Master Thesis:

Algebraic Structures in the Coupling of Gravity to Gauge Theories

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Structure of Local Quantum Field Theories

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Contents

1 Introduction

The theory of General Relativity (GR) and Quantum Field Theory (QFT) are the two great achievements of physics in the 20th century. GR, on the one side, describes nature on very large scales when huge masses are involved. QFT, on the other side, describes nature on very small scales when tiny masses are involved. Being very successful in their regimes, there are situations when both conditions appear at the same time, i.e. when huge masses are compressed into small scales. For example, this situation occurs in models of the big bang and in models of black holes. For these situations a theory of Quantum Gravity (QG) is needed to understand nature. In particular, a theory of QG should be able to clarify how the universe emerged, i.e. if there was a big bang or not. Therefore, it was soon tried to apply the usual techniques of perturbative QFT to the dynamical part of the metric in spacetimes of GR [1]. These works, in [1] called "The covariant line of research", were started by M. Fierz, W. Pauli and L. Rosenfeld in the 1930s. Then, R. Feynman [2] and B. DeWitt [3, 4, 5, 6] calculated the Feynman rules of GR in the 1960s. Next, D. Boulware [7], S. Deser [7], G. t' Hooft [8], P. van Nieuwenhuizen [7] and M. Veltman [9] found evidence of the non-renormalizability in the sense of Z factors of Quantum General Relativity (QGR) in the 1970s. We stress, that by QGR we mean a quantization of GR using QFT methods, whereas by QG we mean any theory of quantized gravitation, such as e.g. Loop Quantum Gravity, String Theory or Supergravity. In this thesis we continue the work on QGR by using the modern techniques of Hopf algebraic renormalization [10, 11, 12], developed by A. Connes and D. Kreimer in the 1990s and 2000s [13, 14, 15].

We start this thesis in Section 2 with basic conventions and notations. Then, in Section 3, we first explain the differential geometric notions needed to understand the Lagrange density of Quantum General Relativity coupled to Quantum Electrodynamics (QGR-QED). Furthermore, we introduce the Lagrange density of QGR-QED,

$$\mathcal{L}_{\text{QGR-QED}} = \left(\frac{1}{\underline{\kappa}^2} R + g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \bar{\Psi} \left(i \nabla\!\!\!\!/^{\,\Sigma M \times U(1)} - m \right) \Psi \right) \mathrm{d}V_g + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}} \, , \tag{1}$$

which consists of the usual Einstein-Hilbert Lagrange density combined with the canonical generalization of the Quantum Electrodynamics (QED) Lagrange density to curved spacetimes. Finally, the Lagrange density, Equation (1), is discussed in detail. Then, in Section 4 we introduce Hopf algebras in general and the Connes-Kreimer renormalization Hopf algebra in particular. In Section 5 we discuss a problem which can occur when associating the Connes-Kreimer renormalization Hopf algebra to a given local QFT. As we remark, this problem occurs already in QED, but also in QGR-QED. More concretely, there can exist divergent Feynman graphs whose residue is not in the residue set of the given local QFT. This problem is solved by either Construction 5.5 or Construction 5.6 combined with Lemma 5.12. Moreover, since their physical interpretation differs, as is discussed in Remark 5.4, both constructions can also be combined to suit the physical needs of the given local QFT. Furthermore, these results are applied to QGR-QED to obtain the corresponding renormalization Hopf algebra of QGR-QED. Therefore, we formulate and prove a generalization of Furry's Theorem in Theorem 5.15, stating that all Feynman amplitudes with an odd number of external photons and an arbitrary number of external gravitons vanish. This is in particular useful, since at least for the calculations done in the realm of this thesis, these are the only Feynman graphs which need to be set to zero when constructing the renormalization Hopf algebra of QGR-QED, besides from pure self-loop Feynman graphs, which vanish in the renormalization process. Then, in Section 6 we present the main part of this thesis. First, we present all combinatorial Green's functions of the one-loop propagator amplitudes, the one-loop three-point amplitudes and the two-loop propagator amplitudes. Then, we present their coproduct structure, for which the coproduct of 155 Feynman graphs has been computed. From this, we conclude that the renormalization Hopf algebra of QGR-QED does not possess Hopf subalgebras for all residues and all multi-indices. This translates into the statement that it is not possible to renormalize all monomials in the Lagrange density using Z factors. However, we remark that it could be possible to improve this situation by imposing the corresponding Slavnov-Taylor identities. In Appendix A we explain how the Feynman graphs for this thesis were obtained. First, we used the program “feyngen” [16] of M. Borinsky to obtain the relevant scalar Feynman graphs with all topologies. Then, we present seven Python [17] programs, all written by the author, used to get all relevant labeled Feynman graphs. In Appendix B is a signed statement of authorship. And finally, in Appendix C all used references are listed.

2 Conventions and Notations

2.1 Einstein Summation Convention

We use the Einstein summation convention, if not stated otherwise.

2.2 Metric as a Bilinear Form and as a Matrix

We write metrics either as bilinear forms $g_{\mu\nu}$ or as matrices g. Then, the matrix g is understood as the bilinear form $g_{\mu\nu}$ with the μ-index naturally raised, i.e.

$$g = g^{\mu}{}_{\nu} \neq g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\nu} \,, \tag{2}$$

where δ^{μ}_{ν} is the Kronecker delta. Concretely, for the Minkowski metric in four dimensions of spacetime, we have

$$\eta = \eta^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \neq \eta^{\mu\rho} \eta_{\rho\nu} = \delta^{\mu}_{\nu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,. \tag{3}$$

2.3 Coupling Constants

We underline the coupling constants for the electric charge, $\underline{e}$, and for the graviton, $\underline{\kappa}$, in order to avoid confusion with Euler's number and vielbein and inverse vielbein.

2.4 Tensor Product

We set the tensor product to be over the rationals if not stated otherwise, i.e.

$$\otimes := \otimes_{\mathbb{Q}} \,. \tag{4}$$

2.5 Oriented Feynman Graph Edges

We draw oriented Feynman graph edges, i.e. fermion, photon-ghost and graviton-ghost edges, without orientation. This is understood as the sum of all Feynman graphs having all possible orientations.

3 Differential Geometric Notions and the Lagrange Density of QGR-QED

3.1 Differential Geometric Notions

Definition 3.1 (Spacetime and the spacetime bundle)**.** We define spacetime to be a four dimensional parallelizable Lorentzian oriented and time-oriented spinnable manifold (M, g), c.f. Remark 3.2. We choose the signature $+, -, -, -$ for g. If the manifold is flat, i.e. when spacetime is the Minkowski spacetime, we denote the metric $g_{\mu\nu}$ by $\eta_{\mu\nu}$, which is given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{5}$$

Finally, the spacetime bundle is the globally trivial bundle $\mathcal{M} := M \times TM \times \Sigma M \times U(1)$, where TM is the tangential bundle, ΣM is the spinor bundle representing fermions and $U(1)$ is the principal bundle representing electrodynamics, c.f. Definition 3.2.

Remark 3.2. The requirement for the spacetime (M, g) being parallelizable is motivated because we want to interpret all fields as particles in the sense of E. Wigner, i.e. as irreducible representations of the Poincaré group. This is only possible if all fields are living on Minkowski spacetime (M, η), since otherwise there does not exist the Poincaré group. This needs to hold in particular for the graviton field $h_{\mu\nu}$, which will be introduced in Definition 3.8, and which relates the Minkowski spacetime (M, η) to the spacetime (M, g). But this requires that the manifold (M, g) is diffeomorphic to the Minkowski spacetime (M, η) and hence is parallelizable. To get a particle interpretation for quantum fields on general curved spacetimes one needs to pass to Algebraic Quantum Field Theory, c.f. [18, 19, 20]. Furthermore, a non-compact manifold (M, g) admits a spin-structure if and only if (M, g) is parallelizable, i.e. has a trivial tangent bundle [21], however it is possible to define fermion fields sufficient for physical needs without a globally defined spin structure [22]. Moreover, the requirement for the spacetime bundle $\mathcal{M} := M \times TM \times \Sigma M \times U(1)$ being trivial is motivated because we need the $U(1)$ bundle to be trivial to get a well-defined connection form A_μ for a given Faraday or electromagnetic field strength tensor $F_{\mu\nu}$, which is the Cartan derivative of A_μ. Otherwise, A_μ would be only well-defined up to the addition of a closed form. Moreover, the tangent bundle TM is trivial since M is chosen to be parallelizable, and thus the spinor bundle ΣM is trivial as well.

Definition 3.3 (Vielbein and inverse vielbein, [23])**.** Let (M, g) be a semi-Riemannian d-dimensional manifold. Then, we can define locally a set of d one-forms or covector fields $\left\{ e^m_\mu \right\}_{m \in \{ 1, \dots, d \}}$, called vielbeins, such that

$$g_{\mu\nu} = \eta_{mn} e^m_\mu e^n_\nu . \tag{6}$$

Furthermore, we can define locally a set of d vector fields $\left\{ e^\mu_m \right\}_{m \in \{ 1, \dots, d \}}$, called inverse vielbeins, such that

$$\eta_{mn} = g_{\mu\nu} e^\mu_m e^\nu_n . \tag{7}$$

Greek indices, here μ and ν, are referred to as curved indices and are raised and lowered using the usual metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, whereas Latin indices, here m and n, are referred to as flat indices and are raised and lowered using the Minkowski metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. Therefore, inverse vielbeins are related to vielbeins via

$$e^\mu_m = g^{\mu\nu} \eta_{mn} e^m_\mu . \tag{8}$$

Moreover, notice that Equation (7) is equivalent to[1]

$$g_{\mu\nu} e^{\nu}_{n} = \eta_{mn} e^{m}_{\mu} = e_{\mu n} \,, \tag{9}$$

which states that, when viewing flat indices as numbers, inverse vielbeins $\{ e^{\mu}_{m} \}_{m \in \{ 1,\ldots,d \}}$ are a set of d eigenvector fields of the metric $g_{\mu\nu}$ with eigenvalues $\eta_{mm} \in \{ \pm 1 \}$, which are normalized to unit length, i.e.

$$\| e^{\mu}_{m} \| = \sqrt{\left| g_{\mu\nu} e^{\mu}_{m} e^{\nu}_{m} \right|} = 1 \,. \tag{10}$$

Remark 3.4. Notice, that the vielbein and the inverse vielbein, as defined in Definition 3.3, are not unique, since for any local Lorentz transformation ${\Lambda^{m}}_{n} \in SO(1,3)$ we have:

$$\begin{aligned} g_{\mu\nu} e^{\mu}_{m} e^{\nu}_{n} &= \eta_{mn} \\ &= \eta_{rs} {\Lambda^{r}}_{m} {\Lambda^{s}}_{n} \\ &= g_{\mu\nu} e^{\mu}_{r} e^{\nu}_{s} {\Lambda^{r}}_{m} {\Lambda^{s}}_{n} \\ &= g_{\mu\nu} \tilde{e}^{\mu}_{m} \tilde{e}^{\nu}_{n} \end{aligned} \tag{11}$$

Here, we denoted the transformed inverse vielbeins as $\tilde{e}^{\mu}_{m} := e^{\mu}_{r} {\Lambda^{r}}_{m}$. Obviously, the same calculations holds also for vielbeins instead of inverse vielbeins. This ambiguity will lead to the first term in the spin connection, c.f. Definition 3.5. In fact, the first term in the spin connection can be viewed as the gauge field associated to local Lorentz transformations.

Definition 3.5 (Connections on the spacetime bundle, [23, 24, 25, 26, 27])**.** We use the following connections on the spacetime bundle: For the tangent bundle TM of the manifold M we use the Levi-Civita connection ∇^{TM}_{μ}, acting on a vector field $x^{\nu} \in \Gamma(TM)$ as

$$\nabla^{TM}_{\mu} x^{\nu} = \partial_{\mu} x^{\nu} + \Gamma^{\nu}_{\mu\lambda} x^{\lambda} \tag{12a}$$

and on covector fields or one-forms as

$$\nabla^{TM}_{\mu} x_{\nu} = \partial_{\mu} x_{\nu} - \Gamma^{\lambda}_{\mu\nu} x_{\lambda} \,, \tag{12b}$$

with the Christoffel symbol

$$\Gamma^{\nu}_{\mu\lambda} = \frac{1}{2} g^{\nu\tau} \left(\partial_{\mu} g_{\lambda\tau} + \partial_{\lambda} g_{\tau\mu} - \partial_{\tau} g_{\mu\lambda} \right) \,. \tag{13}$$

Furthermore, for the spinor bundle we use the covariant derivative $\nabla^{\Sigma M}_{\mu}$, acting on a spinor field $\psi \in \Gamma(\Sigma M)$ as

$$\nabla^{\Sigma M}_{\mu} \psi = \partial_{\mu} \psi + i \omega_{\mu} \psi \,, \tag{14}$$

with the spin connection

$$i \omega_{\mu} = \frac{i}{4} \left(e^{m}_{\nu} \partial_{\mu} e^{\nu n} + e^{m}_{\nu} \Gamma^{\nu}_{\mu\rho} e^{\rho n} \right) \sigma_{mn} \tag{15}$$

and $\sigma_{mn} = \frac{i}{2} [\gamma_m, \gamma_n] \in \mathfrak{spin}(1,3)$. We remark, that the first term in the spin connection, $\frac{i}{4} \left(e^{m}_{\nu} \partial_{\mu} e^{\nu n} \right) \sigma_{mn}$, is due to the ambiguity in the definition of the vielbein and the inverse vielbein, as was discussed in Remark 3.4. Moreover, for the $U(1)$-principle bundle, we use the covariant derivative $\nabla^{U(1)}_{\mu}$, acting on a section $s \in \Gamma\left(U(1)\right)$ as

$$\nabla^{U(1)}_{\mu} s = \partial_{\mu} s + i \underline{e} A_{\mu} s \,, \tag{16}$$

[1] The following Equation (9) should be viewed in the sense of Subsection 2.2. More concretely, the μ-index and the m-index should be viewed as being naturally raised, i.e. without using the inverse metrics $g^{\mu\nu}$ and η^{mn}, such that in general ${g^{\mu}}_{\nu} \neq \delta^{\mu}_{\nu}$ and ${\eta^{m}}_{n} = \pm \delta^{m}_{n} \neq \delta^{m}_{n}$. Then, $g = {g^{\mu}}_{\nu}$ and $\eta = {\eta^{m}}_{n}$ are the matrices corresponding to the metrics $g_{\mu\nu}$ and η_{mn} as in Subsection 2.2. With these definitions, Equation (9) corresponds naturally to the eigenvalue equation ${g^{\mu}}_{\nu} e^{\nu}_{n} = {\eta^{m}}_{n} e^{\mu}_{m} = \pm \delta^{m}_{n} e^{\mu}_{m} = \pm e^{\mu}_{n}$.

with the connection form $i\underline{e}A_\mu$. In the following, we are interested in fermions which are coupled to the electromagnetic field. Mathematically, this is described via sections in the spinor bundle ΣM with a $U(1)$ action on it, i.e. $\Psi \in \Gamma\left(\Sigma M \times U(1)\right)$, c.f. Definition 3.8. Thus, the corresponding covariant derivative reads

$$\nabla_\mu^{\Sigma M\times U(1)}\Psi = \partial_\mu\Psi + i\omega_\mu\Psi + i\underline{e}A_\mu\Psi\,. \tag{17}$$

Definition 3.6 (Dirac operator, [25, 26, 27])**.** We define the Dirac operator $\not\nabla^{\Sigma M\times U(1)}$ on spinors $\Psi \in \Gamma\left(\Sigma M \times U(1)\right)$ such that the following diagram commutes:

$$\begin{tikzcd}
\Gamma\left(\Sigma M \times U(1)\right) \arrow[r, "\not\nabla^{\Sigma M\times U(1)}"] \arrow[d, "\nabla_\mu^{\Sigma M\times U(1)}"'] & \Gamma\left(\Sigma M \times U(1)\right) \\
\Gamma\left(T^*M \otimes_{\mathbb{R}} \left(\Sigma M \times U(1)\right)\right) \arrow[r, "g^{\mu\nu}\otimes_{\mathbb{R}} id"'] & \Gamma\left(TM \otimes_{\mathbb{R}} \left(\Sigma M \times U(1)\right)\right) \arrow[u, "\gamma_\nu"']
\end{tikzcd} \tag{18}$$

Here, $\nabla_\mu^{\Sigma M\times U(1)} := \partial_\mu + i\omega_\mu + i\underline{e}A_\mu$ is the covariant derivative on the bundle $\Sigma M \times U(1)$, and γ_ν the Clifford-multiplication, given locally via $\gamma_\nu := e_\nu^n\gamma_n$, with γ_n being the usual Minkowski spacetime Dirac matrices. Thus, the local description of the Dirac-operator on the bundle $\Sigma M \times U(1)$ is given via[2]

$$\not\nabla^{\Sigma M\times U(1)} = e^{\mu m}\nabla_\mu^{\Sigma M\times U(1)}\gamma_m = e^{\mu m}\left(\partial_\mu + i\omega_\mu + i\underline{e}A_\mu\right)\gamma_m\,. \tag{19}$$

Definition 3.7 (Riemannian volume form)**.** We define the Riemannian volume form for the spacetime (M, g) as

$$\mathrm{d}V_g := \sqrt{-\det(g)}\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,. \tag{20}$$

In particular, the Riemannian volume form for the Minkowski spacetime (M, η) takes the form

$$\mathrm{d}V_\eta = \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,. \tag{21}$$

Definition 3.8 (Fermion, photon and graviton field)**.** The mathematical objects correspond in the following way to physical particles: A four-component Dirac spinor ψ, i.e. a section in the spinor bundle $\psi \in \Gamma(\Sigma M)$, corresponds to a fermion field. In particular, in our case it is a linear combination of the electron and the positron field with charge $\pm\underline{e}$, respectively, inducing the four-current $\underline{e}j^\mu$, given by

$$\underline{e}j^\mu := \underline{e}e^{\mu m}\bar{\psi}\gamma_m\psi\,. \tag{22}$$

Here, $\bar{\psi}$ is the adjoined Fermion field to the fermion field $\psi \in \Gamma(\Sigma M)$, defined via

$$\begin{aligned}
\bar{\psi} &:= (\gamma_0\psi)^\dagger \\
&= \psi^\dagger\gamma_0^\dagger \\
&= \psi^\dagger\gamma_0\,.
\end{aligned} \tag{23}$$

The last equality follows from the fact, that in the representations for the Clifford multiplication of physical interest (Dirac, Weyl and Majorana) we have $\gamma_0^\dagger = \gamma_0$. Furthermore, the connection

[2]Notice, that by the tetrad postulate (which is not actually a postulate since it is always true), stating $\nabla_\mu e_\nu^n = 0$, it does not matter whether we place the vielbein e_ν^n before or after the covariant derivative $\nabla_\mu^{\Sigma M\times U(1)}$ [23].

form A_μ corresponds to the photon field, inducing the Faraday or electromagnetic field strength tensor $F_{\mu\nu}$, given by[3]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{24}$$

In the following, we are interested in the coupling of the fermion field ψ to the photon field A_μ. Mathematically, this is obtained by viewing fermions as sections in the bundle $\Sigma M \times U(1)$, i.e. $\Psi \in \Gamma\left(\Sigma M \times U(1)\right)$. Then, the interaction is given via the covariant derivative

$$\nabla_\mu^{\Sigma M \times U(1)} \Psi = \partial_\mu \Psi + i \omega_\mu \Psi + i \underline{e} A_\mu \Psi \,, \tag{25}$$

c.f. Definition 3.5. Therefore, in the following we mean by spinors or fermions sections in the bundle $\Sigma M \times U(1)$, if not stated otherwise. The adjoined fermion field $\bar{\Psi}$ to the fermion field $\Psi \in \Gamma\left(\Sigma M \times U(1)\right)$ is defined via

$$\bar{\Psi} := \Psi^\dagger \gamma_0 \,, \tag{26}$$

analogously to Equation (23). Moreover, the deviation of the metric $g_{\mu\nu}$ from the Minkowski metric $\eta_{\mu\nu}$ corresponds to the graviton field $h_{\mu\nu}$ with coupling constant $\underline{\kappa}$, i.e.

$$h_{\mu\nu} := \frac{1}{\underline{\kappa}} \left(g_{\mu\nu} - \eta_{\mu\nu}\right) \iff g_{\mu\nu} = \eta_{\mu\nu} + \underline{\kappa} h_{\mu\nu} \,. \tag{27}$$

The graviton field can be thought of as a $(0,2)$-tensor field living on the flat background Minkowski spacetime (M, η). Furthermore, all objects depending on the metric, such as the inverse metric, the vielbein and the inverse vielbein and the Riemannian volume form, need to be expressed in terms of the graviton field to obtain the corresponding Feynman rules.

Remark 3.9 (Feynman rules). To calculate the corresponding gravity Feynman rules, we need to express the inverse metric, the vielbein and inverse vielbein and the prefactor of the Riemannian volume form in terms of the graviton field. Since we don't need actual Feynman rules, we just remark here the corresponding expressions and postpone their detailed treatment to future work. The interested reader finds some of the Feynman rules in [27, 28]. Furthermore, we remark that all following series converge if and only if $\left\| h_{\mu\nu} \right\|_\infty < 1$. The inverse metric in terms of the graviton field is given by the corresponding Neumann series, i.e.

$$g^{\mu\nu} = \sum_{k=0}^{\infty} \left(-\underline{\kappa}\right)^k \left(h^k\right)^{\mu\nu} \,, \tag{28}$$

where $h^{\mu\nu} := \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$, $\left(h^0\right)^{\mu\nu} := \eta^{\mu\nu}$ and $\left(h^k\right)^{\mu\nu} := \underbrace{h^\mu_{\lambda_1} h^{\lambda_1}_{\lambda_2} \cdots h^{\lambda_{k-1}\nu}}_{k\text{-times}}$, for $k \in \mathbb{N}$. Furthermore, the expressions for the vielbein and the inverse vielbein read

$$e_\mu^m = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(h^k\right)^m_\mu \,, \tag{29a}$$

with $h^m_\mu := \eta^{m\nu} h_{\mu\nu}$, and

$$e_m^\mu = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \left(h^k\right)^\mu_m \,, \tag{29b}$$

with $h^\mu_m := \eta^{\mu\nu}\delta^\rho_m h_{\nu\rho}$. Moreover, the prefactor of the Riemannian volume form, $\sqrt{-\det\left(g\right)} = \sqrt{-\det\left(\eta + \underline{\kappa} h\right)}$, can be obtained by first expressing the determinant in terms of traces and

[3] Notice, that the Levi-Civita connections ∇_μ^{TM} reduce to partial derivatives ∂_μ here, since $F_{\mu\nu} = \nabla_\mu^{TM} A_\nu - \nabla_\nu^{TM} A_\mu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho - \partial_\nu A_\mu + \Gamma^\rho_{\nu\mu} A_\rho = \partial_\mu A_\nu - \partial_\nu A_\mu$, as the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ for the Levi-Civita connection ∇_μ^{TM} are symmetric in their lower two indices, c.f. Definition 3.5.

then plug the result into the Taylor series expansion of the square-root. More precisely, the determinant of a 4×4 matrix[4] $\mathfrak{M}$ can be expressed in terms of the trace as

$$\det\left(\mathfrak{M}\right) = \frac{1}{4!} \det \begin{pmatrix} \operatorname{tr}\left(\mathfrak{M}\right) & 1 & 0 & 0 \\ \operatorname{tr}\left(\mathfrak{M}^2\right) & \operatorname{tr}\left(\mathfrak{M}\right) & 2 & 0 \\ \operatorname{tr}\left(\mathfrak{M}^3\right) & \operatorname{tr}\left(\mathfrak{M}^2\right) & \operatorname{tr}\left(\mathfrak{M}\right) & 3 \\ \operatorname{tr}\left(\mathfrak{M}^4\right) & \operatorname{tr}\left(\mathfrak{M}^3\right) & \operatorname{tr}\left(\mathfrak{M}^2\right) & \operatorname{tr}\left(\mathfrak{M}\right) \end{pmatrix} . \tag{30}$$

Now, we set

$$\begin{aligned} \mathfrak{M}_{\mu\nu} &:= g_{\mu\nu} \\ &= \eta_{\mu\nu} + \underline{\kappa} h_{\mu\nu} \, . \end{aligned} \tag{31}$$

Since the trace is linear, we have

$$\begin{aligned} \operatorname{tr}\left(g\right) &= \operatorname{tr}\left(\eta + \underline{\kappa} h\right) \\ &= \operatorname{tr}\left(\eta\right) + \underline{\kappa} \operatorname{tr}\left(h\right) \\ &= -2 + \underline{\kappa} \eta^{\mu\nu} h_{\mu\nu} \, , \end{aligned} \tag{32}$$

and similar expressions for the higher powers in $g = \eta + \underline{\kappa} h$. Thus, the determinant is a polynomial in traces of powers of $\underline{\kappa} h$,[5] i.e.

$$\det\left(\eta + \underline{\kappa} h\right) \in \mathbb{R}\left[\operatorname{tr}\left(h\right), \operatorname{tr}\left(h^2\right), \operatorname{tr}\left(h^3\right), \operatorname{tr}\left(h^4\right)\right] . \tag{33}$$

We separate the constant term, which is -1, and write the non-constant term as $\mathfrak{P}\left(\underline{\kappa} h\right)$, i.e.

$$\mathfrak{P}\left(\underline{\kappa} h\right) := -\det\left(\eta + \underline{\kappa} h\right) - 1 \iff \det\left(\eta + \underline{\kappa} h\right) = -\mathfrak{P}\left(\underline{\kappa} h\right) - 1 \, . \tag{34}$$

Then, we plug this expression into the Taylor series expansion of the square root around 1, i.e.

$$\sqrt{x+1} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k \, , \tag{35}$$

which converges for $|x| < 1$. Finally, we to obtain

$$\begin{aligned} \sqrt{-\det\left(g\right)} &= \sqrt{-\det\left(\eta + \underline{\kappa} h\right)} \\ &= \sqrt{\mathfrak{P}\left(\underline{\kappa} h\right) + 1} \\ &= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(\mathfrak{P}\left(\underline{\kappa} h\right)\right)^k \, . \end{aligned} \tag{36}$$

However, for the realm of this thesis it suffices to know that the inverse metric, the vielbein and inverse vielbein and the prefactor of the Riemannian volume form are all power series in the graviton field $h_{\mu\nu}$.

