Hopf subalgebras from Green’s functions

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Introduction

Since the late 1990s ([3],[4], 1998) there has been much work on the study of Hopf algebras as the underlying mathematical structure of local quantum field theory (QFT). In QFT, Green’s functions are developed as series in coupling constants indexed by Feynman graphs, which have a Hopf algebra structure. Here we work not in Feynman graphs but in the closely related Hopf algebra of decorated rooted trees. The Green’s functions appear as solutions to combinatorial Dyson-Schwinger equations (DSEs), which build families of graphs through the action of a combinatorial grafting operator. The set of graphs constructed in this way generates a subalgebra which, for certain forms of DSE, is itself a Hopf algebra.

The work of Foissy classifies DSEs in Hopf algebras of trees which give rise to Hopf subalgebras ([13],[9],[10],[11],[12]), and in [1], Bergbauer and Kreimer show that the solution of a certain DSE with a coupling constant gives rise to Hopf subalgebras. Here we generalise this result to a finite system of DSEs with a finite number of coupling constants.

Chapter 1 reviews algebraic (bialgebras, Hopf algebras, operads) and graph-theoretical notions (graphs, rooted trees) needed for the rest of the work, with some examples. Chapter 2 introduces the Hopf algebra \( \mathcal{H}_\mathcal{G} \), the grafting operator, and DSEs in \( \mathcal{H}_\mathcal{G} \). Chapter 3 presents and compares the two approaches given for the proof of Theorem 3 in [1], then uses the second approach to extend the theorem from a single Dyson-Schwinger equation to a system of DSEs. The final chapter introduces Feynman graphs and their relation to rooted trees, and gives some examples of DSEs in Feynman graphs.
Chapter 1

Basic definitions

1.1 Hopf algebras

Let $k$ be a field. For two vector spaces $V_1$ and $V_2$, let $\tau_{V_1,V_2}: V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$ denote the twist map $v \otimes w \mapsto w \otimes v$.

**Definition 1.** An (associative) $k$-algebra $(A,m,u)$ is a $k$-vector space $A$ together with two linear maps, $m : A \otimes_k A \rightarrow A$ (multiplication) and $u : k \rightarrow A$ (unit) such that:

1. $m \circ (id \otimes m) = m \circ (m \otimes id)$,
2. $m \circ (u \otimes id) = m \circ (id \otimes u)$.

The multiplication is commutative if $m = m \circ \tau$.

**Definition 2.** A (coassociative) $k$-coalgebra $(C,\Delta,\epsilon)$ is a $k$-vector space $C$ together with two linear maps $\Delta : C \rightarrow C \otimes C$ (comultiplication) and $\epsilon : C \rightarrow k$ (counit) such that:

1. $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$,
2. $(id \otimes \epsilon) \circ \Delta = \tau_{C,A}(\epsilon \otimes id) \circ \Delta$.

The coproduct is cocommutative if $\tau \circ \Delta = \Delta$.

**Remark 1.** A common notation for the coproduct of an element $x \in C$ which will be used here is the Sweedler notation

$$\Delta(x) = x' \otimes x''.$$ 

**Definition 3.** For two algebras $(A_1,m_1,u_1)$ and $(A_2,m_2,u_2)$, a linear map $\phi : A_1 \rightarrow A_2$ is an algebra morphism if

1. $\phi \circ u_1 = u_2$,
2. $\phi \circ m_1 = m_2 \circ (\phi \otimes \phi)$.

---

$^1$The tensor product always written $\otimes$ should be read as $\otimes_k$ ($\otimes_Q$ in chapters 2 and 3).
For two coalgebras \((C_1, \Delta_1, \epsilon_1)\) and \((C_2, \Delta_2, \epsilon_2)\), a linear map \(\psi : C_1 \rightarrow C_2\) is a coalgebra morphism if

\[
\begin{align*}
\epsilon_2 \circ \psi &= \epsilon_1, \\
\Delta_2 \circ \psi &= (\psi \otimes \psi) \circ \Delta_1.
\end{align*}
\]  

\(1.4\)

**Definition 4.** A \(k\)-bialgebra \((B, m, u, \Delta, \epsilon)\) is a \(k\)-vector space \(B\) which has a \(k\)-algebra structure \((m, u)\) and a \(k\)-coalgebra structure \((\Delta, \epsilon)\) such that \((m, u)\) are coalgebra morphisms and \((\Delta, \epsilon)\) are algebra morphisms.

**Theorem 1.** Let \(B\) be a space with \(k\)-algebra structure \((m, u)\) and \(k\)-coalgebra structure \((\Delta, \epsilon)\). Then \((m, u)\) are coalgebra morphisms if and only if \((\Delta, \epsilon)\) are algebra morphisms.

**Proof.** The field \(k\) has a coalgebra structure given by the isomorphisms \(\Delta(1) = 1 \otimes 1\) and \(\epsilon = id_k\). Substitute \(\phi = \Delta : B \rightarrow B \times B\) and \(\phi = \epsilon : B \rightarrow k\) into 1.3, and \(\psi = m\) and \(\psi = u\) into 1.4. The resulting four equations are the same in each case:

\[
\begin{align*}
\Delta \circ u &= (u \otimes u), \\
\Delta \circ m &= (m \otimes m) \circ (\Delta \otimes \Delta), \\
\epsilon \circ u &= 1, \\
\epsilon \circ m &= \epsilon \otimes \epsilon.
\end{align*}
\]

\(\square\)

**Definition 5.** A Hopf algebra \((H, m, u, \Delta, \epsilon, S)\) is a bialgebra \((H, m, u, \Delta, \epsilon)\) endowed with an antipode: a map in \(S \in \text{Hom}_k(H, H)\) which satisfies

\[
m \circ (S \otimes id) \Delta = m \circ (id \otimes S) \Delta = u \circ \epsilon.
\]  

\(1.5\)

**Remark 2.** One can define an algebra \((\text{Hom}_k(H, H), \ast, \epsilon)\), with multiplication and unit defined by

\[
\phi \ast \psi = m \circ (\phi \otimes \psi) \circ \Delta \quad \forall \phi, \psi \in \text{Hom}_k(H, H),
\]

\[
\epsilon = u \circ \epsilon.
\]

Then the antipode of \(H\) can be defined by

\[
S \ast id_H = id_H \ast S = \epsilon.
\]

**Proposition 1.** For any Hopf algebra \(H\), the antipode \(S\) is unique.

**Proof.** Let \(S_1, S_2\) be two possible antipodes. Then, since \((\text{Hom}_k(H, H), \ast, \epsilon)\) satisfies 1.1,

\[
S_1 = S_1 \ast \epsilon = S_1 \ast id_H \ast S_2 = \epsilon \ast S_2 = S_2.
\]

\(\square\)
1.1. HOPF ALGEBRAS

Definition 6. A Hopf algebra over $k$ is graded and connected if there exist $H_i$ such that:

\[ H \cong \bigoplus_{i=0}^{\infty} H_i, \quad (1.6) \]

\[ H_0 \cong k, \quad (1.7) \]

\[ m(H_i \otimes H_j) \subseteq H_{i+j}, \quad (1.8) \]

\[ \Delta(H_i) \subseteq \bigoplus_{j+k=i} H_j \otimes H_k. \quad (1.9) \]

Example 1. For a multiplicative group $(G, \cdot)$ the group algebra $kG$ is the $k$-vector space generated by the elements of $G$. An element $x \in kG$ has the form of a finite sum $x = \sum_{g \in G} \alpha_g g$, with $\alpha_g \in k$. One can construct a Hopf algebra $(kG, m, u, \Delta, \epsilon, S)$ as follows:

- $m(\alpha_1 g_1 \otimes \alpha_2 g_2) = (\alpha_1 \alpha_2)(g_1 \cdot g_2)$.
- $u(\alpha) = \alpha 1_G \forall \alpha \in k$.
- $\Delta(g) = g \otimes g \forall g \in G$.
- $\epsilon(g) = 1_G \forall g \in G$.
- $G$: $S(g) = g^{-1} \in G \forall g \in G$.

