Moduli Spaces of Colored Feynman Graphs

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Abstract

Determining the analytical structure of Feynman integrals is a time-honored problem. Recent considerations have brought to attention that certain moduli spaces of graphs derived from Outer space may hold interesting insights into this matter. This work adds the component of colored edges to these moduli spaces to function as a placeholder for additional physical data. The central part of this work is the explicit calculation of the rational homology of these spaces for the one-loop case with computer assistance. There are many ways to set up such moduli spaces. Three possibilities are explored: Arbitrary colored graphs, holo-colored graphs and the latter with the additional feature of edges remembering their color upon shrinking them to zero length.

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Contents

Chapter 1

Introduction

For decades physicists have tried to understand the analytic structure of Feynman integrals arising in quantum field theory. Many techniques used in practical calculations have yet to be put on rigorous mathematical footing. For example the famous Cutkosky theorem introduced in [Cut60] in the year 1960, relating the imaginary part of Feynman amplitudes to the integral of a graph with edges put on mass-shell, was just recently proven in [DK15]. This paper mentions the idea of studying Outer space and closely related spaces to gain new insights into the properties of Feynman integrals.
Outer space X_n is a topological space that arises in geometrical group theory, used by mathematicians to study the automorphisms of the free group F_n. It is a space in which points are equivalence classes of marked metric graphs and it comes with a group action of $\mathrm{Out}(F_n)$ which changes the markings. Furthermore it contains a deformation retract called the spine of Outer space, which can be naturally grouped into cubes. This leaves a cubical complex in which cubes are represented by pairs of a graph and a spanning forest. This complex captures the combinatorial structure of Cutkosky cut and reduced graphs.
Physicists usually consider graphs with additional structure. They distinguish between different masses or particle types assigned to graph edges. Such additional data can be represented by assigning each edge of a graph a color as a placeholder. This gives rise to slightly generalized spaces that also come with the structure of a cubical complex. This thesis aims to perform the first step for a better understanding of the connection between Feynman amplitudes and moduli spaces of graphs by studying the homology groups of these spaces.
The results of this work are mainly focused on the one-loop case with coefficients in $\mathbb{Q}$. Generators for the homology of non-colored moduli space in this case are calculated explicitly up to five external legs with computer assistance. A set of generators for the highest dimensional non-trivial homology groups is obtained for an arbitrary number of external edges.
Three versions of moduli spaces of colored graphs are under consideration: A space which allows for arbitrary colorings and two spaces of holo-colored graphs in which each edge has a different color, one of which has the additional feature that edges keep the information of the color when shrunken to zero length. The dimensions of their homology groups are also calculated with computer assistance for small numbers of external legs and colors, a specific choice for the generators is given. For the complexes of holo-colored graphs, the dimension of the highest non-trivial homology group is calculated for all one-loop spaces.
The next chapter provides the fundamentals from graph theory, topology, and group theory this thesis makes use of and sets up the necessary notations. Chapter 3 introduces homology theory and a collection of tools necessary for the calculation of homology groups. In chapter 4 Outer space is formally introduced and its most important properties are listed. Several examples are considered in detail. Furthermore, a method used for the computer assisted calculation of the homology groups is described and its results on the non-colored moduli

spaces displayed. Chapter 5 introduces colored graphs and three versions of a colored moduli space of graphs. For each of these the results for the homology groups are displayed. Additionally, the properties of canonically arising maps between the arbitrarily colored complex and the non-colored complex are explored.

Chapter 2

Fundamentals

The following part introduces the necessary mathematical preliminaries. While the subsequent chapter contains the required information on algebraic topology in more detail, this chapter gives a quick overview on graphs, basic topology, and some group theory. First the definition of a graph used throughout the thesis is stated and the frequently occurring notations are introduced. This is followed by a brief review of point set topology with some notes on the fundamental group. The last section introduces the group theoretical machinery needed in this work.

2.1 Graphs

Graphs are very versatile mathematical objects that arise in a variety of fields including discrete mathematics, computer science, and quantum field theory. They prominently arise in the perturbative approach to the latter in form of Feynman graphs, which represent integrals contributing to probability amplitudes in scattering processes (see for example [Wei95]).

There are various possibilities to define a graph, each suited for different purposes. Physicists often use a definition based on half-edges, which allows for a distinction between internal and external edges as needed for Feynman graphs.[1] The definitions set up in this section mainly follow [Ber17].

Definition 2.1.1. *A graph $\Gamma = (V_\Gamma, H_\Gamma, s_\Gamma, c_\Gamma)$ is a 4-tuple, consisting of a set of vertices V_Γ, a set of half-edges H_Γ, a map $s_\Gamma : H_\Gamma \to V_\Gamma$ which connects each half-edge to a source vertex, and a map $c_\Gamma : H_\Gamma \to H_\Gamma$, satisfying $c_\Gamma^2 = 1$, connecting the half-edges with each other.*

Let Γ be a graph, $h_1, h_2 \in H_\Gamma$ two half-edges with $c_\Gamma(h_1) = h_2$. If $h_1 \neq h_2$, the pair $\{h_1, h_2\}$ is called an internal edge of Γ. If $h_1 = h_2$, the half-edge is called an external leg or external edge. We denote the set of internal edges by E_Γ^{int}, the set of external edges by E_Γ^{ext}.

A subgraph $\gamma \subseteq \Gamma$ is a graph such that $V_\gamma \subseteq V_\Gamma$, $H_\gamma \subseteq H_\Gamma$ and $s_\gamma = s_\Gamma|_{H_\gamma}$, $c_\gamma = c_\Gamma|_{H_\gamma}$. If there are no external legs, i.e. $E_\gamma^{ext} = \emptyset$, γ is called an internal subgraph.