Remark 3.10 (Feynman rules 2). For the realm of this thesis where we consider two-loop propagator Feynman graphs and one-loop propagator and three-point Feynman graphs, it is actually sufficient to consider the Lagrange density $\mathcal{L}_{\text{QGR-QED}}$ only up to order $\mathcal{O}\left(\underline{\kappa}^2\right)$, since the higher valent graviton vertices would only contribute to graphs with self-loops. Therefore, the formulas from Remark 3.9 read as follows: For the inverse metric we obtain

$$g^{\mu\nu} = \eta^{\mu\nu} - \underline{\kappa} \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma} + \underline{\kappa}^2 \eta^{\mu\rho} \eta^{\sigma\lambda} \eta^{\tau\nu} h_{\rho\sigma} h_{\lambda\tau} + \mathcal{O}\left(\underline{\kappa}^3\right) . \tag{37}$$

[4] Actually, this works for any $d \times d$ matrix with the obvious generalizations of the following equations.

[5] Notice, that $\operatorname{tr}\left(\eta\right) = \operatorname{tr}\left(\eta^3\right) = -2 \in \mathbb{R}$ and $\operatorname{tr}\left(\eta^2\right) = \operatorname{tr}\left(\eta^4\right) = 4 \in \mathbb{R}$.

Furthermore, the vielbein and inverse vielbein, as defined in Definition 3.3, are given by

$$e_\mu^m = \eta_\mu^m + \frac{1}{2}\underline{\kappa}\eta^{\nu m}h_{\mu\nu} - \frac{1}{8}\underline{\kappa}^2\eta^{\rho\sigma}\eta^{\lambda m}h_{\mu\rho}h_{\sigma\lambda} + \mathcal{O}\left(\underline{\kappa}^3\right) \tag{38a}$$

and

$$e_m^\mu = \eta_m^\mu - \frac{1}{2}\underline{\kappa}\eta^{\mu\nu}\delta_m^\rho h_{\nu\rho} + \frac{3}{8}\underline{\kappa}^2\eta^{\mu\rho}\eta^{\sigma\lambda}\delta_m^\tau h_{\rho\sigma}h_{\lambda\tau} + \mathcal{O}\left(\underline{\kappa}^3\right) . \tag{38b}$$

And finally, the prefactor of the Riemannian volume form, defined in Definition 3.7, is given by

$$\begin{aligned}\sqrt{\det(g)} &= \sqrt{\det(\eta + \underline{\kappa}h)} \\ &= 1 + \frac{\underline{\kappa}}{2}\eta^{\mu\nu}h_{\mu\nu} + \frac{\underline{\kappa}^2}{8}\left(\eta^{\mu\nu}\eta^{\rho\sigma}h_{\mu\nu}h_{\rho\sigma} - 2\eta^{\mu\sigma}\eta^{\nu\rho}h_{\mu\nu}h_{\rho\sigma}\right) + \mathcal{O}\left(\underline{\kappa}^3\right) .\end{aligned} \tag{39}$$

Remark 3.11 (Physical interpretation of the connections on the spacetime bundle). In this remark we stress, that even though the connections on the spacetime bundle are mathematically similarly defined, c.f. Definition 3.5, their physical interpretation is rather different, c.f. Definition 3.8: The Christoffel symbols $\Gamma^\rho_{\mu\nu}$ and the spin connection $i\omega_\mu$ are proportional to a series in the graviton field, i.e. to a sum of arbitrary many particles. Contrary, the connection form $i\underline{e}A_\mu$ on the $U(1)$ principle bundle corresponds directly to the photon field, i.e. to a single particle.

3.2 The Lagrange Density of QGR-QED

In this thesis, we consider the QGR-QED Lagrange density[6]

$$\mathcal{L}_{\text{QGR-QED}} = \left(\frac{1}{\underline{\kappa}^2}R + g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \bar{\Psi}\left(i\nabla\!\!\!\!/^{\Sigma M\times U(1)} - m\right)\Psi\right)\mathrm{d}V_g + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}} \tag{40}$$

on the spacetime bundle $\mathcal{M}$ associated to the spacetime (M, g), where $\mathcal{L}_{\text{GF}}$ are the gauge fixing terms, spelled out in Subsubsection 3.2.3, and $\mathcal{L}_{\text{Ghost}}$ are the ghost terms, spelled out in Subsubsection 3.2.4. This is the canonical way to generalize the QED Lagrange density

$$\mathcal{L}_{\text{QED}} = \left(\eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \bar{\Psi}\left(i\nabla\!\!\!\!/^{\Sigma M\times U(1)} - m\right)\Psi\right)\mathrm{d}V_\eta + \mathcal{L}_{\text{GF}} \tag{41}$$

to curved spacetimes, where here the Dirac operator $\nabla\!\!\!\!/^{\Sigma M\times U(1)}$ on the bundle $\Sigma M \times U(1)$ of the Minkowski spacetime (M, η) simplifies to[7]

$$\nabla\!\!\!\!/^{\Sigma M\times U(1)} = \eta^{\mu m}\left(\partial_\mu + i\omega_\mu + i\underline{e}A_\mu\right)\gamma_m . \tag{42}$$

In the following, we discuss the parts of Equation (40) separately:

3.2.1 Einstein-Hilbert Lagrange Density

The Lagrange density describing gravitation is the Einstein-Hilbert Lagrange density, which we rescale by $\frac{1}{\underline{\kappa}^2}$ such that the propagator is of order $\mathcal{O}\left(\underline{\kappa}^0\right)$. It reads

$$\mathcal{L}_{\text{EH}} = \left(\frac{1}{\underline{\kappa}^2}R\right)\mathrm{d}V_g , \tag{43}$$

[6] We rescale the Einstein-Hilbert part of $\mathcal{L}_{\text{QGR-QED}}$ by $\frac{1}{\underline{\kappa}^2}$ such that the graviton propagator is of order $\mathcal{O}\left(\underline{\kappa}^0\right)$.

[7] Notice, that the first part of the spin connection $i\omega_\mu$ is still necessary since local Lorentz transformations are always possible, i.e. Equation (15) reduces to $i\omega_\mu = \frac{i}{4}\left(e^m_\nu\partial_\mu e^{\nu n}\right)\sigma_{mn}$, c.f. Remark 3.4 and Definition 3.5.

where $R = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}$ is the Ricci scalar, with the Riemann tensor $R_{\mu\nu\rho\sigma}$, given by [23]

$$\begin{aligned} R_{\mu\nu\rho\sigma} &:= g_{\mu\tau} R^{\tau}{}_{\nu\rho\sigma} \\ &= g_{\mu\tau} \left(\partial_\rho \Gamma^\tau_{\sigma\nu} - \partial_\sigma \Gamma^\tau_{\rho\nu} + \Gamma^\tau_{\rho\lambda} \Gamma^\lambda_{\sigma\nu} - \Gamma^\tau_{\sigma\lambda} \Gamma^\lambda_{\rho\nu} \right) . \end{aligned} \tag{44}$$

The Ricci scalar is given in terms of the metric as

$$\begin{aligned} R &= g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma} \\ &= g^{\mu\rho} g^{\nu\sigma} \left(\partial_\mu \partial_\nu g_{\rho\sigma} - \partial_\mu \partial_\rho g_{\nu\sigma} \right) + \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} g^{\lambda\tau} \Big(\left(\partial_\lambda g_{\mu\rho} \right) \left(\partial_\nu g_{\tau\sigma} \right) - \left(\partial_\lambda g_{\mu\nu} \right) \left(\partial_\rho g_{\sigma\tau} \right) \Big) \\ &\quad + \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} g^{\lambda\tau} \Big(\left(\partial_\mu g_{\nu\lambda} \right) \left(\partial_\rho g_{\sigma\tau} \right) - \left(\partial_\lambda g_{\mu\rho} \right) \left(\partial_\tau g_{\nu\sigma} \right) \Big) . \end{aligned} \tag{45}$$

We remark that considering linearized GR is equivalent to neglecting the first bracket in the above equation, i.e. $g^{\mu\rho} g^{\nu\sigma} \left(\partial_\mu \partial_\nu g_{\rho\sigma} - \partial_\mu \partial_\rho g_{\nu\sigma} \right)$. After investigating the de Donder gauge, which will be introduced in Subsubsection 3.2.3, the expression can be simplified to

$$R^{\mathrm{dD}} = -\frac{1}{2} g^{\mu\rho} g^{\nu\sigma} \partial_\mu \partial_\rho g_{\nu\sigma} - \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} g^{\lambda\tau} \left(\partial_\lambda g_{\mu\nu} \right) \left(\partial_\rho g_{\sigma\tau} \right) + \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} g^{\lambda\tau} \left(\partial_\mu g_{\nu\lambda} \right) \left(\partial_\rho g_{\sigma\tau} \right) . \tag{46}$$

After two partial integrations (which are allowed when considering the action integral) and using the de Donder gauge condition on the second term again, the expression can be further simplified to

$$R^{\mathrm{dD,\ pi}} = -\frac{1}{2} g^{\mu\rho} g^{\nu\sigma} \partial_\mu \partial_\rho g_{\nu\sigma} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} g^{\lambda\tau} \left(\partial_\nu g_{\mu\rho} \right) \left(\partial_\sigma g_{\lambda\tau} \right) + \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} g^{\lambda\tau} \left(\partial_\mu g_{\nu\lambda} \right) \left(\partial_\rho g_{\sigma\tau} \right) . \tag{47}$$

Finally, we remark that the first term can be interpreted as a source term. Thus, when considering Feynman rules it would correspond to a graviton half-edge. This would result in the fact that there are infinitely many graviton vertex Feynman rules, since every higher valent graviton vertex could be reduced to a lower valent one, using this graviton half-edge Feynman rule. Again, we remark that this term is neglected when linearized gravity is considered. However, the following investigations are independent of this choice.

3.2.2 Matter Lagrange Density

The Lagrange density describing photons and fermions on the curved spacetime (M, g) is given by

$$\mathcal{L}_{\text{Matter}} = \left(g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \bar{\Psi} \left(i \nabla\!\!\!\!/^{\Sigma M \times U(1)} - m \right) \Psi \right) \mathrm{d}V_g \, , \tag{48}$$

with the Faraday tensor or curvature two-form of the connection form A_μ

$$\begin{aligned} F_{\mu\nu} &= \nabla^{TM}_\mu A_\nu - \nabla^{TM}_\nu A_\mu \\ &= \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho - \partial_\nu A_\mu + \Gamma^\rho_{\nu\mu} A_\rho \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \, . \end{aligned} \tag{49}$$

Observe, that the Levi-Civita connections ∇^{TM}_μ reduce to partial derivatives ∂_μ here, since $F_{\mu\nu} = \nabla^{TM}_\mu A_\nu - \nabla^{TM}_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho - \partial_\nu A_\mu + \Gamma^\rho_{\nu\mu} A_\rho = \partial_\mu A_\nu - \partial_\nu A_\mu$, as the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ for the Levi-Civita connection ∇^{TM}_μ are symmetric in their lower two indices, c.f. Definition 3.5. Moreover, recall from Definition 3.6 that the Dirac operator $\nabla\!\!\!\!/^{\Sigma M \times U(1)}$ is locally given by

$$\nabla\!\!\!\!/^{\Sigma M \times U(1)} = e^{\mu m} \gamma_m \left(\partial_\mu + i \omega_\mu + i \underline{e} A_\mu \right) . \tag{50}$$

Observe, that the coupling of the graviton field to the matter fields is given by the inverse metric, the inverse vielbein and the prefactor of the Riemannian volume form.

3.2.3 Gauge Fixing and Gauge Transformations

The gauge fixing part of the Lagrange density consists of two parts: One for the gravitational part, where we choose the de Donder gauge

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0 \iff g^{\mu\nu}\partial_\mu g_{\nu\rho} = \frac{1}{2} g^{\mu\nu}\partial_\rho g_{\mu\nu} \, , \tag{51}$$

and one for the electrodynamic part, where we choose the Lorenz gauge

$$g^{\mu\nu}\nabla^{TM}_\mu A_\nu = 0 \, . \tag{52}$$

The de Donder gauge is motivated by the fact, that in this gauge the divergence of a vector field $\nabla^{TM}_\mu x^\mu$ or of a covector field $g^{\mu\nu}\nabla^{TM}_\mu x_\nu$ reduces to the simple form

$$\begin{aligned} \nabla^{TM}_\mu x^\mu &= \partial_\mu x^\mu + \Gamma^\mu_{\mu\rho} x^\rho \\ &= g^{\mu\nu}\left(\partial_\mu x_\nu - \Gamma^\sigma_{\mu\nu} x_\sigma\right) \\ &= g^{\mu\nu}\partial_\mu x_\nu \\ &= \partial_\mu x^\mu \, . \end{aligned} \tag{53}$$

Thus, in particular the Beltrami-Laplace operator takes on the simple form

$$\Delta^{TM}_{\text{Beltrami}} = g^{\mu\nu}\partial_\mu\partial_\nu \, . \tag{54}$$

Furthermore, we remark that there does in general there not exist a gauge such that the evolution of the graviton field is governed by a wave-equation because the graviton field is in general non-linear. This comes from the first term in Equations (45) to (47). However, when the linearized Einstein-Hilbert Lagrange density is considered, then there exists a wave equation for the graviton field if the de Donder gauge is considered, a fact which is used in the gravitational wave analysis, c.f. [23]. On the other hand, the Lorenz-Gauge is motivated by the fact, that then the photon field satisfies a wave-equation with a source term, given by the four-current $\underline{e} j^\mu$,

$$\begin{aligned} \Delta^{TM}_{\text{Bochner}} \underline{e} A_\rho &= g^{\mu\nu}\nabla^{TM}_\mu \nabla^{TM}_\nu \underline{e} A_\rho \\ &= g^{\mu\nu}\nabla^{TM}_\mu \left(\nabla^{TM}_\nu \underline{e} A_\rho - \nabla^{TM}_\rho \underline{e} A_\nu + \nabla^{TM}_\rho \underline{e} A_\nu\right) \\ &= g^{\mu\nu}\nabla^{TM}_\mu \left(\nabla^{TM}_\nu \underline{e} A_\rho - \nabla^{TM}_\rho \underline{e} A_\nu\right) + g^{\mu\nu}\nabla^{TM}_\mu \nabla^{TM}_\rho \underline{e} A_\nu \\ &= g^{\mu\nu}\nabla^{TM}_\mu \underline{e} F_{\nu\rho} + g^{\mu\nu}\nabla^{TM}_\rho \nabla^{TM}_\mu \underline{e} A_\nu \\ &= g^{\mu\nu}\nabla^{TM}_\mu \underline{e} F_{\nu\rho} \\ &= g_{\rho\sigma} \underline{e} j^\sigma \, , \end{aligned} \tag{55}$$

where Δ^{TM} is the Bochner-Laplace operator. Observe, that when we apply the de Donder gauge Equation (51), the expression in the Lorenz gauge Equation (52) simplifies to

$$\begin{aligned} g^{\mu\nu}\nabla^{TM}_\mu A_\nu &= g^{\mu\nu}\left(\partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho\right) \\ &= g^{\mu\nu}\partial_\mu A_\nu \, , \end{aligned} \tag{56}$$

as was shown in general in Equation (53). For the following, we write the de Donder gauge, Equation (51), as

$$C^h_\mu := g^{\nu\rho}\partial_\nu g_{\mu\rho} - \frac{1}{2} g^{\nu\rho}\partial_\mu g_{\nu\rho} = 0 \tag{57}$$

and the Lorenz gauge, Equation (52), as

$$C^A := g^{\mu\nu}\nabla^{TM}_\mu A_\nu \, . \tag{58}$$

We implement the de Donder gauge and the Lorenz gauge in the Lagrange density by using Lagrange multipliers $\frac{1}{\xi}$ and $\frac{1}{\zeta}$, i.e. by adding the following Lagrange density:

$$\mathcal{L}_{\mathrm{GF}} = \left(-\frac{1}{2\xi} \left(C^A \right)^2 - \frac{1}{2\zeta} g^{\mu\nu} C^h_\mu C^h_\nu \right) \mathrm{d}V_g \tag{59}$$

If $\frac{1}{\xi}$ and $\frac{1}{\zeta}$ are interpreted as parameters rather than Lagrange multipliers, then $\zeta = 1$ corresponds to the de Donder gauge. Thus, it is similar to the Feynman gauge $\xi = 1$. Now, we consider the residual gauge transformations: Recall that a gauge transformation in GR is an infinitesimal diffeomorphism of the spacetime (M, g) which is induced by a smooth vector field[8] $\underline{\kappa} g^{\mu\nu} \chi^h_\nu \in \Gamma(TM)$ [23, 28]. Then, the residual gauge transformations of the de Donder gauge read [23, 28]

$$\underline{\kappa} h_{\mu\nu} \rightsquigarrow \underline{\kappa} h_{\mu\nu} + \underline{\kappa} \partial_\mu \chi^h_\nu + \underline{\kappa} \partial_\nu \chi^h_\mu \, , \tag{60}$$

where the term $\partial_\mu \chi^h_\nu + \partial_\nu \chi^h_\mu$ corresponds to a Lie derivative of the Minkowski metric $\eta_{\mu\nu}$ with respect to the vector field $\underline{\kappa} g^{\mu\nu} \chi^h_\nu$, c.f. [23, 28]. Furthermore, a gauge transformation of the spacetime (M, g) as in Equation (60) induces also the transformation of the connection form

$$A_\mu \rightsquigarrow A_\mu + \underline{\kappa} g^{\nu\rho} \left(\chi^h_\nu \partial_\rho A_\mu + A_\nu \partial_\mu \chi^h_\rho \right) \, , \tag{61}$$

where the right hand side corresponds to a Lie derivative of the connection form A_μ with respect to the vector field $\underline{\kappa} g^{\mu\nu} \chi^h_\nu$. For the Lorenz gauge, the residual gauge transformations read

$$A_\mu \rightsquigarrow A_\mu + \partial_\mu \chi^A \, , \tag{62}$$

where $\chi^A \in C^\infty(M)$ is a smooth function on the spacetime (M, g). Finally, we remark that only the linearized Riemann tensor is invariant under the gauge transformation of Equation (60) [23]. This is similar to non-abelian gauge theories where the field strength tensor is also not invariant under gauge transformations.

3.2.4 Ghost Lagrange Density

We denote the photon ghost and photon anti-ghost by c^A and $\bar{c}^A$, respectively, and the graviton ghost and graviton anti-ghost by c^h_μ and $\bar{c}^h_\mu$, respectively. The ghost Lagrange density is obtained by varying the gauge fixing condition via a gauge transformation [28, 29]:

$$\begin{aligned} \mathcal{L}_{\mathrm{Ghost}} &= g^{\mu\nu} \bar{c}^h_\mu \left(C^{\tilde{h}}_\nu - C^h_\nu \right) \Big|_{\chi^h_\rho \mapsto c^h_\rho} + \bar{c}^A \left(C^{\tilde{A}} - C^A \right) \Big|_{\substack{\chi^A \mapsto c^A \\ \chi^h_\mu \mapsto 0}} + \bar{c}^A \left(C^{\tilde{A}} - C^A \right) \Big|_{\substack{\chi^A \mapsto 0 \\ \chi^h_\rho \mapsto c^h_\rho}} \\ &= g^{\mu\nu} g^{\rho\sigma} \partial_\mu \bar{c}^h_\rho \partial_\nu c^h_\sigma + g^{\mu\nu} \partial_\mu \bar{c}^A \partial_\nu c^A \\ &\quad + \underline{\kappa} g^{\mu\nu} g^{\rho\sigma} \bar{c}^A \left(\left(\partial_\mu A_\rho \right) \left(\partial_\sigma c^h_\nu \right) + \left(\partial_\mu \partial_\rho A_\nu \right) c^h_\sigma + A_\mu \left(\partial_\rho \partial_\sigma c^h_\nu \right) + \left(\partial_\mu A_\rho \right) \left(\partial_\nu c^h_\sigma \right) \right) \end{aligned} \tag{63}$$

Here, $C^{\tilde{h}}_\mu$ and $C^{\tilde{A}}$ denote the gauge conditions, given in Equations (51) and (52), applied to the gauge transformed fields of Equation (60) and Equation (61) with (62), i.e.

$$\tilde{h}_{\mu\nu} = \underline{\kappa} h_{\mu\nu} + \underline{\kappa} \partial_\mu \chi^h_\nu + \underline{\kappa} \partial_\nu \chi^h_\mu \tag{64}$$

and

$$\tilde{A}_\mu = A_\mu + \partial_\mu \chi^A + \underline{\kappa} g^{\nu\rho} \left(\chi^h_\nu \partial_\rho A_\mu + A_\nu \partial_\mu \chi^h_\rho \right) \tag{65}$$

Notice, that the photon ghost is a scalar particle, whereas the graviton ghost is a spin-one particle.

[8] It will be convenient in the following to dualize the vector field $\underline{\kappa} g^{\mu\nu} \chi^h_\nu \in \Gamma(TM)$ to its covector field $\underline{\kappa} \chi^h_\nu \in \Gamma(T^*M)$.

4 Hopf Algebras and the Connes-Kreimer Renormalization Hopf Algebra

In this section, we introduce Hopf algebras in general and the Connes-Kreimer renormalization Hopf algebra in particular. The intention is to review the basic notions and set the relevant definitions. We refer the reader who wishes a more detailed treatment on Hopf algebras in general and its connection to affine groups to [30] and the reader who wishes a more detailed treatment on the construction of the Connes-Kreimer renormalization Hopf algebra to [13, 14, 15, 31].

4.1 Hopf Algebras

We start by defining Hopf algebras in general. In this thesis, we set k to be a commutative ring with one. Furthermore, by algebra we mean associative algebra with identity and by coalgebra we mean co-associative coalgebra with co-identity. Moreover, algebras, coalgebras, bialgebras and Hopf algebras are all over the ring k.