This Hopf algebra is cocommutative, and is commutative iff $G$ is abelian.

Example 2. For a vector space $V$ over a field $k$, the tensor algebra of $V$ is defined as

\[ T(V) = \bigoplus_{n=0}^{\infty} T^n(V), \]

where $T^n(V) = V^\otimes n$.

Consider arbitrary elements

\[ x \in V, \]

\[ v = v_1 \otimes \ldots \otimes v_n \in T^n(V), \]

\[ w = w_1 \otimes \ldots \otimes w_m \in T^m(V) \]

of $T(V)$. Define a Hopf algebra $(T(V), m, u, \Delta, \epsilon, S)$ by:

- $m(v, w) = v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots w_m \in T^{n+m}(V)$.
- $u(v) = 1_k v$
- $\Delta(x) = 1_k \otimes x + x \otimes 1_k$.
- $\epsilon(x) = \begin{cases} x, & x \in V \\ 0, & f v \in T^n(V), n > 1 \end{cases}$
- $S(x) = -x$.

This Hopf algebra is cocommutative but not commutative, and is graded connected with homogeneous components $T^n(V)$: $T^0(V) \cong k$ and $T^1(V) = V$. 
1.2 Rooted trees

Definition 7. A graph \( G \) consists of a set of vertices \( V(G) \) and a set of edges \( E(G) \subseteq \binom{V}{2} \). 

- \( v \in V(G) \) and \( e \in E(G) \) are said to be incident if \( e = vw \) for some \( w \in V(G) \).
- Two edges \( e_1, e_2 \in E(G) \) are adjacent if they have a common incident vertex: \( e_1 = vw \) and \( e_2 = wx \), for \( v, w, x \in V(G) \).
- Two vertices \( v, w \in V(G) \) are neighbours if \( vx \in E(G) \).
- The degree of a vertex \( v \) is the number of neighbours it has (equivalently, the number of edges incident to it).
- A path is a subset \( P = \{ e_1, e_2, e_3, ... \} \subseteq E(G) \) such that any two edges \( e_i, e_{i+1} \in P \) are adjacent.
- A path \( \{ e_1, ..., e_k \} \subseteq E(G) \) will be denoted by \( (v_1, v_2) \) where \( e_1 = v_1 w, e_2 = w u, ..., e_{k-1} = t v_2, e_k = x v_2 \) for some \( t, u, v_1, v_2, w, x \in V(G) \).
- A cycle is a path of the form \( (v, v) \).
- A graph is connected if there exists a path \( (v, w) \) for any \( v, w \in V(G) \).

Definition 8. A connected graph with no cycles is called a tree.

- A rooted tree is a tree \( T \) along with some distinguished vertex \( r \in V(T) \) which is called the root (here the root will always be drawn at the top of the tree).
- A vertex \( v \in V(T) \setminus \{ r \} \) with degree one is called a leaf of \( T \).
- \( w \in V(T) \) is a child of \( v \in V(T) \) if any path \( (r, w) \) contains the edge \( vw \).
- The fertility of a vertex is the number of children it has (fertility(\( v \)) = degree(\( v \)) - 1).
- A union of (rooted) trees, a graph which has no cycles but is not necessarily connected, is called a (rooted) forest.
- We call the set of all rooted trees \( \mathcal{T} \) and the set of all rooted forests \( \mathcal{F} \).

Definition 9. A decorated rooted tree consists of a rooted tree \( T \), a countable set \( \mathcal{D} \) and a surjective map \( c : \mathcal{D} \rightarrow V(T) \), which assigns an element of \( \mathcal{D} \) to each vertex of \( T \). For any \( v \in V(T) \), we denote \( c^{-1}(v) = d(v) \).

Let \( \mathcal{T}_\mathcal{D} \) and \( \mathcal{F}_\mathcal{D} \) denote respectively the sets of rooted trees and forests decorated by a set \( \mathcal{D} \).

Definition 10. An admissible cut of \( T \) is a nonempty subset \( c \subseteq E(T) \) such that

\[
| c \cap (r, v) | \leq 1 \quad \forall v \in V(T).
\]

\(^2\)We assume graphs here to be simple, i.e. the edges are distinct.
• Call the set of all admissible cuts $\mathcal{C}(T)$.

• Denote by $\mathcal{C}^*(T)$ the union $\mathcal{C}(T) \cup \emptyset$ of admissible cuts and the empty cut.

• Given a cut $c \in \mathcal{C}(T)$, we construct a disconnected graph $G$ with $E(G) = E(T) - c$ and $V(G) = V(T)$.

• The connected component of $G$ which contains the root of $T$ is a tree which we call $R^c(T)$.

• The component(s) of $G$ not connected to the root of $T$ give a forest which we call $P^c(T)$.

Example 3.

$G_1 = \begin{array}{c}
\text{is a graph which is not connected and contains one cycle.}
\end{array}$

$G_2 = \begin{array}{c}
\text{is a subgraph of } G_1 \text{ which is a tree with two leaves. To look at the cuts of } G_2,
\text{we label the edges:}
\end{array}$

\[
\begin{array}{c|c|c|c}
\mathcal{C} & G & P^c(G_2) & R^c(G_2) & \mathcal{C} \text{ Admissible?} \\
\hline
\{1\} & \begin{array}{c}
\bullet
\end{array} & \begin{array}{c}
\bullet
\end{array} & \begin{array}{c}
\bullet
\end{array} & \text{Yes} \\
\{2\} & \begin{array}{c}
\bullet
\end{array} & \begin{array}{c}
\bullet
\end{array} & \begin{array}{c}
\bullet
\end{array} & \text{Yes} \\
\{3\} & \begin{array}{c}
\bullet
\end{array} & \begin{array}{c}
\bullet
\end{array} & \begin{array}{c}
\bullet
\end{array} & \text{Yes} \\
\{1,2\} & \begin{array}{c}
\bullet
\end{array} & \begin{array}{c}
\times
\end{array} & \begin{array}{c}
\times
\end{array} & \text{No} \\
\{1,3\} & \begin{array}{c}
\bullet
\end{array} & \begin{array}{c}
\times
\end{array} & \begin{array}{c}
\times
\end{array} & \text{No} \\
\{2,3\} & \begin{array}{c}
\bullet
\end{array} & \begin{array}{c}
\times
\end{array} & \begin{array}{c}
\times
\end{array} & \text{Yes} \\
\{1,2,3\} & \begin{array}{c}
\bullet
\bullet
\bullet
\end{array} & \begin{array}{c}
\times
\end{array} & \begin{array}{c}
\times
\end{array} & \text{No}
\end{array}
\]
1.3 Operads

Definition 11. An operad $\mathcal{O}^3$ consists of:

- A collection of objects $\{\mathcal{O}(n)\}_{n \in \mathbb{N}}$
- A collection of maps $\{\circ_i\}_i, \circ_i : \mathcal{O}(m) \times \mathcal{O}(n) \to \mathcal{O}(m + n - 1)$

Example 4 (The endomorphism operad). For a set $X$, $\text{Map}(X^n, X)$ is the set of functions from $X^n$ to $X$. The endomorphism operad $\text{End}(X)$ consists of the set of objects

$$\{\text{Map}(X^n, X)\}_{n \in \mathbb{N}},$$

and maps $\circ_i$ defined by

$$(f \circ_i g)(x_1, \ldots, x_{m+n-1}) = f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+m-1}), x_{i+m}, \ldots, x_{m+n-1}),$$

where $f \in \text{Map}(X^n, X)$, $g \in \text{Map}(X^m, X)$ and $x_j \in X \forall j$.

