD) is the theory of the interactions of quarks and gluons. As a combinatorial physical theory it has 5 half-edge types, a half-gluon, a front half-fermion, a back half-fermion, a front half-ghost, and a back half-ghost. There are 3 edge types and 4 vertex types with weights as described in Table 2.1. The dimension of space-time is again 4.

Example 2.10. ϕ^4, a scalar field theory, is the arguably the simplest renormalizable quantum field theory and is often used as an example in quantum field theory textbooks. As a combinatorial theory it consists of one half-edge type, one edge type, — , with weight 2, one vertex type,  , with weight 0, and space-time dimension 4.

Example 2.11. ϕ^3, also a scalar field theory, is another candidate for the simplest renormalizable quantum field theory. It is not as physical since to be renormalizable the dimension of space-time must be 6, and hence it is not as pedagogically popular. However the Feynman graphs in ϕ^3 are a little simpler in some respects and so it will be used here in longer examples such as Example 5.12. ϕ^3 consists of half-edges and edges as in ϕ^4 but the single vertex type, which has weight 0, is 3-valent.

Example 2.12. The final physical theory which we will use for examples is Yukawa theory in 4 dimensions, which has 3 half-edge types, a half-meson edge, a front half-fermion edge, and a back half-fermion edge. The edge types are a meson edge, ---- , with weight 2 and a fermion edge → , with weight 1. There is one vertex type,  , with weight 0. This example arises for us because of [5].

2.2.2 Feynman graphs

Notice that given a graph G, a combinatorial physical theory T, and a map from the half edge of G to the half edge types of T, there is at most one induced map from the internal edges of G to the edge types of T and at most one induced map from the vertices of G to the vertex types of T. Thus we can make the following definition.

Definition 2.13. A *Feynman graph* in a combinatorial physical theory T is

- a graph G,
- a map from the half edges of G to the half edge types of T which is compatible with the edges and vertices of G in the sense that it induces a map from the internal edges of G to the edge types of T and induces a map from the vertices of G to the vertex types of T, and
- a bijection from the external edges of G to $\{1, \dots n\}$ where n is the number of external edges.

The final point serves to fix the external edges of G, which is traditional among physicists.

Lemma 2.14. *Let G be a connected Feynman graph with n half edges. Let m be the number of half edge labelled Feynman graphs (up to isomorphism as labelled Feynman graphs) giving G upon forgetting the labelling, and let* Aut *be the automorphism group of G. Then*

$$\frac{m}{n!} = \frac{1}{|\mathrm{Aut}|}$$

Proof. Aut acts freely on the $n!$ half edge labellings of G. The orbits are the m isomorphism classes of half edge labellings. The result follows by elementary group theory. □

The primary consequence of Lemma 2.14 is that the exponential generating function for half-edge labelled graphs is identical to the generating function for Feynman graphs weighted with $1/|\mathrm{Aut}|$. $1/|\mathrm{Aut}|$ is known as the symmetry factor of the graph. Table 2.2 gives some examples.

We will be concerned from now on with Feynman graphs which are connected and which remain connected upon removal of any one internal edge, a property which physicists call *one particle irreducible* (1PI) and which combinatorialists call *2-edge connected*. Another way to look at this definition is that a 1PI graph is a unions of cycles and external edges. We'll generally be interested in Feynman graphs with each connected component 1PI.

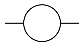
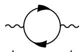
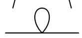
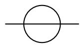
graph	symmetry factor
	$\frac{1}{2}$
	1
	$\frac{1}{4}$
	$\frac{1}{2}$
	$\frac{1}{6}$

Table 2.2: Examples of symmetry factors

2.2.3 Operations

For us subgraphs are always *full* in the sense that all half edges adjacent to a vertex in a subgraph must themselves be in the subgraph.

The most important operations are contraction of subgraphs and insertion of graphs. To set these definitions up cleanly we need a preliminary definition.

Definition 2.15. The set of external edges of a connected Feynman graph is called the *external leg structure* of the Feynman graph. The set of half edge types associated to the external edges of a Feynman graph can be identified with at most one edge or vertex type. This edge or vertex type, if it exists, is also called the *external leg structure*.

Definition 2.16. Let G be a Feynman graph in a theory T, γ a connected subgraph with external leg structure a vertex type V. Then the *contraction* of γ, denoted G/γ is the Feynman graph in T with

- vertex set the vertex set of G with all vertices of γ removed and a new vertex v of type V added,
- half edge set the half edge set of G with all half edges corresponding to internal edges of γ removed,

and with adjacencies induced from G along with the adjacency of the external edges of γ with v.

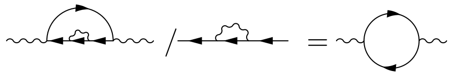
Definition 2.17. Let G be a Feynman graph in a theory T, γ a connected subgraph with external leg structure an edge type E. Then the *contraction* of γ, denoted G/γ is the Feynman graph in T with

- vertex set the vertex set of G with all vertices of γ removed,
- half edge set the half edge set of G with all the half edges of γ removed,

and with the induced adjacencies from G along with the adjacency of the two half edges adjacent to the external edges of γ if they exist.

Definition 2.18. Let G be a Feynman graph in a theory T, γ a not necessarily connected subgraph with the external leg structure of each connected component an edge or vertex type in T. Then the *contraction* of γ, also denoted G/γ is the graph resulting from contracting each connected component of γ.

For example in QED



Also useful is the operation of inserting a subgraph, which is the opposite of contracting a subgraph.

Definition 2.19. Let G and γ be Feynman graphs in a theory T with γ connected. Suppose γ has external leg structure a vertex type and let v be a vertex of G of the same type. Let f be a bijection from the external edges of γ to the half edges adjacent to v preserving half edge type. Then $G \circ_{v,f} \gamma$ is the graph consisting of

- the vertices of G except for v, disjoint union with the vertices of γ,
- the half edges of G and those of γ with the identifications given by f,

with the induced adjacencies from G and γ.

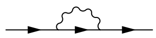
Definition 2.20. Let G and γ be Feynman graphs in a theory T with γ connected. Suppose γ has external leg structure an edge type and let e be an edge of G of the same type. Let f be a bijection from the external edges of γ to the half edges composing e, such that if a is an external edge of G then $(a, f(a))$ is a permissible half edge - half edge adjacency. Then $G \circ_{e,f} \gamma$ is the graph consisting of

- the vertices of G disjoint union with the vertices of γ,
- the half edges of G disjoint union with those of γ,

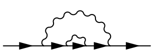
with the adjacency of a and $f(a)$ for each external edge a of γ along with the induced adjacencies from G and γ.

The vertices and edges of G viewed as above are called *insertion places.*

For example if

$G = \gamma =$ 

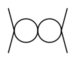
then there is only one possible insertion place for γ in G, namely the bottom internal edge e of G, and there is only one possible map f. Thus

$G \circ_{e,f} \gamma =$  .

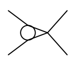
On the other hand if

$G = \gamma =$ 

then there are 2 possible insertion places for γ in G, namely the right vertex and the left vertex. Let e be the left vertex. Then there are also 4! possibilities for f, however 8 of them give

$G \circ_{f,g} \gamma =$ 

and 16 of them give

$G \circ_{f,g} \gamma =$  .

Proposition 2.21. *1. Contracting any subgraph γ of a 1PI graph G results in a 1PI graph.*

2. Inserting a 1PI graph γ into a 1PI graph G results in a 1PI graph.

Proof. 1. Without loss of generality suppose γ is connected. Suppose the result does not hold and e is an internal edge in $\Gamma = G/\gamma$ which disconnects Γ upon removal. Since G is 1PI, e cannot be an internal edge of G and hence must be the insertion place for γ in Γ. However then removing either half edge of e from G would disconnect G which is also impossible.

2. Suppose e is an internal edge in $\Gamma = G \circ \gamma$. Removing e removes at least one internal half edge of G or of γ which cannot disconnect either since both are themselves 1PI, and hence cannot disconnect Γ.

□

2.2.4 Divergence

For a 1PI Feynman graph G and a physical theory T let $w(a)$ be the power counting weight of a where a is an edge or a vertex of G and let D be the dimension of space-time. Then the *superficial degree of divergence* is

$$D\ell - \sum_e w(e) - \sum_v w(v)$$

where ℓ is the loop number of the graph, that is, the number of independent cycles. If the superficial degree of divergence of a graph is nonnegative we say the graph is *divergent*. It is the divergent graphs and subgraphs which we are primarily interested in.

The notion of superficial divergence comes from the fact that the Feynman rules associate to a graph a formal integral, as will be explained in subsection 2.3.1; the corresponding weights $w(a)$ give the degree in the integration variables of the inverse of each factor of the integrand, while the loop number ℓ gives the number of independent integration variables, each running over $\mathbb{R}^D$. Thus the superficial degree of divergence encodes how badly the integral associated to the graph diverges for large values of the integration variables. The adjective *superficial* refers to the fact that the integral may have different, potentially worse, behavior when some subset of the integration variables are large, hence the importance of divergent subgraphs.

In this context we say a theory T (in a given dimension) is *renormalizable* if graph insertion within T does not change the superficial degree of divergence of the graph.

A theory being renormalizable means more than that the integrals associated to the graphs of the theory can be renormalized in the sense of Subsection 2.3.2. In fact even if insertion increases the superficial degree of divergence, and so the theory is called *unrenormalizable*, the individual graphs can typically still be renormalized. Rather, a theory being renormalizable refers to the fact that the theory as a whole can be renormalized, all of its graphs at all loop orders, without introducing more than finitely many new parameters. Combinatorially this translates into the fact that there are finitely many families of divergent graphs, typically indexed by external leg structures. In the unrenormalizable case by contrast there are infinitely many families of divergent graphs and, correspondingly, to renormalize the whole theory would require infinitely many new parameters.

The interplay of renormalizability and dimension explains our choices for the dimension of space-time in our examples. In particular ϕ^4, QED, and QCD are all renormalizable in 4 dimensions and ϕ^3 is renormalizable in 6 dimensions.

By viewing a divergent graph in terms of its divergent subgraphs we see a structural self-similarity. This insight leads to the recursive equations which are the primary object of interest in this thesis.

Another useful definition is

Definition 2.22. Suppose G is a Feynman graph and γ and τ are divergent subgraphs. Then γ and τ are *overlapping* if they have internal edges or vertices in common, but neither contains the other.

2.2.5 The Hopf algebra of Feynman graphs

The algebra structure on divergent 1PI Feynman graphs in a given theory is reasonably simple.

Definition 2.23. Let $\mathcal{H}$ be the vector space formed by the $\mathbb{Q}$ span of disjoint unions of divergent 1PI Feynman graphs including the empty graph denoted $\mathbb{I}$.

Proposition 2.24. *$\mathcal{H}$ has an algebra structure where multiplication $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is given by disjoint union and the unit by $\mathbb{I}$.*

Proof. This multiplication can immediately be checked to be commutative and associative with unit $\mathbb{I}$, and to be a linear map. ☐

Another way to look at this is that as an algebra $\mathcal{H}$ is the polynomial algebra over $\mathbb{Q}$ in divergent 1PI Feynman graphs with the multiplication viewed as disjoint union. Note that we are only considering one graph with no cycles (the empty graph $\mathbb{I}$); from the physical perspective this means we are normalizing all the tree-level graphs to 1.

We will use the notation $e : \mathbb{Q} \to \mathcal{H}$ for the unit map $e(q) = q\mathbb{I}$. Also useful is the notation $\mathcal{H}_{\text{lin}} \subset \mathcal{H}$ for the $\mathbb{Q}$ span of connected nonempty Feynman graphs in $\mathcal{H}$ and $P_{\text{lin}} : \mathcal{H} \to \mathcal{H}_{\text{lin}}$ for the corresponding projection. That is $\mathcal{H}_{\text{lin}}$ is the parts of degree 1. Note that $\mathcal{H}$ is graded by the number of independent cycles in the graph, which is known as the loop number of the graph. This grading, not the degree as a monomial, is the more relevant in most circumstances.

The coalgebra structure encodes, as is common for combinatorial Hopf algebras, how the objects decompose into subobjects.

Definition 2.25. The coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is defined on a connected Feynman graph Γ by

$$\Delta(\Gamma) = \sum_{\substack{\gamma \subseteq \Gamma \\ \gamma \text{ product of divergent} \\ \text{1PI subgraphs}}} \gamma \otimes \Gamma/\gamma$$

and extended to $\mathcal{H}$ as an algebra homomorphism.

Note that the sum in the definition of Δ includes the cases $\gamma = \mathbb{I}$ and $\gamma = \Gamma$, since Γ is divergent and 1PI, hence includes the terms $\mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I}$. Note also that γ may be a product, that is a disjoint union. This is typically intended in presentations of this Hopf algebra, but not always clear.

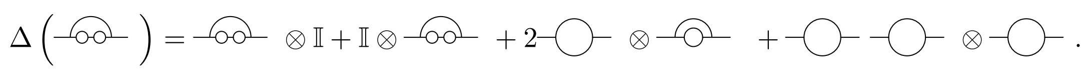
Definition 2.26. Let $\eta : \mathcal{H} \to \mathbb{Q}$ be the algebra homomorphism with $\eta(\mathbb{I}) = 1$ and $\eta(G) = 0$ for G a non-empty connected Feynman graph.

Proposition 2.27. *$\mathcal{H}$ has a coalgebra structure with coproduct Δ and counit η as above.*

Proof. We will verify only coassociativity. Calculate $(\mathrm{id}\otimes\Delta)\Delta\Gamma = \sum_{\gamma'} \gamma' \otimes \Delta(\Gamma/\gamma') = \sum_{\gamma'}\sum_{\gamma} \gamma' \otimes \gamma/\gamma' \otimes \Gamma/\gamma$ where $\gamma' \subseteq \gamma \subseteq \Gamma$ with each connected component of γ' and γ/γ' 1PI divergent. This calculation holds because every subgraph of Γ/γ' is uniquely of the form γ/γ' for some $\gamma' \subseteq \gamma \subseteq \Gamma$. Further by Proposition 2.21 and renormalizability each connected component of γ is 1PI divergent, so we can switch the order of summation to see that the above sum is simply $(\Delta \otimes \mathrm{id})\Delta\Gamma$ giving coassociativity. □

From now on we will only be concerned with the sort of Feynman graphs which appear in $\mathcal{H}$, that is, Feynman graphs with connected components which are divergent and 1PI.

$\mathcal{H}$ is graded by the loop number, that is the first Betti number. $\mathcal{H}$ is commutative but not in general cocommutative. For example in ϕ^3 theory



Definition 2.28. For $f_1, f_2 : \mathcal{H} \to \mathcal{H}$ define the convolution $f_1 \star f_2 = m(f_1 \otimes f_2)\Delta$

We will use the notation id for the identity map $\mathcal{H} \to \mathcal{H}$.

Proposition 2.29. *With antipode $S : \mathcal{H} \to \mathcal{H}$ defined recursively by $S(\mathbb{I}) = \mathbb{I}$ and*

$$S(\Gamma) = -\Gamma - \sum_{\substack{\gamma \subseteq \Gamma \\ \mathbb{I} \neq \gamma \neq \Gamma \\ \gamma \text{ product of divergent} \\ \text{1PI subgraphs}}} S(\gamma)\,\Gamma/\gamma$$

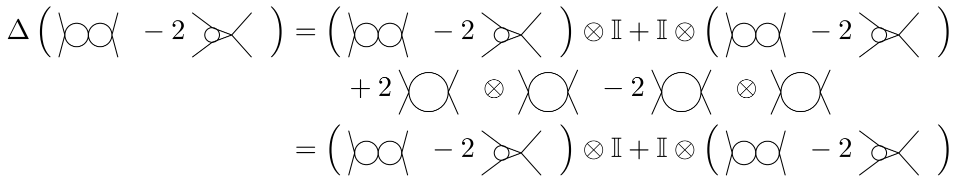
on connected graphs, and extended to all of $\mathcal{H}$ as an antihomomorphism, $\mathcal{H}$ is a Hopf algebra

Proof. The defining property of the antipode is $e\eta = S\star\mathrm{id} = \mathrm{id}\star S$. The first equality gives exactly the proposition in view of the definitions of Δ and $\star$, the second equality is then standard since $\mathcal{H}$ is commutative, see for instance [30, Proposition 4.0.1]. □

Note that since $\mathcal{H}$ is commutative S is in fact a homomorphism. S is not, however, an interesting antipode from the quantum groups perspective since $\mathcal{H}$ is commutative and thus $S \circ S = \mathrm{id}$ (see again [30, Proposition 4.0.1]).

Definition 2.30. An element γ of $\mathcal{H}$ is *primitive* if $\Delta(\gamma) = \gamma \otimes \mathbb{I} + \mathbb{I} \otimes \gamma$.

A single Feynman graph is primitive iff it has no divergent subgraphs. However appropriate sums of nonprimitive graphs are also primitive. For example



This phenomenon will be important in Chapter 5.

We will make sparing but important use of the Hochschild cohomology of $\mathcal{H}$. To define the Hochschild cohomology we will follow the presentation of Bergbauer and Kreimer [1]. The n-cochains are linear maps $L : \mathcal{H} \to \mathcal{H}^{\otimes n}$. The coboundary operator b is defined by

$$bL = (\mathrm{id} \otimes L)\Delta + \sum_{i=1}^{n} (-1)^i \Delta_i L + (-1)^{n+1} L \otimes \mathbb{I}$$

where $\Delta_i = \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta \otimes \mathrm{id} \otimes \cdots \mathrm{id}$ with the Δ appearing in the ith slot. $b^2 = 0$ since Δ is coassociative and so we get a cochain complex and hence cohomology. The only part of the Hochschild cohomology which will be needed below are the 1-cocycles $L : \mathcal{H} \to \mathcal{H}$, whose defining property $bL = 0$ gives

$$\Delta L = (\mathrm{id} \otimes L)\Delta + L \otimes \mathbb{I}. \tag{2.1}$$

2.3 Feynman graphs as physical objects

2.3.1 Feynman rules

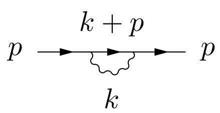
The information in the Feynman rules is the additional piece of analytic information contained in a physical theory, so for us we can define a physical theory to be a combinatorial physical theory along with *Feynman rules*. In the following definition we will use the term *tensor expression* for a tensor written in terms of the standard basis for $\mathbb{R}^D$ where D is the dimension of space-time. Such expressions will be intended to be multiplied and then interpreted with Einstein summation. An example of a tensor expression in indices μ and ν is

$$\frac{g_{\mu,\nu} - \xi \frac{k_\mu k_\nu}{k^2}}{k^2}$$

where g is the Euclidean metric, $k \in \mathbb{R}^4$, k^2 is the standard dot product of k with itself, and ξ is a variable called the *gauge*. Such a tensor expression is meant to be a factor of a larger expression like

$$\gamma_\mu \frac{1}{\not{k} + \not{p} - m} \gamma_\nu \left(\frac{g_{\mu,\nu} - \xi \frac{k_\mu k_\nu}{k^2}}{k^2} \right) \tag{2.2}$$

where the γ_μ are the Dirac gamma matrices, $\not{k}$ is the Feynman slash notation, namely $\not{k} = \gamma^\mu k_\mu$, and m is a variable for the mass. In this example (2.2) is the integrand for the Feynman integral for the graph



Definition 2.31. Let T be a combinatorial physical theory with dimension of space-time D. Let ξ be a real variable. *Feynman rules* consist of 3 maps

1. the first takes a half edge type (viewed as an external edge), an $\mathbb{R}^D$ vector (the momentum), and a tensor index μ to a tensor expression in μ,

2. the second takes an edge type e, an $\mathbb{R}^D$ vector (the momentum), and tensor indices μ, ν for each half edge type making up e to a tensor expression in μ, ν,
3. the third takes a vertex type v and one tensor index $\mu_1, \mu_2, \ldots$ for each half edge type making up v to a tensor expression in $\mu_1, \mu_2, \ldots$.

In each case the tensor expressions may depend on ξ.

If there is a non-trivial dependence on ξ in the Feynman rules then we say we are working in a *gauge theory*. QED and QCD are gauge theories. If the Feynman rules are independent of the tensor indices then we say we are working in a *scalar field theory*. ϕ^4 and ϕ^3 are scalar field theories. Note that unoriented edges have no way to distinguish their two tensor indices and hence must be independent of them. For us the Feynman rules do not include a dependence on a coupling constant x since we wish to use x at the level of Feynman graphs as an indeterminate in which to write power series. This setup ultimately coincides with the more typical situation because there the dependence of the Feynman rules on x is contrived so that it ultimately counts the loop number of the graph and so functions as a counting variable.

Using the Feynman rules we can associate to each graph γ in a theory T a formal integral, that is, an integrand and a space to integrate over but with no assurances that the resulting integral is convergent. We will denote the integrand by Int_γ and take it over a Euclidean space $\mathbb{R}^{D|v_\gamma|}$ where D is the dimension of space time and v_γ is a finite index set corresponding to the set of integration variables appearing in Int_γ. Then the formal integral is given by

$$\int_{\mathbb{R}^{D|v_\gamma|}} \mathrm{Int}_\gamma \prod_{k \in v_\gamma} d^D k$$

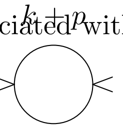
where D is the dimension of space-time in T and where Int_γ and v_γ are defined below.

Associate to each half edge of γ a tensor index. Associate to each internal and external edge of γ a variable (the momentum, with values in $\mathbb{R}^D$) and an orientation of the edge with the restriction that for each vertex v the sum of the momenta of edges entering v equals the sum of the momenta of edges exiting v. Consequently the $\mathbb{R}$-vector space of the edge variables has dimension the loop number of the graph. Let v_γ be a basis of this vector space. Let Int_γ be the product of the Feynman rules applied to the type of each external edge, internal edge, and vertex of γ, along with the assigned tensor indices and the edge variables as the momenta.

Note that Int_γ depends on the momenta $q_1, \ldots, q_n$ for the external edges and that these variables are not "integrated out" in the formal integral. Consequently we may also use the notation $\mathrm{Int}_\gamma(q_1, \ldots, q_n)$ to show this dependence. The factors associated to internal edges are called *propagators*.

In practice the integrals we obtain in this way are not arbitrarily bad in their divergence. In fact for arbitrary $\Lambda < \infty$ each will converge when integrated over a box consisting of all parameters running from $-\Lambda$ to Λ.

For example consider ϕ^4 with Euclidean Feynman rules, see [18, p.268]. The Feynman rules in this case say that an edge labelled with momentum k is associated to the factor $1/(k^2 + m^2)$, where the square of a vector means the usual dot product with itself and m is the mass of the particle. The Feynman rules say that the vertex is associated to -1 (if the coupling constant λ was included in the Feynman rules the vertex would be associated with $-\lambda$.) Consider

$\gamma =$ 

oriented from left to right with the momenta associated to the two right hand external edges summing to p and hence the momenta associated to the two left hand external edges also summing to p. Then the integral associated to γ is

$$\int d^4k \frac{1}{(k^2+m^2)((p+k)^2+m^2)}$$

where $d^4k = dk_0dk_1dk_2dk_3$ with $k = (k_0, k_1, k_2, k_3)$ and squares stand for the standard dot product.

The above discussion of Feynman rules is likely to appear either unmotivated or glib depending on one's background, particularly the rather crass gloss of gauge theories, so it is worth briefly mentioning a few important words of context.

More typically a physical theory might be defined by its Lagrangian $\mathcal{L}$. For example for ϕ^4

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

There is one term for each vertex and edge of the theory and for massive particles an additional term. In this case $\frac{1}{2}\partial^\mu\phi\partial_\mu\phi$ is the term for the edge of ϕ^4, $-\frac{1}{2}m^2\phi^2$ is the mass term, and $-\frac{\lambda}{4!}\phi^4$ is the term for the vertex. One of the many important properties of the Lagrangian is that it is Lorentz invariant.

The Feynman rules can be derived from the Lagrangian in a variety of ways to suit different tastes, for instance directly [12, p.16], or by expanding the path integral in the coupling constant.

Gauge theories are a bit more complicated since they are defined on a fibre bundle over space-time rather than directly on space-time. The structure group of the fibre bundle is called the gauge group. A *gauge field* (for example the photon in QED or the gluon in QCD) is a connection. A *gauge* is a local section. Choosing a gauge brings us back to something closer to the above situation.