Definition 4.1 (Algebra, [30])**.** An algebra is the triple $(A, \mu, \mathbb{I})$. A is a k-module, $\mu : A \otimes_k A \to A$ an associative multiplication map, i.e. the following diagram commutes

$$\begin{tikzcd} A \otimes_k A \otimes_k A \arrow[r, "\mu \otimes_k id"] \arrow[d, "id \otimes_k \mu"'] & A \otimes_k A \arrow[d, "\mu"] \\ A \otimes_k A \arrow[r, "\mu"'] & A \end{tikzcd} \, , \tag{66}$$

and $\mathbb{I} : k \to A$ the identity with respect to μ, i.e. the following diagram commutes

$$\begin{tikzcd} k \otimes_k A \arrow[r, "\mathbb{I} \otimes_k id"] \arrow[dr, "\cong"'] & A \otimes_k A \arrow[d, "\mu"] & A \otimes_k k \arrow[l, "id \otimes_k \mathbb{I}"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd} \, . \tag{67}$$

Definition 4.2 ((Connected) graded algebra, [32])**.** An algebra A is called graded, if the k-module A can be written as a direct sum

$$A = \bigoplus_{m=0}^{\infty} A_m \, , \tag{68}$$

which is respected by the multiplication μ, i.e.

$$\mu \left(A_m \otimes_k A_n \right) \subseteq A_{m+n} \, , \; \forall m, n \in \mathbb{N}_0 \, , \tag{69}$$

and we have

$$\mathbb{I} \in A_0 \, . \tag{70}$$

Furthermore, a graded algebra A is called connected, if the grade zero component is isomorphic to the base ring, i.e.

$$A_0 \cong k \, . \tag{71}$$

Definition 4.3 (Homomorphism of (graded) algebras)**.** Let A_1 and A_2 be two algebras. Then, a map $f : A_1 \to A_2$ is called a homomorphism of algebras, if f is compatible with the products μ_1 on A_1 and μ_2 on A_2, i.e. the following diagram commutes

$$\begin{tikzcd} A_1 \otimes_k A_1 \arrow[r, "\mu_1"] \arrow[d, "f\otimes_k f"'] & A_1 \arrow[d, "f"] \\ A_2 \otimes_k A_2 \arrow[r, "\mu_2"'] & A_2 \end{tikzcd} , \tag{72}$$

and f maps the identity $\mathbb{I}_1$ on A_1 to the identity $\mathbb{I}_2$ on A_2, i.e. the following diagram commutes

$$\begin{tikzcd} & k \arrow[dl, "\mathbb{I}_1"'] \arrow[dr, "\mathbb{I}_2"] & \\ A_1 \arrow[rr, "f"'] & & A_2 \end{tikzcd} . \tag{73}$$

If the algebras A_1 and A_2 are both graded, then f additionally has to respect this structure to be a homomorphism of graded algebras, i.e.

$$f\left((A_1)_m\right) \subseteq (A_2)_m \ , \ \forall m \in \mathbb{N}_0 \, . \tag{74}$$

Definition 4.4 (Coalgebra, [30])**.** A coalgebra is the triple $(C, \Delta, \hat{\mathbb{I}})$. C is a k-module, $\Delta : C \to C \otimes_k C$ a co-associative comultiplication map, i.e. the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_k C \arrow[d, "id\otimes_k\Delta"] \\ C \otimes_k C \arrow[r, "\Delta\otimes_k id"'] & C \otimes_k C \otimes_k C \end{tikzcd} , \tag{75}$$

and $\hat{\mathbb{I}} : C \to k$ the co-identity with respect to Δ, i.e. the following diagram commutes

$$\begin{tikzcd} & C \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ k \otimes_k C & C \otimes_k C \arrow[l, "\hat{\mathbb{I}}\otimes_k id"] \arrow[r, "id\otimes_k\hat{\mathbb{I}}"'] & C \otimes_k k \end{tikzcd} . \tag{76}$$

Definition 4.5 ((Connected) graded coalgebra)**.** A coalgebra C is called graded, if the k-module C can be written as a direct sum

$$C = \bigoplus_{m=0}^{\infty} C_m \, , \tag{77}$$

which is respected by the co-multiplication Δ, i.e.

$$\Delta\left(C_m\right) \subseteq \sum_{n=0}^{m} C_n \otimes_k C_{m-n} \, , \ \forall m \in \mathbb{N}_0 \, , \tag{78}$$

and we have

$$\hat{\mathbb{I}}\left(C_m\right) = \delta_{0,m} C_m \,, \tag{79}$$

where $\delta_{\cdot,\cdot}$ is the Kronecker delta. Furthermore, a graded coalgebra C is called connected, if the grade zero component is isomorphic to the base ring, i.e.

$$C_0 \cong k \,. \tag{80}$$

Definition 4.6 (Homomorphism of (graded) coalgebras, [30])**.** Let C_1 and C_2 be two coalgebras. Then, a map $g\,:\,C_1 \to C_2$ is called a homomorphism of coalgebras, if g is compatible with the two coproducts Δ_1 on C_1 and Δ_2 on C_2, i.e. the following diagram commutes

$$\begin{tikzcd} C_1 \arrow[r, "\Delta_1"] \arrow[d, "g"'] & C_1 \otimes_k C_1 \arrow[d, "g \otimes_k g"] \\ C_2 \arrow[r, "\Delta_2"'] & C_2 \otimes_k C_2 \end{tikzcd} \,, \tag{81}$$

and g is compatible with the two co-identities $\hat{\mathbb{I}}_1$ on C_1 and $\hat{\mathbb{I}}_2$ on C_2, i.e. the following diagram commutes

$$\begin{tikzcd} C_1 \arrow[rr, "g"] \arrow[dr, "\hat{\mathbb{I}}_1"'] & & C_2 \arrow[dl, "\hat{\mathbb{I}}_2"] \\ & k & \end{tikzcd} \,. \tag{82}$$

If the coalgebras C_1 and C_2 are both graded, then g additionally has to respect this structure to be a homomorphism of graded coalgebras, i.e.

$$g\left(\left(C_1\right)_m\right) \subseteq \left(C_2\right)_m \,,\ \forall m \in \mathbb{N}_0 \,. \tag{83}$$

Remark 4.7 (Relation between algebra and coalgebra, [30]). A ((connected) graded) algebra A is related to a ((connected) graded) coalgebra C via dualization, i.e. applying the functor $\mathrm{Hom}_{\mathsf{Alg}}\left(\cdot, k\right)$.

Definition 4.8 ((Connected graded) bialgebra, [30])**.** A bialgebra is the quintuple $(B, \mu, \mathbb{I}, \Delta, \hat{\mathbb{I}})$. The triple $(B, \mu, \mathbb{I})$ is an algebra and the triple $(B, \Delta, \hat{\mathbb{I}})$ is a coalgebra. Furthermore, the coproduct Δ and the co-identity $\hat{\mathbb{I}}$ are homomorphisms of the graded algebra $(B, \mu, \mathbb{I})$, or, equivalently, the multiplication μ and the identity $\mathbb{I}$ are homomorphisms of the graded coalgebra $(B, \Delta, \hat{\mathbb{I}})$. Furthermore, a bialgebra B is called graded, if it is graded as an algebra and a coalgebra. Moreover, a bialgebra B is called connected, if it is connected as an algebra, or, equivalently, as a coalgebra.

Definition 4.9 ((Connected graded) Hopf algebra, [30, 32])**.** A Hopf algebra is the sextuple $(H, \mu, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S)$. The quintuple $(H, \mu, \mathbb{I}, \Delta, \hat{\mathbb{I}})$ is a bialgebra and $S\,:\,H \to H$ is the antipode, defined such that the following diagram commutes:

$$
\begin{tikzcd}
& H \otimes_k H \arrow[rr, "S \otimes_k id"] & & H \otimes_k H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\hat{\mathbb{I}}"] & & k \arrow[rr, "\mathbb{I}"] & & H \\
& H \otimes_k H \arrow[rr, "id \otimes_k S"'] & & H \otimes_k H \arrow[ur, "\mu"'] &
\end{tikzcd}
\tag{84}
$$

Furthermore, a Hopf algebra H is called graded, if it is graded as a bialgebra.[9] Moreover, a Hopf algebra H is called connected, if it is connected as a bialgebra.

Definition 4.10 (Hopf subalgebras of a (connected graded) Hopf algebra)**.** Let H be a Hopf algebra. A k-submodule $h \subset H$ is called a Hopf subalgebra of H, if $(h, \mu\big|_h, \mathbb{I}\big|_h, \Delta\big|_h, \hat{\mathbb{I}}\big|_h, S\big|_h)$ is a Hopf algebra itself, i.e. $\mathbb{I}\big|_h = \mathbb{I}_h \in h$ and h is stable under the restricted actions of the product $\mu\big|_h$, the coproduct $\Delta\big|_h$, the co-identity $\hat{\mathbb{I}}\big|_h$ and the antipode $S\big|_h$. If H is graded, then h inherits a grading from H via

$$
h = H \cap h = \left(\bigoplus_{m=0}^{\infty} H_m \right) \cap h = \bigoplus_{m=0}^{\infty} \left(H_m \cap h \right) = \bigoplus_{m=0}^{\infty} h_m \, , \tag{85}
$$

i.e. we define the grade m subspace of h as $h_m := (H_m \cap h)$ for all $m \in \mathbb{N}_0$. If H is connected, then h is also connected, since $\mathbb{I}_h \in h_0 \neq \{ 0 \}$ and we have

$$
h_0 = H_0 \cap h_0 \cong k \cap h_0 = k \, . \tag{86}
$$

4.2 The Connes-Kreimer Renormalization Hopf algebra

From now on, we consider the Connes-Kreimer renormalization Hopf algebra, which is a Hopf algebra over[10] $k = \mathbb{Q}$.

Definition 4.11 (Weighted residue set of a local QFT)**.** Let $\mathcal{Q}$ be a local QFT. Then, $\mathcal{Q}$ is either given via a Lagrange density $\mathcal{L}_\mathcal{Q}$ or via a set of residues $\mathcal{R}_\mathcal{Q}$ together with a weight function $\omega_\mathcal{Q} \, : \, \mathcal{R}_\mathcal{Q} \to \mathbb{N}_0$. The set of residues is a disjoint union of all vertex-types $\mathcal{R}_\mathcal{Q}^{[0]}$ and all edge-types $\mathcal{R}_\mathcal{Q}^{[1]}$ of $\mathcal{Q}$, i.e. $\mathcal{R}_\mathcal{Q} = \mathcal{R}_\mathcal{Q}^{[0]} \amalg \mathcal{R}_\mathcal{Q}^{[1]}$. If $\mathcal{Q}$ is given via a Lagrange density, then the set of residues $\mathcal{R}_\mathcal{Q}$ and the weight $\omega_\mathcal{Q}(R)$ of each residue $R \in \mathcal{R}_\mathcal{Q}$ is given as follows: Each field in the Lagrange density $\mathcal{L}_\mathcal{Q}$ corresponds to a particle type of $\mathcal{Q}$. Therefore, every monomial in $\mathcal{L}_\mathcal{Q}$ consisting of one field corresponds to a source-term, i.e. a vertex-residue in $\mathcal{R}_\mathcal{Q}^{[0]}$ consisting of a half-edge of that particle type. Every monomial in $\mathcal{L}_\mathcal{Q}$ consisting of two equivalent fields corresponds to a propagation term, i.e. an edge-residue in $\mathcal{R}_\mathcal{Q}^{[1]}$ consisting of an edge of that particle type. And every monomial in $\mathcal{L}_\mathcal{Q}$ consisting of different fields corresponds to an interaction term, i.e. a vertex-residue in $\mathcal{R}_\mathcal{Q}^{[0]}$ consisting of half-edges of that particle types. Then, the weight $\omega_\mathcal{Q}(R) \in \mathbb{N}_0$ of a residue $R \in \mathcal{R}_\mathcal{Q}$ is set to be the number of derivative operators involved in the corresponding field monomial in $\mathcal{L}_\mathcal{Q}$.

[9] Observe, that if a Hopf algebra H is graded as a bialgebra, then the antipode S is automatically an endomorphism of graded algebras and an endomorphism of graded coalgebras.

[10] Actually, the physical needs require k only to be a field with characteristic 0. Since $k = \mathbb{Q}$ is the smallest such field, it is the canonical choice.

Definition 4.12 (Feynman graphs generated by residue sets)**.** Let $\mathcal{Q}$ be a local QFT with residue set $\mathcal{R}_{\mathcal{Q}}$. Then, we denote by $\mathscr{G}_{\mathcal{Q}}$ the set of all one particle irreducible (1PI) Feynman graphs[11] that can be generated by the residue set $\mathcal{R}_{\mathcal{Q}}$ of $\mathcal{Q}$.

Definition 4.13 (Residue of a Feynman graph)**.** Let $\mathcal{Q}$ be a local QFT with residue set $\mathcal{R}_{\mathcal{Q}}$ and Feynman graph set $\mathscr{G}_{\mathcal{Q}}$. Then, the residue of a Feynman graph $\Gamma \in \mathscr{G}_{\mathcal{Q}}$, denoted by $\operatorname{res}(\Gamma)$, is the vertex residue or edge residue $\operatorname{res}(\Gamma)$, not necessary in the residue set $\mathcal{R}_{\mathcal{Q}}$, obtained by shrinking all internal edges of Γ to a single vertex.

Definition 4.14 (First Betti number of a Feynman graph, [32])**.** Let $\mathcal{Q}$ be a local QFT and $\mathscr{G}_{\mathcal{Q}}$ the set of its Feynman graphs. Let furthermore $\Gamma \in \mathscr{G}_{\mathcal{Q}}$ be a Feynman graph. Then, we define the first Betti number of Γ as

$$b_1(\Gamma) := \# H_1(\Gamma) \, , \tag{87}$$

where $\# H_1(\Gamma)$ is the rank of the first singular homology group of Γ.

Definition 4.15 (Superficial degree of divergence)**.** Let $\mathcal{Q}$ be a local QFT with weighted residue set $\mathcal{R}_{\mathcal{Q}}$ and Feynman graph set $\mathscr{G}_{\mathcal{Q}}$. We turn $\mathscr{G}_{\mathcal{Q}}$ into a weighted set as well by declaring the function

$$\omega_{\mathcal{Q}} : \quad \mathscr{G}_{\mathcal{Q}} \to \mathbb{Z} \, , \quad \Gamma \mapsto d b_1(\Gamma) + \sum_{v \in \Gamma^{[0]}} \omega_{\mathcal{Q}}(v) - \sum_{e \in \Gamma^{[1]}} \omega_{\mathcal{Q}}(e) \, , \tag{88}$$

where d is the dimension of spacetime of $\mathcal{Q}$ and $b_1(\Gamma)$ the first Betti number of the Feynman graph $\Gamma \in \mathscr{G}_{\mathcal{Q}}$. Then, the weight $\omega_{\mathcal{Q}}(\Gamma)$ of a Feynman graph $\Gamma \in \mathscr{G}_{\mathcal{Q}}$ is called the superficial degree of divergence of Γ. A Feynman graph $\Gamma \in \mathcal{G}_{\mathcal{Q}}$ is called superficially divergent if $\omega_{\mathcal{Q}}(\Gamma) \geq 0$, otherwise it is called superficially convergent if $\omega_{\mathcal{Q}}(\Gamma) < 0$.

Remark 4.16. The definition of the superficial degree of divergence of a Feynman graph, Definition 4.15, is motivated by the fact, that the Feynman integral corresponding to a given Feynman graph via the Feynman rules converges, if the Feynman graph itself and all its subgraphs are superficially convergent.

Definition 4.17 (Set of superficially divergent subgraphs of a Feynman graph)**.** Let $\mathcal{Q}$ be a local QFT and $\Gamma \in \mathscr{G}_{\mathcal{Q}}$ a Feynman graph of $\mathcal{Q}$. Then, we denote by $\mathscr{D}_{\mathcal{Q}}(\Gamma)$ the set of superficially divergent subgraphs of Γ, i.e.

$$\mathscr{D}_{\mathcal{Q}}(\Gamma) := \left\{ \gamma \, \middle| \, \gamma \subseteq \Gamma \, : \, \gamma = \coprod_{m=1}^{M} \gamma_m \, , \; M \in \mathbb{N} \, : \, \omega_{\mathcal{Q}}(\gamma_m) \geq 0 \, , \; \forall \gamma_m \right\} . \tag{89a}$$

Furthermore, we define the set $\mathscr{D}'_{\mathcal{Q}}(\Gamma)$ of superficially divergent proper subgraphs of Γ, i.e.

$$\mathscr{D}'_{\mathcal{Q}}(\Gamma) := \left\{ \gamma \, \middle| \, \gamma \in \mathscr{D}_{\mathcal{Q}}(\Gamma) \, : \, \gamma \subsetneq \Gamma \right\} . \tag{89b}$$

Definition 4.18 (Renormalization Hopf algebra of a local QFT)**.** Let $\mathcal{Q}$ be a local QFT, $\mathcal{R}_{\mathcal{Q}}$ the set of its weighted residues and $\mathscr{G}_{\mathcal{Q}}$ the set of its weighted Feynman graphs. We assume $\mathcal{Q}$ to be such, that the residues of all superficially divergent Feynman graphs are in the residue set $\mathcal{R}_{\mathcal{Q}}$, i.e. $\left\{ \Gamma \, | \, \Gamma \in \mathscr{G}_{\mathcal{Q}} \, : \, \omega_{\mathcal{Q}}(\Gamma) \geq 0 \, : \, \operatorname{res}(\Gamma) \notin \mathcal{R}_{\mathcal{Q}} \right\} = \emptyset$. The general and more involving case is discussed in Subsection 5.1. Then, the connected graded, c.f. Definition 4.22, renormalization Hopf algebra $(\mathscr{H}_{\mathcal{Q}}, \mu, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S)$ is defined as follows: We set $\mathscr{H}_{\mathcal{Q}}$ to be the vector space over $\mathbb{Q}$

[11] The use of 1PI Feynman graphs, rather than connected Feynman graphs, is justified by Theorem 5.15, as is discussed in Remark 5.16.

generated by the set $\mathscr{G}_{\mathcal{Q}}$. The associative multiplication $\mu : \mathscr{H}_{\mathcal{Q}} \otimes_{\mathbb{Q}} \mathscr{H}_{\mathcal{Q}} \to \mathscr{H}_{\mathcal{Q}}$ is defined as the disjoint union of Feynman graphs, i.e.

$$\mu : \quad \mathscr{H}_{\mathcal{Q}} \otimes_{\mathbb{Q}} \mathscr{H}_{\mathcal{Q}} \to \mathscr{H}_{\mathcal{Q}} , \quad \gamma \otimes_{\mathbb{Q}} \Gamma \mapsto \gamma \Gamma . \tag{90}$$

Then, the identity $\mathbb{I} : \mathbb{Q} \to \mathscr{H}_{\mathcal{Q}}$ is set to be the empty graph, i.e.

$$\mathbb{I} := \emptyset . \tag{91}$$

Moreover, we define the coproduct of a Feynman graph Γ such that it maps to the following sum over all possible combinations of divergent subgraphs of Γ: The left-hand side of the tensor product of each summand is given by a superficially divergent subgraph of the Feynman graph Γ, while the right-hand-side of the tensor product is given by returning the Feynman graph Γ with the corresponding subgraph shrunken to zero length, i.e.

$$\Delta : \quad \mathscr{H}_{\mathcal{Q}} \to \mathscr{H}_{\mathcal{Q}} \otimes_{\mathbb{Q}} \mathscr{H}_{\mathcal{Q}} , \quad \Gamma \mapsto \sum_{\gamma \in \mathscr{D}_{\mathcal{Q}}(\Gamma)} \gamma \otimes_{\mathbb{Q}} \Gamma / \gamma , \tag{92}$$

where the quotient Γ/γ is defined as follows: If γ is a proper subgraph of Γ, then Γ/γ is defined by shrinking all internal edges of γ in Γ to a single vertex for each connected component of γ. Otherwise, if $\gamma = \Gamma$ we define the quotient to be the identity, i.e. $\Gamma/\Gamma := \mathbb{I}$. The co-identity $\hat{\mathbb{I}} : \mathscr{H}_{\mathcal{Q}} \to \mathbb{Q}$ is set such, that its kernel is the Hopf algebra without the associative subalgebra with identity generated by $\mathbb{I}$, i.e.

$$\hat{\mathbb{I}} : \quad \mathscr{H}_{\mathcal{Q}} \to \mathbb{Q} , \quad \Gamma \mapsto \begin{cases} q & \text{if } \Gamma = q\mathbb{I}, \text{ with } q \in \mathbb{Q} \\ 0 & \text{else} \end{cases} . \tag{93}$$

Finally, we define the antipode recursively via

$$S : \quad \mathscr{H}_{\mathcal{Q}} \to \mathscr{H}_{\mathcal{Q}} , \quad \Gamma \mapsto - \sum_{\gamma \in \mathscr{D}_{\mathcal{Q}}(\Gamma)} S(\gamma) \Gamma / \gamma , \tag{94}$$

where the quotient Γ/γ is defined as in the definition of the coproduct after Equation (92).

Definition 4.19 (Reduced coproduct)**.** Let $\mathcal{Q}$ be a local QFT as in Definition 4.18, $\mathcal{R}_{\mathcal{Q}}$ the set of its weighted residues and $(\mathscr{H}_{\mathcal{Q}}, \mu, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S)$ its renormalization Hopf algebra. Then, we define the reduced coproduct as the non-trivial part of the coproduct, i.e.

$$\Delta' : \quad \mathscr{H}_{\mathcal{Q}} \to \mathscr{H}_{\mathcal{Q}} \otimes_{\mathbb{Q}} \mathscr{H}_{\mathcal{Q}} , \quad \Gamma \mapsto \sum_{\gamma \in \mathscr{D}'_{\mathcal{Q}}(\Gamma)} \gamma \otimes_{\mathbb{Q}} \Gamma / \gamma . \tag{95}$$

Definition 4.20 (Product of coupling constants of a Feynman graph)**.** Let $\mathcal{Q}$ be a local QFT and $\mathscr{G}_{\mathcal{Q}}$ the set of its Feynman graphs. Let furthermore $\Gamma = \prod_{m=1}^{M} \Gamma_m$ be a product of $M \in \mathbb{N}$ connected Feynman graphs $\Gamma_m \in \mathscr{G}_{\mathcal{Q}}$, $1 \leq m \leq M$. Then, we define the product of coupling constants of Γ as

$$\operatorname{coupling}(\Gamma) := \prod_{m=1}^{M} \left(\frac{1}{\operatorname{coupling}\big(\operatorname{res}(\Gamma_m)\big)} \prod_{v \in \Gamma_m^{[0]}} \operatorname{coupling}(v) \right) , \tag{96}$$

with

$$\operatorname{coupling}\big(\operatorname{res}(\Gamma_m)\big) := \begin{cases} \text{coupling constant of the vertex } \operatorname{res}(\Gamma_m) & \text{if } \operatorname{res}(\Gamma_m) \in \mathcal{R}_{\mathcal{Q}}^{[0]} \\ 1 & \text{else} \end{cases} . \tag{97}$$

Definition 4.21 (Multi-index corresponding to the product of coupling constants of a Feynman graph)**.** Let $\mathcal{Q}$ be a local QFT and $\mathscr{G}_\mathcal{Q}$ the set of its Feynman graphs. Let furthermore $\Gamma = \prod_{m=1}^{M} \Gamma_m$ be a product of $M \in \mathbb{N}$ connected Feynman graphs $\Gamma_m \in \mathscr{G}_\mathcal{Q}$, $1 \leq m \leq M$ and $\operatorname{coupling}(\Gamma)$ the product of its coupling constants. Then, we define the multi-index $\mathbf{C} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}$ corresponding to $\operatorname{coupling}(\Gamma)$ as the vector counting the multiplicities of the several coupling constants in $\operatorname{coupling}(\Gamma)$. Sums and direct sums over multi-indices are understood componentwise, e.g. let $N := \# \mathcal{R}_\mathcal{Q}^{[0]}$ and $\mathbf{C}^+ = (C_1^+, \cdots, C_N^+) \in \mathbb{Z}^N$ and $\mathbf{C}^- = (C_1^-, \cdots, C_N^-) \in \mathbb{Z}^N$, then we set

$$\sum_{\mathbf{c}=\mathbf{C}^-}^{\mathbf{C}^+} := \sum_{c_1=C_1^-}^{C_1^+} \cdots \sum_{c_N=C_N^-}^{C_N^+} \quad \text{and} \quad \bigoplus_{\mathbf{c}=\mathbf{C}^-}^{\mathbf{C}^+} := \bigoplus_{c_1=C_1^-}^{C_1^+} \cdots \bigoplus_{c_N=C_N^-}^{C_N^+} . \tag{98}$$

Furthermore, we set $\mathbf{0} := (0_1, \cdots, 0_N)$, $-\infty := (-\infty_1, \cdots, -\infty_N)$ and $\infty := (\infty_1, \cdots, \infty_N)$.

Definition 4.22 (Connectedness and gradings of the renormalization Hopf algebra)**.** Let $\mathcal{Q}$ be a local QFT as in Definition 4.18, $\mathcal{R}_\mathcal{Q}$ the set of its residues and $\mathscr{H}_\mathcal{Q}$ its renormalization Hopf algebra. Then, $\mathscr{H}_\mathcal{Q}$ possesses two gradings as a Hopf algebra: The first grading comes from the first Betti number, i.e.

$$\mathscr{H}_\mathcal{Q} = \bigoplus_{L=0}^{\infty} \left(\mathscr{H}_\mathcal{Q}\right)_L . \tag{99}$$

The second grading is a further refinement of the first one, where the grading comes from the multi-index corresponding to the product of coupling constants of a Feynman graph, i.e.

$$\mathscr{H}_\mathcal{Q} = \bigoplus_{\mathbf{C}=-\infty}^{\infty} \left(\mathscr{H}_\mathcal{Q}\right)_\mathbf{C} . \tag{100}$$

Clearly, $\left(\mathscr{H}_\mathcal{Q}\right)_0 \cong \left(\mathscr{H}_\mathcal{Q}\right)_\mathbf{0} \cong \mathbb{Q}$, and thus $\mathscr{H}_\mathcal{Q}$ is connected in both gradings. In this thesis, only the second grading is used since it is more general and implies the first one.

Definition 4.23 (Symmetry factor of a Feynman graph)**.** Let $\mathcal{Q}$ be a local QFT and $\mathscr{G}_\mathcal{Q}$ the set of its Feynman graphs. Let furthermore $\Gamma \in \mathscr{G}_\mathcal{Q}$ be a Feynman graph. Then, we define the symmetry factor of Γ as

$$\operatorname{sym}(\Gamma) := \# \operatorname{aut}(\Gamma) , \tag{101}$$

where $\# \operatorname{aut}(\Gamma)$ is the rank of the automorphism group of Γ, leaving its external leg structure fixed and respecting its vertex and edge types $v \in \mathcal{R}_\mathcal{Q}$ and $e \in \mathcal{R}_\mathcal{Q}$ for all $v \in \Gamma^{[0]}$ and $e \in \Gamma^{[1]}$, respectively.