Example 5 (The rooted trees operad). Take the set $\mathcal{T}(n)$ of rooted trees $T$ with $n$ leaves $\{l_1, \ldots, l_n\} \subset V(T)$. Then there exists an operad $\mathcal{T}$ with the set of objects

$$\{\mathcal{T}(n)\}_{n \in \mathbb{N}},$$

and the maps $\circ_i$, where $T_1 \circ_i T_2$ means “graft the root of $T_2$ to the root $l_i \in V(T_1)$”.

---

3For a formal introduction to and definition of algebraic operads, see [7].
Chapter 2

The Hopf algebra of rooted trees

We consider a Hopf algebra $H_D$ over $\mathbb{Q}$ generated by $T_D$, where $D = \{d_n\}_{n \in \mathbb{N}}$ is a set which we may take, without loss of generality, to be $\mathbb{N}$. Define a Hopf algebra structure $(H_D, m, I, \Delta, \hat{I}, S)$:

- $T_1T_2 := m(T_1 \otimes T_2)$ is given by the forest $T_1 \cup T_2$.
- The unit map sends $q \in \mathbb{Q}$ to $qI \in H_D$.
- The coproduct on $H_D$ is defined on trees using admissible cuts:
  \[ \Delta(T) = I \otimes T + T \otimes I + \sum_{c \in C(T)} P_c(T) \otimes R_c(T) \]  
  \[ \text{(2.1)} \]
  and extended over all of $H_D$ by $\Delta(T_1T_2) = \Delta(T_1)\Delta(T_2)$.
- The counit $\hat{I}$ is given by $\hat{I}(T) = \begin{cases} 1, & T = I \\ 0, & T \neq I \end{cases}$.
- Using these definitions of $m$ and $\Delta$ along with the defining property 1.5 for the antipode, and $S(I) = \hat{I}$ obtain an expression for $S$ acting on a tree $T \in H$:
  \[ m(S \otimes id_H)(\Delta(T)) = S(I)T + S(T)I + \sum_c S(P_c(T))R_c(T) = \hat{I}(T) = 0 \]
  \[ \Rightarrow S(T) = -T - \sum_c S(P_c(T))R_c(T). \]  
  \[ \text{(2.2)} \]
  This can be extended to all of $H_D$ by $S(T_1, \ldots, T_k) = S(T_k) \ldots S(T_1)$.

Example 6.

\[ \Delta \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \otimes I + I \otimes \begin{array}{c} 2 \\ 3 \\ 4 \end{array} + \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \otimes \begin{array}{c} 1 \\ 3 \\ 4 \end{array} + \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \otimes \begin{array}{c} 1 \\ 2 \\ 3 \end{array} + \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \]  

\[ ^1 \text{It is a standard abuse of notation to use $I$ for both the unit map and the "empty tree" defined by $V(I) = \varnothing$.} \]
Theorem 2.

Proof. S is associative: \( \mathbb{H} \) is an algebra morphism: satisfies 1.5 by construction.

For any tree \( T \in \mathcal{H}_D \), define \( n(T) = (n_1, n_2, \ldots) \) by

\[
n_i := \{ v \in V(T) : d(v) = d_n \in \mathcal{D} \}.
\]

For a forest \( F = T_1 \ldots T_k \), \( n_i(F) = \sum_{i=1}^k n_i(T_i) \). Define \( H^n = \text{span}_Q \{ F \in \mathcal{F}_D : n(F) = \mathbf{n} \} \). Then

\[
\mathcal{H}_D = \bigoplus_{n \in \mathbb{N}^\infty} H^n
\]

is a grading on \( \mathcal{H}_D \).

\( \text{Aug}_{\mathcal{H}_D} := \bigoplus_{n \in \mathbb{N}^\infty \setminus \{0\}} H^n \) is the augmentation ideal of \( \mathcal{H}_D \). It contains all elements in \( \mathcal{H}_D \) which vanish under the action of the counit.

Theorem 2. \( (\mathcal{H}_D, m, \mathbb{I}, \Delta, \hat{\Delta}, S) \) as defined above gives a graded connected, commutative, non-cocommutative Hopf algebra.

Proof. We need to show that this construction satisfies all the defining equations 1.1 -1.9 from the previous chapter. Let all \( F_i \) and \( T_i \) be arbitrary forests and trees (respectively) in \( \mathcal{H}_D \), and \( q \in Q \).

\begin{itemize}
  \item \( m \) is associative: \( m(id \otimes m)(F_1 \otimes F_2 \otimes F_3) = F_1(F_2F_3) = (F_1F_2)F_3 = m(m \otimes id)(F_1 \otimes F_2 \otimes F_3) \)
  \item \( \mathbb{I} \) is a unit: \( m(\mathbb{I} \otimes id)(q \otimes F_1) = q \mathbb{I} F_1 = F_1q \mathbb{I} = m(id \otimes \mathbb{I})(F_1 \otimes q) \).
  \item \( \hat{\Delta} \) is a counit: \( (id \otimes \hat{\Delta}) \Delta(T) = T \otimes 1 = \tau_{C,k}(1 \otimes T) = (\hat{\mathbb{I}} \otimes id) \Delta(T) \).
  \item \( \Delta \) is an algebra morphism:
    \[
    \Delta(\hat{\mathbb{I}}(q)) = q \mathbb{I} \otimes \mathbb{I} = (\mathbb{I} \otimes \mathbb{I})(q \mathbb{I} \otimes \mathbb{I}),
    \]
    \[
    \Delta(m(F_1 \otimes F_2)) = \Delta(F_1) \Delta(F_2) = m(\Delta \otimes \Delta)(F_1 \otimes F_2))
    \]
  \item \( \hat{\mathbb{I}} \) is an algebra morphism:
    \[
    \hat{\mathbb{I}}(\mathbb{I}(q)) = \hat{\mathbb{I}}(q \mathbb{I}) = q = u(q) q,
    \]
    \[
    \hat{\mathbb{I}}(m(F_1 \otimes F_2)) = \hat{\mathbb{I}}(F_1F_2) = m(\hat{\mathbb{I}}(F_1) \otimes \hat{\mathbb{I}}(F_2)) = m(\hat{\mathbb{I}} \otimes \hat{\mathbb{I}})(F_1 \otimes F_2).
    \]
  \item \( S \) satisfies 1.5 by construction.
  \item \( \mathcal{H} = \bigoplus H_i \) gives a grading:
    \[
    H^0 = \text{span}_Q \{ F \in \mathcal{F}_D : V(F) = \emptyset \} = Q \mathbb{I} \cong Q,
    \]
    \[
    m(H^j \otimes H^k) \subseteq \{ F \in \mathcal{F}_D : n_i = j_i + k_i \} = H^{j_i+k_i},
    \]
    \[
    \Delta(H^j) = \{ F : F = P^e(T), T \in H^j \} \otimes \{ T' : T' = R^e(T), T \in H^j \}
    \]
    \[
    \subseteq \bigoplus_{k_i + i_i = j_i} H^k \otimes H^j.
    \]
\end{itemize}
Commutativity: $F_1 F_2 := F_1 \cup F_2 = F_2 \cup F_1 =: F_2 F_1$.