There are many ways to choose a gauge each with different advantages and disadvantages. For the present purpose we're interested in a 1-parameter family of Lorentz covariant gauges called the R_ξ gauges. The parameter for the family is denoted ξ and is the ξ which we have called the gauge in the above. The R_ξ gauges can be put into the Lagrangian in the sense that in these gauges we can write a Lagrangian for the theory which depends on ξ. For example, for QED in the R_ξ gauges we have (see for example [6, p.504])

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\psi$$

where the γ^μ are the Dirac gamma matrices. Whence ξ also appears in the Feynman rules, giving the definition of gauge theory used above.

Another perspective, perhaps clearer to many mathematicians is Polyak [27].

2.3.2 Renormalization

Definition 2.32. Let

$$I = \int_{\mathbb{R}^{D|v|}} \text{Int} \prod_{k\in v} d^Dk$$

be a formal integral. I is *logarithmically divergent* if the net degree (that is the degree of the numerator minus the degree of the denominator) of the integration variables in Int is $-D|v|$. I *diverges like an nth power* (or, is *linearly divergent*, *quadratically divergent*, etc.) if the net degree of the integration variables in Int is $-D|v| + n$.

Let ϕ be the Feynman rules viewed as map which associates formal integrals to elements of $\mathcal{H}$. Next we need a method (called renormalization) which can convert the formal integrals for primitive graphs into convergent integrals. There are many possible choices; commonly first a regularization scheme is chosen to introduce one or more additional variables which convert the formal integrals to meromorphic expressions with a pole at the original point. For instance one may raise propagators to non-integer powers (analytic regularization) or take the dimension of space-time to be complex (dimensional regularization, see for instance [8] on setting up the appropriate definitions). Then a map such as minimal subtraction is chosen to remove the pole part.

We will take a slightly different approach. First we will set

$$\int (k^2)^r = 0 \tag{2.3}$$

for all r. This is the result which is obtained, for instance, from dimensional regularization and from analytic regularization, but simply taking it as true allows us to remain agnostic about the choice of regularization scheme. To see the origin of this peculiar identity consider the following computation with $q \in \mathbb{R}^D$ and the square of an element of $\mathbb{R}^D$ denoting its dot product with itself.

$$\begin{aligned}
&\int d^Dk \frac{1}{(k^2)^r((k+q)^2)^s} \\
&= \int d^Dk \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 dx \frac{x^{r-1}(1-x)^{s-1}}{(xk^2+(1-x)(k+q)^2)^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 dx x^{r-1}(1-x)^{s-1} \int d^Dk \frac{1}{\left(xk^2+(1-x)(k+q)^2\right)^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 dx x^{r-1}(1-x)^{s-1} \int d^Dk \frac{1}{\left((k+q(1-x))^2+q^2(x-x^2)\right)^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 dx x^{r-1}(1-x)^{s-1} \int d^Dk \frac{1}{(k^2+q^2(x-x^2))^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 dx x^{r-1}(1-x)^{s-1} \int_0^\infty d|k| \frac{|k|^{D-1}}{(|k|^2+q^2(x-x^2))^{r+s}} \int d\Omega_k \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \int_0^1 dx x^{r-1}(1-x)^{s-1} \int_0^\infty d|k| \frac{|k|^{D-1}}{(|k|^2+q^2(x-x^2))^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \frac{\Gamma(r+s-\frac{D}{2})\Gamma(\frac{D}{2})}{2\Gamma(r+s)} (q^2)^{\frac{D}{2}-r-s} \int_0^1 dx x^{\frac{D}{2}-1-s}(1-x)^{\frac{D}{2}-1-r} \\
&= \frac{\pi^{\frac{D}{2}}\Gamma(r+s-\frac{D}{2})}{\Gamma(r)\Gamma(s)} (q^2)^{\frac{D}{2}-r-s} \frac{\Gamma(\frac{D}{2}-r)\Gamma(\frac{D}{2}-s)}{\Gamma(D-r-s)}
\end{aligned}$$

when $2r+2s > D > 0$, $D > 2r > 0$, and $D > 2s > 0$, and where the first equality is by Feynman parameters:

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(ax+b(1-x))^{\alpha+\beta}} \qquad \text{for } \alpha, \beta > 0$$

and where $d\Omega_k$ refers to the angular integration over the unit $D-1$-sphere in $\mathbb{R}^D$. Now consider just the final line and suppose $s = 0$, then since Γ has simple poles precisely at the nonpositive integers, is never 0, and

$$\Gamma(x)\Gamma(-x) = \frac{-\pi}{x\sin(\pi x)}$$

we see that for $D > 0$ the result is 0 for $s = 0$ and r not a half-integer. If we view the original integral as a function of complex variables r and s for fixed integer D (analytic regularization), or as a function of complex D (dimensional regularization), then by analytic continuation the above calculations gives (2.3).

Returning to the question of renormalization, in view of (2.3) we need only consider logarithmically divergent integrals since by subtracting off 0 in the form of a power of k^2 which is equally divergent to the original integral the whole expression becomes less divergent. Logarithmically divergent integrals with no subdivergences can then be made finite simply by subtracting the same formal integral evaluated at fixed external momenta.

Let R be the map which given a formal integral returns the formal integral evaluated at the subtraction point. In our case then R has as domain and range the algebra of formal integrals where relations are generated by evaluating convergent integrals and (2.3). Let ϕ be the Feynman rules, the algebra homomorphism which given a graph G returns the formal integral $\phi(G)$. We suppose $\phi(\mathbb{I}) = 1$ and $R(1) = 1$.

If instead we had chosen to use a regulator and corresponding renormalization scheme then ϕ would give the regularized integral of a graph, and R would implement the scheme itself. One such example would be dimensional regularization with the minimal subtraction scheme. In that case ϕ would take values in the space of Laurent series in the small parameter ϵ and R would take such a Laurent series and return only the part with negative degree in ϵ. That is $R\phi(\Gamma)$ is the singular part of $\phi(\Gamma)$, the part one wishes to ignore. Note that in this case $R(1) = 0$. The key requirement in general is that R be a Rota-Baxter operator see [14], [15].

To deal with graphs containing subdivergences, define S_R^ϕ recursively by $S_R^\phi(\mathbb{I}) = 1$,

$$S_R^\phi(\Gamma) = -R(\phi(\Gamma)) - \sum_{\substack{\mathbb{I}\neq\gamma\subsetneq\Gamma \\ \gamma \text{ product of divergent} \\ \text{1PI subgraphs}}} S_R^\phi(\gamma)R(\phi(\Gamma/\gamma))$$

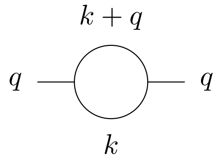
for connected Feynman graphs Γ extended to all of $\mathcal{H}$ as an algebra homomorphism. S_R^ϕ can be thought of as a twisted antipode; the defining recursion says that $S_R^\phi \star R\phi = \eta$. Use S_R^ϕ to define the *renormalized Feynman rules* by

$$\phi_R = S_R^\phi \star \phi.$$

When Γ contains no subdivergences, $\phi_R(\Gamma) = \phi(\Gamma) - R\phi(\Gamma)$; in view of Subsection 2.3.2 we may assume that $\phi(\Gamma)$ is log divergent and so $\phi_R(\Gamma)$ is a convergent integral. Inductively one can show that ϕ_R maps $\mathcal{H}$ to convergent integrals. This result is the original purpose of the Hopf algebraic approach to renormalization. It gives a consistent algebraic framework to the long-known but ad-hoc renormalization procedures of physicists. For more details and more history see for instance instance the survey [15] and the references therein.

These integrals lead to interesting transcendental numbers, but that is very much another story [2], [22], [3].

Example 2.33. To illustrate the conversion to log divergence and renormalization by subtraction consider the following graph in massless ϕ^3



The Feynman rules associate to it the integral

$$I = \frac{1}{q^2}\int d^6k \frac{1}{k^2(k+q)^2}.$$

The factor of $1/q^2$ is there because our conventions have that the graphs with no cycles are all normalized to 1. This integral is quadratically divergent and so can not be renormalized by a simple subtraction. However we take

$$\int d^6k \frac{1}{(k^2)^2} = 0,$$

so

$$\begin{aligned} I =& \frac{1}{q^2}\int d^6k \frac{1}{k^2(k+q)^2} - \frac{1}{q^2}\int d^6k \frac{1}{(k^2)^2} \\ =& -\frac{2}{q^2}\int d^6k \frac{k\cdot q}{(k^2)^2(k+q)^2} - \int d^6k \frac{1}{(k^2)^2(k+q)^2} \\ =& -2I_1 - I_2. \end{aligned}$$

Each of the two resulting terms are now less divergent.

To illustrate renormalization by subtraction consider the integral from the second of the above terms. As formal integrals (or carrying along the subtraction which we will add below), using the same tricks as the calculation earlier this section,

$$\begin{aligned} I_2 = \int d^6k \frac{1}{(k^2)^2(k+q)^2} =& \int d^6k \int_0^1 dx \frac{2x}{(xk^2 + (1-x)(k+q)^2)^3} \\ =& 2\int_0^1 dx x \int d^6k \frac{1}{\left(xk^2 + (1-x)(k+q)^2\right)^3} \\ =& 2\int_0^1 dx x \int d^6k \frac{1}{\left((k+q(1-x))^2 + q^2(x-x^2)\right)^3} \\ =& 2\int_0^1 dx x \int d^6k \frac{1}{(k^2 + q^2(x-x^2))^3} \\ =& 2\int_0^1 dx x \int_0^\infty d|k| \frac{|k|^5}{(|k|^2 + q^2(x-x^2))^3} \int d\Omega_k \\ =& 2\pi^3 \int_0^1 dx x \int_0^\infty d|k| \frac{|k|^5}{(|k|^2 + q^2(x-x^2))^3} \end{aligned}$$

Now consider the result of subtracting at $q^2 = \mu^2$. By Maple

$$I_2 - RI_2 = 2\pi^3 \int_0^1 dx x \int_0^\infty d|k| \frac{|k|^5}{(|k|^2 + q^2(x-x^2))^3} - \frac{|k|^5}{(|k|^2 + \mu^2(x-x^2))^3}$$

$$
\begin{aligned}
&= 2\pi^3 \int_0^1 dxx \left(-\frac{1}{2} \log(q^2(x-x^2)) + \frac{1}{2} \log(\mu^2(x-x^2)) \right) \\
&= -\frac{\pi^3}{2} \log(q^2/\mu^2)
\end{aligned}
$$

giving us a finite value.

To finish the example we need to consider the integral

$$
I_1 = \frac{1}{q^2} \int d^6k \frac{2k \cdot q}{(k^2)^2(k+q)^2}.
$$

This integral is linearly divergent so it needs another subtraction of 0. However, this time we only need

$$
\int d^6k \frac{2k \cdot q}{(k^2)^3} = 0.
$$

which we can derive from (2.3). Write $k = k_\perp + k_\parallel$ where $k_\parallel$ is the orthogonal projection of k onto $\mathrm{span}(q)$ and $k_\perp$ is the orthogonal complement, and notice that

$$
\begin{aligned}
\int d^6k \frac{2k \cdot q}{(k^2)^3} &= \int d^6k \frac{2k_\parallel |q|}{(k_\parallel^2 + k_\perp^2)^3} \\
&= \int d^5k_\perp \int_0^\infty dk_\parallel \frac{2k_\parallel |q|}{(k_\parallel^2 + k_\perp^2)^3} + \int d^5k_\perp \int_{-\infty}^0 dk_\parallel \frac{2k_\parallel |q|}{(k_\parallel^2 + k_\perp^2)^3} \\
&= \frac{|q|}{2} \int d^5k_\perp \frac{1}{(k_\perp^2)^2} - \frac{|q|}{2} \int d^5k_\perp \frac{1}{(k_\perp^2)^2} \\
&= 0 - 0 = 0.
\end{aligned}
$$

So returning to I_1, as formal integrals,

$$
\begin{aligned}
I_1 &= \frac{1}{q^2} \int d^6k \frac{2k \cdot q}{(k^2)^2(k+q)^2} - \frac{1}{q^2} \int d^6k \frac{2k \cdot q}{(k^2)^3} \\
&= -\frac{4}{q^2} \int d^6k \frac{(k \cdot q)^2}{(k^2)^3(k+q)^2} - \int d^6k \frac{k \cdot q}{(k^2)^3(k+q)^2}.
\end{aligned}
$$

The second term is convergent and so needs no further consideration. The first term is now log divergent, call it $-4I_3$. Writing $k = k_\perp + k_\parallel$ as above, we get

$$
\begin{aligned}
I_3 &= \frac{1}{q^2} \int d^6k \frac{(k \cdot q)^2}{(k^2)^3(k+q)^2} \\
&= \frac{1}{q^2} \int d^5k_\perp \int_{-\infty}^\infty dk_\parallel \frac{(k_\parallel |q|)^2}{(k_\parallel^2 + k_\perp^2)^3(k_\perp^2 + (k_\parallel + q)^2)} \\
&= \int_0^\infty d|k_\perp| \int_{-\infty}^\infty dk_\parallel \frac{k_\parallel^2 |k_\perp|^4}{(k_\parallel^2 + |k_\perp|^2)^3(|k_\perp|^2 + (k_\parallel + q)^2)} \int d\Omega_{k_\perp} \\
&= \frac{2\pi^{5/2}}{\Gamma(5/2)} \int_0^\infty d|k_\perp| \int_{-\infty}^\infty dk_\parallel \frac{k_\parallel^2 k_\perp^4}{(k_\parallel^2 + |k_\perp|^2)^3(|k_\perp|^2 + (k_\parallel + q)^2)}.
\end{aligned}
$$

The inner integral Maple can do, and then subtracting at $q^2 = \mu^2$ the outer integral is again within Maple's powers and we finally get a finite answer

$$I_3 - RI_3 = -\frac{1}{16}\pi \log(q^2/\mu^2).$$

Combining these various terms together we have finally computed $I - RI$. This completes this example.

Example 2.34. Subtracting 0 in this way also plays nicely with analytic regularization, and is less messy on top of it. Consider the example

$$\int d^4k \frac{k \cdot q}{(k^2)^{1+\rho_1}((k+q)^2)^{1+\rho_2}}$$

Then

$$\begin{aligned}\int d^4k \frac{k \cdot q}{(k^2)^{1+\rho_1}((k+q)^2)^{1+\rho_2}} &= \int d^4k \frac{k \cdot q}{(k^2)^{1+\rho_1}((k+q)^2)^{1+\rho_2}} - \frac{k \cdot q}{(k^2)^{2+\rho_1+\rho_2}} \\ &= \int d^4k \frac{k \cdot q\Big((k^2)^{1+\rho_2} - ((k+q)^2)^{1+\rho_2}\Big)}{(k^2)^{2+\rho_1+\rho_2}((k+q)^2)^{1+\rho_2}}\end{aligned}$$

which is merely log divergent and so can be renormalized by subtracting the same integrand at $q^2 = \mu^2$. This sort of example will be important later on, as we can simply take this integral with $q = 1$ as the Mellin transform which we need in Section 3.2.

Subtracting off zero in its various forms and subtracting at fixed momenta should not be confused. The former consists just of adding and subtracting zero and so can be done in whatever way is convenient. In the following we will assume that it has been done, and so that all integrals are log divergent. The latter, however, we will always explicitly keep track of. It is our choice of renormalization scheme and a different choice would give different results.

2.3.3 Symmetric insertion

For one of the upcoming reductions we will need to define a symmetric insertion with a single external momentum q^2. Let p be a primitive of $\mathcal{H}$, not necessarily connected. For the purposes of symmetric insertion define the Mellin transform F_p of p (see Section 3.2) as

$$F_p(\rho) = (q^2)^\rho \int \mathrm{Int}_p(q^2) \left(\frac{1}{|p|} \sum_{i=1}^{|p|} (k_i^2)^{-\rho} \right) \prod_{i=1}^{|p|} d^4k_i,$$

where $\mathrm{Int}_p(q^2)$ is the integrand determined by p. We'll renormalize by subtraction at $q^2 = \mu^2$ and let

$$\mathrm{Int}_p^-(q^2) = \mathrm{Int}_p(q^2) - \mathrm{Int}_p(\mu^2).$$

So define renormalized Feynman rules for this symmetric scheme with subtractions at $q^2 = \mu^2$ by

$$\phi_R(B_+^p(X))(q^2/\mu^2) = \int \mathrm{Int}_p^-(q^2) \left(\frac{1}{|p|} \sum_{i=1}^{|p|} \phi_R(X)(-k_i^2/\mu^2) \right) \prod_{i=1}^{|p|} d^4k_i.$$

We have

$$\phi_R(B_+^p(X))(q^2/\mu^2) = \lim_{\rho\to 0} \phi_R(X)(\partial_{-\rho})F_p(\rho)\left((q^2/\mu^2)^{-\rho} - 1\right),$$

where $\partial_{-\rho} = -\frac{\partial}{\partial\rho}$.

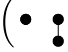
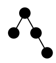
Chapter 3

Dyson-Schwinger equations

3.1 B_+

For γ a primitive Feynman graph, B_+^γ denotes the operation of insertion into γ. There are, however, a few subtleties which we need to address.

In the closely related Connes-Kreimer Hopf algebra of rooted trees [10], see Chapter 5, $B_+(F)$ applied to a forest F denotes the operation of constructing a new tree by adding a new root with children the roots of each tree from F. For example

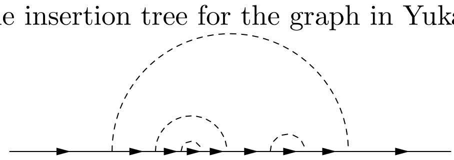
$B_+\Big($  $\Big) =$  .

B_+ in rooted trees is a Hochschild 1-cocycle [10, Theorem 2],

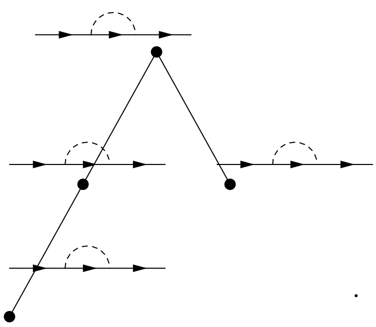
$$\Delta B_+ = (\mathrm{id} \otimes B_+)\Delta + B_+ \otimes \mathbb{I}.$$

This 1-cocycle property is key to many of the arguments below. The corresponding property which is desired of the various B_+ appearing in the Hopf algebras of Feynman graphs is that the sum of all B_+ associated to primitives of the same loop number and the same external leg structures is a Hochschild 1-cocycle.

In the case where all subdivergences are nested rather than overlapping, and where there is only one way to make each insertion, a 1PI Feynman graph Γ can be uniquely represented by a rooted tree with labels on each vertex corresponding to the associated subdivergence. Call such a tree an *insertion tree*. For example the insertion tree for the graph in Yukawa theory

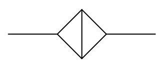


is

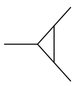


In such cases B_+^γ is the same operation as the B_+ for rooted trees (with the new root labelled by the new graph). So the 1-cocycle identity holds for B_+^γ too.

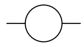
However in general there are many possible ways to insert one graph into another so the tree must also contain the information of which insertion place to use. Also when there are overlapping subdivergences different tree structures of insertions can give rise to the same graph. For example in ϕ^3 the graph



can be obtained by inserting



into



either at the right vertex or at the left vertex giving two different insertion trees. Provided any overlaps are made by multiple copies of the same graph, as in the previous example, then, since γ is primitive, the same tensor products of graphs appear on both sides of (2.1) but potentially with different coefficients. Note that this only requires γ to be primitive, not necessarily connected. Fortunately it is possible to make a choice of coefficients in the definition of B_+^γ which fixes this problem. This is discussed in the first and second sections of [21], and the result is the definition

Definition 3.1. For γ a connected Feynman graph define

$$B_+^\gamma(X) = \sum_{\Gamma \in \mathcal{H}_{\text{lin}}} \frac{\mathbf{bij}(\gamma, X, \Gamma)}{|X|_\vee} \frac{1}{\text{maxf}(\Gamma)} \frac{1}{(\gamma|X)} \Gamma$$

where $\text{maxf}(\Gamma)$ is the number of insertion trees corresponding to Γ, $|X|_\vee$ is the number of distinct graphs obtainable by permuting the external edges of X, $\mathbf{bij}(\gamma, X, \Gamma)$ is the number of bijections of the external edges of X with an insertion place of γ such that the resulting insertion gives Γ, and $(\gamma|X)$ is the number of insertion places for X in γ.

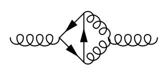
Extend B_+^γ linearly to all primitives γ.

Note that $B_+^\gamma(\mathbb{I}) = \gamma$. Also with the above definition we have B_+^γ defined even for nonprimitive graphs, but this was merely our approach to make the definition for primitives which are sums; now that the definitions are settled we will only consider B_+^γ for primitives.

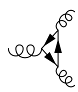
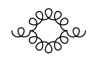
The messy coefficient in the definition of B_+^γ assures that if we sum all B_+^γ running over γ primitive 1PI with a given external leg structure (that is, over all primitives of the Hopf algebra which are single graphs and which have the given external leg structure), inserting into all insertion places of each γ, then each 1PI graph with that external leg structure occurs and is weighted by its symmetry factor. This property is [21, Theorem 4] and is illustrated in Example 3.4.

Gauge theories are more general in one way; there may be overlapping subdivergences with different external leg structures. Consequently we may be able to form a graph G by inserting one graph into another but in the coproduct of G there may be subgraphs and cographs completely different from those which we used to form G as in the following example.

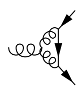
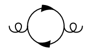
Example 3.2. In QCD



can be obtained by inserting

 into 

or by inserting

 into  .

This makes it impossible for every B_+^γ for γ primitive to be a Hochschild 1-cocycle since there may be graphs appearing on the right hand side of (2.1) which do not appear on the left. In these cases there are identities between graphs, known as Ward identities for QED and Slavnov-Taylor identities for QCD, which guarantee that $\sum B_+^\gamma$ is a 1-cocycle where the sum is over all γ with a given loop number and external leg structure. This phenomenon is discussed in [21] and the result is proved for QED and QCD by van Suijlekom [32].

For our purposes we will consider sets of B_+ operators,

$$\{B_+^{k,i;r}\}_{i=0}^{t_k^r}$$

where k is the loop number, r is an index for the external leg structure, and i is an additional index running over the primitive graphs under consideration with k loops and external leg structure r. In the case where there is only one r under consideration write $\{B_+^{k,i}\}_{i=0}^{t_k}$. Now assume that in this more general case, as in QED and QCD, that the required identities form a Hopf ideal so that by working in a suitable quotient Hopf algebra we get

Assumption 3.3. $\sum_{i=0}^{t_k^r} B_+^{k,i;r}$ is a Hochschild 1-cocycle.

3.2 Dyson-Schwinger equations

Consider power series in the indeterminate x with coefficients in $\mathcal{H}$ where x counts the loop number, that is the coefficient of x^k lives in the kth graded piece of $\mathcal{H}$. By *combinatorial Dyson-Schwinger equations* we will mean a recursive equation, or system of recursive equations, in such power series written in terms of insertion operations B_+. The particular form of combinatorial Dyson-Schwinger equation which we will be able to analyze in detail will be discussed further in section 3.3.

One of the most important examples is the case where the system of equations expresses the series of graphs with a given external leg structure in terms of insertion into all connected primitive graphs with that external leg structure. More specifically for a given primitive we insert into each of its vertices the series for that vertex and for each edge all possible powers of the series for that edge, that is, a geometric series in the series for that edge. The system of such equations generates all 1PI graphs of the theory.

Example 3.4. For QED the system to generate all divergent 1PI graphs in the theory is

$$X^{\sim\!\!<} = \mathbb{I} + \sum_{\substack{\gamma \text{ primitive with} \\ \text{external leg structure } \sim\!\!<}} x^{|\gamma|} B_+^{\gamma}\left(\frac{\left(X^{\sim\!\!<}\right)^{1+2k}}{\left(X^{\sim\sim\sim}\right)^{k}\left(X^{\rightarrow}\right)^{2k}}\right)$$

$$X^{\sim\sim\sim} = \mathbb{I} - xB_+^{\sim\bigcirc\sim}\left(\frac{\left(X^{\sim\!\!<}\right)^{2}}{\left(X^{\rightarrow}\right)^{2}}\right)$$

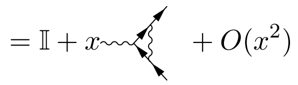
$$X^{\rightarrow} = \mathbb{I} - xB_+^{\rightarrow\frown\rightarrow}\left(\frac{\left(X^{\sim\!\!<}\right)^{2}}{X^{\sim\sim\sim}\,X^{\rightarrow}}\right).$$

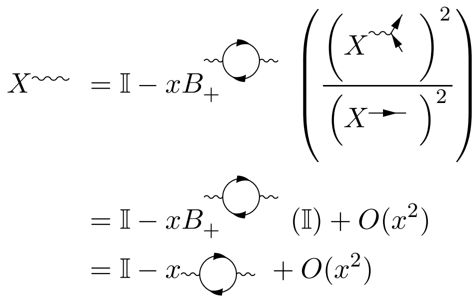
where $|\gamma|$ is the loop number of γ.