Definition 4.24 (Combinatorial Green's functions)**.** Let $\mathcal{Q}$ be a local QFT and $\mathscr{G}_\mathcal{Q}$ the set of its Feynman graphs. Then, the combinatorial Green's function with residue $R \in \mathcal{R}_\mathcal{Q}$ and first Betti number L is defined as the sum

$$c^R_L := \sum_{\substack{\Gamma \in \mathscr{G}_\mathcal{Q} : \\ \operatorname{res}(\Gamma) = R , \\ b_1(\Gamma) = L}} \frac{\operatorname{coupling}(\Gamma)}{\operatorname{sym}(\Gamma)} \Gamma , \tag{102}$$

where we set $c^R_0 := \mathbb{I}$ for all residues R, even if $R \notin \mathcal{R}_\mathcal{Q}$. Furthermore, we set $c^R_L := 0$ if there does not exist a Feynman graph Γ with residue R and first Betti number L. Moreover, the combinatorial Green's function with residue $R \in \mathcal{R}_\mathcal{Q}$ and multi-index of coupling constants $\mathbf{C} \in \mathbb{Z}^{\# \mathcal{R}_\mathcal{Q}^{[0]}}$ is defined as the sum

$$c^R_\mathbf{C} := \sum_{\substack{\Gamma \in \mathscr{G}_\mathcal{Q} : \\ \operatorname{res}(\Gamma) = R , \\ \text{multi-index}(\Gamma) = \mathbf{C}}} \frac{\operatorname{coupling}(\Gamma)}{\operatorname{sym}(\Gamma)} \Gamma , \tag{103}$$

where we set $c^R_{\mathbf{0}} := \mathbb{I}$ for all residues R, even if $R \notin \mathcal{R}_{\mathcal{Q}}$. Furthermore, we set $c^R_{\mathbf{C}} := 0$ if there does not exist a Feynman graph Γ with residue R and multi-index of coupling constants $\mathbf{C}$. Then, the full combinatorial Green's function with residue $R \in \mathcal{R}_{\mathcal{Q}}$ is defined as the sum

$$c^R := \sum_{L=0}^{\infty} c^R_L = \sum_{\mathbf{C}=-\infty}^{\infty} c^R_{\mathbf{C}} \,. \tag{104}$$

Remark 4.25 (Hopf subalgebras in the renormalization Hopf algebra of a local QFT). Let $\mathcal{Q}$ be a local QFT as in Definition 4.18, $\mathcal{R}_{\mathcal{Q}}$ its weighted residue set, $\mathscr{H}_{\mathcal{Q}}$ its renormalization Hopf algebra and $c^R_L, c^R_{\mathbf{C}} \in \mathscr{H}_{\mathcal{Q}}$ its Green's functions. We consider first the case that $\mathcal{Q}$ has only one coupling constant. Then, $\mathscr{H}_{\mathcal{Q}}$ possesses Hopf subalgebras for the residue $R \in \mathcal{R}_{\mathcal{Q}}$ if the following equation is satisfied for all first Betti numbers $L \in \mathbb{N}$

$$\Delta\left(c^R_L\right) = \sum_{l=0}^{L} \mathfrak{P}_l\left(c^R_L\right) \otimes c^R_{L-l} \,, \ \forall L \in \mathbb{N} \,, \tag{105}$$

where $\mathfrak{P}_l\left(c^R_L\right) \in \mathscr{H}_{\mathcal{Q}}$ is a polynomial in graphs such that each summand has first Betti number l. Now we consider the general case, i.e. that $\mathcal{Q}$ has more than one coupling constant. Then, $\mathscr{H}_{\mathcal{Q}}$ possesses Hopf subalgebras for the residue $R \in \mathcal{R}_{\mathcal{Q}}$ if the following equation is satisfied for all multi-indices $\mathbf{C} \in \mathbb{Z}^{\# \mathcal{R}^{[0]}_{\mathcal{Q}}}$

$$\Delta\left(c^R_{\mathbf{C}}\right) = \sum_{\mathbf{c}=\mathbf{0}}^{\mathbf{C}} \mathfrak{P}_{\mathbf{c}}\left(c^R_{\mathbf{C}}\right) \otimes c^R_{\mathbf{C}-\mathbf{c}} \,, \ \forall \mathbf{C} \in \mathbb{Z}^{\# \mathcal{R}^{[0]}_{\mathcal{Q}}} \,, \tag{106}$$

where $\mathfrak{P}_{\mathbf{c}}\left(c^R_{\mathbf{C}}\right) \in \mathscr{H}_{\mathcal{Q}}$ is a polynomial in graphs such that each summand has multi-index of coupling constants $\mathbf{c}$. Notice, that when $\mathcal{Q}$ has only one coupling constant Equations (105) and (106) agree.

5 Associating the Renormalization Hopf Algebra to a local QFT and the Renormalization Hopf Algebra of QGR-QED

5.1 Associating the Renormalization Hopf Algebra to a local QFT

In this subsection, we first describe a problem in the construction of the renormalization Hopf algebra $\mathscr{H}_{\mathcal{Q}}$ to a given local QFT $\mathcal{Q}$ in Remark 5.4. Then, in the following, we describe two different constructions to overcome this problem in Construction 5.5 and in Construction 5.6. The main part of this section is devoted to Construction 5.6, as it is more technical and needs some justification, culminating in Lemma 5.12. Furthermore, it is more general, as it can be applied to every local QFT, whereas Construction 5.5 only works in special cases. As an example, this problem occurs in QED, as is explained in Example 5.7, as well as in QGR-QED. First, we start with a remark concerning the notation used in the following:

Remark 5.1. This remark aims to clarify notation before we start with the actual constructions. In the following, $\mathscr{H}_{\mathcal{Q}}$ is constructed as a $\mathbb{Q}$-algebra and it might or might not be a Hopf algebra over $\mathbb{Q}$ as well. Furthermore, $\mathscr{H}_{\mathscr{Q}}$ and $\mathcal{H}_{\mathcal{Q}}$ are constructed to be Hopf algebras over $\mathbb{Q}$. If $\mathscr{H}_{\mathcal{Q}}$ happens to be already a Hopf algebra, then it is isomorphic as a Hopf algebra to the other two constructed Hopf algebras, i.e. $\mathscr{H}_{\mathscr{Q}} \cong \mathscr{H}_{\mathcal{Q}} \cong \mathcal{H}_{\mathcal{Q}}$. If $\mathscr{H}_{\mathcal{Q}}$ is not a Hopf algebra, then we have the following inclusion of proper $\mathbb{Q}$-subalgebras $\mathscr{H}_{\mathscr{Q}} \subsetneq \mathscr{H}_{\mathcal{Q}} \subsetneq \mathcal{H}_{\mathcal{Q}}$. Furthermore, the script letters $\mathscr{D}_{\mathcal{Q}}(\cdot)$, $\mathscr{G}_{\mathcal{Q}}$ and $\mathscr{H}_{\mathcal{Q}}$ refer to the usual constructions given in Subsection 4.2, whereas their calligraphic partners $\mathcal{D}_{\mathcal{Q}}(\cdot)$, $\mathcal{G}_{\mathcal{Q}}$ and $\mathcal{H}_{\mathcal{Q}}$ refer to the objects obtained with Construction 5.5 or Construction 5.6, respectively. In the following sections of this thesis we only use the calligraphic letters, since they correspond to the Hopf algebra $\mathcal{H}_{\mathcal{Q}}$ associated to the local QFT $\mathcal{Q}$, as defined in Definition 5.13.

Definition 5.2 (Feynman graphs generated by residue sets 2)**.** Let $\mathcal{Q}$ be a local QFT with residue set $\mathcal{R}_{\mathcal{Q}}$. Recall from Definition 4.12 that we denote by $\mathscr{G}_{\mathcal{Q}}$ the set of all one particle irreducible (1PI) Feynman graphs[12] that can be generated by the residue set $\mathcal{R}_{\mathcal{Q}}$ of $\mathcal{Q}$. Moreover, we define the set $\mathcal{G}_{\mathcal{Q}}$ of all 1PI Feynman graphs of $\mathcal{Q}$ which does not contain superficially divergent subgraphs whose residue is not in the residue set $\mathcal{R}_{\mathcal{Q}}$, i.e.

$$\mathcal{G}_{\mathcal{Q}} := \left\{ \Gamma \;\middle|\; \Gamma \in \mathscr{G}_{\mathcal{Q}} \,:\, \Gamma = \coprod_{m=1}^{M} \Gamma_m \,,\; M \in \mathbb{N} \,:\, \operatorname{res}(\Gamma_m) \in \mathcal{R}_{\mathcal{Q}} \,,\; \forall \Gamma_m : \omega_{\mathcal{Q}}(\Gamma_m) \geq 0 \right\} . \tag{107}$$

This set will be used in Construction 5.5 and Construction 5.6.

Definition 5.3 (Set of superficially divergent subgraphs of a Feynman graph 2)**.** Let $\mathcal{Q}$ be a local QFT and $\Gamma \in \mathscr{G}_{\mathcal{Q}}$ a Feynman graph of $\mathcal{Q}$. Recall from Definition 4.17 that we denote by $\mathscr{D}_{\mathcal{Q}}(\Gamma)$ the set of superficially divergent subgraphs of Γ, i.e.

$$\mathscr{D}_{\mathcal{Q}}(\Gamma) := \left\{ \gamma \;\middle|\; \gamma \subseteq \Gamma \,:\, \gamma = \coprod_{m=1}^{M} \gamma_m \,,\; M \in \mathbb{N} \,:\, \omega_{\mathcal{Q}}(\gamma_m) \geq 0 \,,\; \forall \gamma_m \right\} . \tag{108a}$$

Furthermore, we have defined the set $\mathscr{D}'_{\mathcal{Q}}(\Gamma)$ of superficially divergent proper subgraphs of Γ, i.e.

$$\mathscr{D}'_{\mathcal{Q}}(\Gamma) := \left\{ \gamma \;\middle|\; \gamma \in \mathscr{D}_{\mathcal{Q}}(\Gamma) \,:\, \gamma \subsetneq \Gamma \right\} . \tag{108b}$$

[12] Again, we remark that the use of 1PI Feynman graphs, rather than connected Feynman graphs, is justified by Theorem 5.15, as is discussed in Remark 5.16.

Moreover, we define the two additional sets $\mathcal{D}_{\mathcal{Q}}(\Gamma)$ and $\mathcal{D}'_{\mathcal{Q}}(\Gamma)$, corresponding to $\mathscr{D}_{\mathcal{Q}}(\Gamma)$ and $\mathscr{D}'_{\mathcal{Q}}(\Gamma)$, respectively, which do not contain Feynman graphs with superficially divergent subgraphs whose residue is not in the residue set $\mathcal{R}_{\mathcal{Q}}$, i.e.

$$\mathcal{D}_{\mathcal{Q}}(\Gamma) := \left\{ \gamma \;\middle|\; \gamma \in \mathscr{D}_{\mathcal{Q}}(\Gamma) \, : \, \gamma = \coprod_{m=1}^{M} \gamma_m \, , \; M \in \mathbb{N} \, : \, \operatorname{res}(\gamma_m) \in \mathcal{R}_{\mathcal{Q}} \, , \; \forall \gamma_m \right\} \tag{109a}$$

and

$$\mathcal{D}'_{\mathcal{Q}}(\Gamma) := \left\{ \gamma \;\middle|\; \gamma \in \mathcal{D}_{\mathcal{Q}}(\Gamma) \, : \, \gamma \subsetneq \Gamma \right\} , \tag{109b}$$

These sets will be used in Construction 5.5 and Construction 5.6.

Remark 5.4. When constructing the free $\mathbb{Q}$-algebra $\mathscr{H}_{\mathcal{Q}}$, generated by the set $\mathscr{G}_{\mathcal{Q}}$ over $\mathbb{Q}$ to a given local QFT $\mathcal{Q}$, it might happen that $\mathscr{H}_{\mathcal{Q}}$ fails to be a Hopf algebra. More precisely, the coproduct $\Delta_{\mathscr{H}_{\mathcal{Q}}}$ and the antipode $S_{\mathscr{H}_{\mathcal{Q}}}$, as defined by the usual expression for the renormalization Hopf algebra in Equation (92) and Equation (94), respectively, may be ill-defined: There might exist Feynman graphs $\gamma \in \mathscr{H}_{\mathcal{Q}}$ which are superficially divergent by the superficial degree of divergence, i.e. $\omega_{\mathcal{Q}}(\gamma) \geq 0$, but there is no corresponding residue in $\mathcal{Q}$, i.e. $\operatorname{res}(\gamma) \notin \mathcal{R}_{\mathcal{Q}}$. Thus, the contraction of γ in any Feynman graph $\Gamma \in \mathscr{H}_{\mathcal{Q}}$ having γ as a proper subgraph, i.e. $\gamma \subsetneq \Gamma$, is an ill-defined operation. The following two constructions, Construction 5.5 and Construction 5.6, describe two different solutions to overcome this problem. The difference between these two constructions lies in their physical interpretation: In Construction 5.5 the Feynman graphs are set convergent but not to zero, i.e. still contribute to physical amplitudes, whereas in Construction 5.6 the Feynman graphs are directly set to zero. Therefore, Construction 5.5 makes only sense for Feynman graphs which are convergent by themselves, or for a set of Feynman graphs whose sum is convergent, despite their superficial degrees of divergence suggesting otherwise. On the other hand, Construction 5.6 should be used for Feynman graphs which are zero by themselves, for a set of Feynman graphs whose sum is zero, or for a Feynman graph or a set of Feynman graphs which one wishes to remove from the local QFT $\mathcal{Q}$ for any reason. Furthermore, we stress that both constructions could also be combined. A good example for this is QED where both situations appear, as described in Example 5.7. However, in the following sections we are only interested in the existence of a Hopf algebra $\mathcal{H}_{\mathcal{Q}}$ associated to the local QFT $\mathcal{Q}$, independent of the used construction. Therefore, we use the same symbol $\mathcal{H}_{\mathcal{Q}}$ for both constructions, since a distinction is not necessary.

Construction 5.5 (Redefining the set of superficially divergent subgraphs for a given Feynman graph)**.** The first solution to the problem described in Remark 5.4 is simply to declare all superficially divergent Feynman graphs whose residue is not in $\mathcal{R}_{\mathcal{Q}}$, i.e. all Feynman graphs in the set $\left(\mathscr{G}_{\mathcal{Q}} \setminus \mathcal{G}_{\mathcal{Q}}\right)$, to be convergent, where the sets $\mathscr{G}_{\mathcal{Q}}$ and $\mathcal{G}_{\mathcal{Q}}$ were defined in Definition 5.2. More precisely, let $\mathcal{H}_{\mathcal{Q}}$ be the $\mathbb{Q}$-algebra generated by the set $\mathscr{G}_{\mathcal{Q}}$. Then, we define the coproduct $\Delta_{\mathcal{H}_{\mathcal{Q}}}$ and the antipode $S_{\mathcal{H}_{\mathcal{Q}}}$ on a Feynman graph $\Gamma \in \mathcal{H}_{\mathcal{Q}}$ by using the sets $\mathcal{D}_{\mathcal{Q}}(\Gamma)$ instead of $\mathscr{D}_{\mathcal{Q}}(\Gamma)$, where the sets $\mathscr{D}_{\mathcal{Q}}(\Gamma)$ and $\mathcal{D}_{\mathcal{Q}}(\Gamma)$ were defined in Definition 5.3. Observe, that then $\mathcal{H}_{\mathcal{Q}}$, defined as the $\mathbb{Q}$-algebra generated by the set $\mathscr{G}_{\mathcal{Q}}$ is a Hopf algebra, since then the coproduct and the antipode are well-defined.

Construction 5.6 (Dividing the extended Hopf algebra by a Hopf ideal)**.** The second, and more involving, solution to the problem described in Remark 5.4 is to extend the local QFT $\mathcal{Q}$ to $\mathscr{Q}$ by enlarging the residue set to

$$\mathcal{R}_{\mathscr{Q}} = \left(\mathcal{R}_{\mathcal{Q}} \amalg \left\{ R \;\middle|\; R = \operatorname{res}(\Gamma) \, : \, \Gamma \in \mathscr{G}_{\mathcal{Q}} \, : \, \omega_{\mathcal{Q}}(\Gamma) \geq 0 : \operatorname{res}(\Gamma) \notin \mathcal{R}_{\mathcal{Q}} \right\} \right) . \tag{110}$$

Furthermore, we set the weight of any extension residue $R \in \left(\mathcal{R}_{\mathscr{Q}} \setminus \mathcal{R}_{\mathcal{Q}}\right)$ to be the same as a superficially divergent Feynman graph γ having residue $\operatorname{res}(\gamma) = R$, i.e. $\omega_{\mathcal{Q}}(R) := \omega_{\mathcal{Q}}(\gamma)$. This may not be well-defined, since Feynman graphs of the same residue can have different weights. In this case we choose an arbitrary such weight, since the following construction is independent of that choice, as long as the weight is divergent, i.e. $\omega_{\mathcal{Q}}(R) \geq 0$, and there exists a Feynman graph $\gamma \in \mathscr{H}_{\mathcal{Q}}$ having this residue $\operatorname{res}(\gamma) = R$ and this weight $\omega_{\mathcal{Q}}(\gamma) = \omega_{\mathcal{Q}}(R)$. Observe, that the free $\mathbb{Q}$-algebra generated by all Feynman graphs $\mathscr{G}_{\mathscr{Q}}$ is a Hopf algebra by construction. Indeed, all fractions Γ/γ for all $\Gamma \in \mathscr{H}_{\mathscr{Q}}$ and all $\gamma \in \mathscr{D}_{\mathscr{Q}}(\Gamma)$ are themselves elements of the $\mathbb{Q}$-algebra $\mathscr{H}_{\mathscr{Q}}$, i.e. $\Gamma/\gamma \in \mathscr{H}_{\mathscr{Q}}$. Then, we define the coproduct $\Delta_{\mathscr{H}_{\mathscr{Q}}}$ and the antipode $S_{\mathscr{H}_{\mathscr{Q}}}$ on a Feynman graph $\Gamma \in \mathscr{H}_{\mathscr{Q}}$ by using the set $\mathscr{D}_{\mathscr{Q}}(\Gamma)$, where the set $\mathscr{D}_{\mathscr{Q}}(\Gamma)$ was defined in Definition 5.3. However, usually the interesting object to study is the local QFT $\mathcal{Q}$, rather than $\mathscr{Q}$. Therefore, we prove in Proposition 5.8 and Proposition 5.11 properties about Hopf ideals and show in Lemma 5.12 that there exist a Hopf ideal $\mathfrak{i}_{N_{\mathscr{Q}}} \in \mathscr{H}_{\mathscr{Q}}$ in the Hopf algebra $\mathscr{H}_{\mathscr{Q}}$, such that the quotient Hopf algebra $\mathcal{H}_{\mathcal{Q}} := \mathscr{H}_{\mathscr{Q}}/\mathfrak{i}_{N_{\mathscr{Q}}}$ is the renormalization Hopf algebra associated to the local QFT $\mathcal{Q}$. More precisely, it is the $\mathbb{Q}$-algebra generated by the set $\mathcal{G}_{\mathcal{Q}}$ with coproduct $\Delta_{\mathcal{H}_{\mathcal{Q}}}$ and antipode $S_{\mathcal{H}_{\mathcal{Q}}}$, defined via the sets $\mathcal{D}_{\mathcal{Q}}(\Gamma)$ for all Feynman graphs $\Gamma \in \mathcal{H}_{\mathcal{Q}}$, making $\mathcal{H}_{\mathcal{Q}}$ into a Hopf algebra. Moreover, we stress the fact, that as in Construction 5.5 there is no need to enlarge the residue set $\mathcal{R}_{\mathcal{Q}}$ for $\mathcal{H}_{\mathcal{Q}}$ to be a Hopf algebra.

Example 5.7. For example, this problem occurs in QED, where the superficial degree of divergence counts the three-point photon function and the four-point photon function as superficially divergent, but there are no corresponding residues in $\mathcal{R}_{\text{QED}}$. Since it turns out, that the three-point photon functions are zero by Furry's Theorem and the four-point photon functions are finite by gauge invariance, cf. [33, 34, 35], it would make no sense to enlarge the set $\mathcal{R}_{\text{QED}}$. Rather, one should set all Feynman graphs having three external photon edges to zero via Construction 5.6 and all Feynman graphs having four external photon edges to be convergent via Construction 5.5. Plenty more such examples can be found in QGR-QED.

Proposition 5.8 (Ideals generated by residues are Hopf ideals)**.** *Let $\mathscr{Q}$ be a local QFT with residue set $\mathcal{R}_{\mathscr{Q}}$, such that $\mathscr{H}_{\mathscr{Q}}$ really is a Hopf algebra, as described in Remark 5.4. Let $R \in \mathcal{R}_{\mathcal{Q}}$ be a residue, then we denote by $\mathfrak{i}_R \neq \emptyset$ the ideal in the Hopf algebra $\mathscr{H}_{\mathscr{Q}}$, generated by all Feynman graphs in $\mathscr{H}_{\mathscr{Q}}$ having residue R or having a subgraph with residue R,*[13] *i.e.*

$$\mathfrak{i}_R := \left\langle \Gamma \;\middle|\; \Gamma \in \mathscr{G}_{\mathscr{Q}} \, : \, \exists \gamma \subseteq \Gamma \, : \, \operatorname{res}(\gamma) = R \right\rangle_{\mathscr{H}_{\mathscr{Q}}} . \tag{111}$$

Then, $\mathfrak{i}_R$ is a Hopf ideal in $\mathscr{H}_{\mathscr{Q}}$, i.e. we have:

1. $\Delta(\mathfrak{i}_R) \subseteq \mathscr{H}_{\mathscr{Q}} \otimes \mathfrak{i}_R + \mathfrak{i}_R \otimes \mathscr{H}_{\mathscr{Q}}$
2. $\hat{\mathbb{I}}(\mathfrak{i}_R) = 0$
3. $S(\mathfrak{i}_R) \subseteq \mathfrak{i}_R$

Proof. For the proof we consider an element $\mathfrak{G} \in \mathfrak{i}_R$ in the ideal $\mathfrak{i}_R$. We decompose $\mathfrak{G}$ into a product of two Feynman graphs, such that $\mathfrak{G} = \Gamma G$ for Feynman graphs $G \in \mathfrak{i}_R$ and $\Gamma \in \mathscr{H}_{\mathscr{Q}}$. Furthermore, we choose G to be a connected Feynman graph which is a generator of the ideal $\mathfrak{i}_R$. Then, Γ is the remaining product of Feynman graphs, which could also be the empty graph in case $\mathfrak{G} = G$. Then, for the first condition, we have

$$\begin{aligned} \Delta(\Gamma G) &= \Delta(\Gamma)\Delta(G) \\ &= \left(\sum_{\gamma \in \mathscr{D}_{\mathscr{Q}}(\Gamma)} \gamma \otimes \Gamma/\gamma \right) \left(\sum_{g \in \mathscr{D}_{\mathscr{Q}}(G)} g \otimes G/g \right) \\ &= \sum_{\gamma \in \mathscr{D}_{\mathscr{Q}}(\Gamma)} \sum_{g \in \mathscr{D}_{\mathscr{Q}}(G)} (\gamma g) \otimes \left(\Gamma/\gamma G/g \right) , \end{aligned} \tag{112}$$

[13] Notice that in general $\mathfrak{i}_R$ is not finitely generated.