Non-cocommutativity: in general $\tau_{\mathcal{H}_D, \mathcal{H}_D}(\Delta(F)) \neq \Delta(F)$, as $\Delta(T) \in \mathcal{H}_D \otimes \mathcal{T}_D$ and $\Delta \circ \tau_{\mathcal{H}_D, \mathcal{H}_D}(T) \in \mathcal{T}_D \otimes \mathcal{H}_D$ for any tree $T$.

For proof that $\Delta$ is coassociative, see proposition 3.

**Definition 12.** For each $d \in \mathcal{D}$, the grafting operator $B^d_+$ on $\mathcal{H}_D$ is a linear map $B^d_+ : \mathcal{H}_D \rightarrow \text{span}_Q(\mathcal{T}_D)$. $B^d_+$ acts on a rooted forest $T_1, \ldots, T_k$ by joining the roots of $T_1, \ldots, T_k$ to a new root decorated by $d$ giving a tree:

$$B^d_+(T_1 \ldots T_k) = \begin{array}{c}
\begin{array}{c}
\ldots \\
T_1 \quad T_2 \\
\ldots \\
T_3
\end{array}
\end{array}.$$

**Example 7.** Let $\mathcal{D} = \mathbb{N}$. $B^1_+(\begin{array}{c}
\begin{array}{c}
\ldots \\
1 \\
\ldots \\
2 \quad 3
\end{array}
\end{array}) = \begin{array}{c}
\begin{array}{c}
\ldots \\
1 \\
\ldots \\
2
\end{array}
\end{array}.$

**Proposition 2.** Each grafting operator $B^d_+$ satisfies

$$\Delta \circ B^d_+ = B^d_+ \otimes 1 + (id \otimes B^d_+) \circ \Delta. \quad (2.3)$$

**Proof.** Consider a forest $F = T_1 \ldots T_k \in \mathcal{H}_D$. $B^d_+(F) = T$ is a tree in $\mathcal{H}$. Let $C^*(T) = C(T) \cup \emptyset$ for any tree.

For any cut $c \in C(T)$, the restriction of $c$ to any subtree $T_i$ is either in $C^*(T_i)$ or is the total cut on $T_i$. Consider any set of cuts $c_1, \ldots, c_k$ such that $c_i \in C^*(t_i)$. Then there exists a unique cut $c \in C^*(T)$ such that the restriction of $c$ to any $T_i$ gives the corresponding $c_i$.

For this cut, we have

$$P^c(T) = \prod_{i=1}^k P^{c_i}(T_i),$$

$$R^c(T) = B^{dn}_+ \left( \prod_{i=1}^k R^{c_i}(T_i) \right).$$
So:

\[ \Delta(T) = T \otimes \mathbb{I} + \mathbb{I} \otimes T + \sum_{c \in \mathcal{C}(T)} P^c(T) \otimes R^c(T) \]

\[ = B^d_+(F) \otimes \mathbb{I} + \sum_{c \in \mathcal{C}^+(T)} P^c(T) \otimes R^c(T) \]

\[ = B^d_+(F) \otimes \mathbb{I} + \sum_{c_i \in \mathcal{C}^+(T)} \left( \prod_{i=1}^k P^{c_i}(T_i) \otimes B_+^d(R^{c_i}(T_i)) \right) \]

\[ = B^d_+(F) \otimes \mathbb{I} + (id \otimes B^d_+) \left( \prod_{i=1}^k \sum_{c_i \in \mathcal{C}^*} P^{c_i}(T_i) \otimes R^{c_i}(T_i) \right) \]

\[ = B^d_+(F) \otimes \mathbb{I} + (id \otimes B^d_+) \Delta(F). \]

\[ \square \]

**Remark 3.** This proposition is, in fact, saying that the operator \( B_+ \) is a 1-cocycle in the Hochschild cohomology of \( \mathcal{H}_D \).

**Proposition 3.** The coproduct \( \Delta \) is coassociative.

**Proof.** Show that \( \Delta \) satisfies 1.2. Clearly

\[ (\Delta \otimes id_{\mathcal{H}})\Delta(\mathbb{I}) = \Delta(\mathbb{I}) \otimes \mathbb{I} = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} = \mathbb{I} \otimes \Delta(\mathbb{I}) = (id_{\mathcal{H}} \otimes \Delta)\Delta(\mathbb{I}). \]

We proceed by induction on \( |V(F)| \) for any \( F \in \mathcal{H}_D \).

Assume that for all \( |V(F)| < n \), the action of \( \Delta \) on \( F \) is coassociative, and consider \( F \in \mathcal{H}_D \) with \( |V(F)| = n \).

1. If \( F \) is not a tree, \( F = T_1 \ldots T_k \) and each \( |V(T_i)| < n \), so by assumption \( \Delta \) is coassociative on each \( T_i \) and therefore on \( F \).

2. If \( F \) is a tree, then \( F = B^d_+(X) \) for some \( X \in \mathcal{H} \) and \( d \in \mathcal{D} \), with \( |V(X)| = n - 1 \). By assumption \( \Delta \) acts coassociatively on \( X \), and applying equation 2.3 gives:

\[ (id \otimes \Delta)\Delta(B^d_+(X)) = (id \otimes \Delta)(B^d_+(X) \otimes \mathbb{I} + (id \otimes B^d_+)(\Delta(X)) \]

\[ = B^d_+(X) \otimes \mathbb{I} \otimes \mathbb{I} + (id \otimes B^d_+)(\Delta(X)) \]

\[ = B^d_+(X) \otimes \mathbb{I} \otimes \mathbb{I} + (id \otimes B^d_+)(\Delta(X)) \]

\[ = B^d_+(X) \otimes \mathbb{I} \otimes \mathbb{I} + (id \otimes B^d_+)(\Delta(X)) \]

The terms labelled \( \ast \) are immediately equal, the terms marked \( \ast \ast \) cancel under the assumption on \( X \).

\[ \square \]
2.1 Hopf subalgebras

Definition 13. A sub-Hopf algebra of a graded Hopf algebra \( (H, m, u, \Delta, \epsilon, S) \), 
\( H = \bigoplus_i H_i \), is a subspace \( H' \subset H \) with Hopf algebra structure \( (H', m, u, \Delta, \epsilon, S) \) 
such that \( H' = \bigoplus_i (H' \cap H_i) \) gives a grading on \( H' \).

Example 8. The ladder on \( n \) vertices is the tree given by
\[
\lambda_n := (B_+)^n(I).
\]

Let \( H^L \) be the subalgebra of \( H \) generated by the set of all ladders, and \( H^L_n \subset H_n \) 
those elements which have \( n \) vertices. \( \Delta(\lambda_n) = \sum_{m \leq n} \lambda_m \otimes \lambda_{n-m} \). \( H^L \) is a 
Hopf subalgebra of \( H \).

Example 9. More generally, the subalgebra generated by all trees whose vertices 
have fertility bounded from above by \( n \) is a Hopf subalgebra (the ladders are the 
case \( n = 1 \)).