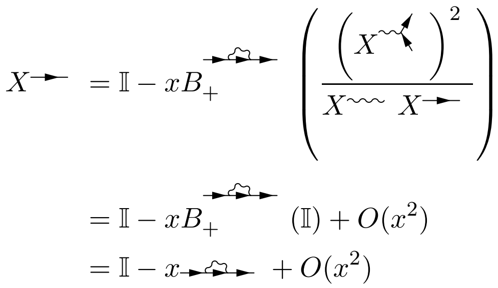
$X^{\sim\!\!<}$ is the vertex series. The coefficient of x^n in $X^{\sim\!\!<}$ is the sum of all 1PI QED Feynman graphs with external leg structure $\sim\!\!<$ and n independent cycles. In QED all graphs have symmetry factor 1 so this example hides the fact that in general each graph will appear weighted with its symmetry factor. $X^{\sim\sim\sim}$ and $X^{\rightarrow}$ are the two edge series. The coefficient of x^n for $n > 0$ in $X^{\sim\sim\sim}$ is minus the sum of all 1PI QED Feynman graphs with external leg structure $\sim\sim\sim$ and n independent cycles. The negative sign appears in the edge series because when we use these series we want their inverses; that is, we are interested in the series where the coefficient of x^n consists of products of graphs each with a given edge as external leg structure and with total loop number n. The arguments to each B_+^{γ} consist of a factor of the vertex series in the numerator for each vertex of γ, a factor of the photon edge series in the denominator for each photon edge of γ, and a factor of the electron edge series in the denominator for each electron edge of γ.

To illustrate these features lets work out the first few coefficients of each series. First work out the coefficient of x.

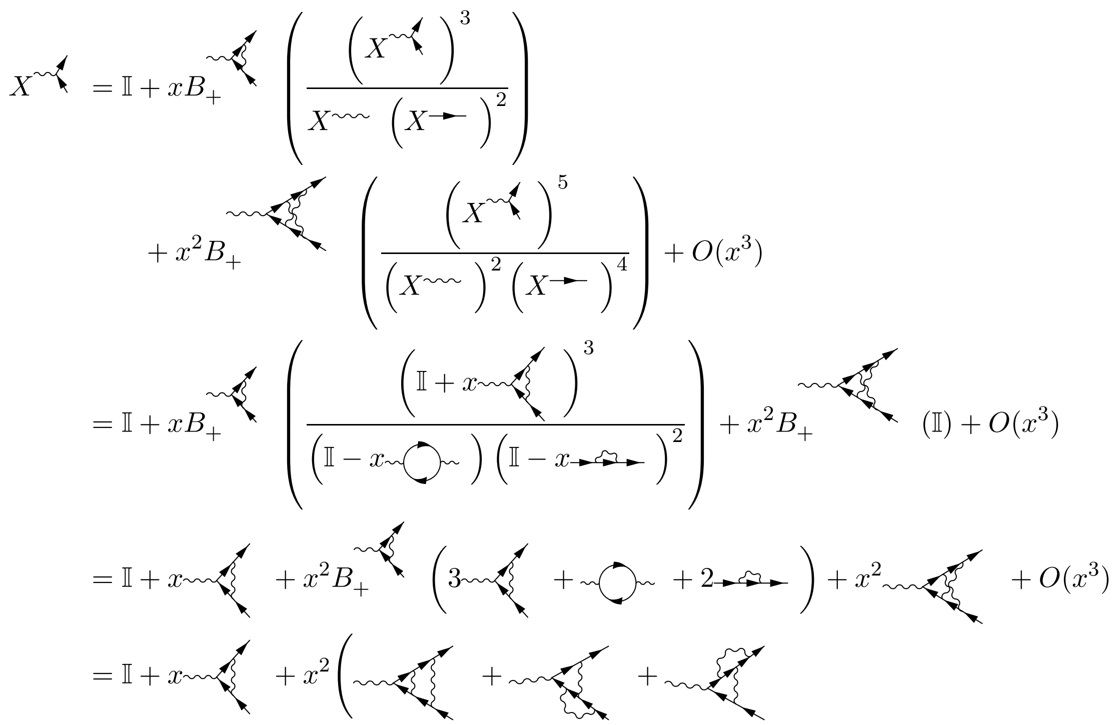
$$\begin{aligned} X^{\sim\!\!<} &= \mathbb{I} + xB_+^{\sim\!\!\lessdot}\left(\frac{\left(X^{\sim\!\!<}\right)^{3}}{X^{\sim\sim\sim}\left(X^{\rightarrow}\right)^{2}}\right) + O(x^2) \\ &= \mathbb{I} + xB_+^{\sim\!\!\lessdot}(\mathbb{I}) + O(x^2) \end{aligned}$$

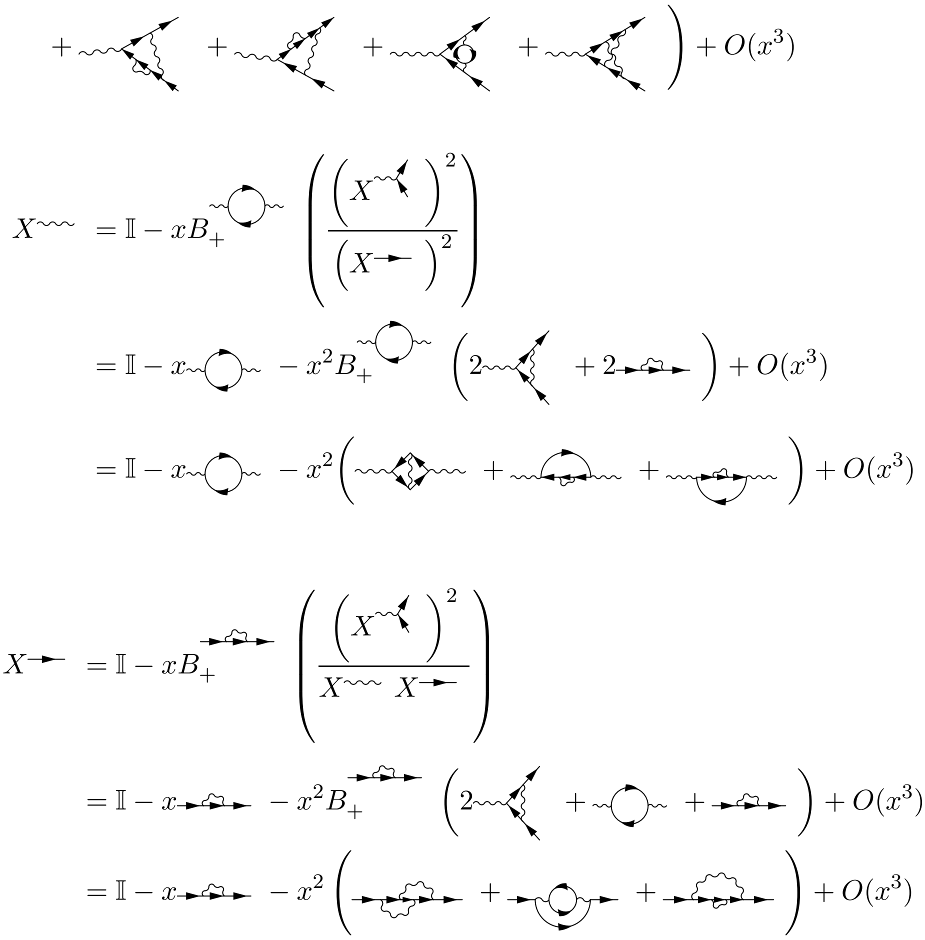






Next work out the coefficient of x^2.





The fact that

appear with coefficient 1 and not 2 is due to the two insertion trees contributing a 2 to the denominator in Definition 3.1.

By *analytic Dyson-Schwinger equations* we will mean the result of applying the renormalized Feynman rules to combinatorial Dyson-Schwinger equations. These are the Dyson-Schwinger equations which a physicist would recognize. The counting variable x becomes the physicists' coupling constant (which we will also denote x, but which might be more typically denoted α or g^2 depending on the theory). The Feynman rules also introduce one or more scale variables L_j which come from the external momenta q_i and the fixed momentum values μ_i used to renormalize by subtracting. In the case of one scale variable we have $L = \log q^2/\mu^2$. See Example 3.5. Note that in the case of more than one scale the L_j are not just $\log q_i^2/\mu^2$, but also include other expressions in the q_i and the μ_i, such as ratios of the q_i (such ratios are not properly speaking scales, but there is no need for a more appropriate name for them since we will quickly move to the case of one scale where this problem does not come up).

The functions of L_j and x appearing in analytic Dyson-Schwinger equations are called Green functions, particularly in the case where the Green functions are the result of applying the renormalized Feynman rules to the series of all graphs with a given external leg structure.

We can begin to disentangle the analytic and combinatorial information in the following way. Suppose we have a combinatorial Dyson-Schwinger equation, potentially a system. Suppose the series in Feynman graphs appearing in the Dyson-Schwinger equation are denoted X^r with $r \in \mathcal{R}$ some index set. Denote G^r the corresponding Green functions.

For each factor $(X^r)^s$ in the argument to some B_+^γ take the formal integrand and multiply it by $(G^r)^s$. For the scale arguments to these G^r use the momenta of the edges where the graphs of X^r are inserted. Then subtract this integral at the fixed external momenta μ_i as when renormalizing a single Feynman integral. Then the analytic Dyson-Schwinger equation has the same form as the combinatorial one but with G^r replacing X^r and with the expression described above replacing B_+^γ. Example 3.5 illustrates this procedure.

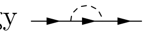
In the case with more than one scale the Green functions may depend on ratios of the different momenta, and we can progress no further in simplifying the setup. Fortunately, in the case with only one scale, which suffices to describe the general case in view of Chapter 5, we can further disentangle the analytic and combinatorial information as follows, see [24] for more details.

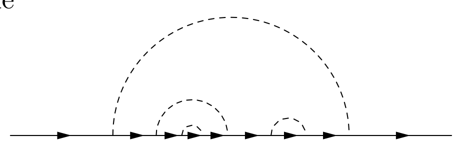
Suppose we have a combinatorial Dyson-Schwinger equation and a single scale. For each primitive graph γ appearing as a B_+^γ we have a formal integral expression

$$\int_{\mathbb{R}^{D|v|}} \mathrm{Int} \prod_{k \in v} d^D k$$

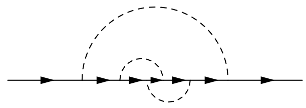
coming from the unrenormalized Feynman rules. Number the edges, say from 1 to n. Raise the factor associated to the ith edge to $1 + \rho_i$ where ρ_i is a new variable. We now have an analytically regularized integral which can be evaluated for suitable values of ρ_i. Finally set all external momenta to 1. Call the resulting function of $\rho_1, \ldots, \rho_n$ the Mellin transform $F_\gamma(\rho_1, \ldots, \rho_n)$ associated to γ. We are interested in F_γ near the origin.

Then, another way to see the analytic Dyson-Schwinger equation as coming from the combinatorial Dyson-Schwinger equation by replacing X^r with G^r and B_+^γ with F_γ. The factor with exponent ρ_i indicates the argument for the recursive appearance of the X^j which is inserted at the insertion place corresponding to edge i. This will be made precise for the cases of interest in the following section, and will be motivated by Example 3.7.

Example 3.5. Broadhurst and Kreimer in [5] discuss the Dyson-Schwinger equation for graphs from massless Yukawa theory where powers of the one loop fermion self energy  are inserted into itself. The result is that they consider any graph made of nestings and chainings of this one primitive, for example

 .

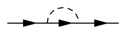
A graph like



is not allowed. These graphs are in one-to-one correspondence with planar rooted trees. The combinatorial Dyson-Schwinger equation is

$$X(x) = \mathbb{I} - xB_+\left(\frac{1}{X(x)}\right).$$

The Mellin transform associated to the single one loop primitive



is, according to the Feynman rules of Yukawa theory,

$$F(\rho_1, \rho_2) = \frac{1}{q^2}\int d^4k \frac{k \cdot q}{(k^2)^{1+\rho_1}((k+q)^2)^{1+\rho_2}}\bigg|_{q^2=1}.$$

However we are only inserting in the insertion place corresponding to ρ_1 so the Mellin transform we're actually interested in is

$$F(\rho) = \frac{1}{q^2}\int d^4k \frac{k \cdot q}{(k^2)^{1+\rho}(k+q)^2}\bigg|_{q^2=1}.$$

Next combine these two facts as described above to get that the Green function satisfies the analytic Dyson-Schwinger equation

$$G(x, L) = 1 - \left(\frac{x}{q^2}\int d^4k \frac{k \cdot q}{k^2 G(x, \log(k^2/\mu^2))(k+q)^2} - \cdots \bigg|_{q^2=\mu^2}\right)$$

where $L = \log(q^2/\mu^2)$ and $\cdots$ stands for the same integrand evaluated as specified. This is the same as what we would have obtained from applying the Feynman rules directly to the combinatorial Dyson-Schwinger equation.

3.3 Setup

We will restrict our attention to Dyson-Schwinger equations of the following form.

3.3.1 Single equations

Fix $s \in \mathbb{Z}$. The case $s = 0$ is not of particular interest since it corresponds to the strictly simpler linear situation discussed in [23]. However, to include $s = 0$ as well, we will make the convention that $\mathrm{sign}(0) = 1$.

Let $Q = X^{-s}$. We call Q the combinatorial invariant charge. Applying the Feynman rules to Q gives the usual physicists' invariant charge.

Consider the Dyson-Schwinger equation

$$X(x) = \mathbb{I} - \operatorname{sign}(s) \sum_{k \geq 1} \sum_{i=0}^{t_k} x^k B_+^{k,i}(XQ^k). \tag{3.1}$$

This includes Example 3.5 where $s = 2$ and there is only one B_+ having $k = 1$.

Let $F_{k,i}(\rho_1, \ldots, \rho_n)$ be the Mellin transform associated to the primitive $B_+^{k,i}(\mathbb{I})$. In view of Chapter 5 we're primarily interested in the case where $n = 1$ at which point we'll assume that the Mellin transforms of the primitives each have a simple pole at $\rho = 0$, which is the case in physical examples. We expand the Green functions in a series in x and in L (which will in general be merely an asymptotic expansion in x) using the following notation

$$G(x, L) = 1 - \operatorname{sign}(s) \sum_{k \geq 1} \gamma_k(x) L^k \qquad \gamma_k(x) = \sum_{j \geq k} \gamma_{k,j} x^j \tag{3.2}$$

The idea is to follow the prescriptions of the previous section to obtain the analytic Dyson-Schwinger equation, then simplify the resulting expression by following the following steps. See Example 3.7 for a worked example. First, expand G as a series in L. Second, convert the resulting logarithms of the integration variables into derivatives via the identity $\partial_\rho^k y^{-\rho}|_{\rho=0} = (-1)^k \log^k(y)$. The choice of name for the new variable ρ is not coincidental. Third, switch the order of integration and derivation. The result then is a complicated expression in derivatives of the Mellin transforms of the primitives.

However, to avoid the need for additional notation and for appropriate assumptions on the $F_{k,i}$, instead of following this path we will instead define our analytic Dyson-Schwinger equations to be the final result of this procedure.

Definition 3.6. For a single scale μ^2, the analytic Dyson-Schwinger equation associated to (3.1) is

$$G(x, L) = 1 - \operatorname{sign}(s) \sum_{k \geq 1} \sum_{i=0}^{t_k} x^k G(x, \partial_{-\rho_1})^{-\operatorname{sign}(s)} \cdots G(x, \partial_{-\rho_{n_k}})^{-\operatorname{sign}(s)}$$
$$(e^{-L(\rho_1 + \cdots + \rho_{n_k})} - 1) F^{k,i}(\rho_1, \ldots, \rho_{n_k}) \Big|_{\rho_1 = \cdots = \rho_{n_k} = 0}$$

where $n_k = \operatorname{sign}(s)(sk - 1)$.

We only need one subtraction because in view of the discussion at the end of Subsection 2.3.2 all the integrals of interest are log divergent.

In view of the following chapters we need not concern ourselves with the complexity of the general definition as we will further reduce to the case where there is only one symmetric insertion place and a single scale giving

$$G(x, L) = 1 - \operatorname{sign}(s) \sum_{k \geq 1} \sum_{i=0}^{t_k} x^k G(x, \partial_{-\rho})^{1-sk} (e^{-L\rho} - 1) F^{k,i}(\rho) \Big|_{\rho=0}$$

or rewritten

$$\gamma \cdot L = \sum_{k \geq 1} x^k (1 - \operatorname{sign}(s) \gamma \cdot \partial_{-\rho})^{1-sk} (e^{-L\rho} - 1) F^k(\rho) \Big|_{\rho=0} \tag{3.3}$$

where $\gamma \cdot U = \sum \gamma_k U^k$, $F^k(\rho) = \sum_{i=0}^{t_k} F^{k,i}(\rho)$.

The connection between the different forms of the analytic Dyson-Schwinger equation and the notational messiness of the original presentation can be explained by a motivating example.

Example 3.7. Let us return to Example 3.5. The analytic Dyson-Schwinger equation is

$$G(x,L) = 1 - \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(x, \log(k^2/\mu^2))(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

where $L = \log(q^2/\mu^2)$.

Substitute in the Ansatz

$$G(x,L) = 1 - \sum_{k\geq 1}\gamma_k(x)L^k$$

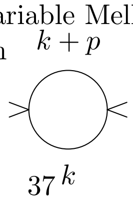
to get

$$\begin{aligned}
\sum_{k\geq 1}\gamma_k(x)L^k &= \frac{x}{q^2}\int d^4k \sum_{\ell_1+\cdots+\ell_s=\ell} \frac{(k\cdot q)\gamma_{\ell_1}(x)\cdots\gamma_{\ell_s}(x)\log^{\ell}(k^2/\mu^2)}{k^2(k+q)^2} - \cdots \Big|_{q^2=\mu^2} \\
&= \frac{x}{q^2}\sum_{\ell_1+\cdots+\ell_s=\ell}\gamma_{\ell_1}(x)\cdots\gamma_{\ell_s}(x)\int d^4k \frac{(k\cdot q)\log^{\ell}(k^2/\mu^2)}{k^2(k+q)^2} - \cdots \Big|_{q^2=\mu^2} \\
&= \frac{x}{q^2}\sum_{\ell_1+\cdots+\ell_s=\ell}\gamma_{\ell_1}(x)\cdots\gamma_{\ell_s}(x)\int d^4k \frac{(k\cdot q)(-1)^{\ell}\partial_\rho^{\ell}(k^2/\mu^2)^{-\rho}|_{\rho=0}}{k^2(k+q)^2} - \cdots \Big|_{q^2=\mu^2} \\
&= \frac{x}{q^2}\sum_{\ell_1+\cdots+\ell_s=\ell}\gamma_{\ell_1}(x)\cdots\gamma_{\ell_s}(x)(-1)^{\ell} \\
&\qquad \cdot\partial_\rho^{\ell}(\mu^2)^{\rho}\int d^4k \frac{k\cdot q}{(k^2)^{1+\rho}(k+q)^2} - \cdots \Big|_{q^2=\mu^2}\Big|_{\rho=0} \\
&= x\left(1 - \sum_{k\geq 1}\gamma_k(x)\partial_{-\rho}^k\right)^{-1}\frac{(\mu^2)^{\rho}}{q^2}\int d^4k \frac{k\cdot q}{(k^2)^{1+\rho}(k+q)^2} - \cdots \Big|_{q^2=\mu^2}\Big|_{\rho=0} \\
&= x\left(1 - \sum_{k\geq 1}\gamma_k(x)\partial_{-\rho}^k\right)^{-1}\frac{(\mu^2)^{\rho}}{(q^2)^{\rho}}\int d^4k_0 \frac{k_0\cdot q_0}{(k_0^2)^{1+\rho}(k_0+q_0)^2} - \cdots \Big|_{q^2=\mu^2}\Big|_{\rho=0} \\
&\qquad \text{where } q = rq_0 \text{ with } r\in\mathbb{R},\ r^2 = q^2,\ q_0^2 = 1 \text{ and } k = rk_0 \\
&= x\left(1 - \sum_{k\geq 1}\gamma_k(x)\partial_{-\rho}^k\right)^{-1}(e^{-L\rho}-1)F(\rho)\Big|_{\rho=0}
\end{aligned}$$

using $\partial_\rho^k y^{-\rho}|_{\rho=0} = (-1)^k\log^k(y)$. Thus using the notation $\gamma\cdot U = \sum\gamma_k U^k$ we can write

$$\gamma\cdot L = x(1-\gamma\cdot\partial_{-\rho})^{-1}(e^{-L\rho}-1)F(\rho)|_{\rho=0}$$

Example 3.8. To see an example of a two variable Mellin transform (a slightly different example can be found in [24]) consider again the graph

$\gamma =$ 

with the momenta associated to the two right hand external edges summing to p. As an integral the Mellin transform of γ is

$$\int d^D k \frac{1}{(k^2)^{1+\rho_1}((k+q)^2)^{1+\rho_2}}\bigg|_{q^2=1}.$$

By the calculations of Subsection 2.3.2

$$\int d^4 k \frac{1}{(k^2)^{1+\rho_1}((k+q)^2)^{1+\rho_2}}\bigg|_{q^2=1} = \frac{\pi^2\Gamma(\rho_1+\rho_2)}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)}(q^2)^{-\rho_1-\rho_2}\frac{\Gamma(-\rho_1)\Gamma(-\rho_2)}{\Gamma(2-\rho_1-\rho_2)}$$

So the Mellin transform is

$$F_\gamma(\rho_1,\rho_2) = \frac{\pi^2\Gamma(\rho_1+\rho_2)}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)}(q^2)^{-\rho_1-\rho_2}\frac{\Gamma(-\rho_1)\Gamma(-\rho_2)}{\Gamma(2-\rho_1-\rho_2)}.$$

Upon subtracting at $q^2=\mu^2$ then we get

$$((q^2)^{-\rho_1-\rho_2} - (\mu^2)^{-\rho_1-\rho_2})F_\gamma(\rho_1,\rho_2) = (e^{-L(\rho_1+\rho_2)}-1)(\mu^2)^{-\rho_1-\rho_2}F_\gamma(\rho_1,\rho_2)$$

So the only dependence on q is the dependence on L which is showing up in the correct form for Definition 3.6. The extra powers of μ^2 would get taken care of by the recursive iteration as in Example 3.7.

3.3.2 Systems

Now suppose we have a system of Dyson-Schwinger equations

$$X^r(x) = \mathbb{I} - \operatorname{sign}(s_r)\sum_{k\geq 1}\sum_{i=0}^{t_k^r} x^k B_+^{k,i;r}(X^r Q^k) \tag{3.4}$$

for $r \in \mathcal{R}$ with $\mathcal{R}$ a finite set and where

$$Q = \prod_{r\in\mathcal{R}} X^r(x)^{-s_r}. \tag{3.5}$$

The fact that the system can be written in terms of the invariant charge Q in this form is typical of realistic quantum field theories. For example, in QED (see Example 3.4)

$$Q = \frac{\left(X^{\leftarrow\!\!\prec}\right)^2}{\left(X^{\sim\sim}\right)\left(X^{\rightarrow}\right)^2}.$$

Suppose a theory T has a single vertex $v \in \mathcal{R}$ with external legs $e_i \in \mathcal{R}$ appearing with multiplicity m_i, $i = 1, \dots, n$ where the external legs (made of half-edges types under our definitions) are viewed as full edge types, hence as being in $\mathcal{R}$, by simply taking the full edge type which contains the given half-edge types (hence ignoring whether the edge is the front or back half of an oriented edge type). Let $\operatorname{val}(v)$ be the valence of the vertex type v. Then we define

$$Q = \left(\frac{(X^v)^2}{\prod_{i=1}^n (X^{e_i})^{m_i}}\right)^{1/(\operatorname{val}(v)-2)} \tag{3.6}$$

For theories with more than one vertex we form such a quotient for each vertex. We are again (see section 3.1) saved by the Slavnov-Taylor identities which tell us that these quotients agree, giving a unique invariant charge [21, section 2]. Then $X^r Q^k$ is exactly what can be inserted into a graph with external leg structure r and k loops.