where $G/g \in \mathfrak{i}_R$ or $g \in \mathfrak{i}_R$, since by assumption either $\operatorname{res}\left(G/g\right) = \operatorname{res}(G) = R$, $\operatorname{res}(g) = R$ or G/g contains a subgraph with residue R, and $\gamma, g, \Gamma/\gamma, G/g \in \mathscr{H}_\mathscr{Q}$. Hence $\Gamma/\gamma G/g \in \mathfrak{i}_R$ and $\gamma g \in \mathscr{H}_\mathscr{Q}$ or $\gamma g \in \mathfrak{i}_R$ and $\Gamma/\gamma G/g \in \mathscr{H}_\mathscr{Q}$, and it follows that

$$\begin{aligned}\Delta(\mathfrak{i}_R) &\subseteq \mathscr{H}_\mathscr{Q} \otimes \mathfrak{i}_R \\ &\subseteq \mathscr{H}_\mathscr{Q} \otimes \mathfrak{i}_R + \mathfrak{i}_R \otimes \mathfrak{i}_R \\ &\subseteq \mathscr{H}_\mathscr{Q} \otimes \mathfrak{i}_R + \mathfrak{i}_R \otimes \mathscr{H}_\mathscr{Q}\end{aligned} \tag{113a}$$

or

$$\begin{aligned}\Delta(\mathfrak{i}_R) &\subseteq \mathfrak{i}_R \otimes \mathscr{H}_\mathscr{Q} \\ &\subseteq \mathfrak{i}_R \otimes \mathscr{H}_\mathscr{Q} + \mathfrak{i}_R \otimes \mathfrak{i}_R \\ &\subseteq \mathfrak{i}_R \otimes \mathscr{H}_\mathscr{Q} + \mathscr{H}_\mathscr{Q} \otimes \mathfrak{i}_R\end{aligned} \tag{113b}$$

and in either way $\mathfrak{i}_R$ satisfies the first condition. The second condition follows directly from $\mathfrak{i}_R \neq \emptyset$, and we have

$$\hat{\mathbb{I}}(\mathfrak{i}_R) = 0\,. \tag{114}$$

For the third condition, we consider again the element $\mathfrak{G} \in \mathfrak{i}_R$ with the decomposition $\mathfrak{G} = \Gamma G$ as above. Then, we have

$$\begin{aligned}S(\Gamma G) &= S(\Gamma) S(G) \\ &= \left(- \sum_{\gamma \in \mathscr{D}_\mathscr{Q}(\Gamma)} S(\gamma) \Gamma/\gamma\right) \left(- \sum_{g \in \mathscr{D}_\mathscr{Q}(G)} S(g) G/g\right) \\ &= \sum_{\gamma \in \mathscr{D}_\mathscr{Q}(\Gamma)} \sum_{g \in \mathscr{D}_\mathscr{Q}(G)} S(\gamma) \Gamma/\gamma S(g) G/g\,,\end{aligned} \tag{115}$$

where again $G/g \in \mathfrak{i}_R$ or $S(g) \in \mathfrak{i}_R$, since by assumption either $\operatorname{res}\left(G/g\right) = \operatorname{res}(G) = R$, $\operatorname{res}(g) = R$ or G/g contains a subgraph with residue R, and $S(\gamma), \Gamma/\gamma, S(g), G/g \in \mathscr{H}_\mathscr{Q}$. Hence $S(\gamma) \Gamma/\gamma S(g) G/g \in \mathfrak{i}_R$ and it follows that

$$S(\mathfrak{i}_R) \subseteq \mathfrak{i}_R\,, \tag{116}$$

and thus $\mathfrak{i}_R$ satisfies the third condition as well. Therefore, $\mathfrak{i}_R$ is a Hopf ideal. ■

Proposition 5.9 (Ideals generated by pure self-loop Feynman graphs are Hopf ideals)**.** *Let $\mathscr{Q}$ be a local QFT with residue set $\mathcal{R}_\mathscr{Q}$, such that $\mathscr{H}_\mathscr{Q}$ really is a Hopf algebra, as described in Remark 5.4. If $\mathcal{Q}$ possesses self-loop graphs, we denote by $\mathfrak{i}_{S_\mathscr{Q}} \neq \emptyset$ the ideal in the Hopf algebra $\mathscr{H}_\mathscr{Q}$, generated by all Feynman graphs in $\mathscr{H}_\mathscr{Q}$ which are pure self-loop Feynman graphs (sometimes such graphs are called "roses"),[14] i.e.*

$$\mathfrak{i}_{S_\mathscr{Q}} := \left\langle \Gamma \;\middle|\; \Gamma \in \mathscr{G}_\mathscr{Q} \, : \, \#\Gamma^{[0]} = 1\,,\ \#\Gamma^{[1]} \geq 1 \right\rangle_{\mathscr{H}_\mathscr{Q}}\,. \tag{117}$$

Then, $\mathfrak{i}_{S_\mathscr{Q}}$ is a Hopf ideal in $\mathscr{H}_\mathscr{Q}$, i.e. we have:

1. $\Delta\left(\mathfrak{i}_{S_\mathscr{Q}}\right) \subseteq \mathscr{H}_\mathscr{Q} \otimes \mathfrak{i}_{S_\mathscr{Q}} + \mathfrak{i}_{S_\mathscr{Q}} \otimes \mathscr{H}_\mathscr{Q}$
2. $\hat{\mathbb{I}}\left(\mathfrak{i}_{S_\mathscr{Q}}\right) = 0$
3. $S\left(\mathfrak{i}_{S_\mathscr{Q}}\right) \subseteq \mathfrak{i}_{S_\mathscr{Q}}$

[14] Notice that in general $\mathfrak{i}_{S_\mathscr{Q}}$ is not finitely generated.

Proof. The proof is quite similar to the proof of Proposition 5.8. Again, for the proof we consider an element in the ideal $\mathfrak{G} \in \mathfrak{i}_{S_\mathscr{Q}}$. We decompose $\mathfrak{G}$ into a product of two Feynman graphs, such that $\mathfrak{G} = \Gamma G$ for Feynman graphs $G \in \mathfrak{i}_{S_\mathscr{Q}}$ and $\Gamma \in \mathscr{H}_\mathscr{Q}$. Furthermore, we choose G to be a connected Feynman graph which is a generator of the ideal $\mathfrak{i}_{S_\mathscr{Q}}$. Then, Γ is the remaining product of Feynman graphs, which could also be the empty graph in case $\mathfrak{G} = G$. Then, for the first condition, we have

$$\begin{aligned} \Delta (\Gamma G) &= \Delta (\Gamma) \Delta (G) \\ &= \left(\sum_{\gamma \in \mathscr{D}_\mathscr{Q}(\Gamma)} \gamma \otimes \Gamma / \gamma \right) \left(\sum_{g \in \mathscr{D}_\mathscr{Q}(G)} g \otimes G / g \right) \\ &= \sum_{\gamma \in \mathscr{D}_\mathscr{Q}(\Gamma)} \sum_{g \in \mathscr{D}_\mathscr{Q}(G)} (\gamma g) \otimes \big(\Gamma / \gamma G / g \big) \, , \end{aligned} \tag{118}$$

where $G/g \in \mathfrak{i}_{S_\mathscr{Q}}$ or $g \in \mathfrak{i}_{S_\mathscr{Q}}$, since by assumption either G/g or g is a pure self-loop Feynman graph, and $\gamma, g, \Gamma/\gamma, G/g \in \mathscr{H}_\mathscr{Q}$. Hence $\Gamma/\gamma G/g \in \mathfrak{i}_{S_\mathscr{Q}}$ and $\gamma g \in \mathscr{H}_\mathscr{Q}$ or $\gamma g \in \mathfrak{i}_{S_\mathscr{Q}}$ and $\Gamma/\gamma G/g \in \mathscr{H}_\mathscr{Q}$, and it follows that

$$\begin{aligned} \Delta \left(\mathfrak{i}_{S_\mathscr{Q}} \right) &\subseteq \mathscr{H}_\mathscr{Q} \otimes \mathfrak{i}_{S_\mathscr{Q}} \\ &\subseteq \mathscr{H}_\mathscr{Q} \otimes \mathfrak{i}_{S_\mathscr{Q}} + \mathfrak{i}_{S_\mathscr{Q}} \otimes \mathfrak{i}_{S_\mathscr{Q}} \\ &\subseteq \mathscr{H}_\mathscr{Q} \otimes \mathfrak{i}_{S_\mathscr{Q}} + \mathfrak{i}_{S_\mathscr{Q}} \otimes \mathscr{H}_\mathscr{Q} \end{aligned} \tag{119a}$$

or

$$\begin{aligned} \Delta \left(\mathfrak{i}_{S_\mathscr{Q}} \right) &\subseteq \mathfrak{i}_{S_\mathscr{Q}} \otimes \mathscr{H}_\mathscr{Q} \\ &\subseteq \mathfrak{i}_{S_\mathscr{Q}} \otimes \mathscr{H}_\mathscr{Q} + \mathfrak{i}_{S_\mathscr{Q}} \otimes \mathfrak{i}_{S_\mathscr{Q}} \\ &\subseteq \mathfrak{i}_{S_\mathscr{Q}} \otimes \mathscr{H}_\mathscr{Q} + \mathscr{H}_\mathscr{Q} \otimes \mathfrak{i}_{S_\mathscr{Q}} \end{aligned} \tag{119b}$$

and in either way $\mathfrak{i}_{S_\mathscr{Q}}$ satisfies the first condition. The second condition follows directly from $\mathfrak{i}_{S_\mathscr{Q}} \neq \emptyset$, and we have

$$\hat{\mathbb{I}} \left(\mathfrak{i}_{S_\mathscr{Q}} \right) = 0 \, . \tag{120}$$

For the third condition, we consider again the element $\mathfrak{G} \in \mathfrak{i}_R$ with the decomposition $\mathfrak{G} = \Gamma G$ as above. Then, we have

$$\begin{aligned} S (\Gamma G) &= S (\Gamma) S (G) \\ &= \left(- \sum_{\gamma \in \mathscr{D}_\mathscr{Q}(\Gamma)} S (\gamma) \Gamma / \gamma \right) \left(- \sum_{g \in \mathscr{D}_\mathscr{Q}(G)} S (g) G / g \right) \\ &= \sum_{\gamma \in \mathscr{D}_\mathscr{Q}(\Gamma)} \sum_{g \in \mathscr{D}_\mathscr{Q}(G)} S (\gamma) \Gamma / \gamma S (g) G / g \, , \end{aligned} \tag{121}$$

where again $G/g \in \mathfrak{i}_{S_\mathscr{Q}}$ or $S(g) \in \mathfrak{i}_{S_\mathscr{Q}}$, since by assumption either G/g or g is a pure self-loop Feynman graph, and $S(\gamma), \Gamma/\gamma, S(g), G/g \in \mathscr{H}_\mathscr{Q}$. Hence $S(\gamma) \Gamma/\gamma S(g) G/g \in \mathfrak{i}_{S_\mathscr{Q}}$ and it follows that

$$S \left(\mathfrak{i}_{S_\mathscr{Q}} \right) \subseteq \mathfrak{i}_{S_\mathscr{Q}} \, , \tag{122}$$

and thus $\mathfrak{i}_{S_\mathscr{Q}}$ satisfies the third condition as well. Therefore, $\mathfrak{i}_{S_\mathscr{Q}}$ is a Hopf ideal. ■

Remark 5.10. Let $\mathscr{Q}$ be a local QFT as in Proposition 5.9 with set of Feynman graphs $\mathscr{G}_\mathscr{Q}$. Then, the Feynman graphs generating the Hopf ideal $\mathfrak{i}_{S_\mathscr{Q}}$ in Proposition 5.9, i.e. pure self-loop Feynman graphs in $\mathscr{G}_\mathscr{Q}$ ("roses"), vanish during the renormalization process. Thus it is physically justified to remove them from the theory because they do not contribute to any scattering amplitude.

Proposition 5.11 (Sums of Hopf ideals are Hopf ideals)**.** *Let $\mathscr{Q}$ be a local QFT with residue set $\mathcal{R}_{\mathscr{Q}}$, such that $\mathscr{H}_{\mathscr{Q}}$ really is a Hopf algebra, as described in Remark 5.4. Let furthermore $\{ \mathfrak{i}_n \}_{n=1}^{N}$ be a set of N non-empty Hopf ideals, where $N \in \mathbb{N} \cup \infty$. Then, their sum*

$$\mathfrak{i}_N := \sum_{n=1}^{N} \mathfrak{i}_n \,, \tag{123}$$

i.e. the ideal $\mathfrak{i}_N$ generated by the union of the generators of all Hopf ideals in the set $\{ \mathfrak{i}_n \}_{n=1}^{N}$, is also a Hopf ideal in $\mathscr{H}_{\mathscr{Q}}$, i.e. we have:

1. $\Delta\left(\mathfrak{i}_N\right) \subseteq \mathscr{H}_{\mathscr{Q}} \otimes \mathfrak{i}_N + \mathfrak{i}_N \otimes \mathscr{H}_{\mathscr{Q}}$
2. $\hat{\mathbb{I}}\left(\mathfrak{i}_N\right) = 0$
3. $S\left(\mathfrak{i}_N\right) \subseteq \mathfrak{i}_N$

Proof. Again, the proof is quite similar to the proof of Proposition 5.8. Again, for the proof we consider an element in the ideal $\mathfrak{G} \in \mathfrak{i}_N$. We decompose $\mathfrak{G}$ into a product of two Feynman graphs, such that $\mathfrak{G} = \Gamma G$ for Feynman graphs $G \in \mathfrak{i}_N$ and $\Gamma \in \mathscr{H}_{\mathscr{Q}}$. Furthermore, we choose G to be a connected Feynman graph which is a generator of the ideal $\mathfrak{i}_N$. Even more, G is then also a generator of at least one of the ideals in the set $\{ \mathfrak{i}_n \}_{n=1}^{N}$, we choose one and denote it by $\mathfrak{i}_k$. Then, Γ is the remaining product of Feynman graphs, which could also be the empty graph in case $\mathfrak{G} = G$. Then, for the first condition, we have

$$\begin{aligned}
\Delta\left(\Gamma G\right) &= \Delta\left(\Gamma\right)\Delta\left(G\right) \\
&= \left(\sum_{\gamma \in \mathscr{D}_{\mathscr{Q}}(\Gamma)} \gamma \otimes \Gamma/\gamma\right)\left(\sum_{g \in \mathscr{D}_{\mathscr{Q}}(G)} g \otimes G/g\right) \\
&= \sum_{\gamma \in \mathscr{D}_{\mathscr{Q}}(\Gamma)} \sum_{g \in \mathscr{D}_{\mathscr{Q}}(G)} (\gamma g) \otimes \big(\Gamma/\gamma G/g\big) \,,
\end{aligned} \tag{124}$$

where $G/g \in \mathfrak{i}_k$ or $g \in \mathfrak{i}_k$, since by assumption $\mathfrak{i}_k$ is a Hopf ideal, and $\gamma, g, \Gamma/\gamma, G/g \in \mathscr{H}_{\mathscr{Q}}$. Hence $\Gamma/\gamma G/g \in \mathfrak{i}_k \subset \mathfrak{i}_N$ and $\gamma g \in \mathscr{H}_{\mathscr{Q}}$ or $\gamma g \in \mathfrak{i}_k \subset \mathfrak{i}_N$ and $\Gamma/\gamma G/g \in \mathscr{H}_{\mathscr{Q}}$, and it follows that

$$\begin{aligned}
\Delta\left(\mathfrak{i}_N\right) &\subseteq \mathscr{H}_{\mathscr{Q}} \otimes \mathfrak{i}_k \\
&\subseteq \mathscr{H}_{\mathscr{Q}} \otimes \mathfrak{i}_N \\
&\subseteq \mathscr{H}_{\mathscr{Q}} \otimes \mathfrak{i}_N + \mathfrak{i}_N \otimes \mathfrak{i}_N \\
&\subseteq \mathscr{H}_{\mathscr{Q}} \otimes \mathfrak{i}_N + \mathfrak{i}_N \otimes \mathscr{H}_{\mathscr{Q}}
\end{aligned} \tag{125a}$$

or

$$\begin{aligned}
\Delta\left(\mathfrak{i}_N\right) &\subseteq \mathfrak{i}_k \otimes \mathscr{H}_{\mathscr{Q}} \\
&\subseteq \mathfrak{i}_N \otimes \mathscr{H}_{\mathscr{Q}} \\
&\subseteq \mathfrak{i}_N \otimes \mathscr{H}_{\mathscr{Q}} + \mathfrak{i}_N \otimes \mathfrak{i}_N \\
&\subseteq \mathfrak{i}_N \otimes \mathscr{H}_{\mathscr{Q}} + \mathscr{H}_{\mathscr{Q}} \otimes \mathfrak{i}_N
\end{aligned} \tag{125b}$$

and in either way $\mathfrak{i}_N$ satisfies the first condition. The second condition follows directly from $\mathfrak{i}_N \neq \emptyset$, and we have

$$\hat{\mathbb{I}}\left(\mathfrak{i}_N\right) = 0 \,. \tag{126}$$

For the third condition, we consider again the element $\mathfrak{G} \in \mathfrak{i}_R$ with the decomposition $\mathfrak{G} = \Gamma G$ as above. Then, we have

$$\begin{aligned} S(\Gamma G) &= S(\Gamma) S(G) \\ &= \left(- \sum_{\gamma \in \mathscr{D}_{\mathscr{Q}}(\Gamma)} S(\gamma) \Gamma/\gamma \right) \left(- \sum_{g \in \mathscr{D}_{\mathscr{Q}}(G)} S(g) G/g \right) \\ &= \sum_{\gamma \in \mathscr{D}_{\mathscr{Q}}(\Gamma)} \sum_{g \in \mathscr{D}_{\mathscr{Q}}(G)} S(\gamma) \Gamma/\gamma S(g) G/g \, , \end{aligned} \tag{127}$$

where again $G/g \in \mathfrak{i}_k$ or $S(g) \in \mathfrak{i}_k$, since by assumption $\mathfrak{i}_k$ is a Hopf ideal, and $S(\gamma), \Gamma/\gamma, S(g)$, $G/g \in \mathscr{H}_{\mathscr{Q}}$. Hence $S(\gamma) \Gamma/\gamma S(g) G/g \in \mathfrak{i}_k \subseteq \mathfrak{i}_N$ and it follows that

$$S(\mathfrak{i}_N) \subseteq \mathfrak{i}_N \, , \tag{128}$$

and thus $\mathfrak{i}_N$ satisfies the third condition as well. Therefore, $\mathfrak{i}_N$ is a Hopf ideal. ■

Lemma 5.12 (Associating the renormalization Hopf algebra to a local QFT)**.** *Let $\mathcal{Q}$ be a local QFT and $\mathcal{R}_{\mathcal{Q}}$ the set of it's residues. Furthermore, let $\mathscr{Q}$ be the extension of $\mathcal{Q}$ with residue set $\mathcal{R}_{\mathscr{Q}} \supset \mathcal{R}_{\mathcal{Q}}$ and Hopf algebra $\mathscr{H}_{\mathscr{Q}}$, as in Construction 5.6. Then, there exists a Hopf ideal $\mathfrak{i}_{N_{\mathscr{Q}}} \subset \mathscr{H}_{\mathscr{Q}}$, such that the quotient Hopf algebra $\mathcal{H}_{\mathcal{Q}} := \mathscr{H}_{\mathscr{Q}} / \mathfrak{i}_{N_{\mathscr{Q}}}$ is the $\mathbb{Q}$-algebra generated by the set $\mathcal{G}_{\mathcal{Q}}$, with coproduct $\Delta_{\mathcal{H}_{\mathcal{Q}}}$ and antipode $S_{\mathcal{H}_{\mathcal{Q}}}$ defined via the sets $\mathcal{D}_{\mathcal{Q}}(\Gamma)$ for all Feynman graphs $\Gamma \in \mathcal{H}_{\mathcal{Q}}$.*

Proof. By Proposition 5.8 the ideals $\mathfrak{i}_{R_n}$ associated to the residues $R_n \in \left(\mathcal{R}_{\mathscr{Q}} \setminus \mathcal{R}_{\mathcal{Q}} \right)$ are Hopf ideals in the Hopf algebra $\mathscr{H}_{\mathscr{Q}}$. Let $N_{\mathcal{Q}} \in \mathbb{N} \cup \infty$ denote the cardinality of the countable set $\left(\mathcal{R}_{\mathscr{Q}} \setminus \mathcal{R}_{\mathcal{Q}} \right)$, i.e. $N_{\mathcal{Q}} := \# \left(\mathcal{R}_{\mathscr{Q}} \setminus \mathcal{R}_{\mathcal{Q}} \right)$. By Proposition 5.11 the ideal $\mathfrak{i}_{N_{\mathscr{Q}}} = \sum_{n=1}^{N_{\mathcal{Q}}} \mathfrak{i}_{R_n}$ is a Hopf ideal in the Hopf algebra $\mathscr{H}_{\mathscr{Q}}$ as well. Then, the quotient Hopf algebra $\mathcal{H}_{\mathcal{Q}} := \mathscr{H}_{\mathscr{Q}} / \mathfrak{i}_{N_{\mathscr{Q}}}$ is as desired. In particular, every Feynman graph $\Gamma \in \left(\mathscr{G}_{\mathcal{Q}} \setminus \mathcal{G}_{\mathcal{Q}} \right)$ is zero in $\mathcal{H}_{\mathcal{Q}}$ and hence $\mathcal{H}_{\mathcal{Q}}$ is the $\mathbb{Q}$-algebra, generated by the set $\mathcal{G}_{\mathcal{Q}}$ and the sets $\mathscr{D}_{\mathcal{Q}}(\Gamma)$ reduce to $\mathcal{D}_{\mathcal{Q}}(\Gamma)$ for all $\Gamma \in \mathcal{H}_{\mathcal{Q}}$. ■

Definition 5.13 (Renormalization Hopf algebra associated to a local QFT)**.** Let $\mathcal{Q}$ be a local QFT with residue set $\mathcal{R}_{\mathcal{Q}}$. We call the Hopf algebra $\mathcal{H}_{\mathcal{Q}}$, as constructed via either Construction 5.5, Construction 5.6 with Lemma 5.12 or a combination of both, the renormalization Hopf algebra associated to the local QFT $\mathcal{Q}$. Additionally, we define $\mathcal{H}_{\mathcal{Q}}$ to be such that the Hopf ideal $\mathfrak{i}_{S_{\mathcal{Q}}}$ from Proposition 5.9 is set to zero, as was justified in Remark 5.10. As already mentioned in Remark 5.4, we stress that in the following sections we are only interested in the existence of a Hopf algebra $\mathcal{H}_{\mathcal{Q}}$ associated to the local QFT $\mathcal{Q}$, independent of which of the two constructions or a combination of both was used. Therefore, we use the same symbol $\mathcal{H}_{\mathcal{Q}}$ for both constructions, since a distinction is not necessary.

5.2 The Renormalization Hopf Algebra of QGR-QED

Now, we assume the associated renormalization Hopf algebra to QGR-QED, as constructed in the previous Subsection 5.1. Furthermore, we proove a generalization of Furry's Theorem [33, 34, 35] in Theorem 5.15, stating that every scattering amplitude of an odd number of photons and an arbitrary number of gravitons vanishes. This is in particular useful, since the calculation shows that for the two-loop propagator combinatorial Green's functions, given explicitly in Subsection 6.1, all divergent subgraphs whose residue is not in the set $\mathcal{R}_{\text{QGR-QED}}$ are either of this type or the divergent subgraphs or the corresponding cographs are pure self-loop Feynman graphs ("roses") and thus their renormalized amplitude vanishes. Thus, we conclude that all graphs that are set to zero in the construction of the renormalization Hopf algebra are really zero when renormalized amplitudes are considered. In the following, fermion edges are denoted by , photon edges by , graviton edges by , photon-ghost edges by and graviton-ghost edges by .