Example 10. For any \( d \in D \), the set of forests with \( n_i(F) = \begin{cases} 
1, & d_i = d \\
0, & d_i \neq d 
\end{cases} \) 
forms a Hopf subalgebra. It is isomorphic to the Hopf algebra \( H \) of undecorated 
rooted trees, which has \( m, I, \Delta, \hat{I} \) and \( S \) ad \( H_D \), but only one grafting operator 
\( B_+ \).

Example 11 (The Connes-Moscovici Hopf algebra). The natural growth oper- 
ator is a linear map \( N : H \rightarrow H \) defined by \( N(I) = \bullet \) and
\[
N(T) := \sum_{v \in V(T)} T_v,
\]
where \( T_v \) is the tree obtained by adding one extra vertex \( w \) to \( V(T) \), and the 
edge \( vw \) to \( E(T) \). \( N \) acts as a derivation on forests.

Define \( \delta_n := N^n(I) \).
Theorem 3. The elements \( \{ \delta_n \} \)\( n \in \mathbb{N} \) generate a Hopf subalgebra \( H_{CM} \) of \( H \).

Proof. We need to show that \( \Delta(\delta_n) \subseteq H_{CM} \otimes H_{CM} \) and \( S(\delta_n) \subseteq H_{CM} \).

If we write \( \delta_n = \sum_i T_i \) and \( \delta_{n+1} = \sum_i T_i' \), we have

\[
\Delta(\delta_0) = \delta_0 \otimes \delta_0,
\]

\[
\Delta(\delta_1) = \delta_1 \otimes \delta_0 + \delta_0 \otimes \delta_1,
\]

\[
\Delta(\delta_n) = \delta_n \otimes \delta_0 + \delta_0 \otimes \delta_n + \sum_i \left( \sum_{c \in C(T_i)} P_c(T_i) \otimes R_c(T_i) \right),
\]

\[
\Delta(\delta_{n+1}) = \delta_{n+1} \otimes \delta_0 + \delta_0 \otimes \delta_{n+1} + n \delta_1 \otimes \delta_n
\]

\[
+ \sum_i \left( \sum_{c \in C(T_i)} N(P_c(T_i)) \otimes R_c(T_i) + P_c(T_i) \otimes N(R_c(T_i)) \right)
\]

\[
+ \sum_i \left( \sum_{c \in C(T_i)} \delta_1 \mid R_c(T_i) \mid P_c(T_i) \otimes R_c(T_i) \right).
\]

By the assumption \( \Delta(\delta_n) \in H_{CM} \otimes H_{CM} \), it immediately follows that \( \Delta(\delta_{n+1}) \in H_{CM} \otimes H_{CM} \) and \( S(\delta_n) \subseteq H_{CM} \).

\[\square\]

2.2 Dyson-Schwinger equations

A (combinatorial) Dyson-Schwinger equation builds a family of forests through the action of the grafting operators \( \{ B_d^2 \} \) on elements of \( H_D \).

Example 12. Consider the Hopf algebra \( H \) of undecorated rooted trees, and the equation

\[ X = \alpha B_+(X^2) \]

for a parameter \( \alpha \) (which is called a coupling constant), and \( X \in H[[\alpha]] \). This equation has unique solution \( X(\alpha) = \sum_{n=0}^{\infty} c_n \alpha^n \), where the family \( \{ c_n \} \) are defined recursively by \( c_0 = \mathbb{I} \) and

\[ c_n = B_+ \left( \sum_{m \leq n-1} c_m c_n \right). \]

The first five members of this family are

\[
\begin{align*}
c_0 &= \mathbb{I} \\
c_1 &= \bullet \\
c_2 &= 2 \\
c_3 &= 4 + \bigtriangleup \\
c_4 &= 8 + \bigtriangleup + 2 
\end{align*}
\]
In the next chapter we study a more general DSE

\[ X = I + \sum_{n=1}^{\infty} \omega_n \alpha^n B_{+}^{d_n} (X^{n+1}), \]

with \( X \in \mathcal{H}_{D}[\alpha] \), and then a system of DSEs given by

\[ X_i = \mathbb{1} + \sum_{\rho} \omega_i^{\rho} \alpha_i^{\rho_1} \ldots \alpha_n^{\rho_n} B_{+}^{d_{\rho}} \left( X_{i_{s_{1}}}^{\zeta_{i_{s_{1}}}} \ldots X_{i_{s_{n}}}^{\zeta_{i_{s_{n}}}} \right), \]

with \( X_i \in \mathcal{H}_{D}[\{\alpha_1, \ldots, \alpha_n\}] \).
Chapter 3

Hopf subalgebras from Dyson-Schwinger equations

In this chapter we consider solutions
\[ X(\alpha) = \sum_n \alpha^n c_n \]
or, more generally,
\[ X_i(\alpha_1, \ldots, \alpha_m) = \sum_{k_1, \ldots, k_m} \alpha_{k_1}^{k_1} \ldots \alpha_{k_m}^{k_m} c_i^{k_1, \ldots, k_m} \]
to DSEs, and the algebra \( \mathcal{H}_D \) generated by the coefficients \( c^{k_1, \ldots, k_m} \in \mathcal{H}_D \).

Let \( \mathcal{H}_D = H^0 \cap \mathcal{H}_D \). The goal of this chapter is to show that these sub-algebras \( \mathcal{H}_D \subset \mathcal{H}_D \) are Hopf. That is, that \( \mathcal{H}_D \) is closed under \( \Delta \) and \( S \), and respects the grading on \( \mathcal{H}_D \):

\[ \Delta(c_i^{k_1}) \in \bigoplus_{n_i + m_i = k_i} H_n^k \otimes H_m^k, \]
\[ S(c_i^{k_1}) \in H_k^k. \]

To this end, we show that \( R(c_i^{k_1}) \in \mathcal{H}_D \) and \( P(c_i^{k_1}) \in \mathcal{H}_D^{-1} \) for some \( l \). Then the conditions 3.1 will follow immediately from the definitions 2.1 and 2.2.

3.1 Single equation

Following the example of \([1]\), we begin by studying the DSE

\[ X = I + \sum_{n=1}^{\infty} \omega_n \alpha^n B^n_{\alpha^*} (X^{n+1}) \]

with solutions \( X(\alpha) \in \mathcal{H}_D[[\alpha]] \).

Lemma 1. Equation 3.2 has a unique solution

\[ X = \sum_n \alpha^n c_n, \]

with \( c_n \in \mathcal{H}_D \) for all \( n \in \mathbb{N} \).
Proof. Substitute the ansatz 3.3 into 3.2 to find

\[ c_0 = \mathbb{1}, \quad (3.4) \]

\[ c_n = \sum_{m=1}^{n} \omega_n B^d_n \left( \sum_{k_1+\ldots+k_{m+1} = n-m} c_{k_1}\ldots c_{k_{m+1}} \right). \]

\[ \Delta(c_n) = \sum_{k=0}^{n} P^n_k \otimes c_k, \quad (3.5) \]

where \( P^n_k \) is a homogeneous polynomial of degree \( n - k \) in the \( c_k \) (\( k \leq n \)).