Proposition 3.9. *Suppose Q is as defined as in the previous paragraph. Let G be a 1PI Feynman graph with external leg structure r and $k > 0$ loops. Then $X^r Q^k$ is exactly what can be inserted into G in the sense that we can write $X^r Q^k = \prod_{j \in \mathcal{R}} (X^j)^{t_j}$ so that G has t_j vertices of type j for j a vertex type and G has $-t_j$ edges of type j for j an edge type.*

Proof. In view of (3.6) for r a vertex type it suffices to prove that we can write

$$Q^{k+(\mathrm{val}(r)-2)/2} = \prod_{j \in \mathcal{R}} (X^j)^{\widetilde{t_j}}$$

so that G has $\widetilde{t_j}$ vertices of type j for j a vertex type and G has $-2\widetilde{t_j}$ half edges in edge type j (including external half edges) for j an edge type. For r an edge type it likewise suffices to prove that we can do the same where we define $\mathrm{val}(r) = 2$.

This holds for some k by viewing a graph as made from a set consisting of vertices each attached to their adjacent half edges.

To see that k is correct note that since G has 1 connected component, $e - v + 1 = \ell$, where e is the number of internal edges of G, v the number of vertices of G, and ℓ the loop number of G. Letting h be the number of half edges (including external half edges) of G we have

$$\frac{h}{2} - v + 1 - \frac{\mathrm{val}(r)}{2} = \ell. \tag{3.7}$$

Each Q contributes $\mathrm{val}(r)/(\mathrm{val}(r) - 2)$ edge insertions and $2/(\mathrm{val}(r) - 2)$ vertex insertions; so each Q contributes

$$\frac{\mathrm{val}(r)}{\mathrm{val}(r) - 2} - \frac{2}{\mathrm{val}(r) - 2} = 1$$

to (3.7). So if k is so that $Q^{k+(\mathrm{val}(r)-2)/2}$ counts the half edges and vertices of G as described above then

$$\ell = k + \frac{\mathrm{val}(r) - 2}{2} + 1 - \frac{\mathrm{val}(r)}{2} = k.$$

So k is the loop number as required.

□

The specific form of Q from (3.6) will only be used in the renormalization group derivation of the first recursion, section 4.1.

Write $F_{k,i;r}(\rho_1, \ldots, \rho_{n_{k,i;r}})$ for the Mellin transform associated to the primitive $B_+^{k,i;r}(\mathbb{I})$. Again assume a simple pole at the origin. We can then write the analytic Dyson-Schwinger equations as in the single equation case.

Definition 3.10. The analytic Dyson-Schwinger equations associated to (3.4) are

$G^r(x, L_1, \ldots, L_j)$

$$= 1 - \operatorname{sign}(s_r) \sum_{k \geq 1} \sum_{i=0}^{t_k^r} x^k G^r(x, \partial_{-\rho_1^r})^{-\operatorname{sign}(s_r)} \cdots G^r(x, \partial_{-\rho^r_{\operatorname{sign}(s_r)(s_r k - 1)}})^{-\operatorname{sign}(s_r)}$$

$$\prod_{t \in \mathcal{R} \setminus \{r\}} G^t(x, \partial_{-\rho_1^t})^{-\operatorname{sign}(s_t)} \cdots G^t(x, \partial_{-\rho^t_{\operatorname{sign}(s_t)(s_t k)}})^{-\operatorname{sign}(s_t)}$$

$$(e^{-L(\rho_1 + \cdots + \rho_{n_{k,i;r}})} - 1) F^{k,i;r}(\rho_1, \ldots, \rho_{n_{k,i;r}}) \Big|_{\rho_1 = \cdots = \rho_{n_{k,i;r}} = 0}$$

where the ρ_i^j run over the ρ_k so that the ith factor of G^j is inserted at ρ_i^j.

The following notation will be used for expanding the analytic Dyson-Schwinger equations as (in general asymptotic) series about the origin,

$$G^r(x, L) = 1 - \operatorname{sign}(s_r) \sum_{k \geq 1} \gamma_k^r(x) L^k \qquad \gamma_k^r(x) = \sum_{j \geq k} \gamma_{k,j}^r x^j \tag{3.8}$$

In view of the following chapters we will reduce to the case

$$G^r(x, L) =$$

$$1 - \operatorname{sign}(s_r) \sum_{k \geq 1} \sum_{i=0}^{t_k} x^k G^r(x, \partial_{-\rho})^{1 - s_r k} \prod_{j \in \mathcal{R} \setminus \{r\}} G^j(x, \partial_{-\rho})^{-s_j k} (e^{-L\rho} - 1) F^{k,i}(\rho) \Big|_{\rho = 0}$$

or rewritten

$$\gamma^r \cdot L = \sum_{k \geq 1} \sum_{i=0}^{t_k} x^k (1 - \operatorname{sign}(s_r) \gamma^r \cdot \partial_{-\rho})^{1 - sk} \prod_{j \in \mathcal{R} \setminus \{r\}} (1 - \operatorname{sign}(s_j) \gamma^j \cdot \partial_{-\rho})^{-s_j k}$$

$$(e^{-L\rho} - 1) F^{k,i}(\rho) \Big|_{\rho = 0} \tag{3.9}$$

where $\gamma^j \cdot U = \sum \gamma_k^j U^k$.

Chapter 4

The first recursion

There are two approaches to deriving the first recursion, neither of which is completely self contained. The first goes directly through the renormalization group equation, and the second through the Connes-Kreimer scattering-type formula [9].

4.1 From the renormalization group equation

This is primarily an exercise in converting from usual physics conventions to ours.

Using the notation of section 3.3.2 the renormalization group equation, see for instance [7, Section 3.4] or [17], reads

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \sum_{e \text{ adjacent to } v} \gamma^e(x)\right) x^{(\mathrm{val}(v)-2)/2} G^v(x, L) = 0 \qquad \text{for } v \text{ a vertex type}$$

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - 2\gamma^e(x)\right) G^e(x, L) = 0 \qquad \text{for } e \text{ an edge type}$$

where $\beta(x)$ is the β function of the theory, $\gamma^e(x)$ is the anomalous dimension for $G^e(x, L)$ (both of which will be defined in our notation below), and $\mathrm{val}(v)$ is the valence of v. The edge case and vertex case can be unified by writing $\mathrm{val}(e) = 2$ and taking the edges adjacent to e to be two copies of e itself (one for each half edge making e). Our scale variable L already has a log taken so ∂_L often appears as $\mu\partial_\mu$ in the literature where μ is the scale before taking logarithms. The use of $x^{(\mathrm{val}(v)-2)/2}G^v(x, L)$ in the vertex case in place of what is more typically simply $G^v(x, L)$ comes about because by taking the coupling constant to count the loop number rather than having the Feynman rules associate a coupling constant factor to each vertex we have divided out by the coupling constant factor for one vertex, that is by $x^{(\mathrm{val}(v)-2)/2}$. As a result our series begin with a constant term even for vertices.

To see that it makes sense that a vertex v contributes a factor of $x^{(\mathrm{val}(v)-2)/2}$ recall that that for a graph G with one connected component and external leg structure r we have (3.7) which reads

$$\frac{h}{2} - t + 1 - \frac{\mathrm{val}(r)}{2} = \ell$$

where h is the number of half edges of G, t the number of vertices, ℓ the loop number, and where we take $\mathrm{val}(r) = 2$ for if r is an edge type. Suppose vertices, but not edges, contribute some power of x. Then a vertex v contributes $\mathrm{val}(v)/2-1$ to the left hand side of (3.7), so it is consistent that v also contribute the same power of x. The whole graph G then has $x^{\ell+\mathrm{val}(r)/2-1}$ as expected. If the power of x associated to a vertex depends only on its valence, then this is the only way to make the counting work.

Returning to β and γ, define

$$\beta(x) = \partial_L x\phi_R(Q)|_{L=0} \tag{4.1}$$

and

$$\gamma^e(x) = -\frac{1}{2}\partial_L G^e(x,L)|_{L=0} = \frac{1}{2}\gamma_1^e \tag{4.2}$$

for e an edge, that is $s_e > 0$ as discussed in subsection 3.3.2. The factor of x in β comes again from our normalization of the coupling constant powers to serve to count the loop number (recall from the discussion surrounding (3.6) that Q contributes a 1 to (3.7) and so, in view of the previous paragraph, Q is short one power of x), while the factor of 1/2 in (4.2) is usual. The sign in (4.2) comes from the fact that our conventions have the Green functions for the edges with a negative sign, while the second equality uses the explicit expansion (3.8). Note that this β-function is not the Euler β function, $\Gamma(x)\Gamma(y)/\Gamma(x+y)$; rather it encodes the flow of the coupling constant depending on the energy scale. Another way to look at matters is that the β-function measures the nonlinearity in a theory, specifically it is essentially the coefficient of L in the invariant charge, as in the definition above.

In the case of an edge type e we obtain quickly from (4.1), (4.2), (3.5), and (3.8), that

$$\begin{aligned}
0 &= \left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \gamma_1^e(x)\right) G^e(x,L) \\
&= \left(\frac{\partial}{\partial L} + \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x) x\frac{\partial}{\partial x} - \gamma_1^e(x)\right) G^e(x,L).
\end{aligned}$$

In the case of a vertex type v compute as follows.

$$\begin{aligned}
0 &= \left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \sum_{e \text{ adjacent to } r} \gamma^e(x)\right) x^{(\mathrm{val}(v)-2)/2} G^v(x,L) \\
&= x^{(\mathrm{val}(v)-2)/2}\frac{\partial}{\partial L}G^v(x,L) + x^{(\mathrm{val}(v)-2)/2}\beta(x)\frac{\partial}{\partial x}G^v(x,L) \\
&\quad + \frac{\mathrm{val}(v)-2}{2}x^{(\mathrm{val}-4)/2}\beta(x)G^v(x,L) - x^{(\mathrm{val}(v)-2)/2}\sum_{e \text{ adjacent to } r}\gamma^e(x)G^v(x,L) \\
&= x^{(\mathrm{val}(v)-2)/2}\frac{\partial}{\partial L}G^r(x,L) + x^{(\mathrm{val}(v)-2)/2}\beta(x)\frac{\partial}{\partial x}G^r(x,L) \\
&\quad + x^{(\mathrm{val}-2)/2}\frac{\mathrm{val}(v)-2}{2}\frac{1}{\mathrm{val}(v)-2}\left(2\gamma_1^v(x) + \sum_{e \text{ adjacent to } r}\gamma^e(x)\right)G^v(x,L) \\
&\quad - x^{(\mathrm{val}(v)-2)/2}\sum_{e \text{ adjacent to } r}\frac{1}{2}\gamma_1^e(x)G^v(x,L)
\end{aligned}$$

from (4.1), (3.6), and (3.8)

$$= x^{(\mathrm{val}(v)-2)/2}\left(\frac{\partial}{\partial L}G^v(x,L) + \beta(x)\frac{\partial}{\partial x}G^v(x,L) + \gamma_1^v G^v(x,L)\right)$$

Dividing by $x^{(\mathrm{val}(v)-2)/2}$ and using (3.5) and (3.8) we have

$$\left(\frac{\partial}{\partial L} + \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)x\frac{\partial}{\partial x} + \gamma_1^v(x)\right) G^v(x,L) = 0.$$

In both cases extracting the coefficient of L^{k-1} and rearranging gives

Theorem 4.1.

$$\gamma_k^r(x) = \frac{1}{k}\left(\mathrm{sign}(s_r)\gamma_1^r(x) - \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)x\partial_x\right)\gamma_{k-1}^r(x).$$

Specializing to the single equation case gives

Theorem 4.2.

$$\gamma_k = \frac{1}{k}\gamma_1(x)(\mathrm{sign}(s) - |s|x\partial_x)\gamma_{k-1}(x).$$

Note that the signs in the above do not match with [25] because here the sign conventions have that the X^r have their graphs appear with a negative sign precisely if r is an edge type, whereas in [25] there was a negative sign in all cases.

4.2 From $S \star Y$

Definition 4.3. *Y is the grading operator on $\mathcal{H}$. $Y(\gamma) = |\gamma|\gamma$ for $\gamma \in \mathcal{H}$.*

Definition 4.4. Let

$$\sigma_1 = \partial_L \phi_R(S \star Y)|_{L=0}$$

and

$$\sigma_n = \frac{1}{n!}m^{n-1}(\underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{n \text{ times}})\Delta^{n-1}$$

Lemma 4.5. *$S \star Y$ is zero off $\mathcal{H}_{\mathrm{lin}}$*

Proof. First $S \star Y(\mathbb{I}) = \mathbb{I}\cdot 0 = 0$. Suppose $\Gamma_1, \Gamma_2 \in \mathcal{H}\setminus \mathbb{Q}\mathbb{I}$. Since S is a homomorphism and Y is a derivation,

$$\begin{aligned}
S \star Y(\Gamma_1\Gamma_2) &= \sum S(\gamma_1'\gamma_2')Y(\gamma_1''\gamma_2'') \\
&= \left(\sum S(\gamma_1')\gamma_1''\right)\left(\sum S(\gamma_2')Y(\gamma_2''\right) + \left(\sum S(\gamma_1')Y(\gamma_1'')\right)\left(\sum S(\gamma_2')\gamma_2''\right) \\
&= 0
\end{aligned}$$

since by definition $S \star \mathrm{id}(\Gamma_1) = S \star \mathrm{id}(\Gamma_2) = 0$. Here we used the Sweedler notation, $\sum \gamma_j' \otimes \gamma_j'' = \Delta(\Gamma_j)$. ☐

Lemma 4.6.

$$\Delta([x^k]X^r) = \sum_{j=0}^{k}[x^j]X^rQ^{k-j} \otimes [x^{k-j}]X^r$$
$$\Delta([x^k]X^rQ^\ell) = \sum_{j=0}^{k}[x^j]X^rQ^{k+\ell-j} \otimes [x^{k-j}]X^rQ^\ell$$

where $[\cdot]$ *denotes the coefficient operator as in Definition 2.2.*

Proof. The proof follows by induction. Note that both equations read $\mathbb{I}\otimes\mathbb{I} = \mathbb{I}\otimes\mathbb{I}$ when $k = 0$. For a given value of $k > 0$ the second equality follows from the first for all $0 \leq \ell \leq k$ using the multiplicativity of Δ and the fact that partitions of k into m parts each part then partitioned into two parts are isomorphic with partitions of k into two parts with each part then partitioned into m parts.

Consider then the first equation with $k > 0$. By Assumption 3.3, for all $1 \leq \ell$, $\sum_{i=0}^{t_\ell^r} B_+^{\ell,i;r}$ is a Hochschild 1-cocycle. Thus using (3.4)

$$\begin{aligned}
\Delta([x^k]X^r) &= \Delta\left(-\text{sign}(s_r)\sum_{1\leq\ell\leq k}\sum_{i=0}^{t_\ell^r} B_+^{\ell,i;r}([x^{k-\ell}]X^rQ^\ell)\right)\\
&= -\text{sign}(s_r)\sum_{1\leq\ell\leq k}\sum_{i=0}^{t_\ell^r}(\text{id}\otimes B_+^{\ell,i;r})\Delta([x^{k-\ell}]X^rQ^\ell)\\
&\quad - \text{sign}(s_r)\sum_{1\leq\ell\leq k}\sum_{i=0}^{t_\ell^r}(B_+^{\ell,i;r}([x^{k-\ell}]X^rQ^\ell)\otimes\mathbb{I})\\
&= -\text{sign}(s_r)\sum_{1\leq\ell\leq k}\sum_{i=0}^{t_\ell^r}(\text{id}\otimes B_+^{\ell,i;r})\left(\sum_{j=0}^{k-\ell}[x^j]X^rQ^{k-j}\otimes[x^{k-\ell-j}]X^rQ^\ell\right)\\
&\quad + [x^k]X^r\otimes\mathbb{I}\\
&= \sum_{j=0}^{k-1}\left([x^j]X^rQ^{k-j}\otimes -\text{sign}(s_r)\sum_{1\leq\ell\leq k-j}\sum_{i=0}^{t_\ell^r}B_+^{\ell,i;r}([x^{k-\ell-j}]X^rQ^\ell)\right)\\
&\quad + [x^k]X^r\otimes\mathbb{I}\\
&= \sum_{j=0}^{k-1}[x^j]X^rQ^{k-j}\otimes[x^{k-j}]X^r + [x^k]X^r\otimes\mathbb{I}
\end{aligned}$$

The result follows. ∎

Note that $\Delta(X^r) = \sum_{k=0}^{\infty} X^rQ^k \otimes (\text{terms of degree } k \text{ in } X^r)$.

4.2.1 Single equations

Proposition 4.7. $\sigma_n(X) = \text{sign}(s)\gamma_n(x)$

Proof. For $n = 1$ this appears as equation (25) of [9] and equation (12) of [4]. For $n > 1$ expand the scattering type formula [9, (14)]. The sign is due to our sign conventions, see (3.2). ☐

Rephrasing Lemma 4.6 we have

Corollary 4.8. *Suppose Γx^k appears in X with coefficient c and $Z \otimes \Gamma x^k$ consists of all terms in ΔX with Γ on the right hand side. Then $Z = cXQ^k$.*

Proposition 4.9.

$$(P_{\text{lin}} \otimes \text{id})\Delta X = X \otimes X - sX \otimes x\partial_x X$$

Proof. The Corollary implies that every graph appearing on the right hand side of ΔX also appears in X and vice versa. Suppose Γx^k appears in X and $Z \otimes \Gamma x^k$ consists of all terms in ΔX with Γ on the right hand side.

By Corollary 4.8 $XQ^k = Z$. So in $(P_{\text{lin}} \otimes \text{id})\Delta X$ we have the corresponding terms $P_{\text{lin}}(XQ^k) \otimes \Gamma$.

Compute

$$\begin{aligned} P_{\text{lin}}(XQ^k) &= P_{\text{lin}}X + P_{\text{lin}}Q^k \\ &= P_{\text{lin}}X + kP_{\text{lin}}Q \\ &= P_{\text{lin}}X - ksP_{\text{lin}}X \\ &= X - ksX \end{aligned}$$

Thus

$$(P_{\text{lin}} \otimes \text{id})\Delta X = X \otimes X - sX \otimes x\partial_x X$$

☐

Theorem 4.10.

$$\gamma_k = \frac{1}{k}\gamma_1(x)(\text{sign}(s) - |s|x\partial_x)\gamma_{k-1}(x)$$

Proof.

$$\begin{aligned} \gamma_k &= \text{sign}(s)\sigma_k(X) \qquad \text{by Proposition 4.7} \\ &= \frac{\text{sign}(s)}{k!} m^{k-1}(\underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{k \text{ times}})\Delta^{k-1}(X) \\ &= \frac{\text{sign}(s)}{k} m \left(\sigma_1 \otimes \frac{1}{(k-1)!} m^{k-2}(\underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{k-1 \text{ times}})\Delta^{k-2} \right) \Delta(X) \\ &= \frac{\text{sign}(s)}{k} m(\sigma_1 P_{\text{lin}} \otimes \sigma_{k-1})\Delta(X) \qquad \text{by Lemma 4.5} \\ &= \frac{1}{k}\text{sign}(s)\sigma_1(X)\sigma_{k-1}(X) - s\sigma_1(X)x\partial_x\sigma_{k-1}(X) \qquad \text{by Proposition 4.9} \\ &= \frac{1}{k}\gamma_1(x)(\text{sign}(s)\gamma_{k-1}(x) - |s|x\partial_x\gamma_{k-1}(x)) \end{aligned}$$

☐

4.2.2 Systems of equations

Proposition 4.11. $\sigma_n(X^r) = \mathrm{sign}(s_r)\gamma_n^r(x)$

Proof. The arguments of [9] and [4] do not depend on how or whether the Green functions depend on other Green functions, so the same arguments as in the single equation case applied. The sign comes from our conventions, see (3.8). Note that β in [9] is the operator associated to the β-function only in the single equation case, otherwise it is simply the anomalous dimension. □

As in the single equation case we can rewrite Lemma 4.6 to get

Corollary 4.12. *Suppose Γx^k appears in X^r with coefficient c and $Z \otimes \Gamma x^k$ consists of all terms in ΔX^r with Γ on the right hand side. Then $Z = cX^r Q^k$.*

Proposition 4.13.

$$(P_{\mathrm{lin}} \otimes \mathrm{id})\Delta X^r = X^r \otimes X^r - \sum_{j\in\mathcal{R}} s_j X^j \otimes x\partial_x X^r$$

Proof. As in the single equation case every graph appearing on the right hand side of ΔX^r also appears in X^r and vice versa. Suppose Γx^k appears in X^r and $Z \otimes \Gamma x^k$ consists of all terms in ΔX with Γ on the right hand side.

By Corollary 4.12 $X^r Q^k = Z$, and

$$\begin{aligned}
P_{\mathrm{lin}}(X^r Q^k) &= P_{\mathrm{lin}} X^r + P_{\mathrm{lin}} Q^k \\
&= P_{\mathrm{lin}} X^r + k P_{\mathrm{lin}} Q \\
&= P_{\mathrm{lin}} X^r - k \sum_{j\in\mathcal{R}} s_j P_{\mathrm{lin}} X^j \\
&= X^r - k \sum_{j\in\mathcal{R}} s_j X^j
\end{aligned}$$

The result follows. □

Theorem 4.14.

$$\gamma_k^r = \frac{1}{k}\left(\mathrm{sign}(s_r)\gamma_1^r(x)^2 - \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x) x\partial_x \gamma_{k-1}^r(x)\right)$$

Proof.

$$\begin{aligned}
\gamma_k^r &= \mathrm{sign}(s_r)\sigma_k(X^r) \qquad \text{by Proposition 4.11} \\
&= \frac{\mathrm{sign}(s_r)}{k} m(\sigma_1 P_{\mathrm{lin}} \otimes \sigma_{k-1})\Delta(X^r) \qquad \text{as in the single equation case} \\
&= \frac{}{k}\left(\mathrm{sign}(s_r)\sigma_1(X^r)\sigma_{k-1}(X^r) - \sum_{j\in\mathcal{R}} s_j \sigma_1(X^j) x\partial_x \mathrm{sign}(s_r)\sigma_{k-1}(X^r)\right) \\
&\qquad \text{by Proposition 4.13} \\
&= \frac{1}{k}\left(\mathrm{sign}(s_r)\gamma_1^r(x)\gamma_{k-1}^r(x) - \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x) x\partial_x \gamma_{k-1}^r(x))\right).
\end{aligned}$$

□

As in the previous section the signs do not match with [25] because here the sign conventions have that the X^r have their graphs appear with a negative sign precisely if r is an edge type, whereas in [25] there was a negative sign in all cases.

4.3 Properties

The following observation is perhaps obvious to the physicists, but worth noticing

Lemma 4.15. *As a series in x, the lowest term in γ^r_k is of order at least k. If $\gamma^r_{1,1} \neq \ell \sum_{j\in\mathcal{R}} s_j \gamma^j_{1,1}$, for $\ell = 0, \ldots, k-1$ then the lowest term in γ^r_k is exactly order k.*

Note that in the single equation case, the condition to get lowest term exactly of order k is simply $\gamma_{1,1} \neq 0$.

Proof. Expanding the combinatorial Dyson-Schwinger equation, (3.1) or (3.4), in x we see immediately that the x^0 term is exactly $\mathbb{I}$. The Feynman rules are independent of x so the x^0 term in the analytic Dyson-Schwinger equation is $1-1=0$ due to the fact that we renormalize by subtractions.

Then inductively the γ^r_k recursion, Theorem 4.1 or 4.14, gives that as a series in x, γ^r_k has no nonzero term before

$$\frac{x^k}{k}\left(\mathrm{sign}(s_r)\gamma^r_{1,1} - (k-1)\sum_{j\in\mathcal{R}} |s_j|\gamma^j_{1,1}\right)\gamma^r_{k-1,k-1}$$

The result follows. ☐

We also can say that $\gamma^r_{k,j}$ is homogeneous in the coefficients of the Mellin transforms in the sense indicated below. This will not be used in the following. For simplicity we will only give it in the single equation case with one insertion place.

Expand $F_{k,i}(\rho) = \sum_{j\geq -1} m_{j,k,i}\rho^j$.