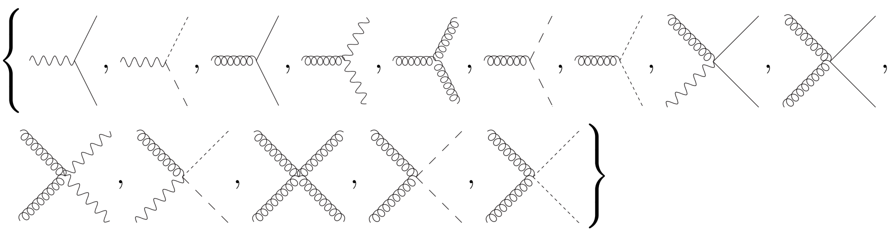
Remark 5.14 (Residue set of QGR-QED). Recall from Definition 4.11, how to obtain the corresponding residue set $\mathcal{R}_{\text{QGR-QED}}$ from the QGR-QED Lagrange density

$$\mathcal{L}_{\text{QGR-QED}} = \left(\frac{1}{\underline{\kappa}^2} R + g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \bar{\Psi} \left(i \nabla\!\!\!\!/^{\Sigma M \times U(1)} - m \right) \Psi \right) \mathrm{d}V_g + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}} \, , \tag{129}$$

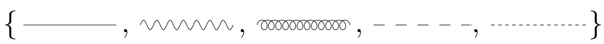
which was introduced in Subsection 3.2. As we restrict ourselves to $\mathcal{O}\left(\underline{\kappa}^2\right)$ in this thesis, we obtain the following finite residue set $\mathcal{R}_{\text{QGR-QED}}$. It splits, as usual, into a disjoint union of vertex residues $\mathcal{R}^{[0]}_{\text{QGR-QED}}$ and edge residues $\mathcal{R}^{[1]}_{\text{QGR-QED}}$, i.e.

$$\mathcal{R}_{\text{QGR-QED}} = \mathcal{R}^{[0]}_{\text{QGR-QED}} \amalg \mathcal{R}^{[1]}_{\text{QGR-QED}} \, . \tag{130}$$

Concretely, we have

$\mathcal{R}^{[0]}_{\text{QGR-QED}} =$  (131a)

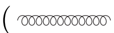
and

$\mathcal{R}^{[1]}_{\text{QGR-QED}} =$  . (131b)

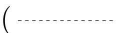
Furthermore, their corresponding weights read:

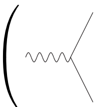
$\omega_{\mathcal{Q}}$ () $= 1$ (132a)

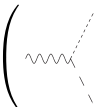
$\omega_{\mathcal{Q}}$ () $= 2$ (132b)

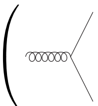
$\omega_{\mathcal{Q}}$ () $= 2$ (132c)

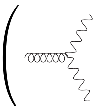
$\omega_{\mathcal{Q}}$ () $= 2$ (132d)

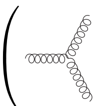
$\omega_{\mathcal{Q}}$ () $= 2$ (132e)

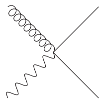
$\omega_{\mathcal{Q}}$ () $= 0$ (132f)

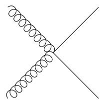
$\omega_{\mathcal{Q}}$ () $= 2$ (132g)

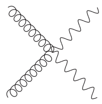
$\omega_{\mathcal{Q}}$ () $= 1$ (132h)

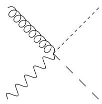
$\omega_{\mathcal{Q}}$ () $= 2$ (132i)

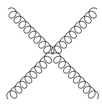
$\omega_{\mathcal{Q}}$ () $= 2$ (132j)

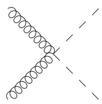
$\omega_{\mathcal{Q}}$ () $= 0$ (132k)

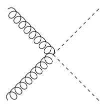
$\omega_{\mathcal{Q}}$ () $= 1$ (132l)

$\omega_{\mathcal{Q}}$ () $= 2$ (132m)

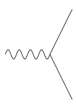
$\omega_{\mathcal{Q}}$ () $= 2$ (132n)

$\omega_{\mathcal{Q}}$ () $= 2$ (132o)

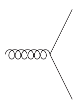
$\omega_{\mathcal{Q}}$ () $= 2$ (132p)

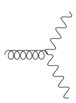
$\omega_{\mathcal{Q}}$ () $= 2$ (132q)

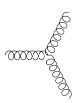
Moreover, the corresponding coupling constants of the vertex residues read:

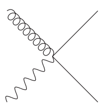
coupling () $= \underline{e}$ (133a)

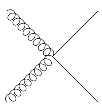
coupling () $= \underline{\kappa}$ (133b)

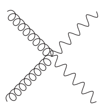
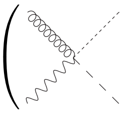
coupling () $= \underline{\kappa}$ (133c)

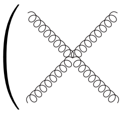
coupling () $= \underline{\kappa}$ (133d)

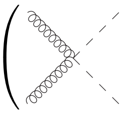
coupling () $= \underline{\kappa}$ (133e)

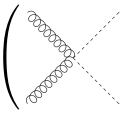
coupling () $= \underline{e\kappa}$ (133f)

coupling () $= \underline{\kappa}^2$ (133g)

coupling () $= \underline{\kappa}^2$ (133h) coupling $\Big($  $\Big) = \underline{\kappa}^2$ (133i)

coupling $\Big($  $\Big) = \underline{\kappa}^2$ (133j)

coupling $\Big($  $\Big) = \underline{\kappa}^2$ (133k)

coupling $\Big($  $\Big) = \underline{\kappa}^2$ (133l)

Theorem 5.15 (Generalized Furry's Theorem)**.** *Consider QGR-QED with the Lagrange density*

$$\mathcal{L}_{QGR\text{-}QED} = \left(\frac{1}{\underline{\kappa}^2} R + g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \bar{\Psi} \left(i \nabla\!\!\!/^{\Sigma M \times U(1)} - m \right) \Psi \right) \mathrm{d}V_g + \mathcal{L}_{GF} + \mathcal{L}_{Ghost} \,. \tag{134}$$

Then, all scattering amplitudes with an odd number of external photons and an arbitrary number of external gravitons vanish identically.

Proof. Let $\mathcal{C}$ denote the charge conjugation operator with inverse $\mathcal{C}^{-1}$. Recall that the vacuum is invariant under charge conjugation [33, 34, 35], i.e.

$$|0\rangle = \mathcal{C} \, |0\rangle \tag{135a}$$

and

$$\langle 0| = \langle 0| \, \mathcal{C}^{-1} \,. \tag{135b}$$

Furthermore, recall that the electromagnetic four-current changes sign under charge conjugation [33, 34, 35], i.e.

$$\underline{e} j^\rho = -\mathcal{C}^{-1} \underline{e} j^\rho \mathcal{C} \,, \tag{136}$$

whereas the graviton field is invariant under charge conjugation since it is it's own antiparticle, i.e.

$$h_{\mu\nu} = \mathcal{C}^{-1} h_{\mu\nu} \mathcal{C} \,. \tag{137}$$

Now, we consider the vacuum expectation values of scattering amplitudes $\mathcal{S}^{\rho_1 \cdots \rho_M}{}_{\mu_1 \nu_1 \cdots \mu_N \nu_N}$ with an odd number $M \in (2\mathbb{N}_0 + 1)$ of external photons and an arbitrary number $N \in \mathbb{N}_0$ of external gravitons:[15]

$$\begin{aligned}
\mathcal{S}^{\rho_1 \cdots \rho_M}{}_{\mu_1 \nu_1 \cdots \mu_N \nu_N} &= \langle 0| \, j_1^{\rho_1} \cdots j_M^{\rho_M} h_{1\,\mu_1 \nu_1} \cdots h_{N\,\mu_N \nu_N} \, |0\rangle \\
&= \langle 0| \, \mathcal{C}^{-1} j_1^{\rho_1} \cdots j_M^{\rho_M} h_{1\,\mu_1 \nu_1} \cdots h_{N\,\mu_N \nu_N} \mathcal{C} \, |0\rangle \\
&= \langle 0| \, \mathcal{C}^{-1} j_1^{\rho_1} \mathcal{C} \mathcal{C}^{-1} \cdots \mathcal{C} \mathcal{C}^{-1} j_M^{\rho_M} \mathcal{C} \mathcal{C}^{-1} h_{1\,\mu_1 \nu_1} \mathcal{C} \mathcal{C}^{-1} \cdots \mathcal{C} \mathcal{C}^{-1} h_{N\,\mu_N \nu_N} \mathcal{C} \, |0\rangle \\
&= (-1)^M \, \langle 0| \, j_1^{\rho_1} \cdots j_M^{\rho_M} h_{1\,\mu_1 \nu_1} \cdots h_{N\,\mu_N \nu_N} \, |0\rangle \\
&= - \, \langle 0| \, j_1^{\rho_1} \cdots j_M^{\rho_M} h_{1\,\mu_1 \nu_1} \cdots h_{N\,\mu_N \nu_N} \, |0\rangle \\
&= -\mathcal{S}^{\rho_1 \cdots \rho_M}{}_{\mu_1 \nu_1 \cdots \mu_N \nu_N} \\
&= 0
\end{aligned} \tag{138}$$

Thus, the scattering amplitudes with an odd number of external photons and an arbitrary number of external gravitons vanish identically, as claimed. ■

[15] Observe, that the following result is independent of the order of the particles in the vacuum expectation value.

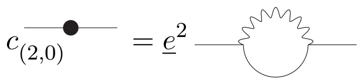
Remark 5.16. Theorem 5.15 also justifies the restriction of connected Feynman graphs to 1PI Feynman graphs in Definition 4.12 and Definition 5.2. More precisely, since the amplitude of the two-point vertex function of a photon and a graviton vanishes, the set of connected Feynman graphs which are not 1PI is a trivial extension of the set of 1PI Feynman graphs.

6 Combinatorial Green's Functions in QGR-QED and their Coproduct Structure

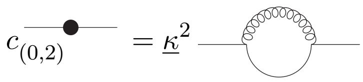
Now, we consider the coproduct structure on the two-loop propagator graphs in QGR-QED. We associate to QGR-QED its renormalization Hopf algebra $\mathcal{H}_{\text{QGR-QED}}$, as was described in Subsection 5.2. Let furthermore $c^R_{\mathbf{C}}$ be the combinatorial Green's function with residue $R \in \mathcal{R}_{\text{QGR-QED}}$ and multindex $\mathbf{C} \in \mathbb{Z}^2$, as was defined in Definition 4.24. The multi-index $\mathbf{C}$, introduced in Definition 4.21, is defined such that $\mathbf{C} = (m, n)$ corresponds to a graphs with coupling constants of order $\mathcal{O}\left(\underline{e}^m \underline{\kappa}^n\right)$, with $m, n \in \mathbb{Z}$. In the following subsections, we present the corresponding Green's functions and their coproducts. We remind, that fermion edges are denoted by ———, photon edges by $\sim\sim\sim$, graviton edges by oooooo, photon-ghost edges by $- - - - -$ and graviton-ghost edges by $\cdots\cdots$.

6.1 Combinatorial Green's Functions in QGR-QED

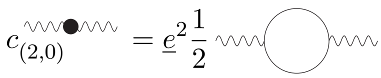
We obtain the following combinatorial Green's functions, where non-symmetric graphs are drawn only once, but with the corresponding multiplicity:



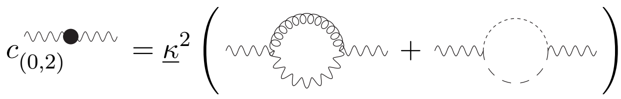
(139)



(140)



(141)



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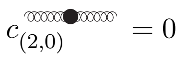
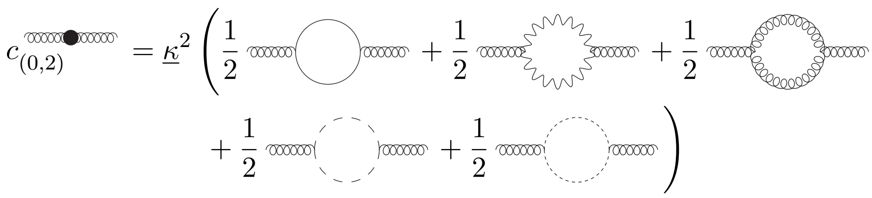
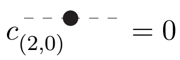


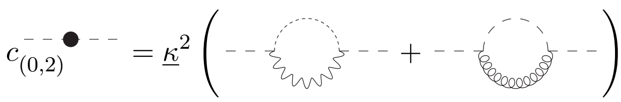
(143)



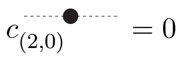
(144)



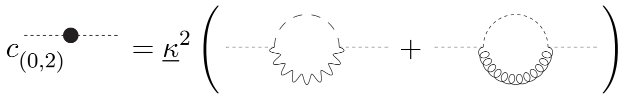
(145)



(146)



(147)



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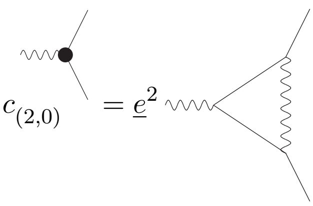

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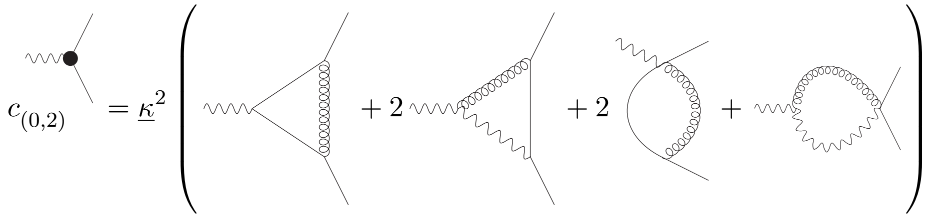
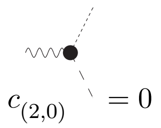


(150)



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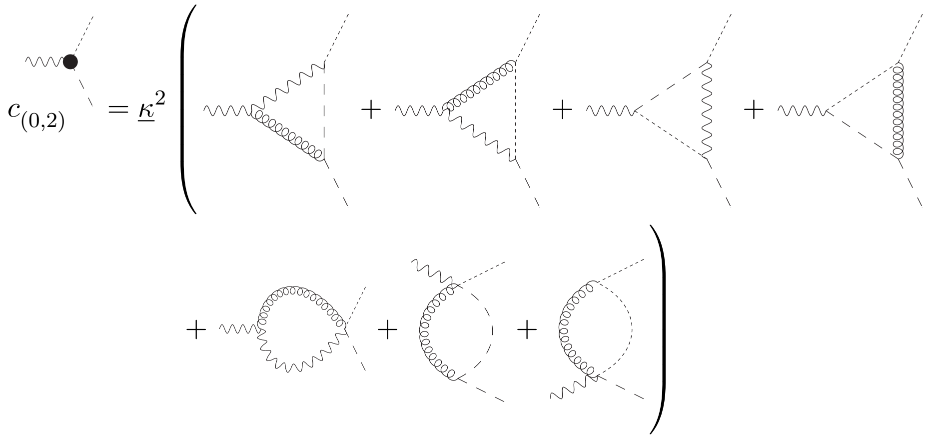

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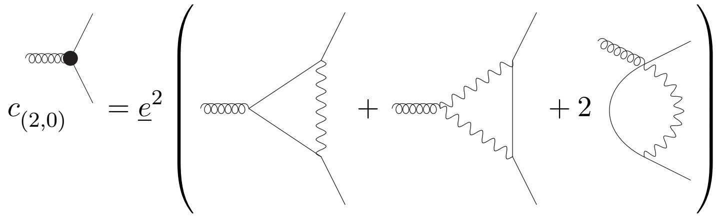
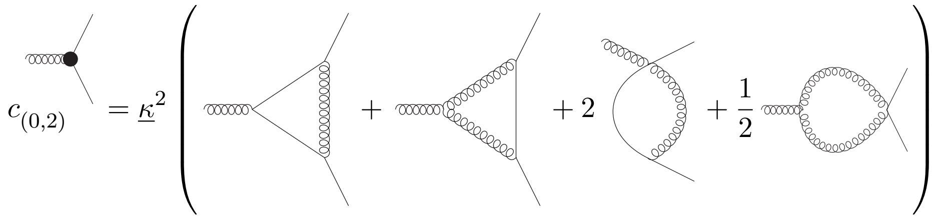


(153)



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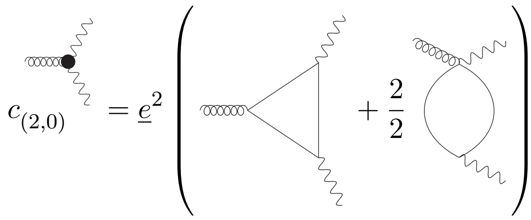
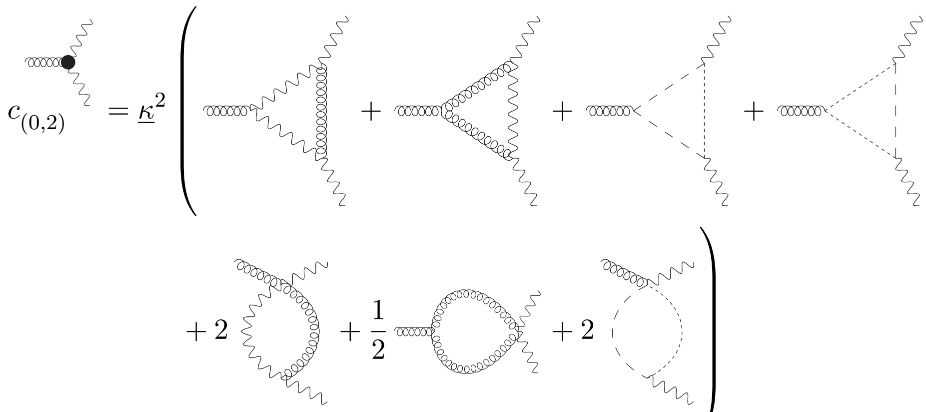


(155)



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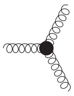
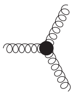
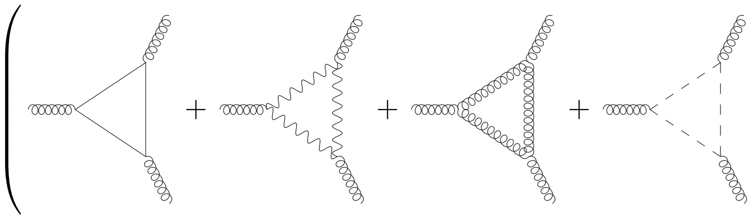
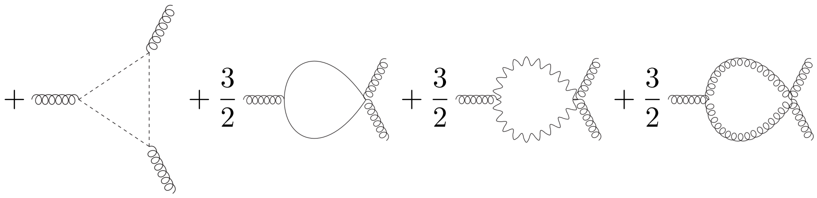


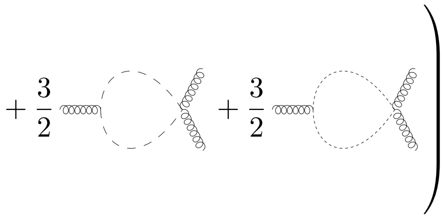
$$c_{(2,0)} = 0 \tag{157}$$



$$c_{(0,2)} = \underline{\kappa}^2$$







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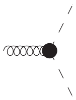
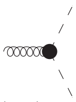
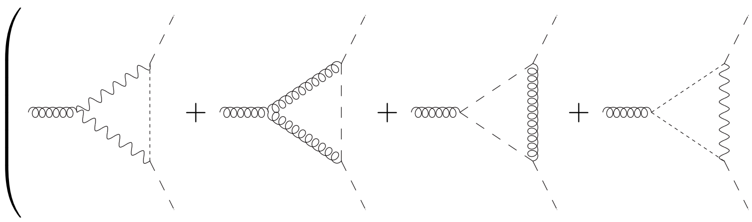
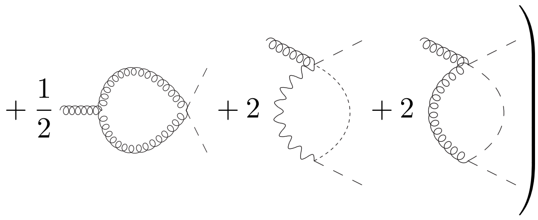


$$c_{(2,0)} = 0 \tag{159}$$

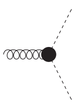


$$c_{(0,2)} = \underline{\kappa}^2$$

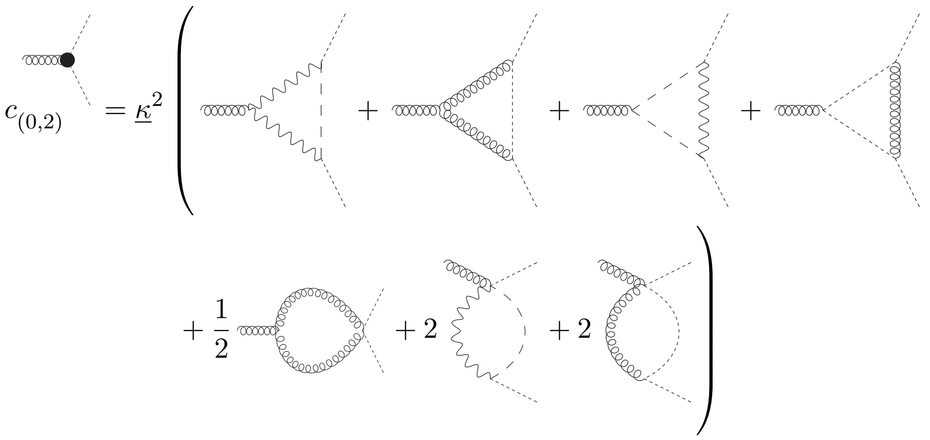




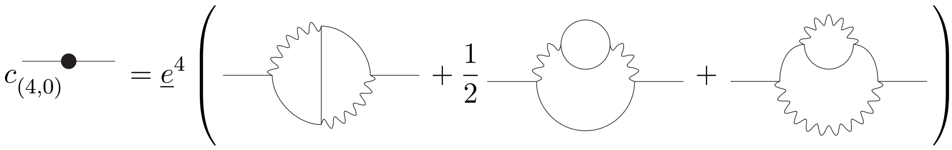
(160)



$$c_{(2,0)} = 0 \tag{161}$$



(162)



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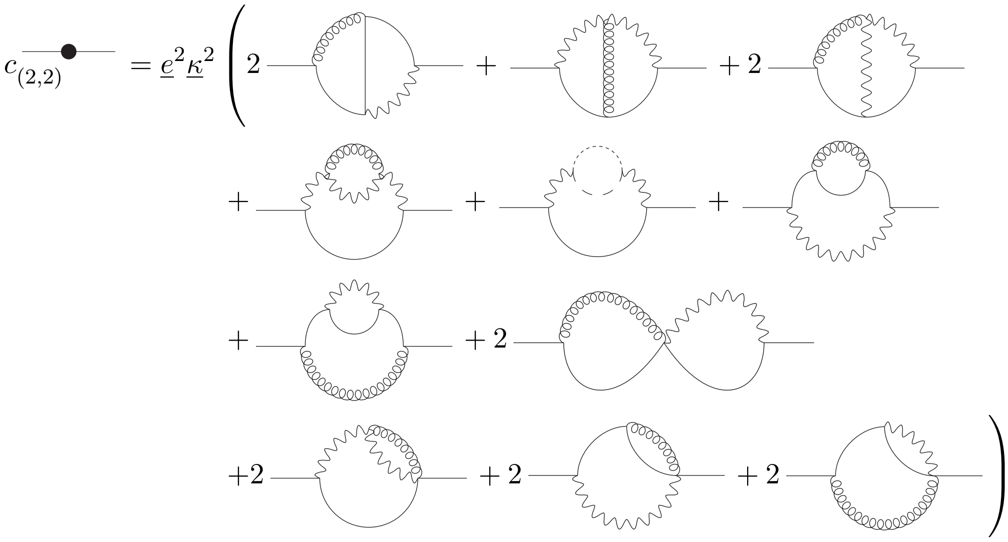

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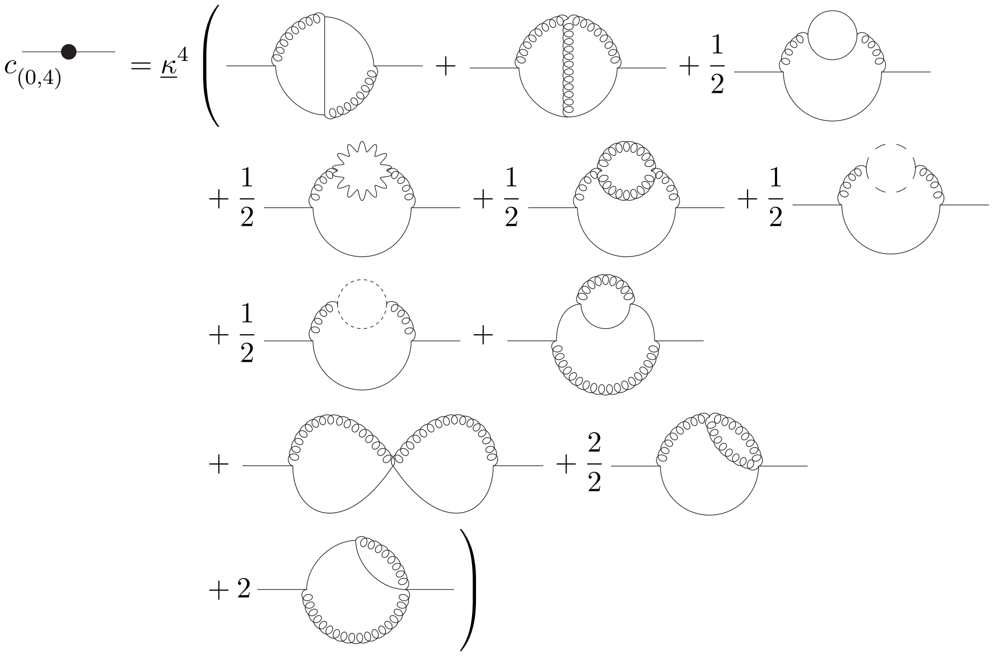
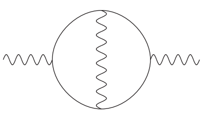
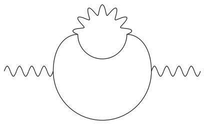
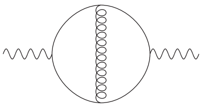
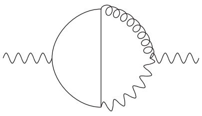
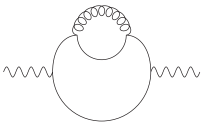
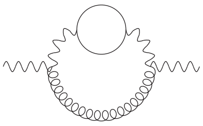
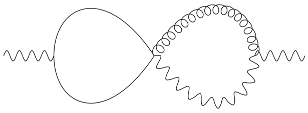
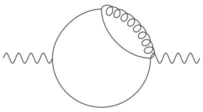
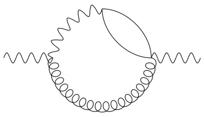