Proof.

\[ \Delta(c_0) = c_0 \otimes c_0, \]

\[ \Delta(c_1) = c_1 \otimes c_0 + c_0 \otimes c_1, \]

Now proceed by induction: assume that 3.5 holds for all \( c_i, i \leq n-1 \). Let * denote the condition \( k_1 + \ldots + k_{m+1} = n - m \). Using the 1-cocycle property (2.3) of the grafting operators, see that

\[
\Delta(c_n) = \sum_{m=1}^{n} \omega_m \Delta \left( B^d_n \left( \sum_{\ast} c_{k_1}\ldots c_{k_{m+1}} \right) \right)
= c_n \otimes \mathbb{1} + \sum_{m} \omega_m (id \otimes B^d_n) \left( \sum_{\ast} \Delta(c_{k_1})\ldots\Delta(c_{k_{m+1}}) \right),
\]

and using the assumption on \( c_i, i \leq n \):

\[
\Delta(c_n) = c_n \otimes \mathbb{1} + \sum_{m} \omega_m (id \otimes B^d_n) \left( \sum_{l_1 \leq k_1} \left( \sum_{l_m+1 \leq k_{m+1}} P^n_{l_1}\ldots P^n_{l_{m+1}} \otimes B^d_n (c_{l_1}\ldots c_{l_{m+1}}) \right) \right)
\]

\[
= c_n \otimes \mathbb{1} + \sum_{m\leq n-1} \omega_m \left( \sum_{l_1 \leq k_1} \left( \sum_{l_m+1 \leq k_{m+1}} P^n_{l_1}\ldots P^n_{l_{m+1}} \otimes B^d_n (c_{l_1}\ldots c_{l_{m+1}}) \right) \right).
\]

For any fixed \( m \), let \( l_1, \ldots, l_{m+1} \) vary. \( \Delta(c_n) \) has a term

\[
\sum_{\ast} \left( P^n_{l_1}\ldots P^n_{l_{m+1}} \otimes B^d_n (c_{l_1}\ldots c_{l_{m+1}}) \right).
\]

By assumption, each product \( P^n_{l_1}\ldots P^n_{l_{m+1}} \) is a polynomial of degree \( n - m - (l_1 + \ldots + l_{m+1}) \), and therefore so is the left hand side of this tensor product. The corresponding right hand side is nothing but a single term in the expression 3.4 for \( c_{n-m} \). Therefore summing over \( m \) gives \( \sum_{k \leq n} P^k \otimes c_k \), since \( k = n - m \) by definition. \( \square \)
3.2 The operadic approach

The next step is to extend theorem 4 to cover a system of DSEs. Continuing with a proof following the method above would be possible, but labourious (and error-prone), with huge products of sums and sets of indices. Instead, we present the operadic proof given by Bergbauer and Kreimer in [1] and use it to reduce the proofs of theorem 4 and the main theorem (theorem 5) to simple counting exercises.

Outline of the approach:

• Consider trees in $\mathcal{H}_{D_c}$ as operadic objects, with inputs at every vertex and an output at the root.

• We consider these operadic objects to be planar, i.e.

![Tree Diagram]

whereas the forests in $\mathcal{H}_D$ are nonplanar:

![Tree Diagram]

• The number of inputs at each vertex is determined by the particular form of the DSE, and there may be different types of in/outputs.

• The operadic maps $\circ_i$ are given composition of the output and the input at the $i^{th}$ leaf (where we can number the leaves, without loss of generality, 1,2,... working left-to-right). For example,

![Tree Diagram]

• Construct the solution (a series in the coupling constants with operadic coefficients $\nu$) to an operadic version of the DSE.

• Translate this solution back into the solution $X$ in terms of the $c \in \mathcal{H}_{D_c}$ by removing the in/outputs and removing the planarity restriction to recover weights.

Example 13. We have already seen the DSE

$$X = \alpha B_+(X^2)$$
and its solution. The operadic version of the coefficient $c_3$ is

$$
\nu_3 = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3} \\
\text{Diagram 4} \\
\text{Diagram 5}
\end{array}
$$

### 3.2.1 Operadic proof of theorem 4

Consider the operadic fixpoint equation:

$$
G = I + \sum_n \alpha^n \mu_{n+1}(G^{\otimes n+1}). \tag{3.6}
$$

Each $\mu_j$ is a map $V^{\otimes j} \to V$ for some vector space $V$, and $G(\alpha)$ is a series in $\alpha$ with coefficients $\nu$:

$$
G(\alpha) = I + \sum_k \alpha^k \nu_k. \tag{3.7}
$$

**Lemma 2.** Each $\nu_k$ is the sum with unit weights over all maps $V^{\otimes k+1} \to V$ obtained by compositions of the “undecomposable” maps $\mu_j$.

**Proof.** By simply inserting 3.7 into 3.6, we find that

$$
\nu_0 = I
$$

and

$$
\nu_1 = \mu_2 \circ I = \mu_2.
$$

Assume for some $n \in \mathbb{N}$ that the lemma holds for all $k < n$. Then we look at the coefficients of $\alpha^n$ and see

$$
\nu_n = \mu_{n+1} \circ I + \mu_n \circ \nu_1 + \mu_n \circ \nu_1 \circ \nu_1 + \mu_1 \circ \nu_{n-1} + \ldots + \mu_1 \circ \nu_{n-1},
$$

and by assumption on $\nu_2, \ldots, \nu_{n-1}$ the lemma holds for $\nu_n$. \hfill \square

For this equation each map $\mu_j$ is a single vertex with $j + 1$ inputs:

$$
\mu_1 = \begin{array}{c}
1
\end{array}, \quad \mu_2 = \begin{array}{c}
2
\end{array}, \quad \mu_3 = \begin{array}{c}
3
\end{array}, \ldots
$$

**Remark 4.** To compute a given $\nu_k$, it is not necessary to know all $\nu_j$ for $j < k$, as for the coefficients $c_k$ above. One can simply construct $\nu_k$ by making the appropriate compositions of the undecomposable $\mu_i$. 
3.2. THE OPERADIC APPROACH

Example 14. Let $\omega_n = 1$ for all $n$, and let $D = \mathbb{N}$.

\[ \nu_3 = 3 + 1 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1 \]

$\Rightarrow c_3 = 3 + 1 + 2 + 4 + 1$.

We look at the action of the coproduct $\Delta$ on the $\nu_n$:

$\Delta(\nu_n) = \nu'_n \otimes \nu''_n$.

$\nu''_n$ is the sum of $R^e(\nu_n)$ terms (the admissible cuts of $\nu_n$ are exactly the admissible cuts of $c_n$). Lemma 2 tells us that $\nu_n$ is the sum over all maps with unit weight from $V^\otimes n + 1$ to $V$. As $\Delta$ takes all admissible cuts, $\nu'_n$ contains all maps with unit weight from any $V^\otimes k + 1$, with $k \leq n - 1$ to $V$, and therefore sums over every $\nu_k : V^\otimes k + 1 \rightarrow V$, with $k \leq n$.

$\nu'_n$ contains all the terms $P^e(\nu_n)$. The left hand side corresponding to a $\nu_k$ on the right hand side of the tensor product is a sum over all forests $(\nu_{i_1})^{r_1} \cdots (\nu_{i_t})^{r_t} : V^\otimes n + 1 \rightarrow V^\otimes k + 1$, with conditions:

\[ r = r_1 + \cdots + r_t = k + 1, \]
\[ r_1 i_1 + \cdots + r_t i_t + r = n + 1, \]
\[ \Rightarrow r_1 i_1 + \cdots + r_t i_t = n - k. \] (3.8)

So we have

$\Delta(\nu_n) = \sum_{k \leq n} Q^n_k \otimes \nu_k$,

with

$Q^n_k = \sum_{3.8} (\nu_{i_1})^{r_1} \cdots (\nu_{i_t})^{r_t}$.

What changes upon translating this equation back into the nonplanar form 3.5? Each term in the sum over $k \leq n$ will pick up some factor $F^n_k$, and to avoid over-counting any monomial $(c_{i_1})^{r_1} \cdots (c_{i_t})^{r_t}$ we add in the condition that $1 \leq i_1 < \cdots < i_t$ (which we denote by $\ast$).