Recall (3.3)

$$\gamma \cdot L = \sum_{k\geq 1} x^k (1 - \mathrm{sign}(s)\gamma\cdot\partial_{-\rho})^{1-sk}(e^{-L\rho}-1)F^k(\rho)\Big|_{\rho=0}$$

Taking one L derivative and setting L to 0 we get

$$\gamma_1 = -\sum_k \sum_i x^k (1-\mathrm{sign}(s)\gamma\cdot\partial_{-\rho})^{1-sk}\rho F_{k,i}(\rho)|_{\rho=0} \tag{4.3}$$

Proposition 4.16. *Writing $\gamma_{k,j} = \sum c_{k,j,j_1,\cdots j_u,\bar{\ell},\bar{i}} m_{j_1,\ell_1,i_1}\cdots m_{j_u,\ell_u,i_u}$ for $j \geq k$ we have that $j_1 + \cdots + j_u = j - k$*

Proof. The proof proceeds by induction. Call $j_1 + \cdots + j_u$ the m-degree of $\gamma_{k,j}$. First note that $\gamma_{1,1} = \sum_i m_{0,1,i}$ from (4.3).

Assume the result holds for $k, j < n$.

Then from Theorem 4.2 or 4.10 $\gamma_{1,n}$ is a sum over s of terms of the form

$$C\gamma_{\ell_1,t_1}\cdots\gamma_{\ell_u,s_u}m_s \tag{4.4}$$

where $\ell_1 + \cdots + \ell_u = s$ and $t_1 + \cdots t_u = n-1$. By the induction hypothesis γ_{ℓ_i,t_i} has m-degree $t_i - \ell_i$, so (4.4) has m-degree $\sum t_i - \sum \ell_i + s = n-1-s+s = n-1$ as desired.

Next from Theorem 4.2 or 4.10 $\gamma_{k,j}$, for $k, j \leq n$, is a sum over $1 \leq i \leq n$ of terms of the form

$$C\gamma_{1,i}\gamma_{k-1,j-i} \tag{4.5}$$

By the induction hypothesis $\gamma_{1,i}$ has m-degree $i-1$ and $\gamma_{k-1,j-1}$ has m-degree $j-i-k+1$ so (4.5) has m-degree $j-k$ as desired. □

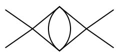
Chapter 5

Reduction to one insertion place

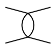
5.1 Colored insertion trees

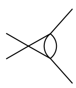
From now on we will need to carry around some additional information with our Feynman graphs. Namely we want to keep track of two different kinds of insertion, normal insertion, and a modified insertion which inserts symmetrically into all insertion places. Symmetric insertion does not analytically create overlapping divergences, but simply marking each subgraph by how it was inserted may be ambiguous as in the example below. We will use insertion trees to retain the information of how a graph was formed by insertions.

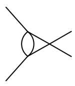
In examples without overlaps, and even in simple overlapping cases, it suffices to label the divergent subgraphs with one of two colors, black for normal insertion and red for symmetric insertion. To see that coloring does not suffice in the general case consider the graph

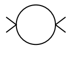
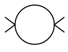


There are three proper subdivergent graphs; give them the following names for easy reference

$A =$ 

$B =$ 

$C =$ 

Then if A is red while B and C are black then this could represent A inserted symmetrically into  or it could represent B inserted into  while B itself is made of A symmetrically inserted into  and likewise for C.

Definition 5.1. A *decorated rooted tree* is a finite rooted tree (not embedded in the plane) with a map from its vertices to a fixed, possibly infinite, set of decorations.

The polynomial algebra over $\mathbb{Q}$ generated by (isomorphism classes) of decorated rooted trees forms a Hopf algebra as follows.

Definition 5.2. The *(Connes-Kreimer) Hopf algebra of decorated rooted trees*, $\mathcal{H}_{CK}$, consists of the $\mathbb{Q}$ span of forests of decorated rooted trees with disjoint union as multiplication, including the empty forest $\mathbb{I}$. The coproduct on $\mathcal{H}_{CK}$ is the algebra homomorphism defined on a tree by

$$\Delta(T) = \sum_c P_c(T) \otimes R_c(T)$$

where the sum runs over ways to cut edges of T so that each path from the root to a leaf is cut at most once, $R_c(T)$ is the connected component of the result connected to the original root, and $P_c(T)$ is the forest of the remaining components. The antipode is defined recursively from $S \star \mathrm{id} = e\eta$ (as in the Feynman graph situation),

See [10] for more details on $\mathcal{H}_{CK}$. Insertion trees are decorated rooted trees where each element in the decoration set consists of an ordered triple of a primitive of $\mathcal{H}$ (potentially a sum), an insertion place in the primitive of the parent of the current vertex, and a bijection from the external edges of the Feynman graph to the half edges of the insertion place. The second and third elements of the triple serve to unambiguously define an insertion as in Subsection 2.2.3. Often the insertion information will be left out if it is unambiguous.

Definition 5.3. For a 1PI Feynman graph G in a given theory let $F(G)$ be the forest of insertion trees which give G.

From $F(G)$, or even just one tree of $F(G)$, we can immediately recover G simply by doing the specified insertions. The result of the insertion defined by a particular parent and child pair of vertices is unambiguous since all the insertion information is included in the decoration. The choice of order to do the insertions defined by an insertion tree does not affect the result due to the coassociativity of the Feynman graph Hopf algebra.

Extend F to $F : \mathcal{H} \to \mathcal{H}_{CK}$ as an algebra homomorphism. In fact it is an injective Hopf algebra morphism by the following proposition.

Proposition 5.4. *$F(\Delta(G)) = \Delta(F(G))$.*

Proof. Let γ be a (not necessarily connected) divergent subgraph of G. Since G can be made by inserting γ into G/γ, then among $F(G)$ we can find each tree of $F(\gamma)$ grafted into each tree of $F(G/\gamma)$. Cutting edges where $F(\gamma)$ is grafted into $F(G/\gamma)$ we see that $F(\gamma) \otimes F(G/\gamma)$ appears in $\Delta(F(G))$. The coefficients are the same since each insertion place for γ in G/γ which gives G we have a grafting with this insertion information and vice versa. Finally every cut of $F(G)$ consists of a forest of insertion trees, which by doing the insertions gives a divergent subgraph of G. The result follows. □

Now we wish to extend this situation by coloring the edges of the insertion trees.

Definition 5.5. Let T be a decorated rooted tree with edge set E. Define an *insertion coloring map* to be a map $f : E \to \{\text{black}, \text{red}\}$. If T is an insertion tree when call T with f a *colored insertion tree.*

Definition 5.6. For a colored insertion tree define the coproduct to be as before with the natural colorings.

To translate back to Feynman graphs think of the edge as coloring the graph defined by the insertion tree below it. The result is a Feynman graph with colored proper subgraphs. The coproduct in the tree case forgets the color of the cut edges. Correspondingly in the Feynman graph case the color of the graphs, but not their subgraphs, on the left hand sides of the tensor product are forgotten.

Proposition 5.7. *Colored insertion trees form a Hopf algebra with the above coproduct which agrees with $\mathcal{H}_{CK}$ upon forgetting the colors.*

Proof. Straightforward. □

Call the Hopf algebra of colored insertion trees $\mathcal{H}_c$. In view of the above $R : \mathcal{H} \hookrightarrow \mathcal{H}_c$ by taking $R : \mathcal{H} \hookrightarrow \mathcal{H}_{CK}$ and coloring all edges black.

Analytically, black insertion follows the usual Feynman rules, red insertion follows the symmetric insertion rules as defined in subsection 2.3.3.

Definition 5.8. For γ a primitive element of $\mathcal{H}$ or $\mathcal{H}_c$, write $R_+^\gamma : \mathcal{H}_c \to \mathcal{H}_c$ for the operation of adding a root decorated with γ with the edges connecting it colored red. Also write $B_+^\gamma : \mathcal{H}_c \to \mathcal{H}_c$ for ordinary insertion of Feynman graphs translated to insertion trees with new edges colored black. Note that this is not the usual B_+ on rooted trees in view of overlapping divergences.

When working directly with Feynman graphs R_+^γ corresponds to insertion with the inserted graphs colored red and no overlapping divergences.

Another way of understanding the importance of Definition 3.1 and Theorem 3.3 is that $\sum_{i=0}^{t_k^r} B_+^{k,i;r}$ is the same whether interpreted as specified above by B_+ on Feynman graphs translated to $\mathcal{H}_c$, or directly on $\mathcal{H}_c$ simply by adding a new root labelled by γ and the corresponding insertion places without consideration for overlapping divergences.

Lemma 5.9. *R_+^γ is a Hochschild 1-cocycle for $\mathcal{H}_c$.*

Proof. The standard B_+ of adding a root is a Hochschild 1-cocycle in $\mathcal{H}_{CK}$, see [10, Theorem 2]. Edges attached to the root on the right hand side of the tensors are red on both sides of the 1-cocycle identity. The remaining edge colors must also satisfy the 1-cocycle property which we can see by attaching this information to the decoration of the node which is further from the root. □

5.2 Dyson-Schwinger equations with one insertion place

To reduce to one insertion place we need to show that we can write Dyson-Schwinger equations in which only involves R_+s but which, results in the same series X^r which contains only black insertions. We can achieve this recursively, while viewing $\mathcal{H} \hookrightarrow \mathcal{H}_c$.

Suppose our combinatorial Dyson-Schwinger equation is as in (3.4)

$$X^r(x) = \mathbb{I} - \operatorname{sign}(s_r) \sum_{k \geq 1} \sum_{i=0}^{t_k^r} x^k B_+^{k,i;r}(X^r Q^k).$$

Then, using $[\cdot]$ to denote the coefficient operator as in Definition 2.2, define

$$q_1^r = -\operatorname{sign}(s_r)[x]X^r = \sum_{i=0}^{t_i^r} B_+^{1,i;r}(\mathbb{I})$$

$$\begin{aligned} q_n^r &= -\mathrm{sign}(s_r)[x^n]X^r + \mathrm{sign}(s_r)\sum_{k=1}^{n-1} R_+^{q_k^r}([x^{n-k}]X^r Q^k) \\ &= \sum_{k=1}^{n}\sum_{i=0}^{t_k^r} B_+^{k,i;r}([x^{n-k}]X^r Q^k) + \mathrm{sign}(s_r)\sum_{k=1}^{n-1} R_+^{q_k^r}([x^{n-k}]X^r Q^k) \end{aligned}$$

In order to know that the q_n^r are well defined we need to know that they are primitive.

Proposition 5.10. *q_n^r is primitive for $r \in \mathcal{R}$ and $n \geq 1$.*

Proof. First note that $B_+(\mathbb{I})$ is primitive for any B_+ and the sum of primitives is primitive, so q_1^r is primitive for each $r \in \mathcal{R}$.

Then inductively for $n > 1$

$$\begin{aligned} \Delta(q_n^r) &= \sum_{k=1}^{n}\sum_{i=0}^{t_k^r}(\mathrm{id}\otimes B_+^{k,i;r})(\Delta[x^{n-k}]X^r Q^k) - \sum_{k=1}^{n-1}(\mathrm{id}\otimes R_+^{q_k^r})(\Delta[x^{n-k}]X^r Q^k) \\ &\quad + \sum_{k=1}^{n}\sum_{i=0}^{t_k^r} B_+^{k,i;r}([x^{n-k}]X^r Q^k)\otimes\mathbb{I} - \sum_{k=1}^{n-1} R_+^{q_k^r}([x^{n-k}]X^r Q^k)\otimes\mathbb{I} \\ &= \sum_{k=1}^{n}\sum_{i=0}^{t_k^r}\sum_{\ell=0}^{n-k}\Big([x^\ell]X^r Q^k \otimes B_+^{k,i;r}([x^{n-\ell-k}]X^r Q^k)\Big) \\ &\quad - \sum_{k=1}^{n-1}\sum_{\ell=0}^{n-k}\Big([x^\ell]X^r Q^k \otimes R_+^{q_k^r}([x^{n-\ell-k}]X^r Q^k)\Big) + q_n^r\otimes\mathbb{I} \\ &= \mathbb{I}\otimes q_n^r + q_n^r\otimes\mathbb{I} \\ &+ \sum_{\ell=1}^{n-1}\sum_{k=1}^{n-\ell}\left([x^\ell]X^r Q^{n-\ell}\otimes\left(\sum_{i=0}^{t_k^r} B_+^{k,i;r}([x^{n-\ell-k}]X^r Q^k) - R_+^{q_k^r}([x^{n-\ell-k}]X^r Q^k)\right)\right) \\ &= \mathbb{I}\otimes q_n^r + q_n^r\otimes\mathbb{I} - \sum_{\ell=1}^{n-1}\Big([x^\ell]X^r Q^{n-\ell}\otimes(q_\ell^r - q_\ell^r)\Big) \\ &= \mathbb{I}\otimes q_n^r + q_n^r\otimes\mathbb{I}. \end{aligned}$$

□

Theorem 5.11.

$$X^r = 1 - \mathrm{sign}(s_r)\sum_{k\geq 1} x^k R_+^{q_k^r}(X^r Q^k).$$

Proof. The constant terms of both sides of the equation match and for $n \geq 1$

$$\begin{aligned} -\mathrm{sign}(s_r)[x^n]\sum_{k\geq 1} x^k R_+^{q_k^r}(X^r Q^k) &= -\mathrm{sign}(s_r)\sum_{k=1}^{n} x^k R_+^{q_k^r}([x^{n-k}]X^r Q^k) \\ &= -\mathrm{sign}(s_r)q_k^r - \mathrm{sign}(s_r)\sum_{k=1}^{n-1} x^k R_+^{q_k^r}([x^{n-k}]X^r Q^k) \end{aligned}$$

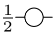
$$= [x^n]X^r.$$

□

The interpretation of the Theorem is that we can reduce to considering only red insertion, that is to a single symmetric insertion place.

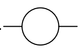
In simple cases we can avoid the not only the insertion trees, but also the subgraph coloring, and literally reduce to a single insertion place in the original Hopf algebra. However this cannot work with different types of insertions or with vertex insertions where each vertex can not take an arbitrary number of inserted graphs. Consequently such simple examples can only arise with a single type of edge insertion as in the following example.

Example 5.12. Suppose we have the Dyson-Schwinger equation

$$X = 1 - xB_+^{\frac{1}{2}}$$  $$\left(\frac{1}{X^2}\right).$$

where we insert into both internal edges. In this case we need not resort to red insertion in order to reduce to one insertion place.

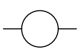
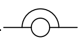
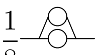
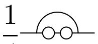
Let

$$q_1 = \frac{1}{2}$$ 

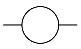
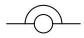
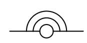
where we only insert into the bottom edge and let

$$X_1 = 1 - xB_+^{q_1}\left(\frac{1}{X_1^2}\right)$$

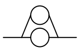
Then to order x^3 we have that

$$X = 1 - x\frac{1}{2}$$  $$- x^2\frac{1}{2}$$  $$- x^3\left(\frac{1}{8}\right.$$  $$+\frac{1}{2}$$  $$\left.+\frac{1}{4}\right.$$  $$\left.\right)$$

and

$$X_1 = 1 - x\frac{1}{2}$$  $$- x^2\frac{1}{2}$$  $$- x^3\left(\frac{3}{8}\right.$$  $$\left.+\frac{1}{2}\right.$$  $$\left.\right)$$

so

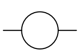
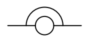
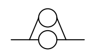
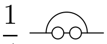
$$q_2 = 0 \qquad \text{and} \qquad q_3 = \frac{1}{8}$$  $$-\frac{1}{16}$$  $$-\frac{1}{16}$$ 

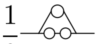
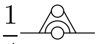
where in the first graph of q_3 we insert only in the bottom edge of the bottom inserted bubble, in the second graph we insert only in the bottom edge of the leftmost inserted bubble, and in the third graph we insert only in the bottom edge of the rightmost inserted bubble.

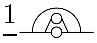
Note that q_3 is primitive. Let

$$X_2 = 1 - xB_+^{q_1}\left(\frac{1}{X_1^2}\right) - x^3B_+^{q_3}\left(\frac{1}{X^8}\right)$$

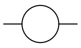
The order x^4 we have

$$X = 1 - x\frac{1}{2}$$  $$- x^2\frac{1}{2}$$  $$- x^3\left(\frac{1}{8}\right.$$  $$+\frac{1}{2}$$  $$\left.+\frac{1}{4}\right.$$  $$\left.\right)$$

$-x^4\Big(\frac{1}{8}$  $+\frac{1}{4}$  $+\frac{1}{2}$  $+\frac{1}{8}$  $+\frac{1}{4}$ 

$+\frac{1}{8}$  $+\frac{1}{4}$  $+\frac{1}{4}$  $\Big)$

and

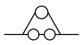
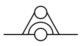
$X_2 = 1 - x\frac{1}{2}$  $- x^2\frac{1}{2}$  $- x^3\Big(\frac{1}{2}$  $+\frac{3}{8}$  $\Big)$

$- x^4\Big(\frac{3}{8}$  $+\frac{1}{4}$  $+\frac{1}{2}$  $+\frac{3}{8}$  $+\frac{3}{8}$ 

$+\frac{1}{8}$  $-\frac{1}{8}$  $\Big)$

where the first 2 lines come from inserting X_2 into q_1 and the third line comes from inserting X_2 into q_3.

Consequently let

$q_4 = \frac{1}{8}$  $-\frac{1}{8}$  $-\frac{1}{4}$  $+\frac{1}{8}$  $+\frac{1}{8}$ 

which we can check is primitive. Continue likewise.

Chapter 6

Reduction to geometric series

6.1 Single equations

like our Yukawa example

Let $D = \text{sign}(s)\gamma \cdot \partial_{-\rho}$ and $F_k(\rho) = \sum_{i=0}^{t_k} F_{k,i}(\rho)$ so the Dyson-Schwinger equation (3.3) reads

$$\gamma \cdot L = \sum_{k \geq 1} x^k (1-D)^{1-sk}(e^{-L\rho} - 1) F_k(\rho) \Big|_{\rho=0}$$

Only terms $L^j x^k$ with $k \geq j \geq 1$ occur by Lemma 4.15 so this series lies in $(\mathbb{R}[L])[[x]]$. Then we have the following

Theorem 6.1. *There exists unique $r_k, r_{k,i} \in \mathbb{R}$, $k \geq 1$, $1 \leq i < k$ such that*

Yesterday: $F = \frac{-1}{\rho(2-\rho)}$

$$\sum_k x^k (1-D)^{1-sk}(e^{-L\rho} - 1) F_k(\rho) \Big|_{\rho=0}$$
$$= \sum_k x^k (1-D)^{1-sk}(e^{-L\rho} - 1) \left(\frac{r_k}{\rho(1-\rho)} + \sum_{1 \leq i < k} \frac{r_{k,i} L^i}{\rho} \right) \Big|_{\rho=0}$$

Proof. For $\ell \geq 0$ the series in x

$$x^k (1-D)^{1-sk} \rho^\ell |_{\rho=0}$$

has no term of degree less than $k + \ell$ since $\gamma_i(x)$ has no term of degree less than i by Lemma 4.15. It follows that

$$x^k (1-D)^{1-sk}(e^{-L\rho} - 1) \frac{1}{\rho(1-\rho)} \Big|_{\rho=0} = -Lx^k + O(x^{k+1})$$

and

$$x^k (1-D)^{1-sk}(e^{-L\rho} - 1) \frac{L^i}{\rho} \Big|_{\rho=0} = -L^{i+1} x^k + O(x^{k+1})$$

Now expand

$$F_{k,i} = \sum_{j=-1}^{\infty} f_{k,i,j} \rho^j.$$

and define r_n, $r_{n,i}$ recursively in n so

$$\sum_k x^k(1-D)^{1-sk}(e^{-L\rho}-1)F_k(\rho)\Big|_{\rho=0}$$
$$= \sum_k x^k(1-D)^{1-sk}(e^{-L\rho}-1)\left(\frac{r_k}{\rho(1-\rho)} + \sum_{1\leq i<k}\frac{r_{k,i}L^i}{\rho}\right)\Bigg|_{\rho=0} + O(x^{n+1}).$$

This is possible since as noted above the coefficient of x^n in $\sum_k x^k(1-D)^{1-sk}(e^{-L\rho}-1)F_k(\rho)|_{\rho=0}$ is a polynomial in L with degree at most $n-1$. □

The meaning of this theorem is that we can modify the Mellin transforms of the primitives to be geometric series at order L. The higher powers of ρ in the Mellin transform of a primitive at k loops become part of the coefficients of primitives at higher loops. Note that there are now terms at each loop order even if this was not originally the case.

Example 6.2. Consider the case $s = 2$ with a single B_+ at order 1 as in Example 3.5. Write

$$F = \sum_{j=-1}^{\infty} f_j\rho^j.$$

then computation gives

$$\begin{aligned}
r_1 &= f_{-1}\\
r_2 &= f_{-1}^2 - f_{-1}f_0\\
r_{2,1} &= 0\\
r_3 &= 2f_{-1}^3 + f_{-1}^2(-4f_0 + f_1) + f_{-1}f_0^2\\
r_{3,1} &= -f_{-1}^3 + f_{-1}^2 f_0\\
r_{3,2} &= 0\\
r_4 &= 2f_{-1}^4 + f_{-1}^3(-12f_0 + 6f_1 - f_2) + f_{-1}^2(9f_0^2 - 3f_0f_1) - f_{-1}f_0^3\\
r_{4,1} &= -f_{-1}^4 + f_{-1}^3(6f_0 - 2f_1) - 3f_{-1}^2f_0^2\\
r_{4,2} &= \frac{7}{6}f_{-1}^4 - \frac{7}{6}f_{-1}^3f_0\\
r_{4,3} &= 0\\
r_5 &= -10f_{-1}^5 + f_{-1}^4(-6f_0 + 18f_1 - 8f_2 + f_3) + f_{-1}^3(40f_0^2 - 32f_0f_1 + 4f_0f_2 + 2f_1^2)\\
&\quad + f_{-1}^2(-16f_0^3 + 6f_0^2f_1) + f_{-1}f_0^4\\
&\ \ \vdots
\end{aligned}$$

These identities are at present still a mystery. Even the coefficients of f_{-1}^k in r_k do not appear in Sloane's encyclopedia of integer sequences [28] in any straightforward manner. In the case

$$F(\rho) = \frac{-1}{\rho(1-\rho)(2-\rho)(3-\rho)},$$

as in the ϕ^3 example from [5], the above specializes to the also mysterious sequence

$$\begin{aligned}
r_1 &= -\frac{1}{6} \\
r_2 &= -\frac{5}{6^3} & r_{2,1} &= 0 \\
r_3 &= -\frac{14}{6^5} & r_{3,1} &= \frac{-5}{6^4} & r_{3,2} &= 0 \\
r_4 &= \frac{563}{6^7} & r_{4,1} &= \frac{-173}{6^6} & r_{4,2} &= \frac{-35}{6^6} \\
r_5 &= \frac{13030}{6^9} & &\vdots & &\vdots \\
r_6 &= -\frac{194178}{6^{11}}
\end{aligned}$$

Note that even if the coefficients of the original Mellin transforms are all of one sign the r_k may unfortunately not be so.

6.2 Systems

As in the single equation case we can reduce to geometric series Mellin transforms at order L.