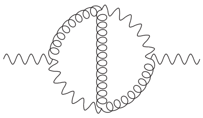
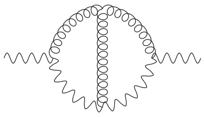
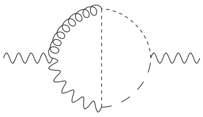
(165)

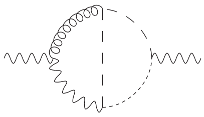
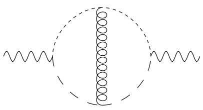
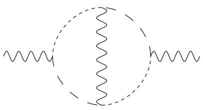
$c_{(4,0)}$  $= \underline{e}^4 \Bigg(\frac{1}{2}$  $+$  $\Bigg)$ (166)

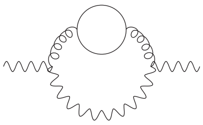
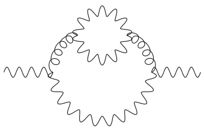
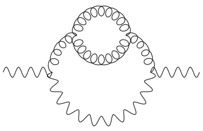
$c_{(2,2)}$  $= \underline{e}^2 \underline{\kappa}^2 \Bigg(\frac{1}{2}$  $+ 2$  $+$ 

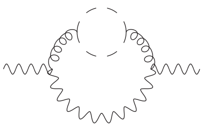
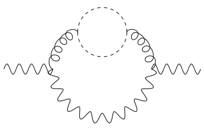
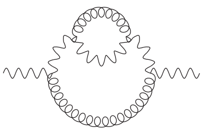
$+ \frac{1}{2}$  $+ \frac{2}{2}$  (167)

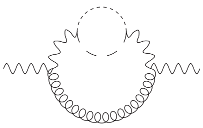
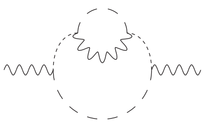
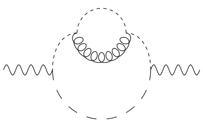
$+2$  $+ \frac{2}{2}$  $\Bigg)$

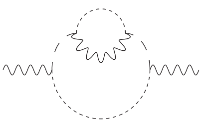
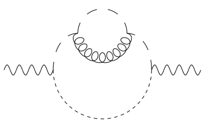
$c_{(0,4)}$  $= \underline{\kappa}^4 \Bigg($  $+$  $+ 2$ 

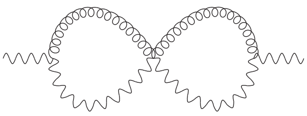
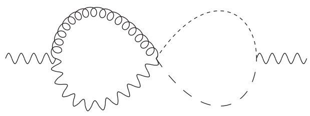
$+ 2$  $+$  $+$ 

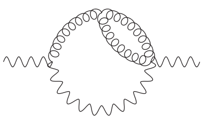
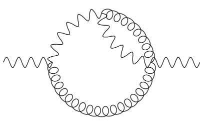
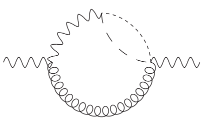
$+ \frac{1}{2}$  $+ \frac{1}{2}$  $+ \frac{1}{2}$ 

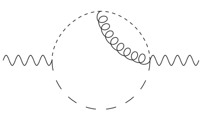
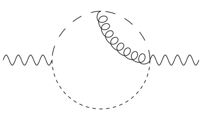
$+ \frac{1}{2}$  $+ \frac{1}{2}$  $+$ 

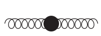
$+$  $+$  $+$  (168)

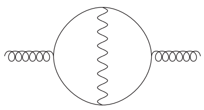
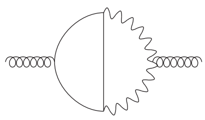
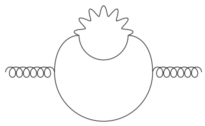
$+$  $+$ 

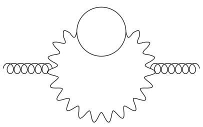
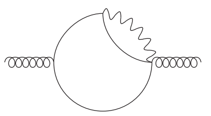
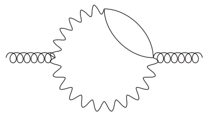
 $+ 2$ 

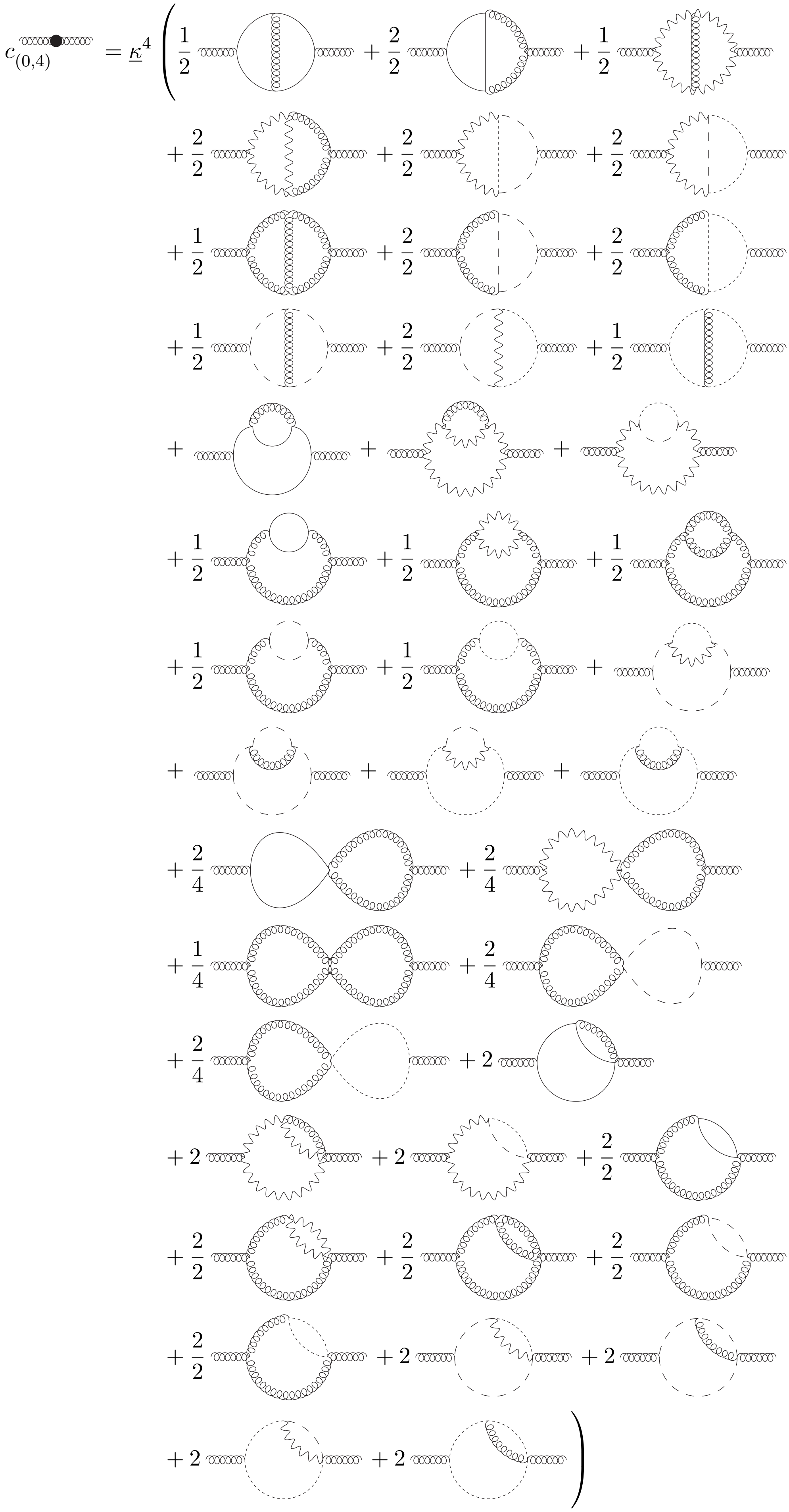
$+ \frac{2}{2}$  $+ 2$  $+ 2$ 

$+ 2$  $+ 2$  $\Bigg)$

$c_{(4,0)}$  $= 0$ (169)

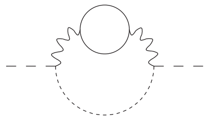
$c_{(2,2)}$  $= \underline{e}^2\,\underline{\kappa}^2 \Bigg(\frac{1}{2}$  $+ \frac{2}{2}$  $+$ 

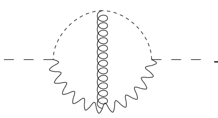
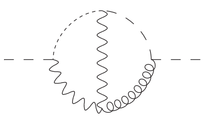
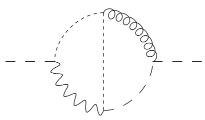
$+ \frac{1}{2}$  $+ 2$  $+ \frac{2}{2}$  $\Bigg)$ (170)

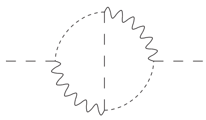
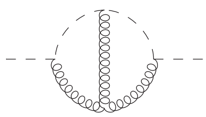
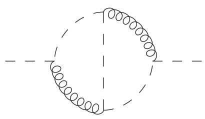


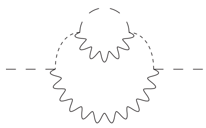
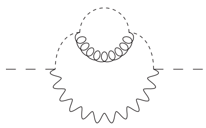
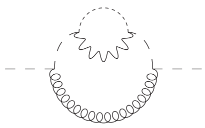
(171)

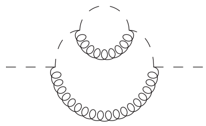
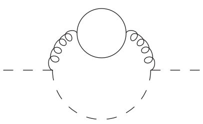
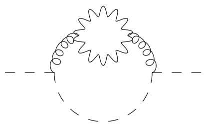
$$c_{(4,0)} = 0 \qquad (172)$$

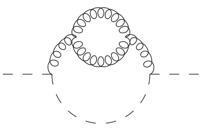
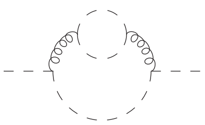
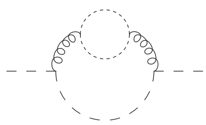
$$c_{(2,2)} = \underline{e}^2\underline{\kappa}^2\frac{1}{2}$$  (173)

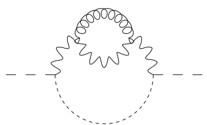
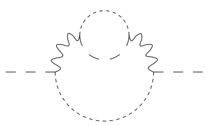
$$c_{(0,4)} = \underline{\kappa}^4 \Bigg($$  $+\,2$  $+\,2$ 

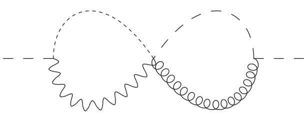
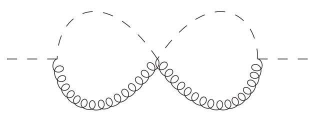
$+$  $+$  $+$ 

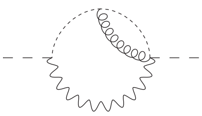
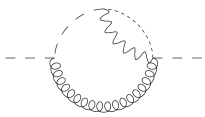
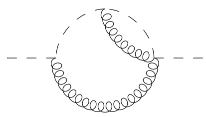
$+$  $+$  $+$ 

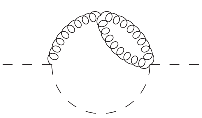
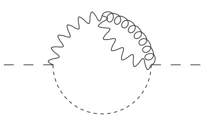
$+$  $+\,\frac{1}{2}$  $+\,\frac{1}{2}$ 

$+\,\frac{1}{2}$  $+\,\frac{1}{2}$  $+\,\frac{1}{2}$  (174)

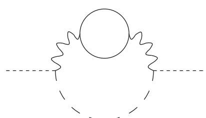
$+$  $+$ 

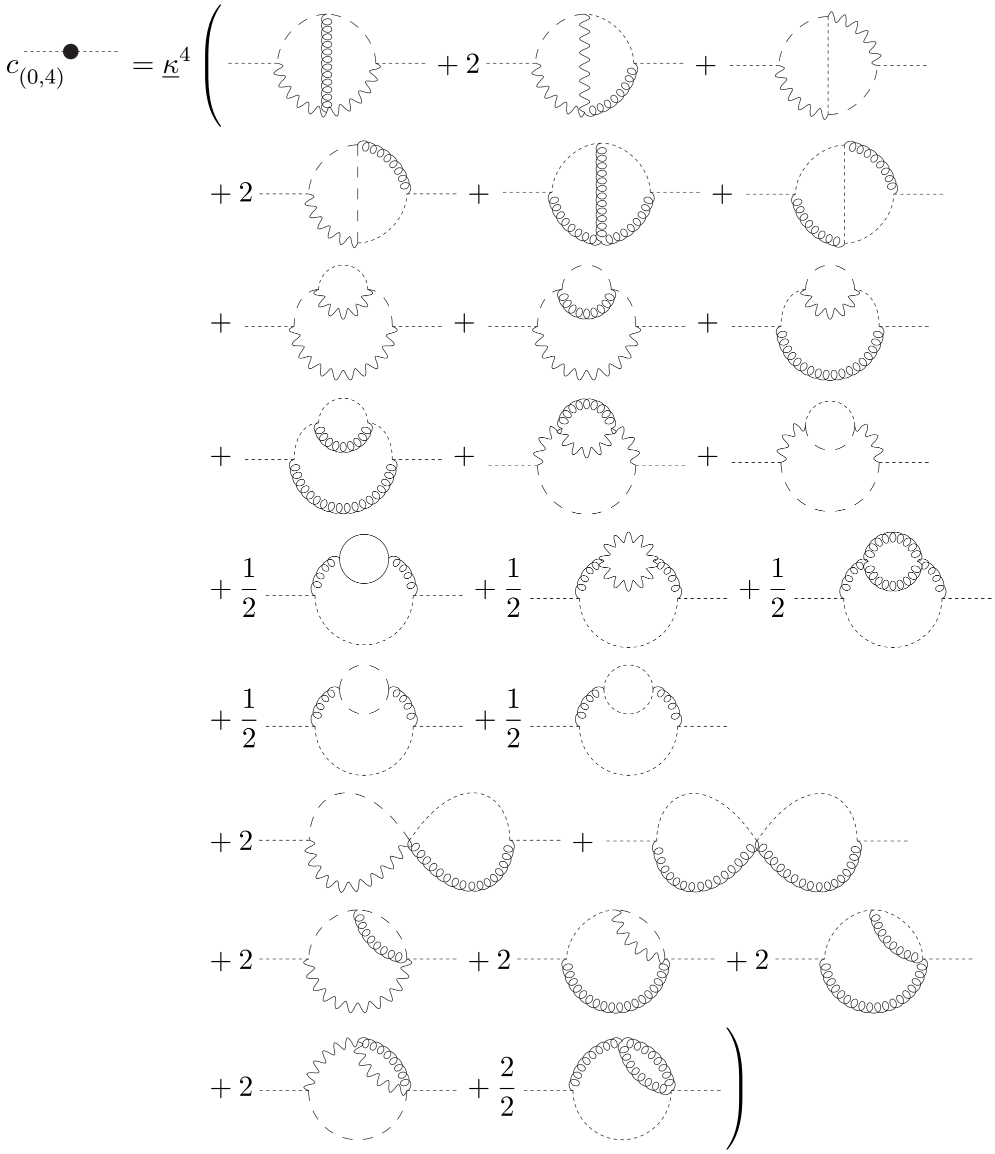
$+\,2$  $+$ 

$+\,2$  $+\,2$  $+\,2$ 

$+\,\frac{2}{2}$  $+\,2$  $\Bigg)$

$$c_{(4,0)} = 0 \qquad (175)$$

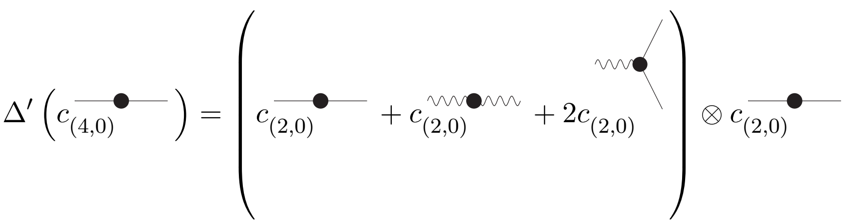
$$c_{(2,2)} = \underline{e}^2\underline{\kappa}^2\frac{1}{2}$$  (176)



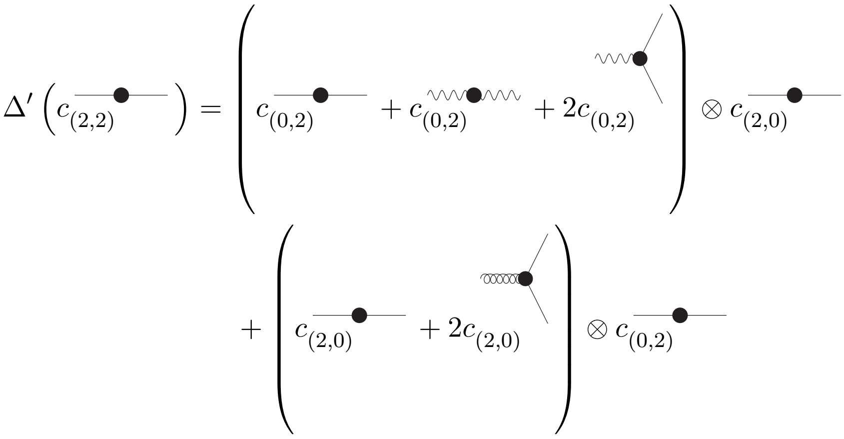
(177)

6.2 The Coproduct Structure of combinatorial Green's Functions in QGR-QED

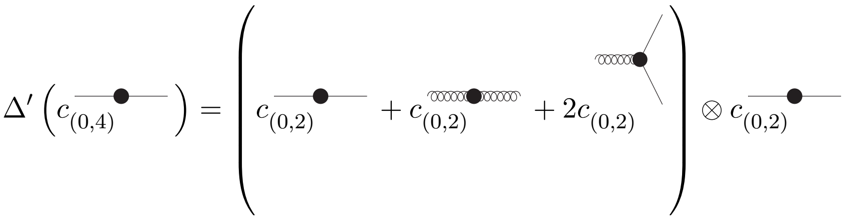
We obtain the following reduced coproduct structure of the two-loop propagator combinatorial Green's functions:



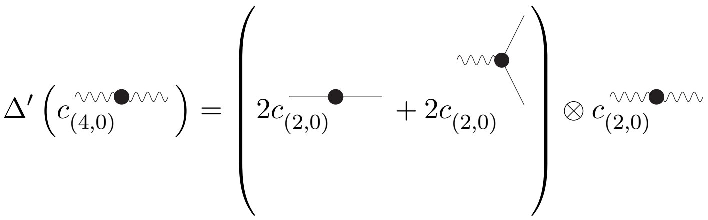
(178)



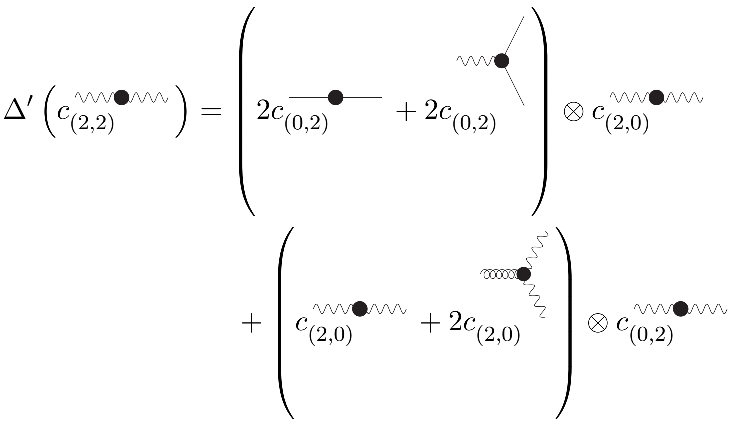
(179)



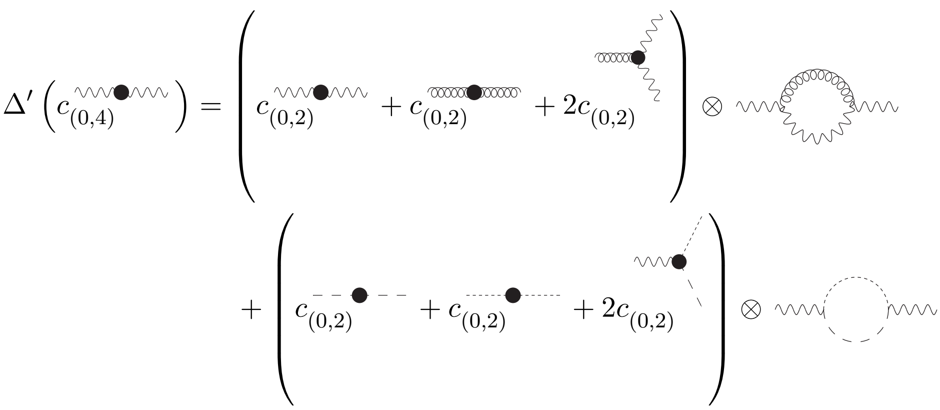
(180)



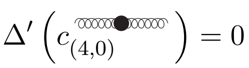
(181)



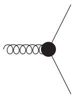
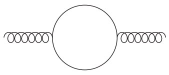
(182)

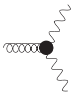
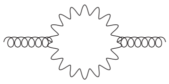


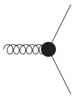
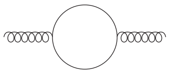
(183)

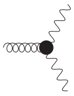
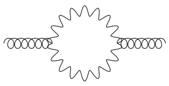


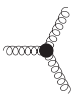
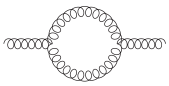
(184)

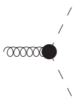
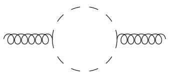
$\Delta'($ $c_{(2,2)}$  $) = ($ $2c_{(2,0)}$  $+ 2c_{(2,0)}$  $) \otimes \frac{1}{2}$ 

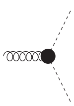
$+ ($ $2c_{(2,0)}$  $+ 2c_{(2,0)}$  $) \otimes \frac{1}{2}$  (185)

$\Delta'($ $c_{(0,4)}$  $) = ($ $2c_{(0,2)}$  $+ 2c_{(0,2)}$  $) \otimes \frac{1}{2}$ 

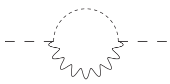
$+ ($ $2c_{(0,2)}$  $+ 2c_{(0,2)}$  $) \otimes \frac{1}{2}$ 

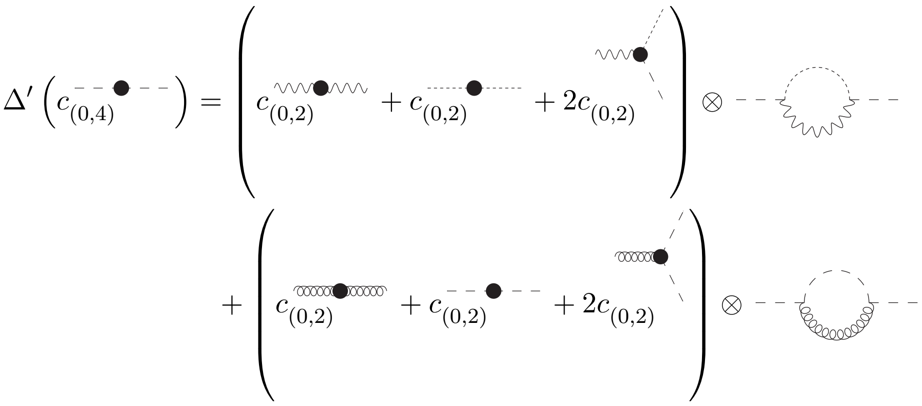
$+ ($ $2c_{(0,2)}$  $+ 2c_{(0,2)}$  $) \otimes \frac{1}{2}$  (186)

$+ ($ $2c_{(0,2)}$  $+ 2c_{(0,2)}$  $) \otimes \frac{1}{2}$ 

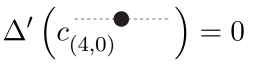
$+ ($ $2c_{(0,2)}$  $+ 2c_{(0,2)}$  $) \otimes \frac{1}{2}$ 

$\Delta'($ $c_{(4,0)}$  $) = 0$ (187)

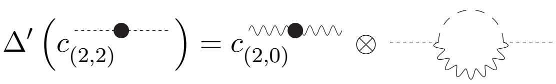
$\Delta'($ $c_{(2,2)}$  $) = c_{(2,0)}$  $\otimes$  (188)



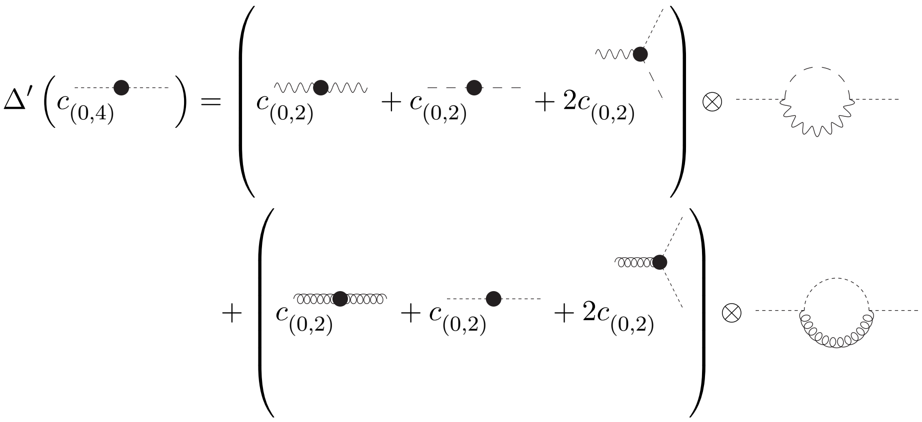
(189)



(190)



(191)



(192)

Thus, using Remark 4.25, we conclude: The two-loop combinatorial Green's functions for the fermion propagator with any multi-index and the photon propagator with multi-indices $(4,0)$ and $(2,2)$ does possess Hopf subalgebras. Contrary, the two-loop combinatorial Green's functions for the photon propagator with multi-index $(0,4)$, the graviton propagator with any multi-index, the photon ghost propagator with any multi-index and the graviton ghost propagator with any multi-index does not possess Hopf subalgebras. The existence of Hopf subalgebras in the renormalization Hopf algebra for a given residue is directly related to the possibility of Z factor renormalization for the corresponding monomial in the Lagrange density. Thus, it is not possible to renormalize all monomials in the Lagrange density of QGR-QED using Z factors. This situation may be improved by imposing the corresponding Slavnov-Taylor identities [10, 11, 12, 33, 34, 35]. However, their validity needs to be checked by applying the Feynman rules to the corresponding combinatorial Green's functions. This discussion is postponed to future work, since it goes beyond the scope of this thesis.