To compute $F^n_k$, we need to count how many planar forests $(\nu_{i_1})^{r_1} \cdots (\nu_{i_t})^{r_t}$ will map to the same nonplanar forest when the planarity restriction is lifted. This is the same as counting how many ways a given $(\nu_{i_1})^{r_1} \cdots (\nu_{i_t})^{r_t} : V^\otimes n + 1 \rightarrow V^\otimes k + 1$ can be composed with a corresponding $\nu_k : V^\otimes k + 1 \rightarrow V$. 
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There are \((k + 1)!\) possibilities for switching the positions of any the \(\nu_l\) appearing in a monomial, and \(r_1!r_2!...r_t!\) of them give the same result, even in the planar setting. So
\[
F_n^k = \frac{(k + 1)!}{r_1!...r_t!}.
\]
This means that \(\Delta(c_n) = \sum_{k=0}^n P_n^k \otimes c_k\), with
\[
P_n^k = \sum_{s,t} \frac{(k + 1)!}{r_1!...r_t!} c_{r_1}^{i_1}...c_{r_t}^{i_t}.
\]

3.3 Main theorem

For ease of notation, we may write multi-indices as \(k = k_1, ..., k_n\) and so on.

Consider a system of \(n\) DSEs, with \(n\) coupling constants \(\alpha_1, ..., \alpha_n\) given by:
\[
X_i = I + \sum_{\rho} \omega_\rho^i \alpha_1^{i_1}...\alpha_n^{i_n} B_\rho^{i_1} \left( X_1^{\zeta_1 s_1}...X_i^{\zeta_i s_i + 1}...X_n^{\zeta_n s_n} \right), \tag{3.9}
\]
where
- The vector \(\rho\) is defined by \(\rho_k = \zeta_k s_k\).
- \(X_i = X_i(\alpha_1, ..., \alpha_n) \in H_D[[\{\alpha_1, ..., \alpha_n\}]]\)
- The powers \(\zeta_1, ..., \zeta_n \in \mathbb{Z}\) vary
- \(s_1, ..., s_n \in \mathbb{Z}\), \(s_i\) is fixed for each \(X_i\).

Lemma 3. The equation 3.9 has a unique solution
\[
X_i = \sum_{k_j \in S} \alpha_1^{k_1}...\alpha_n^{k_n} c_{k_1}^{i_1}...c_{k_n}^{i_n}, \tag{3.10}
\]

Theorem 5. The coefficients \(c_k^i\) generate a Hopf subalgebra of \(H_D\):
\[
\Delta(c_k^i) = \sum_{k_j \leq k_i} P_{k_j}^{k_i} \otimes c_k^i,
\]

Remark 5. This new result, appearing here for the first time, is a direct generalisation of work by Loïc Foissy, allowing for several coupling constants.

Proof. Consider the operadic equation
\[
G_i = I + \sum_{\rho} \alpha_1^{i_1}...\alpha_n^{i_n} \beta_{\rho}^i \left( G_1^{\otimes \zeta_1 s_2}...G_i^{\otimes \zeta_i s_i + 1}...G_n^{\otimes \zeta_n s_n} \right), \tag{3.11}
\]
with irreducible maps \(\beta_{\rho}^i : V_1^{\otimes \zeta_1} \otimes ... \otimes V_n^{\otimes \zeta_n} \rightarrow V_i\), for some vector spaces \(V_1, ..., V_n\). This has solutions
\[
G_i = G_i = I + \sum_{k \in \mathbb{Z}^n} \alpha_1^{k_1}...\alpha_n^{k_n} c_k^i.
\]
Lemma 4. Each \( \nu_k^i \) is the sum with unit weights over all maps \( V_{k_1}^{\otimes k_1} \otimes \cdots \otimes V_{k_m}^{\otimes k_m} \rightarrow V^i \) obtained by compositions of undecomposable maps \( \mu_k^l \).

As for lemma 2, this can be proved by induction. This structure of the \( \nu_k^i \) is the basis for the proof of theorem 5.

Consider the form of \( \Delta(\nu_k^i) = \nu' \otimes \nu'' \). The side \( \nu'' \) contains all possible trees obtained by removing some (non-root) forest in \( \nu_k^i \) (which by lemma 4 means the sum of all \( \nu_1^i : V_{l_1}^{\otimes l_1} \otimes \cdots \otimes V_{l_m}^{\otimes l_m} \rightarrow V^i \) where \( l_i \leq k_i \)). The part of \( \nu' \) corresponding to each \( \nu_k^i \) contains the forests removed from \( \nu_k^i \) to obtain \( \nu_k^j \), products which form maps \( V_{k_1}^{\otimes k_1} \otimes \cdots \otimes V_{k_m}^{\otimes k_m} \rightarrow V_{l_1}^{\otimes l_1} \otimes \cdots \otimes V_{l_m}^{\otimes l_m} \).

Consider one such term: a monomial \( (\nu_1^{i_1})^{r_1} \cdots (\nu_t^{i_t})^{r_t} \), where

\[
\begin{align*}
    r_1 l_1^1 + \cdots + r_t l_t^t &= \begin{cases} s_1 k_j, j \neq i \\ s_1 k_i + 1, j = i \end{cases} \\
    r_j &= \sum_{i=1}^{k_i} r_k = \begin{cases} s_1 l_j, j \neq i \\ s_1 l_i + 1, j = i \end{cases}.
\end{align*}
\] (3.12) (3.13)

Now we have an expression

\[
\Delta(\nu_k^i) = \sum_{\nu \leq k_i} Q_{k_1}^{k_1} \otimes (\nu_1^i),
\]

where

\[
Q_{k}^{k_1} = \sum_{3,12,3,13} (\nu_1^{i_1})^{r_1} \cdots (\nu_t^{i_t})^{r_t}.
\]

Again, translation back to the planar picture adds a factor \( F_{k_1}^{k_1} \) to each term and the constraint (call it \( * \)) that the upper indices \( i_j \) are distinct. To compute \( F_{k_1}^{k_1} \), count the number of terms in \( Q_{k}^{k_1} \) which translate to the same monomial in \( P_{k}^{k_1} \): there are \( \prod_j r_j! \) ways to switch the order of the \( \nu_i \), and \( \prod_j r_j! \) are equivalent even in the operadic setting. This gives:

\[
F_{k_1}^{k_1} = \frac{r_1! \cdots r_t!}{r_1! \cdots r_t!},
\]

so that

\[
P_{k_1}^{k_1} = \sum_{3,12,3,13,*} \frac{r_1! \cdots r_t!}{r_1! \cdots r_t!} (c_1^{i_1})^{r_1} \cdots (c_t^{i_t})^{r_t}.
\]
Chapter 4

Application in physics

Definition 14. A Feynman graph is a connected multigraph\(^1\) \(\Gamma\) which has two kinds of vertices, called internal and external. We write this as \(V(\Gamma) = V^{\text{int}} \cup V^{\text{ext}}\). An external vertex is simply any vertex of degree 1. The unique edge incident to an external vertex is called an external edge. All other vertices and edges are called internal. A Feynman graph has the following properties:

- \(|V^{\text{int}}| \geq 2\)
- There are several types of edge and vertex (i.e. there are decorations on both \(V\) and \(E\)). The different edge types represent different types of particle, while the vertex types represent possible interactions of the theory in question.
- \(\Gamma\) must be "1-particle-irreducible"(1PI): \(\Gamma\) remains connected after deletion of any single internal edge.
- The internal edges and vertices may be weighted.