Theorem 6.3. *There exists unique $r_k^j, r_{k,i}^j \in \mathbb{R}$, $k \geq 1$, $1 \leq i < k$, $j \in \mathcal{R}$ such that*

$$\begin{aligned}
&\sum_{k\geq 1}\sum_{i=0}^{t_k} x^k(1-\mathrm{sign}(s_r)\gamma^r\cdot\partial_{-\rho})^{1-s_rk}\prod_{j\in\mathcal{R}\smallsetminus\{r\}}(1-\mathrm{sign}(s_j)\gamma^j\partial_{-\rho})^{-s_jk}(e^{-L\rho}-1)F^{k,i}(\rho)\Big|_{\rho=0} \\
&=\sum_{k\geq 1}\sum_{i=0}^{t_k} x^k(1-\mathrm{sign}(s_r)\gamma^r\cdot\partial_{-\rho})^{1-s_rk}\prod_{j\in\mathcal{R}\smallsetminus\{r\}}(1-\mathrm{sign}(s_j)\gamma^j\partial_{-\rho})^{-s_jk} \\
&\qquad (e^{-L\rho}-1)\left(\frac{r_{k,i}^r}{\rho(1-\rho)}+\sum_{1\leq i<k}\frac{r_{k,i}^r L^i}{\rho}\right)\Bigg|_{\rho=0}
\end{aligned}$$

Proof. The proof follows as in the single equation case with the observation that for $\ell \geq 0$

$$x^k\prod_{j\in\mathcal{R}}(1+\gamma^j\cdot\partial_{-\rho})^{-s_jk+1}\rho^\ell|_{\rho=0}$$

still has lowest term $x^{k+\ell}$. □

Chapter 7

The second recursion

7.1 Single equations

Reducing to geometric series Mellin transforms at order L allows us to write a tidy recursion for γ_1. Again let $D = \text{sign}(s)\gamma \cdot \partial_{-\rho}$ and $F_k(\rho) = \sum_{i=0}^{t_k} F_{k,i}(\rho)$. By Theorem 6.1 we have

$$\gamma \cdot L = \sum_k x^k (1-D)^{1-sk}(e^{-L\rho}-1)\left(\frac{r_k}{\rho(1-\rho)} + \sum_{1\leq i<k} \frac{r_{k,i}L^i}{\rho}\right)\Bigg|_{\rho=0} \tag{7.1}$$

Taking the coefficients of L and L^2 gives

$$\gamma_1 = \sum_k x^k (1-D)^{1-sk}\left(\frac{-r_k}{1-\rho}\right)\Bigg|_{\rho=0}$$

$$\gamma_2 = \sum_k x^k (1-D)^{1-sk}\left(\rho\frac{r_k}{2(1-\rho)} - r_{k,1}\right)\Bigg|_{\rho=0}$$

So

$$\gamma_1 + 2\gamma_2 = \sum_k x^k (1-D)^{1-sk}(-r_k - 2r_{k,1})\Big|_{\rho=0} = \sum_k p(k)x^k = P(x)$$

where $p(k) = -r_k - 2r_{k,1}$. Then from Theorem 4.2 or 4.10

$$\gamma_1 = P(x) - 2\gamma_2 = P(x) - \gamma_1(\text{sign}(s) - |s|x\partial_x)\gamma_1$$

giving

Theorem 7.1.

$$\gamma_1(x) = P(x) - \gamma_1(x)(\text{sign}(s) - |s|x\partial_x)\gamma_1(x)$$

or at the level of coefficients

$$\gamma_{1,n} = p(n) + \sum_{j=1}^{n-1}(|s|j - \text{sign}(s))\gamma_{1,j}\gamma_{1,n-j}$$

Notice that in defining the r_k and $r_{k,i}$ we only used a geometric series in the first case. Specifically, we used $1/(\rho(1-\rho))$ for r_k but $1/\rho$ for $r_{k,i}$. We could have used $1/\rho$ in all cases; then one L derivative would give $\gamma_1(x) = \sum r_k x^k$ so all the information of γ_1 is in the r_k, we learn nothing recursively. The choice of a geometric series at order L was made to capture the fact that conformal invariance tells us that the Mellin transform will be symmetrical when $\rho \mapsto 1-\rho$, and it also entirely captures examples such as the Yukawa example from [5] and Example 3.5. On the other hand choosing to use a geometric series for the $r_{k,i}$ as well would not have resulted in a tidy recursion for γ_1 using these techniques. We hope that the choice here gives an appropriate balance between representing the underlying physics and giving tractable results all without putting too much of the information into $P(x)$.

Another important question is how to interpret $P(x)$. In cases like the Yukawa example of [5] where the various reductions are unnecessary, then $P(x)$ is simply the renormalized Feynman rules applied to the primitives. In that particular example there is only one primitive, and $P(x) = cx$ for appropriate c. In the general case we would like to interpret $P(x)$ as a modified version of the renormalized Feynman rules applied to the primitives. For the first reduction this is a reasonable interpretation since that reduction simply makes new primitives, either within the Hopf algebra of Feynman graphs or more generally. For the second reduction the idea is that the geometric series part of each Mellin transform is the primary part due to conformal invariance. At order L the rest of the Mellin transform gets pushed into higher loop orders, while at order L^2 the reduction is a bit more crass. This information together gives the r_k and the $r_{k,1}$ and hence gives $P(x)$. So again, in view of the previous paragraph, we view this as a modified version of the Feynman rules applied to the primitives.

7.2 Systems

Theorem 6.3 gives us

$$\begin{aligned}
&\gamma^r \cdot L \\
&= \sum_{k\geq 1}\sum_{i=0}^{t_k} x^k (1-\mathrm{sign}(s_r)\gamma^r\cdot\partial_{-\rho})^{1-s_r k} \prod_{j\in\mathcal{R}\smallsetminus\{r\}} (1-\mathrm{sign}(s_j)\gamma^j\partial_{-\rho})^{-s_j k} \\
&\quad (e^{-L\rho}-1)\left(\frac{r^r_{k,i}}{\rho(1-\rho)} + \prod_{1\leq i<k}\frac{r^r_{k,i}L^i}{\rho}\right)\Bigg|_{\rho=0}
\end{aligned}$$

As in the single equation case we can find tidy recursions for the γ^r_1 by comparing the coefficients of L and L^2 in the above. We get

$$\gamma^r_1 = -\sum_k x^k(1-\mathrm{sign}(s_r)\gamma^r\cdot\partial_{-\rho})^{1-s_r k}\prod_{j\in\mathcal{R}\smallsetminus\{r\}}(1-\mathrm{sign}(s_j)\gamma^j\cdot\partial_{-\rho})^{-s_j k}\frac{-r^r_k}{1-\rho}|_{\rho=0}$$

and

$$\begin{aligned}
2\gamma^r_2 = &\sum_k x^k(1-\mathrm{sign}(s_r)\gamma^r\cdot\partial_{-\rho})^{1-s_r k}\prod_{j\in\mathcal{R}\smallsetminus\{r\}}(1-\mathrm{sign}(s_j)\gamma^j\cdot\partial_{-\rho})^{-s_j k} \\
&\left(\frac{\rho r^r_k}{1-\rho} - 2r^r_{k,1}\right)|_{\rho=0}
\end{aligned}$$

$$= -\gamma_1^r - \sum_{k\geq 1}(r_k^r + 2r_{k,1}^r)x^k$$

Thus letting $p^r(k) = -r_k^r - 2r_{k,1}^r$ and using the first recursion (Theorem 4.1 or 4.14)

$$\gamma_1^r = \sum_{k\geq 1} p^k(k)x^k - 2\gamma_2^r = \sum_{k\geq 1} p^r(k)x^k - \mathrm{sign}(s_r)\gamma_1^r(x)^2 + \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)x\partial_x\gamma_1^r(x)$$

giving

Theorem 7.2.

$$\gamma_1^r = \sum_{k\geq 1} p^r(k)x^k - \mathrm{sign}(s_r)\gamma_1^r(x)^2 + \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)x\partial_x\gamma_1^r(x)$$

or at the level of coefficients

$$\gamma_{1,n}^r = p^r(n) + \sum_{i=1}^{n-1}(|s_r|i - \mathrm{sign}(s_r))\gamma_{1,i}^r\gamma_{1,n-i}^r + \sum_{\substack{j\in\mathcal{R}\\ j\neq r}}\sum_{i=1}^{n-1}(|s_j|i)\gamma_{1,n-i}^j\gamma_{1,i}^r$$

7.3 Variants

The value of the reduction to geometric series is that if $F(\rho) = r/(\rho(1-\rho))$ then $\rho^2 F(\rho) = \rho F(\rho) - 1$. However this reduction is rather crass, particularly for higher orders of L, so it is worth considering other special forms of F as in the following example.

Example 7.3. Consider again the ϕ^3 example from [5] as setup in Example 6.2. We have $s = 2$ and

$$F(\rho) = \frac{-1}{\rho(1-\rho)(2-\rho)(3-\rho)},$$

so

$$\begin{aligned}\rho F(\rho) =& \frac{-1}{(1-\rho)(2-\rho)(3-\rho)}\\ =& -\frac{1}{6}\left(1 + \frac{\rho - \frac{11}{6}\rho^2 + \frac{1}{6}\rho^3}{(1-\rho)(2-\rho)(3-\rho)}\right)\\ =& -\frac{1}{6} + \rho^2 F(\rho) - \frac{11}{6}\rho^3 F(\rho) + \frac{1}{6}\rho^4 F(\rho)\end{aligned}$$

This gives that

$$\begin{aligned}\gamma_1 =& -x(1-\gamma\cdot\partial_{-\rho})^{-1}\rho F(\rho)|_{\rho=0}\\ =& \frac{1}{6}x(1-\gamma\cdot\partial_{-\rho})^{-1}1|_{\rho=0} - x(1-\gamma\cdot\partial_{-\rho})^{-1}\rho^2 F(\rho)|_{\rho=0}\\ &+ \frac{11}{6}x(1-\gamma\cdot\partial_{-\rho})^{-1}\rho^3 F(\rho)|_{\rho=0} - \frac{1}{6}x(1-\gamma\cdot\partial_{-\rho})^{-1}\rho^4 F(\rho)|_{\rho=0}\\ =& \frac{x}{6} - 2\gamma_2 - 11\gamma_3 - 4\gamma_4.\end{aligned}$$

In view of Theorem 4.2 or 4.10, which in this case reads

$$\gamma_k = \frac{1}{k}\gamma_1(x)(1 - 2x\partial_x)\gamma_{k-1}(x),$$

we thus get a fourth order differential equation for γ_1 which contains no infinite series and for which we completely understand the signs of the coefficients.

Chapter 8

The radius of convergence

8.1 Single equations

We see from the second recursion, Theorem 7.1, that if $\sum p(k)x^k$ is Gevrey-n but not Gevrey-m for any $m < n$, then γ_1 is at best Gevrey-n.

Of most interest for quantum field theory applications is the case where only finitely many $p(k)$ are nonzero but all are nonnegative and the case where $p(k) = c^k k!$ giving the Lipatov bound. In both cases $\sum p(k)x^k$ is Gevrey-1. Also for positivity reasons we are interested in $s \geq 1$ or $s < 0$. Thus for the remainder of this section the following assumptions are in effect.

Assumption 8.1. Assume $|s| \geq 1$ or $s < 0$. Assume $p(k) \geq 0$ for $k \geq 1$ and

$$\sum_{k \geq 1} x^k \frac{p(k)}{k!} = f(x)$$

has radius of convergence $0 < \rho \leq \infty$ and is not identically zero.

Under these assumptions γ_1 is also Gevrey-1 and the radius is the minimum of ρ and $1/(sa_1)$ (where we view $1/(sa_1)$ as $+\infty$ in the case $a_1 = 0$) the proof of which is the content of this section.

Definition 8.2. Let $a_n = \gamma_{1,n}/n!$, $\mathbf{A}(x) = \sum_{n \geq 1} a_n x^n$, and let ρ_a be the radius of convergence of $\mathbf{A}(x)$.

Then $a_1 = \gamma_{1,1} = p(1)$ and

$$\begin{aligned} a_n =& \frac{p(n)}{n!} + \sum_{j=1}^{n-1} (|s|j - \operatorname{sign}(s)) \binom{n}{j}^{-1} a_j a_{n-j} \\ =& \frac{p(n)}{n!} + \frac{1}{2} \sum_{j=1}^{n-1} (|s|j - \operatorname{sign}(s) + |s|(n-j) - \operatorname{sign}(s)) \binom{n}{j}^{-1} a_j a_{n-j} \\ =& \frac{p(n)}{n!} + \left(|s|\frac{n}{2} - \operatorname{sign}(s)\right) \sum_{j=1}^{n-1} \binom{n}{j}^{-1} a_j a_{n-j} \end{aligned} \tag{8.1}$$

Inductively, we see that $a_1, a_2, \ldots$ are all nonnegative

Note that if $s = 1$, $p(1) > 0$, and $p(n) = 0$ for $n > 1$ then $a_1 = p(1)$, $a_n = 0$ for $n > 1$ solves the recursion. In this case $\rho_a = \rho = \infty$, but $0 < 1/(|s|a_1) < \infty$. This boundary case is the only case with this behavior as we see in the following Proposition.

Proposition 8.3. *Suppose that either $s \neq 1$, or $p(n) > 0$ for some $n > 1$. Then $\rho_a \leq \min\{\rho, 1/(|s|a_1)\}$ where $1/(|s|a_1) = \infty$ when $a_1 = 0$*

Proof. Take the first and last terms of the sum (8.1) to get

$$a_n \geq \frac{p(n)}{n!} + |s|\frac{n-2}{n}a_1 a_{n-1} \tag{8.2}$$

for $n \geq 2$. In particular

$$a_n \geq \frac{p(n)}{n!}$$

so $\rho_a \leq \rho$. Further if a_1 and at least one a_j, $j > 1$ are nonzero then by (8.2) $a_n > 0$ for all $n > j$, since the $p(n)$ are assumed nonnegative. In this case, then, we also have

$$\frac{a_{n-1}}{a_n} \leq \frac{n}{(n-2)a_1|s|}$$

and so $\rho_a \leq 1/(|s|a_1)$. The inequality $\rho_a \leq 1/(|s|a_1)$ also holds by convention if $a_1 = 0$. Finally suppose $a_1 \neq 0$ but $a_n = 0$ for all $n > 1$. Then $p(n) = 0$ for all $n > 1$, and, from (8.1) for $n = 2$, $s = 1$. This is the case we have excluded. The result follows. ☐

For the lower bound on the radius we need a few preliminary results. First, some simple combinatorial facts.

Lemma 8.4.

$$\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$$

for $n, k \in \mathbb{Z}$, $n \geq k \geq 0$.

Proof.

$$\binom{n}{k} = \frac{n}{k}\frac{n-1}{k-1}\cdots\frac{n-k+1}{1} \geq \frac{n}{k}\frac{n}{k}\cdots\frac{n}{k} = \left(\frac{n}{k}\right)^k$$

☐

Lemma 8.5. *Given $0 < \theta < 1$*

$$\frac{1}{n}\binom{n}{j} \geq \frac{\theta^{-j+1}}{j}$$

for $1 \leq j \leq \theta n$ and $n \geq 2$.

Proof. Fix n. Write $j = \lambda n$, $0 < \lambda \leq \theta$. Then using Lemma 8.4

$$\frac{1}{n}\binom{n}{j} = \frac{1}{n}\binom{n}{\lambda n} \geq \frac{n^{\lambda n - 1}}{(\lambda n)^{\lambda n}} = \frac{\lambda^{-\lambda n+1}}{\lambda n} \geq \frac{\theta^{-j+1}}{j}$$

☐

Second, we need to understand the behavior of $\sum a_n x^n$ at the radius of convergence.

Lemma 8.6.

$$\mathbf{A}(x) \leq f(x) + x|s|(1+\epsilon)\mathbf{A}'(\theta x)\mathbf{A}(x) + \frac{|s|}{2x}\frac{d}{dx}\left(x^2\mathbf{A}^2(\theta^\theta x)\right) + P_\epsilon(x)$$

for all $0 < \theta < 1/e$*,* $\epsilon > 0$*, and* $0 < x < \rho_a$*, where* $P_\epsilon(x)$ *is a polynomial in* x *with nonnegative coefficients.*

Proof. Take $0 < \theta < 1/e$ and $\epsilon > 0$.

$$\begin{aligned}
a_n =& \frac{p(n)}{n!} + \left(|s|\frac{n}{2} - \operatorname{sign}(s)\right)\sum_{j=1}^{n-1}\binom{n}{j}^{-1} a_j a_{n-j} \\
\leq& \frac{p(n)}{n!} + |s|(n+2)\sum_{1\leq j\leq \theta n}\binom{n}{j}^{-1} a_j a_{n-j} + |s|\frac{n+2}{2}\sum_{\theta n\leq j\leq n-\theta n}\binom{n}{j}^{-1} a_j a_{n-j} \\
\leq& \frac{p(n)}{n!} + |s|\frac{n+2}{n}\sum_{1\leq j\leq \theta n} j\theta^{j-1} a_j a_{n-j} + |s|\binom{n}{\lceil \theta n\rceil}^{-1}\frac{n+2}{2}\sum_{\theta n\leq j\leq n-\theta n} a_j a_{n-j} \\
& \text{by Lemma 8.5} \\
\leq& \frac{p(n)}{n!} + |s|\frac{n+2}{n}\sum_{1\leq j\leq \theta n} j\theta^{j-1} a_j a_{n-j} + \frac{|s|}{2}(n+2)\theta^{\theta n}\sum_{\theta n\leq j\leq n-\theta n} a_j a_{n-j} \\
& \text{by Lemma 8.5 and since } (x/n)^x \text{ is decreasing for } 0 < x < n/e
\end{aligned}$$

Thus for n sufficiently large that $(n+2)/n \leq 1+\epsilon$ the coefficients of $\mathbf{A}(x)$ are bounded above by the coefficients of

$$f(x) + x|s|(1+\epsilon)\mathbf{A}'(\theta x)\mathbf{A}(x) + \frac{|s|}{2x}\frac{d}{dx}\left(x^2\mathbf{A}^2(\theta^\theta x)\right).$$

Adding a polynomial to dominate the earlier coefficients of $\mathbf{A}(x)$ we get that the coefficients of $\mathbf{A}(x)$ are bounded above by the coefficients of

$$f(x) + x|s|(1+\epsilon)\mathbf{A}'(\theta x)\mathbf{A}(x) + \frac{|s|}{2x}\frac{d}{dx}\left(x^2\mathbf{A}^2(\theta^\theta x)\right) + P_\epsilon(x).$$

Since all coefficients are nonnegative, for any $0 < x < \rho_a$ we have

$$\mathbf{A}(x) \leq f(x) + x|s|(1+\epsilon)\mathbf{A}'(\theta x)\mathbf{A}(x) + \frac{|s|}{2x}\frac{d}{dx}\left(x^2\mathbf{A}^2(\theta^\theta x)\right) + P_\epsilon(x).$$

□

Lemma 8.7. *If* $\rho_a < \rho$ *and* $\rho_a < 1/(|s|a_1)$ *then* $\mathbf{A}(\rho_a) < \infty$*.*

Proof. Consider Lemma 8.6. Choose $\theta > 0$ and $\epsilon > 0$ so that

$$\rho_a < \frac{1}{|s|(1+\epsilon)\mathbf{A}'(\theta\rho_a)} \tag{8.3}$$

which is possible since $\lim_{\theta\to 0} \mathbf{A}'(\theta x) = a_1$ and $\rho_a < 1/(|s|a_1)$. Letting $x \to \rho_a$ we see that

$$\lim_{x\to\rho_a} \mathbf{A}(x) \le C + \rho_a|s|(1+\epsilon)\mathbf{A}'(\theta\rho_a) \lim_{x\to\rho_a} \mathbf{A}(x)$$

where C is constant, since $\theta^\theta < 1$, and $\rho_a < \rho$. So

$$(1 - \rho_a|s|(1+\epsilon)\mathbf{A}'(\theta\rho_a)) \lim_{x\to\rho_a} \mathbf{A}(x) \le C.$$

But by (8.3), $1 - \rho_a|s|(1+\epsilon)\mathbf{A}'(\theta\rho_a) > 0$, so $\mathbf{A}(\rho_a) < \infty$. □

Lemma 8.8. *If $\rho_a < \rho$ and $\rho_a < 1/(|s|a_1)$ then $\mathbf{A}(x)$ is unbounded on $0 < x < \rho_a$.*

Proof. Take any $\epsilon > 0$. Then there exists an $N > 0$ such that for $n > N$

$$a_n \le \frac{p(n)}{n!} + |s|a_1 a_{n-1} + \epsilon \sum_{j=1}^{n-1} a_j a_{n-j}$$

Define

$$c_n = \begin{cases} a_n & \text{if } a_n > \frac{p(n)}{n!} + |s|c_1c_{n-1} + \epsilon\sum_{j=1}^{n-1} c_jc_{n-j} \\ \frac{p(n)}{n!} + |s|c_1c_{n-1} + \epsilon\sum_{j=1}^{n-1} c_jc_{n-j} & \text{otherwise (in particular when } n > N) \end{cases}$$

In particular $c_1 = a_1$. Let $\mathbf{C}(x) = \sum_{x\ge 1} c_n x^n$ (which implicitly depends on ϵ) have radius ρ_ϵ. Since $a_n \le c_n$, $\rho_a \ge \rho_\epsilon$. Rewriting with generating series

$$\mathbf{C}(x) = f(x) + |s|a_1 x\mathbf{C}(x) + \epsilon\mathbf{C}^2(x) + P_\epsilon(x)$$

where $P_\epsilon(x)$ is some polynomial. This equation can be solved by the quadratic formula. The discriminant is

$$\Delta_\epsilon = (1 - |s|a_1x)^2 - 4\epsilon(f(x) + P_\epsilon(x)).$$

ρ_ϵ is the closest root to 0 of Δ_ϵ.

By construction, the coefficient of x^n in $P_\epsilon(x)$ is bounded by a_n. Suppose $\mathbf{A}(\rho_a) < \infty$. Thus $f(\rho_a)+P_\epsilon(\rho_a) \le f(\rho_a)+\mathbf{A}(\rho_a)$. By the nonnegativity of the coefficients of f and P_ϵ then $f(x)+P_\epsilon(x)$ independently of ϵ for for $0 < x \le \rho_a$. Thus

$$\lim_{\epsilon\to 0} \Delta_\epsilon = (1 - |s|a_1\rho_a)^2$$

for $0 < x \le \rho_a$. So

$$\frac{1}{|s|a_1} > \rho_a \ge \rho_\epsilon \to \frac{1}{|s|a_1}$$

as $\epsilon \to 0$ which is a contradiction, giving that $\mathbf{A}(x)$ is unbounded on $0 < x < \rho_a$. □

Proposition 8.9. *$\rho_a \ge \min\{\rho, 1/(|s|a_1)\}$, where $1/(|s|a_1) = \infty$ when $a_1 = 0$.*

Proof. Suppose on the contrary that $\rho_a < \rho$ and $\rho_a < 1/(|s|a_1)$ then Lemmas 8.7 and 8.8 contradict each other so this cannot be the case. □

Taking the two bounds together we get the final result

Theorem 8.10. *Assume $\sum_{k\ge 1} x^k p(k)/k!$ has radius ρ. Then $\sum x^n\gamma_{1,n}/n!$ converges with radius of convergence $\min\{\rho, 1/(s\gamma_{1,1})\}$, where $1/(|s|\gamma_{1,1}) = \infty$ if $\gamma_{1,1} = 0$.*

Proof. Immediate from Lemmas 8.3 and 8.9. □

8.2 Systems

Now suppose we have a system of Dyson-Schwinger equations as in (3.4)

$$X^r(x) = \mathbb{I} - \text{sign}(s_r)\sum_{k\geq 1}\sum_{i=0}^{t_k^r} x^k B_+^{k,i;r}(X^r Q^k)$$

for $r \in \mathcal{R}$ with $\mathcal{R}$ a finite set and where

$$Q = \prod_{r\in\mathcal{R}} X^r(x)^{-s_r}$$

for all $r \in \mathcal{R}$.

To attack the growth of the γ_1^r we will again assume that the series of primitives is Gevrey-1 and that the s_r give nonnegative series.

Assumption 8.11. Assume $s_r \geq 1$ or $s_r < 0$ for each $r \in \mathcal{R}$. Assume that

$$\sum_{k\geq 1} x^k \frac{p^r(k)}{k!} = f^r(x)$$

has radius $0 < \rho_r \leq \infty$, $p^r(k) > 0$ for $k \geq 1$, and the $f^r(x)$ are not identically 0.

We'll proceed by similar bounds to before.

Definition 8.12. Let $a_n^r = \gamma_{1,n}^r/n!$ and $\mathbf{A}^r(x) = \sum_{n\geq 1} a_n^r x^n$.

Again the a_i^r are all nonnegative.

Then

$$a_n^r = \frac{p^r(n)}{n!} + \sum_{i=1}^{n-1}(|s_r|i - \text{sign}(s_r))a_i^r a_{n-i}^r \binom{n}{i}^{-1} + \sum_{\substack{j\in\mathcal{R}\\ j\neq r}}\sum_{i=1}^{n-1}(|s_j|i)a_{n-i}^j a_i^r \binom{n}{i}^{-1} \tag{8.4}$$

Proposition 8.13. *For all $r \in \mathcal{R}$, the radius of convergence of $\mathbf{A}^r(x)$ is at most*

$$\min\left\{\rho_r, \frac{1}{\sum_{j\in\mathcal{R}}|s_j|a_1^j}\right\}$$

interpreting the second possibility to be ∞ when $\sum_{j\in\mathcal{R}}|s_j|a_1^j = 0$.