7 Conclusion

In this thesis, we considered Quantum General Relativity coupled to Quantum Electrodynamics (QGR-QED). First, we stated basic conventions and notations in Section 2. Then, in Section 3 we introduced the necessary differential geometric notions and the Lagrange density of QGR-QED. In Section 4 we introduced Hopf algebras in general and the Connes-Kreimer renormalization Hopf algebra in particular. Next, in Section 5 we discussed a problem which can occur when associating the renormalization Hopf algebra to a given local QFT. In particular, two constructions, Construction 5.5 and Construction 5.6 with Lemma 5.12, were worked out to overcome this problem. Moreover, their physical interpretation is discussed in Remark 5.4. Next, the application of these general results to QGR-QED is discussed. In particular, a generalization of Furry's Theorem is formulated and proved in Theorem 5.15, stating that all amplitudes with an odd number of external photons and an arbitrary number of external gravitons vanish. This is in particular useful, since the calculations showed that, besides from pure self-loop Feynman graphs which vanish in the renormalization process, these are the only graphs which need to be set to zero when constructing the renormalization Hopf algebra of QGR-QED. Then, in Section 6 the main part of this thesis is presented. First, all combinatorial Green's functions of the one-loop propagator amplitudes, the one-loop three-point amplitudes and the two-loop propagator amplitudes are presented. Then, their coproduct structure is presented, for which the coproduct of 155 Feynman graphs has been computed. From the result we conclude that the renormalization Hopf algebra of QGR-QED does not possess Hopf subalgebras for all residues and multi-indices. This translates into the statement that Z factor renormalization is not possible for all monomials in the Lagrange density of QGR-QED.

A couple of points were postponed to future work, since they went beyond the scope of this thesis. This includes a detailed treatment of the Feynman rules of QGR-QED, as was mentioned in Remark 3.9. Additionally, the coproduct structure of the two-loop propagator combinatorial Green's functions, given in Subsection 6.2, may be simplified by imposing the corresponding Slavnov-Taylor identities such that more combinatorial Green's functions possess Hopf subalgebras. Their validity and the corresponding consequences will be investigated in future work. Finally, it is also interesting if it is possible to define a Corolla polynomial [36, 37, 38] creating the Feynman rules for QGR-QED. This should also simplify the discussion of the corresponding Slavnov-Taylor identities, and more importantly, it would show that QGR-QED is renormalizable in a Hopf algebraic sense.

A Feynman Graph Generation Programs

The Feynman graphs needed for this thesis were generated in two steps. First, the various topologies of the scalar QFT Feynman graphs were created using the program "feyngen" by M. Borinsky [16]. Then, the corresponding QGR-QED Feynman graphs were created through seven small Python [17] programs written by the author.

A.1 Generation of the scalar QFT Feynman Graphs

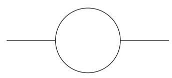
The topologies for the scalar Feynman diagrams were created using the program "feyngen" by M. Borinsky. In the following, we list the corresponding commands and graphs:

A.1.1 One-Loop Propagator Graph Γ_1

The one-loop propagator graphs were created using the command:

```
./feyngen -p --phi34 -j2 1
```

We obtain the following graph

$$\Gamma_1 :=$$  . (193)

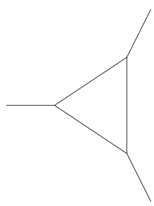
Furthermore, there is also an additional pure self-loop graph which we ignore, since its amplitude vanishes during the renormalization process.

A.1.2 One-Loop Three-Point Graphs Γ_2 and Γ_3

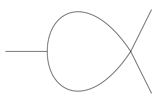
The one-loop three-point graphs were created using the command:

```
./feyngen -p --phi34 -j3 1
```

We obtain the following graphs

$$\Gamma_2 :=$$  (194)

and

$$\Gamma_3 :=$$  (195)

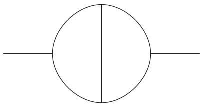
Moreover, the last graph comes with two additional twists.

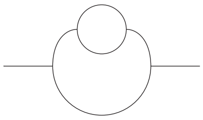
A.1.3 Two-Loop Propagator Graphs Γ_4, Γ_5, Γ_6 and Γ_7

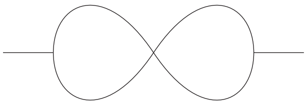
And the two-loop propagator graphs were created using the command:

```
./feyngen -p --phi34 -j2 2
```

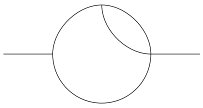
We obtain the following graphs

$\Gamma_4 :=$  , (196)

$\Gamma_5 :=$  , (197)

$\Gamma_6 :=$  (198)

and

$\Gamma_7 :=$  (199)

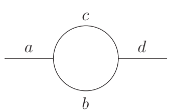
Furthermore, there are also four additional graphs whose coproduct structure consists only of pure self-loop graphs which we ignore, since their amplitude vanishes during the renormalization process. Moreover, the last graph comes also mirrored.

A.2 Generation of the QGR-QED Feynman Graphs

Now, we present the seven Python [17] programs, all written by the author, to obtain all possible QGR-QED labelings for the seven scalar QFT Feynman graphs of Subsection A.1. The idea behind these programs is the following: Since we restrict ourselves to at most four-valent vertices in this thesis, we assign to every particle type a power of 4. Thus, when we sum the weights of all particle types of a vertex, we obtain an injective function from vertex types to the natural numbers $\mathbb{N}$. Therefore, in the program we first create a list of all possible labelings and then pick all allowed labelings by checking that every vertex is in the vertex residue of QGR-QED. Finally, we check for possible isomorphic graphs in the set. We even consider graphs as isomorphic when they are isomorphic in the usual sense combined with an action of the corresponding Dieder group, by introducing corresponding multiplicities. Additionally, we consider a labeling of the edges of the graphs, as shown in Equations (200) to (206). Then, the output of the programs are mappings from the corresponding edge labels $\{ a, b, c, d(, e, f, g) \}$ to the set $\{ 1, 2, 3, 4, 5 \}$, where the numbers correspond to the edge residues of QGR-QED, given in Equation (131b), in the following way:

Number	Assigned particle type
1	Fermion
2	Photon
3	Graviton
4	Photon Ghost
5	Graviton Ghost

A.2.1 Labeling of the Graph Γ_1

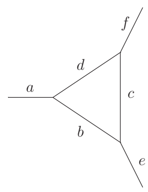
$\Gamma_1 :=$ 

(200)

```python
r=5 #number of particles
interactions = [6,18,22,24,34,40,48,64,144,160,324,340,528,544] #residue set of QGR-QED
j=0
m=0
n=0

combinations = [[0]*5 for i in range(r**4)] #create the powerset of all labelings
for a in xrange(0, r):
  for b in xrange(0, r):
    for c in xrange(0, r):
      for d in xrange(0, r):
          combinations[j]=[a,b,c,d,0]
          v = 4**a+4**b+4**c
          w = 4**b+4**c+4**d
          if ((v in interactions) and (w in interactions)): #set the last variable to 1 if the labeling is allowed
            combinations[j]=[a,b,c,d,1]
            j=j+1

for k in xrange(0, r**4): #check for isomorphic labelings
  if combinations[k][4]==1:
    a0=combinations[k][0]
    b1=combinations[k][1]
    c2=combinations[k][2]
    d3=combinations[k][3]
    n=n+1
    print "%i.) a=%i, b=%i, c=%i, d=%i" % (n, combinations[k][0], combinations[k][1], combinations[k][2], combinations[k][3])
    for l in xrange(k, r**4):
      if ((combinations[l][4]==1) and (((a0==combinations[l][0]) and (d3==combinations[l][3])) or ((a0==combinations[l][3]) and (d3==combinations[l][0]))) and (((b1==combinations[l][1]) and (c2==combinations[l][2])) or ((b1==combinations[l][2]) and (c2==combinations[l][1])))):
        combinations[l][4]=0

print "In total %i options, and %i non-isomorphic ones." % (j, n) #print the result; 0 := fermion edge, 1 := photon edge, 2 := graviton edge, 3 := photon ghost edge and 4 := graviton ghost edge
```

Listing 1: Labeling of the graph Γ_1

A.2.2 Labeling of the Graph Γ_2

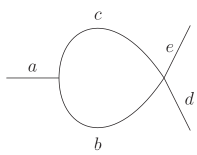
$\Gamma_2 :=$  (201)


```python
r=5 #number of particles
interactions = [6,18,22,24,34,40,48,64,144,160,324,340,528,544] #residue set of QGR-QED
j=0
m=0
n=0

combinations = [[0]*7 for i in range(r**6)] #create the powerset of all labelings
for a in xrange(0, r):
  for b in xrange(0, r):
```

```
    for c in xrange(0, r):
      for d in xrange(0, r):
        for e in xrange(0, r):
          for f in xrange(0, r):
            combinations[j]=[a,b,c,d,e,f,0]
            v = 4**a+4**b+4**d
            w = 4**b+4**c+4**e
            x = 4**c+4**d+4**f
            if ((v in interactions) and (w in interactions) and (x in interactions)): #set the last variable to 1 if the labeling is allowed
              combinations[j][6]=1
              j=j+1

for k in xrange(0, r**6): #check for isomorphic labelings
  if combinations[k][6]==1:
    a0=combinations[k][0]
    b1=combinations[k][1]
    c2=combinations[k][2]
    d3=combinations[k][3]
    e4=combinations[k][4]
    f5=combinations[k][5]
    n=n+1
    print "%i.) a=%i, b=%i, c=%i, d=%i, e=%i, f=%i" % (n, combinations[k][0], combinations[k][1], combinations[k][2], combinations[k][3], combinations[k][4], combinations[k][5])
    for l in xrange(k, r**6):

      if ((combinations[l][6]==1) and (((a0==combinations[l][0]) and (b1==combinations[l][3]) and (c2==combinations[l][2]) and (d3==combinations[l][1]) and (e4==combinations[l][5]) and (f5==combinations[l][4])) or ((a0==combinations[l][4]) and (b1==combinations[l][1]) and (c2==combinations[l][3]) and (d3==combinations[l][2]) and (e4==combinations[l][0]) and (f5==combinations[l][5])) or ((a0==combinations[l][4]) and (b1==combinations[l][2]) and (c2==combinations[l][3]) and (d3==combinations[l][1]) and (e4==combinations[l][5]) and (f5==combinations[l][0])) or ((a0==combinations[l][5]) and (b1==combinations[l][2]) and (c2==combinations[l][1]) and (d3==combinations[l][3]) and (e4==combinations[l][4]) and (f5==combinations[l][0])) or ((a0==combinations[l][5]) and (b1==combinations[l][3]) and (c2==combinations[l][1]) and (d3==combinations[l][2]) and (e4==combinations[l][0]) and (f5==combinations[l][4])))):
        combinations[l][6]=0

print "In total %i options, and %i non-isomorphic ones." % (j, n) #print the result; 0 := fermion edge, 1 := photon edge, 2 := graviton edge, 3 := photon ghost edge and 4 := graviton ghost edge
```

Listing 2: Labeling of the graph Γ_2

A.2.3 Labeling of the Graph Γ_3

$\Gamma_3 :=$  (202)

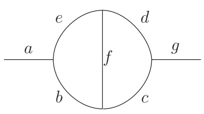

```
r=5 #number of particles
interactions = [6,18,22,24,34,40,48,64,144,160,324,340,528,544] #residue set of QGR-QED
j=0
m=0
n=0
```

```python

combinations = [[0]*6 for i in range(r**5)] #create the powerset of all labelings
for a in xrange(0, r):
  for b in xrange(0, r):
    for c in xrange(0, r):
      for d in xrange(0, r):
        for e in xrange(0, r):
          combinations[j]=[a,b,c,d,e,0]
          v = 4**a+4**b+4**c
          w = 4**b+4**c+4**d+4**e
          if ((v in interactions) and (w in interactions)): #set the last variable to 1 if the labeling is allowed
            combinations[j]=[a,b,c,d,e,1]
            j=j+1

for k in xrange(0, r**5): #check for isomorphic labelings
  if combinations[k][5]==1:
    a0=combinations[k][0]
    b1=combinations[k][1]
    c2=combinations[k][2]
    d3=combinations[k][3]
    e4=combinations[k][4]
    n=n+1
    print "%i.) a=%i, b=%i, c=%i, d=%i, e=%i" % (n, combinations[k][0], combinations[k][1], combinations[k][2], combinations[k][3], combinations[k][4])
    for l in xrange(k, r**5):
      if ((combinations[l][5]==1) and (a0==combinations[l][0]) and (((b1==combinations[l][1]) and (c2==combinations[l][2])) or (b1==combinations[l][2]) and (c2==combinations[l][1])) and (((d3==combinations[l][3]) and (e4==combinations[l][4])) or ((d3==combinations[l][4]) and (e4==combinations[l][3])))):
        combinations[l][5]=0

print "In total %i options, and %i non-isomorphic ones." % (j, n) #print the result; 0 := fermion edge, 1 := photon edge, 2 := graviton edge, 3 := photon ghost and 4 := graviton ghost edge
```

Listing 3: Labeling of the graph Γ_3

A.2.4 Labeling of the Graph Γ_4

$\Gamma_4 :=$



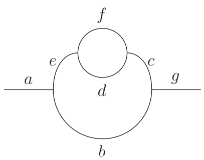
(203)

```python
r=5 #number of particles
interactions = [6,18,22,24,34,40,48,64,144,160,324,340,528,544] #residue set of QGR-QED
j=0
m=0
n=0

combinations = [[0]*8 for i in range(r**7)] #create the powerset of all labelings
for a in xrange(0, r):
  for b in xrange(0, r):
    for c in xrange(0, r):
      for d in xrange(0, r):
        for e in xrange(0, r):
          for f in xrange(0, r):
            for g in xrange(0, r):
```

```
                combinations[j]=[a,b,c,d,e,f,g,0]
                v = 4**a+4**b+4**e
                w = 4**b+4**c+4**f
                x = 4**c+4**d+4**g
                y = 4**d+4**e+4**f
                if ((v in interactions) and (w in interactions) and (x in interactions) and (y in interactions)): #set the last variable to 1 if the labeling is allowed
                  combinations[j][7]=1
                  j=j+1

for k in xrange(0, r**7): #check for isomorphic labelings
  if combinations[k][7]==1:
    a0=combinations[k][0]
    b1=combinations[k][1]
    c2=combinations[k][2]
    d3=combinations[k][3]
    e4=combinations[k][4]
    f5=combinations[k][5]
    g6=combinations[k][6]
    n=n+1
    print "%i.) a=%i, b=%i, c=%i, d=%i, e=%i, f=%i, g=%i" % (n, combinations[k][0], combinations[k][1], combinations[k][2], combinations[k][3], combinations[k][4], combinations[k][5], combinations[k][6])
    for l in xrange(k, r**7):
      if ((combinations[l][7]==1) and (((a0==combinations[l][0]) and (b1==combinations[l][4]) and (c2==combinations[l][3]) and (d3==combinations[l][2]) and (e4==combinations[l][1]) and (g6==combinations[l][6])) or ((a0==combinations[l][6]) and (b1==combinations[l][2]) and (c2==combinations[l][1]) and (d3==combinations[l][4]) and (e4==combinations[l][3]) and (g6==combinations[l][0])) or ((a0==combinations[l][6]) and (b1==combinations[l][3]) and (c2==combinations[l][4]) and (d3==combinations[l][1]) and (e4==combinations[l][2]) and (g6==combinations[l][0]))) and (f5==combinations[l][5])):
        combinations[l][7]=0

print "In total %i options, and %i non-isomorphic ones." % (j, n) #print the result; 0 := fermion edge, 1 := photon edge, 2 := graviton edge, 3 := photon ghost edge and 4 := graviton ghost edge
```

Listing 4: Labeling of the graph Γ_4

A.2.5 Labeling of the Graph Γ_5

$\Gamma_5 :=$  (204)

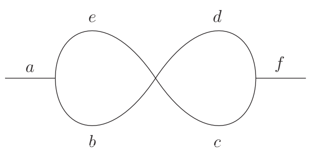

```
r=5 #number of particles
interactions = [6,18,22,24,34,40,48,64,144,160,324,340,528,544] #residue set of QGR-QED
j=0
m=0
n=0

combinations = [[0]*8 for i in range(r**7)] #create the powerset of all labelings
for a in xrange(0, r):
  for b in xrange(0, r):
    for c in xrange(0, r):
```

```
      for d in xrange(0, r):
        for e in xrange(0, r):
          for f in xrange(0, r):
            for g in xrange(0, r):
              combinations[j]=[a,b,c,d,e,f,g,0]
              v = 4**a+4**b+4**e
              w = 4**b+4**c+4**g
              x = 4**c+4**d+4**f
              y = 4**d+4**e+4**f
              if ((v in interactions) and (w in interactions) and (y in interactions) and (x in interactions)): #set the last variable to 1 if the labeling is allowed
                combinations[j][7]=1
                j=j+1

for k in xrange(0, r**7): #check for isomorphic labelings
  if combinations[k][7]==1:
    a0=combinations[k][0]
    b1=combinations[k][1]
    c2=combinations[k][2]
    d3=combinations[k][3]
    e4=combinations[k][4]
    f5=combinations[k][5]
    g6=combinations[k][6]
    n=n+1
    print "%i.) a=%i, b=%i, c=%i, d=%i, e=%i, f=%i, g=%i" % (n, combinations[k][0], combinations[k][1], combinations[k][2], combinations[k][3], combinations[k][4], combinations[k][5], combinations[k][6])
    for l in xrange(k, r**7):
      if ((combinations[l][7]==1) and (((a0==combinations[l][0]) and (c2==combinations[l][2]) and (e4==combinations[l][4]) and (g6==combinations[l][6])) or ((a0==combinations[l][6]) and (c2==combinations[l][4]) and (e4==combinations[l][2]) and (g6==combinations[l][0]))) and (b1==combinations[l][1]) and (((d3==combinations[l][3]) and (f5==combinations[l][5])) or ((d3==combinations[l][5]) and (f5==combinations[l][3])))):
        combinations[l][7]=0

print "In total %i options, and %i non-isomorphic ones." % (j, n) #print the result; 0 := fermion edge, 1 := photon edge, 2 := graviton edge, 3 := photon ghost edge and 4 := graviton ghost edge
```

Listing 5: Labeling of the graph Γ_5

A.2.6 Labeling of the Graph Γ_6

$\Gamma_6 :=$



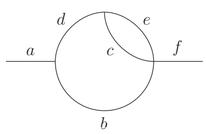
(205)

```
r=5 #number of particles
interactions = [6,18,22,24,34,40,48,64,144,160,324,340,528,544] #residue set of QGR-QED
j=0
m=0
n=0

combinations = [[0]*7 for i in range(r**6)] #create the powerset of all labelings
for a in xrange(0, r):
  for b in xrange(0, r):
    for c in xrange(0, r):
```

```
        for d in xrange(0, r):
          for e in xrange(0, r):
            for f in xrange(0, r):
              combinations[j]=[a,b,c,d,e,f,0]
              v = 4**a+4**b+4**e
              w = 4**b+4**c+4**d+4**e
              x = 4**c+4**d+4**f
              if ((v in interactions) and (w in interactions) and (x in interactions
    )): #set the last variable to 1 if the labeling is allowed
                combinations[j][6]=1
                j=j+1

for k in xrange(0, r**6): #check for isomorphic labelings
  if combinations[k][6]==1:
    a0=combinations[k][0]
    b1=combinations[k][1]
    c2=combinations[k][2]
    d3=combinations[k][3]
    e4=combinations[k][4]
    f5=combinations[k][5]
    n=n+1
    print "%i.) a=%i, b=%i, c=%i, d=%i, e=%i, f=%i" % (n, combinations[k][0],
    combinations[k][1], combinations[k][2], combinations[k][3], combinations[k
    ][4], combinations[k][5])
    for l in xrange(k, r**6):
      if ((combinations[l][6]==1) and (((a0==combinations[l][0]) and (((b1==
    combinations[l][1]) and (e4==combinations[l][4])) or ((b1==combinations[l][4])
     and (e4==combinations[l][1]))) and (((c2==combinations[l][2]) and (d3==
    combinations[l][3])) or ((c2==combinations[l][3]) and (d3==combinations[l][2])
    )) and (f5==combinations[l][5])) or ((a0==combinations[l][5]) and (((b1==
    combinations[l][2]) and (e4==combinations[l][3])) or ((b1==combinations[l][3])
     and (e4==combinations[l][2]))) and (((c2==combinations[l][1]) and (d3==
    combinations[l][4])) or ((c2==combinations[l][4]) and (d3==combinations[l][1])
    )) and (f5==combinations[l][0])))):
        combinations[l][6]=0

print "In total %i options, and %i non-isomorphic ones." % (j, n) #print the
    result; 0 := fermion edge, 1 := photon edge, 2 := graviton edge, 3 := photon
    ghost edge and 4 := graviton ghost edge
```

Listing 6: Labeling of the graph Γ_6

A.2.7 Labeling of the Graph Γ_7

$\Gamma_7 :=$ 

(206)

```
r=5 #number of particles
interactions = [6,18,22,24,34,40,48,64,144,160,324,340,528,544] #residue set of
    QGR-QED
j=0
m=0
n=0

combinations = [[0]*7 for i in range(r**6)] #create the powerset of all labelings
for a in xrange(0, r):
  for b in xrange(0, r):
    for c in xrange(0, r):
      for d in xrange(0, r):
```

```
          for e in xrange(0, r):
            for f in xrange(0, r):
              combinations[j]=[a,b,c,d,e,f,0]
              v = 4**a+4**b+4**d
              w = 4**b+4**c+4**e+4**f
              x = 4**c+4**d+4**e
              if ((v in interactions) and (w in interactions) and (x in interactions
    )): #set the last variable to 1 if the labeling is allowed
                combinations[j][6]=1
                j=j+1

for k in xrange(0, r**6): #check for isomorphic labelings
  if combinations[k][6]==1:
    a0=combinations[k][0]
    b1=combinations[k][1]
    c2=combinations[k][2]
    d3=combinations[k][3]
    e4=combinations[k][4]
    f5=combinations[k][5]
    n=n+1
    print "%i.) a=%i, b=%i, c=%i, d=%i, e=%i, f=%i" % (n, combinations[k][0],
    combinations[k][1], combinations[k][2], combinations[k][3], combinations[k
    ][4], combinations[k][5])
    for l in xrange(k, r**6):
      if ((combinations[l][6]==1) and (a0==combinations[l][0]) and (b1==
    combinations[l][1]) and (c2==combinations[l][4]) and (d3==combinations[l][3])
    and (e4==combinations[l][2]) and (f5==combinations[l][5])):
        combinations[l][6]=0

print "In total %i options, and %i non-isomorphic ones." % (j, n) #print the
    result; 0 := fermion edge, 1 := photon edge, 2 := graviton edge, 3 := photon
    ghost edge and 4 := graviton ghost edge
```

Listing 7: Labeling of the graph Γ_7

B Statement of Authorship

I hereby declare that I noticed the 2014 version of the "Studien- und Prüfungsordnung für den Masterstudiengang Mathematik". The present master thesis and the results therein are worked out by me and only me, if not indicated otherwise. I used nothing but the indicated references in Appendix C. Furthermore, this master thesis is the first which is submitted by me personally in the subject "Mathematik" and has not been submitted to any other scientific institution.

Berlin, April 10, 2017:

David Prinz

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