We refer to the sets \(V^{\text{ext}}\) and \(E^{\text{ext}}\) as the external structure of \(\Gamma\).

The set of Feynman graphs forms a graded Hopf algebra \((\mathcal{H}_{FG}, m, \mathbb{I}, \Delta, \hat{1}, S)\) ([5]). Just as the coproduct of \(\mathcal{H}_D\) sums over ways to decompose a forest into smaller forests, the coproduct of \(\mathcal{H}_{FG}\) keeps track of all cycles in a Feynman graph:

\[
\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum \gamma \otimes \frac{\Gamma}{\gamma},
\]

where \(\gamma\) are the subgraphs of \(\Gamma\) whose connected components are 1PI (and whose external structure and weight may be constrained), and \(\frac{\Gamma}{\gamma}\) is the graph obtained by shrinking \(\gamma\) in \(\Gamma\) to a single vertex. The insertion operator \(B_\gamma\) on Feynman graphs is a linear map in \(B_\gamma : \mathcal{H}_{FG} \rightarrow \mathcal{H}_{FG}\) which acts on \(\Gamma\) by replacing some internal edge of \(\Gamma\) by the graph \(\gamma\). In perturbative QFT, Feynman graphs are

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\(^1\) Edges are not necessarily distinct.

\(^2\) The Hochschild 1-cocycles \(B^1_D\) of \(\mathcal{H}_D\) and \(B_\gamma\) of \(\mathcal{H}_{FG}\) are linked by a Hopf algebra morphism, see theorem 5 of [13].
associated via diffeomorphisms (called Feynman rules) to integrals which form a perturbation series describing a given interaction. Unfortunately for quantum field theorists, the integrals encountered are often divergent (a divergence in an integral corresponds to a cycle in its associated Feynman graph). The art of renormalisation exists to tame these integrals and produce a usable field theory. In the Hopf-algebraic setting, one associates a decorated rooted tree to each graph $\Gamma$, which represents the structure of the divergences of $\Gamma$ ([8],[5],[6]).

**Example 15.** In $\phi^3$-theory, the graph $\Gamma = \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \draw [thick, ->] (.3,-.2) -- (.3,.2); \end{tikzpicture}$ has two sub-divergences: $\gamma_1 = \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \end{tikzpicture}$, $\gamma_2 = \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (.3,-.2) -- (.3,.2); \end{tikzpicture}$. This gives

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \gamma_1 \otimes \frac{\Gamma}{\gamma_1} + \gamma_2 \otimes \frac{\Gamma}{\gamma_2} + \gamma_1 \cup \gamma_2 \otimes \frac{\Gamma}{\gamma_1 \cup \gamma_2}$$

$$= \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \draw [thick, ->] (.3,-.2) -- (.3,.2); \end{tikzpicture} + \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \draw [thick, ->] (.3,-.2) -- (.3,.2); \draw [thick, ->] (.3,-.2) -- (.3,.2); \end{tikzpicture} + \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \draw [thick, ->] (.3,-.2) -- (.3,.2); \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \end{tikzpicture} + \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \draw [thick, ->] (.3,-.2) -- (.3,.2); \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \end{tikzpicture}$$

We can associate to each sub-divergence a vertex in a rooted tree $T$. The tree associated to $\Gamma$ is $\begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \draw [thick, ->] (.3,-.2) -- (.3,.2); \draw [thick, ->] (.3,-.2) -- (.3,.2); \end{tikzpicture}$.

Dyson-Schwinger equations are equations of motion for Green’s functions in QFT. They take the form of combinatorial equations based on the action of $B_\gamma$ in the Hopf-algebraic setting, and of analytic integral equations which result from applying Feynman rules to the combinatorial DSEs. The Green’s functions themselves appear as infinite sums in coupling constants, indexed by Feynman graphs with a certain external structure.

**Example 16.** In quantum electrodynamics (QED) there are two possible edge types

$\begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \end{tikzpicture}$, $\begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (.3,-.2) -- (.3,.2); \end{tikzpicture}$, and three possible external structures for any graph. We let

$X_1 = \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \end{tikzpicture}$, $X_2 = \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \end{tikzpicture}$, $X_3 = \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \end{tikzpicture}$
denote the infinite sums of graphs with each particular external structure, where $\begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \end{tikzpicture}$ denotes all possible internal structures. The Dyson-Schwinger equations of this theory are

$$X_1 = B \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \end{tikzpicture} \left( \frac{(1 + X_3)^2}{(1 - X_1)(1 - X_2)} \right)$$

$$X_2 = B \begin{tikzpicture} [baseline=0pt] \draw [thick, ->] (-.3,-.2) -- (-.3,.2); \end{tikzpicture} \left( \frac{(1 + X_3)^2}{1 - X_1}(1 - X_2) \right)$$

$$X_3 = \sum_\gamma B_\gamma \left( \frac{(1 + X_3)^{1+2(\gamma)}}{(1 - X_1)^{2(\gamma)}(1 - X_2)^{2(\gamma)}} \right)$$

---

3An explanation of the concepts and methods of perturbative QFT can be found in, for example, the early chapters of [15].

4Described in great detail in, for example, [2].
where \( l(\gamma) \) is the number of cycles in \( \gamma \).

**Example 17.** Consider a theory with two edge types and three allowed vertex types

\[
\begin{align*}
v_1 = & \quad , \\
v_2 = & \quad , \\
v_3 = & \quad .
\end{align*}
\]

The Feynman graphs of this theory with one cycle are

\[
\begin{align*}
\gamma_1 = & \quad , \\
\gamma_2 = & \quad + \\
\gamma_3 = & \quad , \\
\gamma_4 = & \quad , \\
\gamma_5 = & \quad , \\
\gamma_6 = & \quad , \\
\gamma_7 = & \quad , \\
\gamma_8 = & \quad , \\
\gamma_9 = & \quad .
\end{align*}
\]

Associate to each vertex type \( v_i \) a coupling constant \( \alpha_i \). Again, denote

\[
\begin{align*}
X_1 = & \quad , \\
X_2 = & \quad , \\
X_3 = & \quad .
\end{align*}
\]

After a truncation, the Dyson-Schwinger equations are

\[
\begin{align*}
X_1 = & \quad I + \alpha_1^2 B_{\gamma_1}(X_1^3) + \alpha_2 B_{\gamma_2}(X_1 X_2) + \alpha_3 B_{\gamma_3}(X_1 X_3), \\
X_2 = & \quad I + \alpha_1^2 \alpha_3 \frac{1}{\alpha_2} B_{\gamma_4}(X_2^3) + \alpha_3 B_{\gamma_5}(X_2 X_3) + \alpha_1^2 B_{\gamma_6}(X_2^2 X_2), \\
X_3 = & \quad I + \alpha_3 B_{\gamma_7}(X_3^2) + \alpha_1^2 B_{\gamma_8}(X_3^3 X_3) + \alpha_1^2 \frac{1}{\alpha_3} B_{\gamma_9}(X_1^4).
\end{align*}
\]

This system is of the form 3.9, so it does indeed generate a Hopf subalgebra of the algebra \( \mathcal{H}_{FG} \) generated by all Feynman graphs of this theory.
Bibliography


Selbstständigkeitserklärung

Ich habe die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt, und ich reiche zum erstenmal eine Masterarbeit in diesem Studiengang ein.

Berlin,

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