Proof. Taking the last term in each sum of (8.4) we have

$$a_n^r \geq \frac{p^r(n)}{n!} + \left(\sum_{j\in\mathcal{R}}|s_j|a_1^j\right)\frac{n-2}{n}a_{n-1}^r$$

Let b_n^r be the series defined by $b_1^r = a_1^r$ and equality in the above recursion. Then argue as in the single equation case, Proposition 8.3, to get that the radius of $\mathbf{A}^r(x)$ is at most

$$\min\left\{\rho_r, \frac{1}{\sum_{j\in\mathcal{R}}|s_j|a_1^j}\right\}.$$

□

Proposition 8.14. *The radius of convergence of* $\sum_{r\in\mathcal{R}} \mathbf{A}^r(x)$ *is at least*

$$\min_{r\in\mathcal{R}} \left\{ \rho_r, \frac{1}{\sum_{j\in\mathcal{R}} |s_j| a_1^j} \right\}$$

interpreting the second possibility to be ∞ *when* $\sum_{j\in\mathcal{R}} |s_j| a_1^j = 0$.

Proof. The overall structure of the argument is as in the single equation case.

The equivalent of Lemma 8.6 for this case follows from

$$\begin{aligned}
\sum_{r\in\mathcal{R}} a_n^r &\le \sum_{r\in\mathcal{R}} \frac{p^r(n)}{n!} + \frac{n+2}{n} \sum_{j\in\mathcal{R}} |s_j| a_1^j \sum_{r\in\mathcal{R}} a_{n-1}^r + \sum_{r,j\in\mathcal{R}} \sum_{i=1}^{n-1} (|s_j|(i+1)) a_{n-i}^j a_i^r \binom{n}{i}^{-1} \\
&\le \sum_{r\in\mathcal{R}} \frac{p^r(n)}{n!} + \frac{n+2}{n} \sum_{j\in\mathcal{R}} |s_j| a_1^j \sum_{r\in\mathcal{R}} a_{n-1}^r \\
&\quad + \max_j(|s_j|) \sum_{i=2}^{n-2} (i+1) \binom{n}{i}^{-1} \left(\sum_{r\in\mathcal{R}} a_{n-i}^r \right) \left(\sum_{r\in\mathcal{R}} a_i^r \right) \\
&= \sum_{r\in\mathcal{R}} \frac{p^r(n)}{n!} + \frac{n+2}{n} \sum_{j\in\mathcal{R}} |s_j| a_1^j \sum_{r\in\mathcal{R}} a_{n-1}^r \\
&\quad + \max_j(|s_j|)(n+2) \sum_{2\le i\le \theta n} \binom{n}{i}^{-1} \left(\sum_{r\in\mathcal{R}} a_{n-i}^r \right) \left(\sum_{r\in\mathcal{R}} a_i^r \right) \\
&\quad + \max_j(|s_j|) \frac{n+2}{2} \sum_{\theta n\le i\le n-\theta n} \binom{n}{i}^{-1} \left(\sum_{r\in\mathcal{R}} a_{n-i}^r \right) \left(\sum_{r\in\mathcal{R}} a_i^r \right)
\end{aligned}$$

for θ as in Lemma 8.6 with $\sum_{r\in\mathcal{R}} \mathbf{A}^r(x)$ in place of $\mathbf{A}(x)$, where $\mathbf{A}^r(x) = \sum a^r(n) x^n$. Then continue the argument as in Lemma 8.6 with $\sum_{r\in\mathcal{R}} f^r(x)$ in place of $f(x)$ and $\max_j(s_j)$ in place of s, and using the second term to get the correct linear part.

For the argument as in Lemma 8.8 Take any $\epsilon > 0$ then there exists an $N > 0$ such that for $n > N$ we get

$$a_n^r \le \frac{p^r(n)}{n!} + \left(\sum_{j\in\mathcal{R}} |s_j| a_1^j \right) a_{n-1}^r + \epsilon \sum_{i=1}^{n-1} \sum_{j\in\mathcal{R}} a_i^r a_{n-i}^j$$

Taking $\mathbf{C}^r(x)$ to be the series whose coefficients satisfy the above recursion with equality in the cases when this gives a result $\ge a_n^r$ and equal to a_n^r otherwise we get

$$\mathbf{C}^r(x) = f^r(x) + \left(\sum_{j\in\mathcal{R}} |s_j| a_1^j \right) x\mathbf{C}^r(x) + \epsilon \sum_{j\in\mathcal{R}} \mathbf{C}^r(x) \mathbf{C}^j(x) + P_\epsilon^r(x)$$

where P_ϵ^r is a polynomial.

Summing over r we get a recursive equation for $\sum_{r\in\mathcal{R}} \mathbf{C}^r(x)$ of the same form as in the single equation case. Note that since each $\mathbf{C}^r$ is a series with nonnegative coefficients there can be no cancellation of singularities and hence the radius of convergence of each $\mathbf{C}^r$ is at least that of the sum. Thus by the analysis of the single equation case we get a lower bound on the radius of $\sum_r \mathbf{A}^r(x)$ of $\min_{s\in\mathcal{R}}\{\rho_s, 1/\sum_{j\in\mathcal{R}} |s_j| a_1^j\}$. ☐

Proposition 8.15. *Each $\mathbf{A}^s(x)$, $s \in \mathcal{R}$, has the same radius of convergence.*

Proof. Suppose the radius of $\mathbf{A}^s(x)$ was strictly greater than that of $\mathbf{A}^r(x)$. Then we can find $\beta > \delta > 0$ such that

$$a_n^r > \beta^n > \delta^n > a_n^s$$

for n sufficiently large. Pick a $k \geq 1$ such that $a_k^s > 0$. Then

$$\delta^n > a_n^s \geq \frac{|s_r|k!a_k^s}{n\cdots(n-k+1)}a_{n-k}^r > \frac{|s_r|k!a_k^s}{n\cdots(n-k+1)}\beta^{n-k}$$

so

$$\frac{\delta^k}{|s_r|a_k^s}\left(\frac{\delta}{\beta}\right)^{n-k} > \frac{k!}{n\cdots(n-k+1)}$$

which is false for n sufficiently large, giving a contradiction. □

Theorem 8.16. *For all $r \in \mathcal{R}$, $\sum x^n\gamma_{1,n}^r/n!$ converges with radius*

$$\min_{r\in\mathcal{R}}\{\rho_r, 1/\sum_{j\in\mathcal{R}}|s_j|\gamma_{1,1}^j\},$$

where the second possibility is interpreted as ∞ when $\sum_{j\in\mathcal{R}}|s_j|\gamma_{1,1}^j = 0$.

Proof. Take $s \in \mathcal{R}$ such that ρ_s is minimal.

Since we are working with nonnegative series the radius of $\mathbf{A}^s(x)$ is at least that of $\sum_{r\in\mathcal{R}}\mathbf{A}^r(x)$. Hence by Lemmas 8.13 and 8.14 $\mathbf{A}^s(x)$ has radius exactly

$$\min_{r\in\mathcal{R}}\{\rho_r, 1/\sum_{j\in\mathcal{R}}|s_j|a_1^j\}.$$

Thus by Lemma 8.15 all the $\sum a_n^s x^n$ have the same radius $\min_{r\in\mathcal{R}}\{\rho_r, 1/\sum_{j\in\mathcal{R}}|s_j|a_1^j\}$ □

8.3 Possibly negative systems

Let us relax the restriction that $p^r(n) \geq 0$. It is now difficult to make general statements concerning the radius of convergence of the $\mathbf{A}^r(x)$. For example consider the system

$$a_n^1 = \frac{p^1(n)}{n!} + \sum_{j=1}^{n-1}(2j-1)a_j^1a_{n-j}^1\binom{n}{j}^{-1} + \sum_{j=1}^{n-1}ja_j^1a_{n-j}^2\binom{n}{j}^{-1}$$
$$a_n^2 = \frac{p^2(n)}{n!} + \sum_{j=1}^{n-1}(j+1)a_j^2a_{n-j}^2\binom{n}{j}^{-1} + \sum_{j=1}^{n-1}2ja_j^2a_{n-j}^1\binom{n}{j}^{-1}$$

so $s_1 = 2$ and $s_2 = -1$. Suppose also that

$$p^2(2) = -4(a_1^2)^2$$
$$a_1^1 = a_1^2$$
$$p^2(n) = -2(n-1)!a_1^2a_{n-1}^1$$

Then $a_2^2 = 0$ and inductively $a_n^2 = 0$ for $n \geq 2$ so the system degenerates to

$$a_n^1 = \frac{p^1(n)}{n!} + \sum_{j=1}^{n-1}(2j-1)a_j^1 a_{n-j}^1 \binom{n}{j}^{-1} - \frac{n-1}{n}a_1^1 a_{n-1}^1$$
$$a_n^2 = \begin{cases} a_1^1 & \text{if n = 1} \\ 0 & \text{otherwise} \end{cases}$$

We still have a free choice of $p^1(n)$, and hence control of the radius of the a^1 series. On the other hand the a^2 series trivially has infinite radius of convergence.

Generally, finding a lower bound on the radii of the solution series, remains approachable by the preceding methods while control of the radii from above is no longer apparent.

Precisely,

Theorem 8.17. *The radius of convergence of $\sum_{n\geq 1} x^n \gamma_{1,n}^r / n!$ is at least*

$$\min_{r\in\mathcal{R}} \left\{ \rho_r, \frac{1}{\left|\sum_{j\in\mathcal{R}} |s_j|\gamma_{1,1}^j\right|} \right\}$$

where the second possibility is interpreted as ∞ when $\sum_{j\in\mathcal{R}} |s_j|\gamma_{1,1}^j = 0$.

Proof. for any $\epsilon > 0$

$$\begin{aligned} |a_n^r| &\leq \frac{|p^r(n)|}{n!} + \left|\sum_{j\in\mathcal{R}} |s_j| a_1^j\right| |a_{n-1}^r| + \sum_{i=1}^{n-2} |(|s_r|i - \text{sign}(s_r))||a_i^r||a_{n-i}^r| \binom{n}{i}^{-1} \\ &\quad + \sum_{\substack{j\in\mathcal{R} \\ j\neq r}} \sum_{i=1}^{n-2} |s_j||i|a_{n-i}^j||a_i^r| \binom{n}{i}^{-1} \\ &\leq \frac{|p^r(n)|}{n!} + \left|\sum_{j\in\mathcal{R}} |s_j| a_1^j\right| |a_{n-1}^r| + \epsilon \sum_{i=1}^{n-1} \sum_{j\in\mathcal{R}} |a_i^r||a_{n-i}^j| \end{aligned}$$

So, for a lower bound on the radius we may proceed as in the nonnegative case using the absolute value of the coefficients. □

Chapter 9

The second recursion as a differential equation

In this final chapter let us consider the second recursion derived in Chapter 7 as a differential equation rather than as a recursive equation. That is, in the system case

$$\gamma_1^r(x) = P_r(x) - \operatorname{sign}(s_r)\gamma_1^r(x)^2 + \left(\sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)\right) x\partial_x\gamma_1^r(x) \tag{9.1}$$

as r runs over $\mathcal{R}$, the residues of the theory. While in the single equation case

$$m\gamma_1(x) = P(x) - \operatorname{sign}(s)\gamma_1(x)^2 + |s|\gamma_1(x)x\partial_x\gamma_1(x) \tag{9.2}$$

The parameter m was added to keep the QED example in the most natural form, however it is not interesting since since we can remove it by the transformation $\gamma_1(x) \mapsto m\gamma_1(x)$, $P(x) \mapsto m^2P(x)$.

No non-trivial results will be proved in this chapter, we will simply discuss some features of some important examples. More substantial results will appear in [31].

As a consequence of the renormalization group origin of the first recursion discussed in section 4.1 the β-function for the system shows up as the coefficient of $(\gamma_1^r)'(x)$, namely

$$\beta(x) = x\sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)$$

in the system case and

$$\beta(x) = x|s|\gamma_1(x)$$

in the single equation case. Consequently this differential equation is well suited to improving our understanding of the β-function.

In particular in the single equation case, we see immediately from (9.2) that any zeroes of $\beta(x)$ must occur either where $P(x) = 0$ or where $\gamma_1'(x)$ is infinite. The second of these possibilities does not turn out to be physically reasonable as we will discuss in more detail below. The system case is not quite so simple. Assume $\beta(x) = 0$. If we rule out infinite $(\gamma_1^r)'(x)$, then we can only conclude that for each $r \in \mathcal{R}$

$$\gamma_1^r(x) + \operatorname{sign}(s_r)\gamma_1^r(x)^2 - P_r(x) = 0.$$

In order to extract further information in both the single equation and the system case we will proceed to examine plots of the vector field of $(\gamma_1^r)'(x)$, first in some toy single equation cases, second in the case of QED reduced to one equation, and finally in the 2 equation example of ϕ^4.

Shirkov and collaborators on analytic continuation of the anom. dim. of gluon.

$\frac{1}{g^2(1-\Sigma(g^2,L))}$

Λ_{QCD}

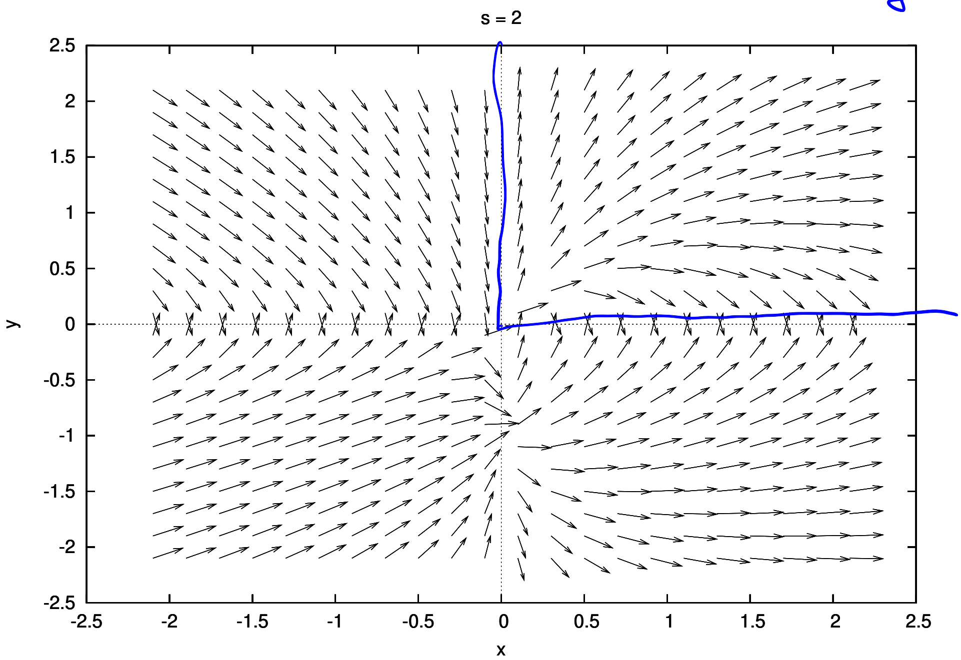


Figure 9.1: The vector field of $\gamma_1'(x)$ with $s = 2$, $m = 1$, and $P(x) = x$.

9.1 Toys

First let us consider a family of examples which are simpler than those which occur in full quantum field theories, namely the family where $m = 1$ and $P(x) = x$.

9.1.1 The case $s = 2$

If we set $s = 2$ we get the situation explored in [5] which describes the piece of massless Yukawa theory consisting of nestings and chainings of the one loop fermion self energy into itself as discussed in Example 3.5. The second recursion viewed as a differential equation is

$G''G = 1 \rightarrow$

$$\gamma_1(x) = x - \gamma_1(x)(1 - 2x\partial_x)\gamma_1(x).$$

Broadhurst and Kreimer [5] solved this Dyson-Schwinger equation by clever rearranging and recognizing the resulting asymptotic expansion. The solution, written in a slightly different form, is given implicitly by

$$\exp\left(\frac{(1+\gamma_1(x))^2}{2x}\right)\sqrt{-x} + \operatorname{erf}\left(\frac{1+\gamma_1(x)}{\sqrt{-2x}}\right)\frac{\sqrt{\pi}}{\sqrt{2}} = C$$

with integration constant C.

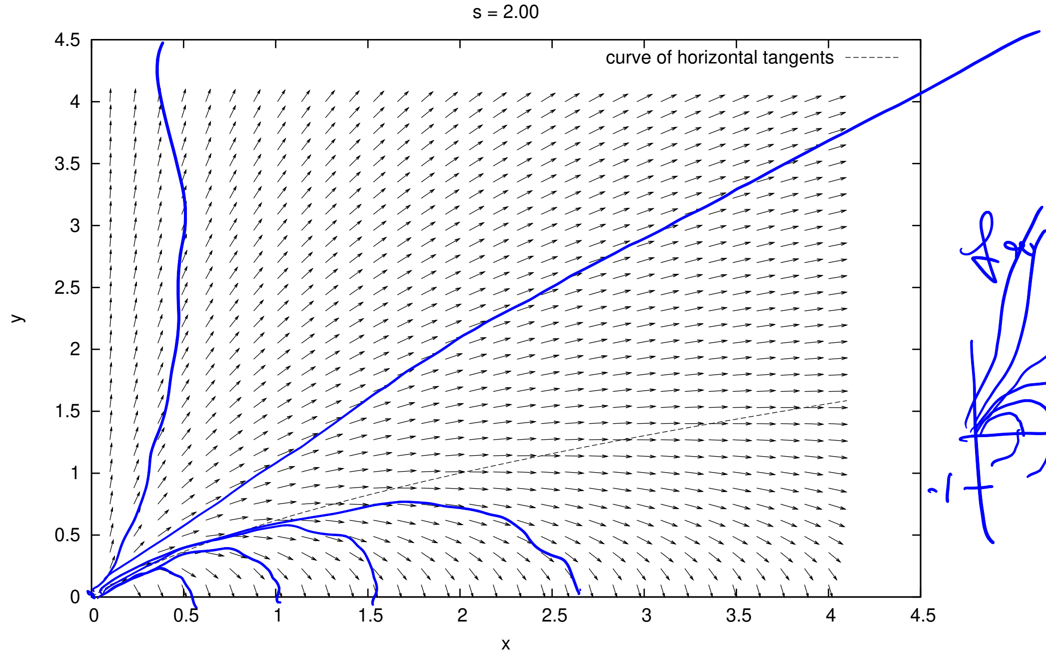
We can proceed to look at the vector field of $\gamma_1'(x)$, see Figure 9.1.

We are primarily interested in the behavior in the first quadrant. Of particular interest are possible zeros of solutions since, in this simple single equation situation, $x\gamma_1(x) = \beta(x)$ where $\beta(x)$ is the β-function of the system.

For QED and QCD, we have the case s = 1. (instead of s = 2)

And P(x) is more complicated.

$P(x) = x + c_2 x^2 + c_3 x^3 + \ldots$



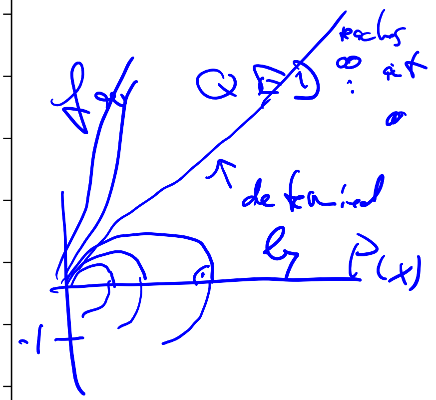
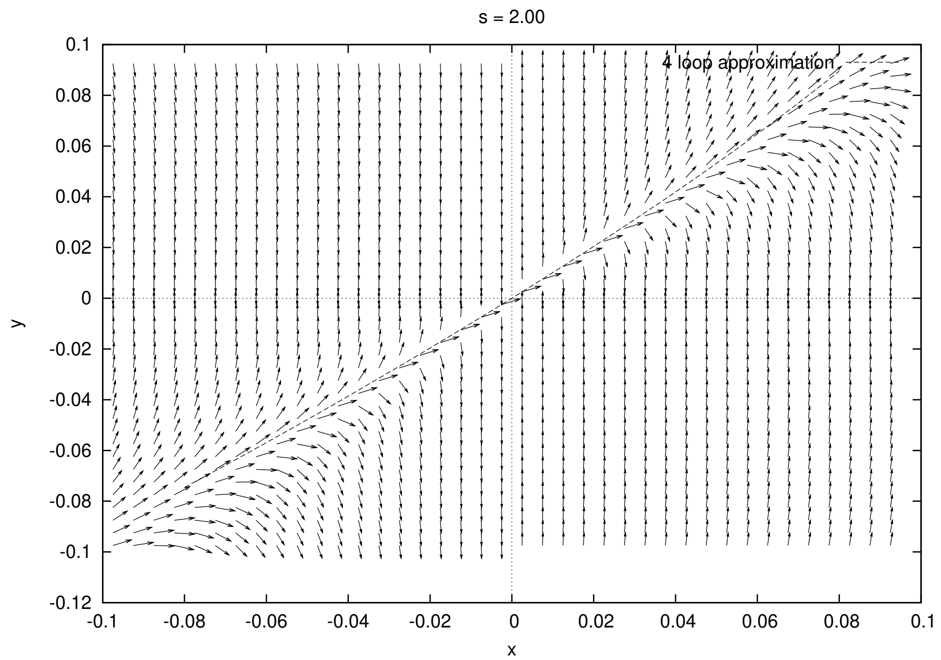


Figure 9.2: Solutions which die in finite time along with the curve where $\gamma_1'(x) = 0$.



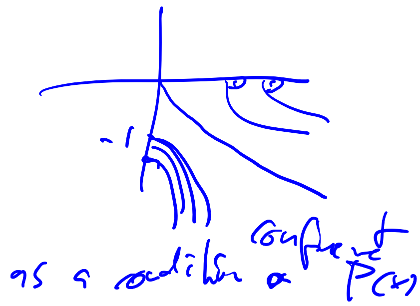


Figure 9.3: The four loop approximation near the origin.

From the figures we notice a family of solutions which come down to hit the x axis with vertical tangent. These solutions have no real continuation past this point. These solutions are consequently unphysical. It is not clear from the figure whether all solutions have this behavior. One of the major goals of [31] is to find conditions guaranteeing the existence of a separatrix.

Viewing the vector field near the origin can be quite misleading, since it appears to have both types of behavior simply because all the solutions have the same asymptotic expansion at the origin. Additionally this implies that the apparent, but potentially false, separatrix is well matched by the first four terms of the asymptotic expansion as illustrated in Figure 9.3. Of course given that we have a recursive equation and an implicit solution we can easily calculate the asymptotic series out to hundreds of terms [5], and the use of a four loop approximation is merely meant to be illustrative.

Another simple observation is that we can derive the equation for the curve where the solutions are horizontal by solving for $\gamma_1'(x)$

$$\gamma_1'(x) = \frac{\gamma_1(x) + \gamma_1^2(x) - x}{2x\gamma_1(x)}$$

and then solving the numerator to get the curve

$$y = \frac{-1 + \sqrt{1+4x}}{2}$$

illustrated in Figure 9.2.

9.1.2 Other cases

Let us return to general s while maintaining the assumption $m = 1$, $P(x) = x$.

The case $s = 0$ is degenerate, giving the algebraic equation $\gamma_1(x) = x - \gamma_1(x)^2$ with solutions

$$\gamma_1(x) = \frac{-1 \pm \sqrt{1+4x}}{2}.$$

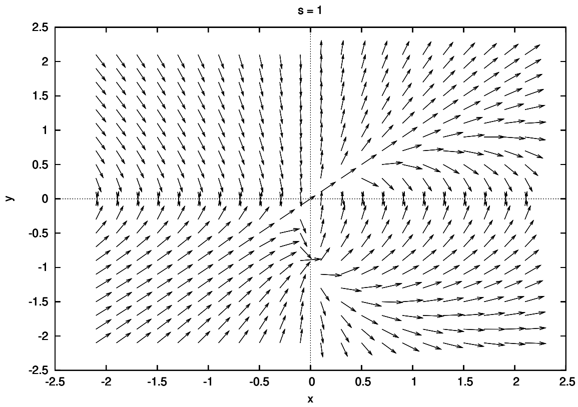
From now on we will assume $s \neq 0$.

We can obtain implicit solutions for a few other isolated values of s using Maple

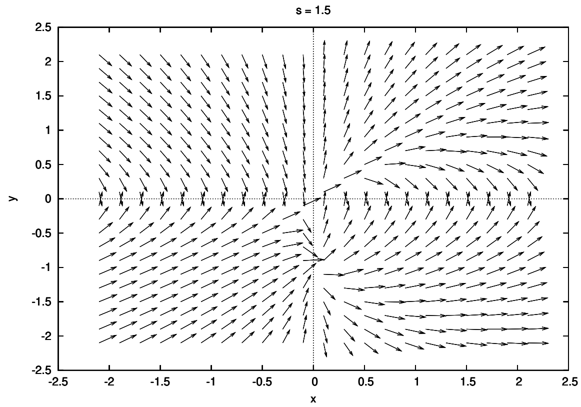
$$\begin{aligned}
s = 1 :&\gamma_1(x) = x + xW\left(C\exp\left(-\frac{1+x}{x}\right)\right),\\
s = \frac{3}{2} :&A\left(X\right) - x^{1/3}2^{1/3}A'\left(X\right) = C\left(B\left(X\right) - x^{1/3}2^{1/3}B'\left(X\right)\right) \text{ where } X = \frac{1+\gamma_1(x)}{2^{2/3}x^{2/3}},\\
s = 2 :&\exp\left(\frac{(1+\gamma_1(x))^2}{2x}\right)\sqrt{-x} + \operatorname{erf}\left(\frac{1+\gamma_1(x)}{\sqrt{-2x}}\right)\frac{\sqrt{\pi}}{\sqrt{2}} = C,\\
s = 3 :&(\gamma_1(x)+1)A\left(X\right) - 2^{2/3}A'\left(X\right) = C\left((\gamma_1(x)+1)B\left(X\right) - 2^{2/3}B'\left(X\right)\right)\\
&\qquad \text{where } X = \frac{(1+\gamma_1(x))^2 + 2x}{2^{4/3}x^{2/3}},
\end{aligned}$$

where A is the Airy Ai function, B the Airy Bi function and W the Lambert W function.

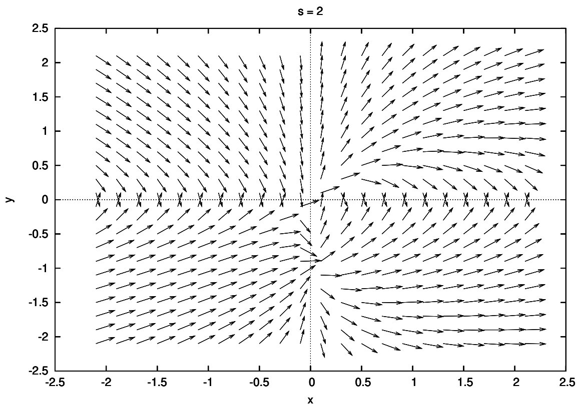
Qualitatively the vector fields are rather similar, see Figure 9.4. The same qualitative picture also remains for values of $s > 0$ where we do not have exact solutions. For $s < 0$ the picture is



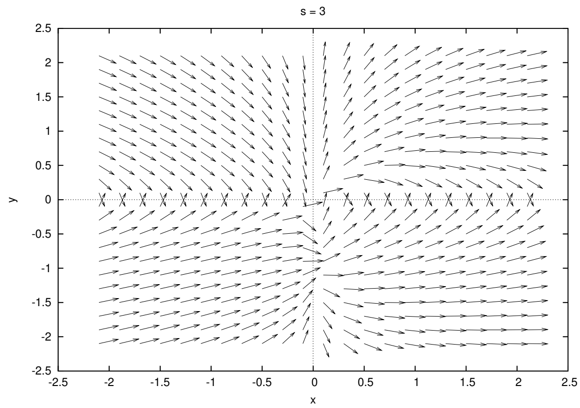
(a) $s = 1$



(b) $s = 3/2$



(c) $s = 2$



(d) $s = 3$

Figure 9.4: The vector field of $\gamma_1'(x)$ with $m = 1$ and $P(x) = x$, showing the dependence on $s > 0$.

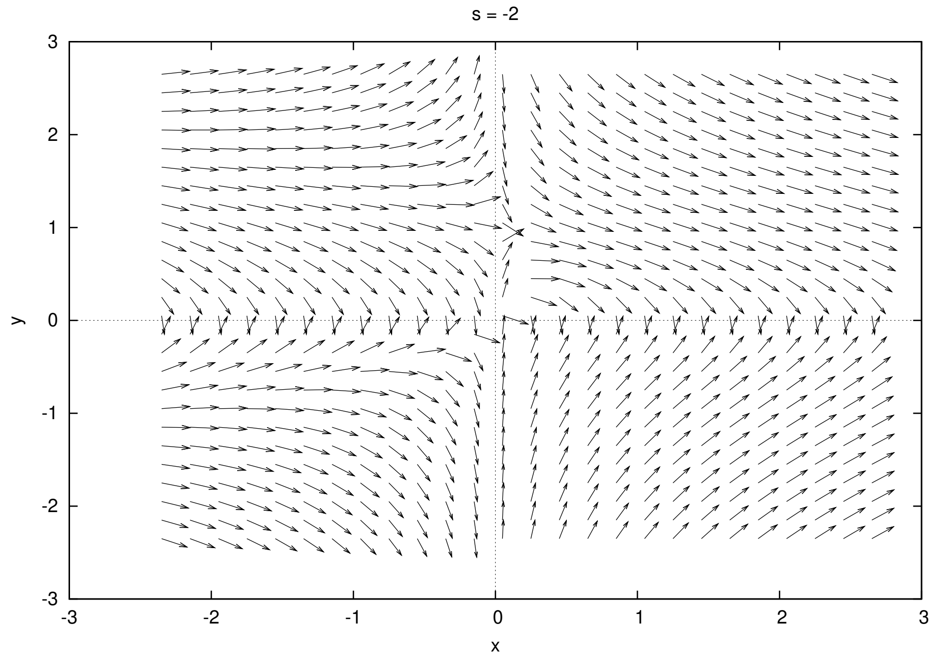


Figure 9.5: The case $P(x) = x$ and $s = -2$. A typical example with $s < 0$.

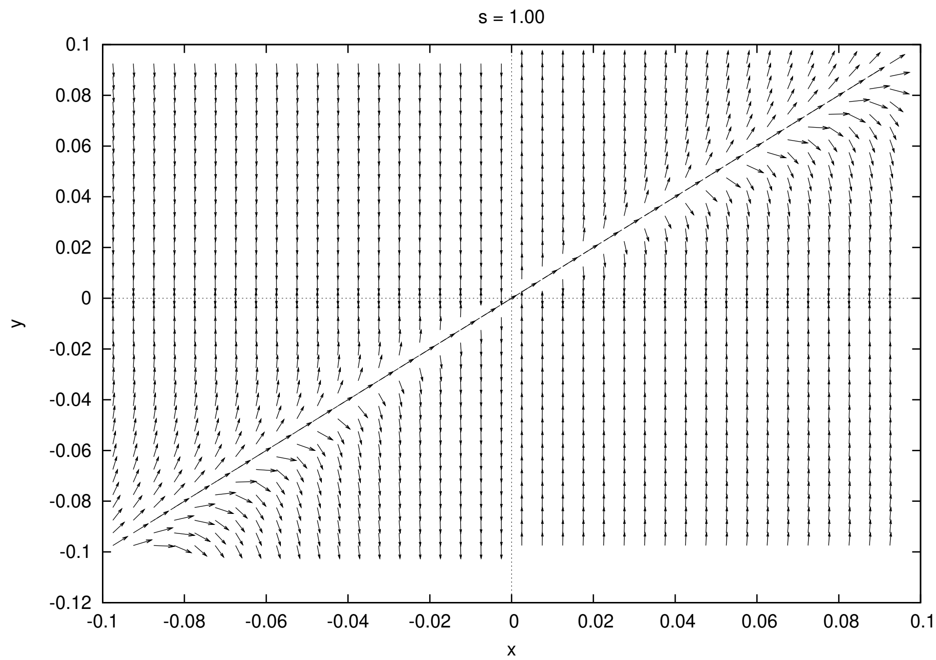


Figure 9.6: The case $s = 1$ compared to the curve $\gamma_1(x) = x$

somewhat different, see Figure 9.5, but we still see solutions which die and can still ask whether there are solutions which exist for all $x > 0$

In the case $s = 1$, $\gamma_1(x) = x$ is manifestly a solution, so there are solutions which exist for all x for some values of s. $\gamma_1(x) = x$ is illustrated in Figure 9.6.

Note also that we can, as before, calculate the curve where solutions are flat for general s, and it depends only on the sign of s since

$$\gamma_1'(x) = \frac{\gamma_1(x) + \mathrm{sign}(s)\gamma_1^2(x) - x}{|s|x\gamma_1(x)}$$

giving the curve

$$y = \frac{-1 + \sqrt{1 + \mathrm{sign}(s)4x}}{2}.$$

9.2 QED as a single equation

In this section we are interested in the case where $m = 2$, and $s = 1$ in (9.2). In view of the Ward identities and the work of Johnson, Baker, and Willey [19] the QED system can be reduced by a suitable choice of gauge to the single equation with those values of m and s describing the photon propagator.

The first question is how to choose $P(x)$. To 2 loops

$$P(x) = \frac{x}{3} + \frac{x^2}{4}$$

To 4 loops we need to correct the primitives in view of the reductions of the previous chapters. Values are from [16].

$$P(x) = \frac{x}{3} + \frac{x^2}{4} + (-0.0312 + 0.06037)x^3 + (-0.6755 + 0.05074)x^4$$

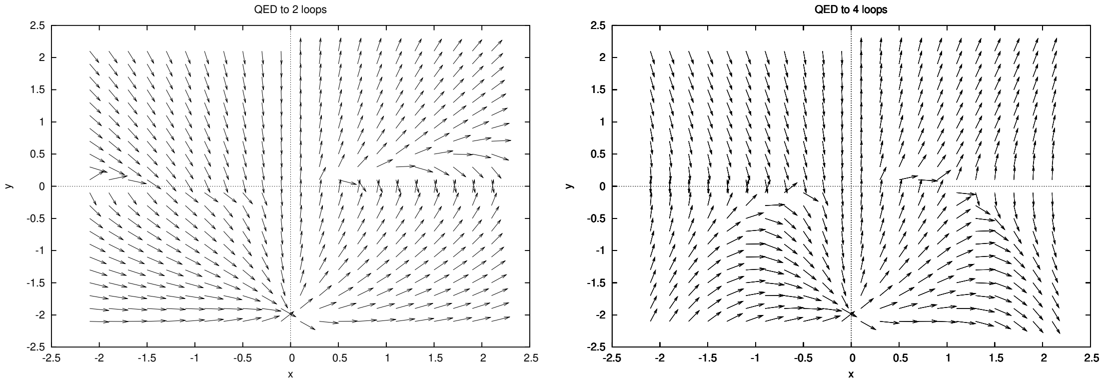
In the first of these cases little has changed from the simple examples of the previous sections. At 4 loops, however, $P(0.992\ldots) = 0$ which causes substantial changes to the overall picture, see Figure 9.7.

This zero in $P(x)$ is expected to be spurious, due only to taking the 4 loop approximation out beyond where it is valid, and the qualitative behavior of the solutions looks much more familiar if we restrict our attention to $0 \leq x < 0.992\ldots$, see Figure 9.8.

Note that if $P(x) > 0$ for $x > 0$ then by the same analysis as in the $P(x) = x$ case we can determine the curve where the solutions are flat. The curve is

$$y = \frac{-1 + \sqrt{1 + 4P(x)}}{2}.$$

The first four loops of perturbation theory give a good approximation to reality, and also as expected match the apparent separatrix for small values of x, which is illustrated quite strikingly in Figure 9.9.



(a) $P(x)$ taken to 2 loops.

(b) $P(x)$ taken to 4 loops.

Figure 9.7: The vector field of $\gamma_1'(x)$ for QED with different choices for $P(x)$.

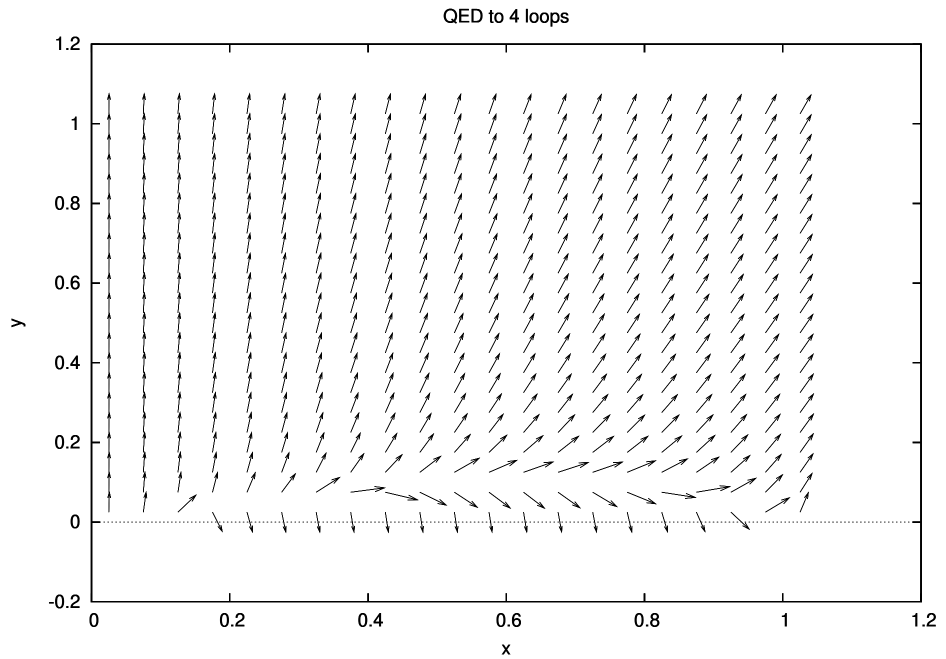
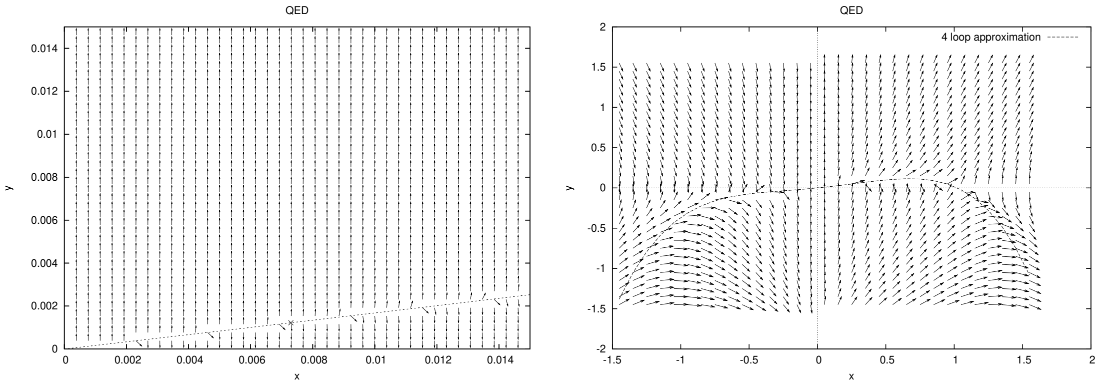


Figure 9.8: The region between $x = 0$ and $x = 1$ in the vector field of $\gamma_1'(x)$ for QED with $P(x)$ taken to 4 loops.



(a) Close to the origin, with $x = 1/137\ldots$ marked. (b) The limits of the validity of the four loop approximation.

Figure 9.9: The four loop approximation to $\gamma_1(x)$ for QED.

9.3 ϕ^4

Let us now consider ϕ^4 as an example which legitimately leads to a system of equations, but for which it remains possible to create illustrations, and perhaps even to analyze. Taking advantage of the graphical similarity between the vertex and propagator in ϕ^4 and the symbols $+$ and $-$ respectively we will write the specialization of (9.1) for ϕ^4 as the system

$$\gamma_1^+(x) = P^+(x) + \gamma_1^+(x)^2 + (\gamma_1^+(x) + 2\gamma_1^-(x))x\partial_x\gamma_1^+(x)$$
$$\gamma_1^-(x) = P^-(x) - \gamma_1^-(x)^2 + (\gamma_1^+(x) + 2\gamma_1^-(x))x\partial_x\gamma_1^-(x)$$

The values of γ_1^+ and γ_1^- up to order x^5 can be obtained from [20] and hence so can those of P^+ and P^-. Close to the origin we see a distinguished solution, see Figure 9.10. As in subsection 9.1.1, this may not indicate a solution which exists for all x, but we hope that this solution is physical.

There are many tantalizing features appearing in these examples which will hopefully be the genesis for future work linking to different fields. The equations derived in Chapter 7 seem considerably more tractable than the original Dyson-Schwinger equations when viewed either as recursive equations or as differential equations. They have already led to physically interesting results as in Chapter 8 and hold much promise for the future.

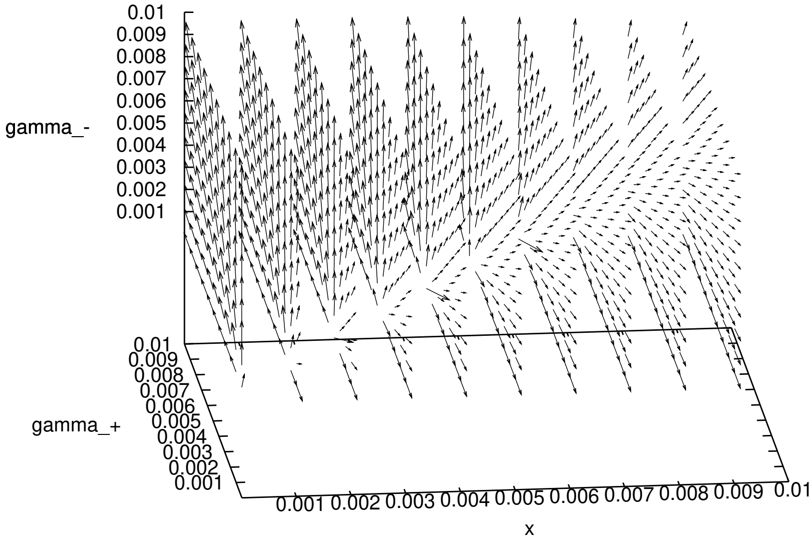


Figure 9.10: ϕ^4 near the origin.

List of Journal Abbreviations

Adv. Math.	Advances in Mathematics
Annals Phys.	Annals of Physics
Commun. Math. Phys.	Communications in Mathematical Physics
IRMA Lect. Math. Theor. Phys.	Institut de Recherche Mathématique Avancée Lectures in Mathematics and Theoretical Physics
J. Phys. A	Journal of Physics A: Mathematical and Theoretical
Nucl. Phys. B	Nuclear Physics B: Particle physics, field theory and statistical systems, physical mathematics
Nucl. Phys. B Proc. Suppl.	Nuclear Physics B - Proceedings Supplements
Phys. Lett. B	Physics Letters B: Nuclear Physics and Particle Physics
Phys. Rev. B	Physical Review B: Condensed Matter and Materials Physics

Bibliography

[1] Christoph Bergbauer and Dirk Kreimer. Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology. *IRMA Lect. Math. Theor. Phys.*, 10:133–164, 2006. arXiv:hep-th/0506190.

[2] Spencer Bloch, Hélène Esnault, and Dirk Kreimer. On motives associated to graph polynomials. *Commun. Math. Phys.*, 267:181–225, 2006. arXiv:math/0510011v1 [math.AG].

[3] D. J. Broadhurst and D. Kreimer. Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops. *Phys. Lett. B*, B393:403–412, 1997. arXiv:hep-th/9609128.

[4] D.J. Broadhurst and D. Kreimer. Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Pade-Borel resummation. *Phys. Lett. B*, 475:63–70, 2000. arXiv:hep-th/9912093.

[5] D.J. Broadhurst and D. Kreimer. Exact solutions of Dyson-Schwinger equations for iterated one-loop integrals and propagator-coupling duality. *Nucl. Phys. B*, 600:403–422, 2001. arXiv:hep-th/0012146.

[6] Ta-Pei Cheng and Ling-Fong Li. *Gauge theory of elementary particle physics.* Oxford University Press, 1984.

[7] Sidney Coleman. *Aspects of Symmetry*, chapter Dilatations, pages 67–98. Cambridge University Press, 1985.

[8] John C. Collins. *Renormalization.* Cambridge monographs on mathematical physics. Cambridge University Press, 1984.

[9] A. Connes and D. Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. II: The beta-function, diffeomorphisms and the renormalization group. *Commun. Math. Phys.*, 216:215–241, 2001. arXiv:hep-th/0003188.

[10] Alain Connes and Dirk Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Commun. Math. Phys.*, 199:203–242, 1998. arXiv:hep-th/9808042.

[11] Predrag Cvitanović. Asymptotic estimates and gauge invariance. *Nucl. Phys. B*, 127:176–188, 1977.

[12] Predrag Cvitanović. *Field Theory.* Nordita Lecture Notes, 1983.

[13] F. David, J. Feldman, and V. Rivasseau. On the large order behaviour of ϕ_4^4. *Commun. Math. Phys.*, 116:215–233, 1988.

[14] Kurusch Ebrahimi-Fard, Li Guo, and Dirk Kreimer. Integrable renormalization ii: the general case. *Annales Henri Poincare*, 6:369–395, 2005. arXiv:hep-th/0403118v1.

[15] Kurusch Ebrahimi-Fard and Dirk Kreimer. Hopf algebra approach to Feynman diagram calculations. *J. Phys. A*, 38:R285–R406, 2005. arXiv:hep-th/0510202.

[16] S.G. Gorishny, A.L. Kataev, S.A. Larin, and L.R. Surguladze. The analytic four-loop corrections to the QED β-function in the MS scheme and the QED ψ-function. Total reevaluation. *Phys. Lett. B*, 256(1):81–86, 1991.

[17] David J. Gross. Applications of the renormalization group to high-energy physics. In Roger Balian and Jean Zinn-Justin, editors, *Methods in field theory*, Les Houches, pages 141–250. North-Holland, 1975.

[18] Claude Itzykson and Jean-Bernard Zuber. *Quantum Field Theory*. McGraw-Hill, 1980. Dover edition 2005.

[19] K. Johnson, M. Baker, and R. Willey. Self-energy of the electron. *Phys. Rev B*, 136:B1111–B1119, 1964.

[20] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin, and S.A. Larin. Five-loop renormalization group functions of $O(n)$-symmetric ϕ^4-theory and ϵ-expansions of critical exponents up to ϵ^5. *Phys. Lett. B*, 272:39–44, 1991. Erratum Phys. Lett. B 319 (1993) 545, arXiv:hep-th/9503230.

[21] Dirk Kreimer. Anatomy of a gauge theory. *Annals Phys.*, 321:2757–2781, 2006. arXiv:hep-th/0509135v3.

[22] Dirk Kreimer. The residues of quantum field theory - numbers we should know. In C. Consani and M. Marcolli, editors, *Noncommutative Geometry and Number Theory*, pages 187–204. Vieweg, 2006. arXiv:hep-th/0404090.

[23] Dirk Kreimer. Etude for linear Dyson-Schwinger equations. In Sergio Albeverio, Matilde Marcolli, Sylvie Paycha, and Jorge Plazas, editors, *Traces in Geometry, Number Theory and Quantum Fields*, number E 38 in Aspects of Mathematics, pages 155–160. Vieweg Verlag, 2008.

[24] Dirk Kreimer and Karen Yeats. An étude in non-linear Dyson-Schwinger equations. *Nucl. Phys. B Proc. Suppl.*, 160:116–121, 2006. arXiv:hep-th/0605096.

[25] Dirk Kreimer and Karen Yeats. Recursion and growth estimates in renormalizable quantum field theory. *Commun. Math. Phys.*, 279(2):401–427, 2008. arXiv:hep-th/0612179.

[26] J. Magnen, F. Nicolo, V. Rivasseau, and R. Seneor. A Lipatov bound for ϕ^4 in four-dimensions Euclidean field theory. *Commun. Math. Phys.*, 108:257–289, 1987.

[27] Michael Polyak. Feynman diagrams for pedestrians and mathematicians. In Mikhail Lyubich and Leon Takhtajan, editors, *Graphs and patterns in mathematics and theoretical physics*, volume 73 of *Proceedings of Symposia in Pure Mathematics*. American Mathematical Society, 2005.

[28] N. J. A. Sloane. The on-line encyclopedia of integer sequences. www.research.att.com/~njas/sequences/, 2008.

[29] M. Stingl. Field-theory amplitudes as resurgent functions. arXiv:hep-ph/0207349.

[30] Moss E. Sweedler. *Hopf Algebras*. W. A. Benjamin, 1969.

[31] Guillaume van Baalen, Dirk Kreimer, David Uminsky, and Karen Yeats. The QED beta-function from global solutions to Dyson-Schwinger equations. *Ann. Phys.*, 2008. (in press). arXiv:0805.0826.

[32] Walter D. van Suijlekom. Renormalization of gauge fields: A Hopf algebra approach. *Commun. Math. Phys.*, 276:773–798, 2007. arXiv:hep-th/0610137.