The following notations occur frequently throughout the thesis:

- The number of connected components of Γ is denoted by $h_0(\Gamma)$ (also called the zeroth Betti number).

[1] In mathematical textbooks the word graph often refers to objects that allow for at most one edge between each pair of vertices. In this thesis we do not demand this restriction and work with what is often called multigraphs.

- The number of loops in Γ is denoted by $|\Gamma|$ or $h_1(\Gamma)$ (also called the first Betti number).
- For any vertex $v \in V_\Gamma$ we denote its valency by $|v| := |s^{-1}(v)|$.
- Γ is called one-particle irreducible (1PI) or bridge-free if it is connected and still connected upon removal of any edge $e \in E_\Gamma^{int}$. In case Γ is not 1PI, the edges leaving the graph disconnected upon removal are called bridges or separating edges.
- A graph F is called a k-forest if $E_F^{\text{ext}} = \emptyset$, $|F| = 0$ and $h_0(F) = k$. In particular, a 1-forest is called a tree. A subgraph $F \subseteq \Gamma$ is called a spanning k-forest if F is a k-forest and $V_F = V_\Gamma$. In particular, if $k = 1$ then F is said to be a spanning tree.
- A graph with $n \in \mathbb{N}$ internal edges and one single vertex is called a rose with n petals and is denoted by R_n.

If an internal subgraph γ of a graph Γ contains no vertex of valency zero, it is completely determined by the set of internal edges it contains. The vertices of γ in this case are all vertices incident to at least one of the edges.
Graphs as defined in definition 2.1.1 are purely combinatorial objects. To obtain more structure, a graph can be endowed with a length function, assigning each edge a real number greater than or equal to zero.

Definition 2.1.2. *A graph Γ together with a map $\lambda : E_\Gamma^{int} \to \mathbb{R}^+$ is called a metric graph.*

The sum of all edge length $\sum_{e \in E_\Gamma} \lambda(e)$ is called the volume of Γ. The distance between two vertices can then be defined as the shortest path along the edges of the graph connecting them, endowing the set of vertices V_Γ with the structure of a metric space.
The graphs occuring in this text almost exclusively belong to a particularly nice class of graphs.

Definition 2.1.3. *A graph Γ is called admissible if it is 1PI and $|v| \geq 3$ for all $v \in V_\Gamma$.*

The following example is meant to illustrate the previous notions.

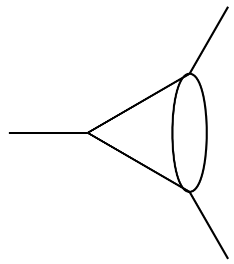
Example 2.1.4. *Consider the graph Γ with $V_\Gamma = \{v_1, v_2, v_3\}$, $H_\Gamma = h_1, ..., h_{11}$, $s : H_\Gamma \to V_\Gamma$ defined by*

$$s(h_i) = \begin{cases} v_1, & \textit{if } 1 \leq i \leq 3 \\ v_2, & \textit{if } 4 \leq i \leq 7 \\ v_3, & \textit{if } 8 \leq i \leq 11 \end{cases}$$

and $c : H_\Gamma \to H_\Gamma$ defined by

i	1	2	3	4	5	6	7	8	9	10	11
$c(h_i)$	h_1	h_5	h_9	h_4	h_2	h_{10}	h_{11}	h_8	h_3	h_6	h_7

The graph Γ can be graphically represented by drawing disjoint points for all vertices in V_Γ and drawing lines between any two points if they are connected by an internal edge. The *external edges are drawn as lines sticking out from the vertices they are attached to.*



This graph is often referred to as the Dunce's cap graph due to its visual resemblance to a pointed hat (when ignoring the external legs). There are four internal edges contained in Γ*, namely*

$$e_1 := \{h_2, h_5\}, \quad e_2 := \{h_3, h_9\}, \quad e_3 := \{h_6, h_{10}\}, \quad e_4 := \{h_7, h_{11}\}.$$

It has two loops and is admissible. The vertex v_1 *has valency* $|v_1| = 3$*, while the vertices* v_2 *and* v_3 *have valency* $|v_2| = |v_3| = 4$*. The graph* Γ *admits five distinct spanning trees* $T_1, \ldots, T_5$ *which are uniquely determined by their set of edges*

$$E_{T_1} = \{e_1, e_2\}, \quad E_{T_2} = \{e_1, e_3\}, \quad E_{T_3} = \{e_1, e_4\}, \quad E_{T_4} = \{e_2, e_3\}, \quad E_{T_5} = \{e_2, e_4\}.$$

There are numerous operations that can be performed on graphs. The following definition establishes the two operations primarily used in this thesis: removing and shrinking edges of a graph.

Definition 2.1.5. *Let* Γ *be a graph,* $\gamma \subseteq \Gamma$ *an internal subgraph.*

1. $\Gamma \backslash \gamma$ *denotes the graph with all edges of* γ *removed, i.e.* $V_{\Gamma\backslash\gamma} = V_\Gamma$ *and* $E^{int}_{\Gamma\backslash\gamma} = E^{int}_\Gamma \backslash E^{int}_\gamma$.
2. *For* γ *connected,* Γ_γ *denotes the graph with* γ *shrunken to a single vertex, i.e.* γ *is replaced by a vertex with all edges attached to both, a vertex in* V_γ *and a vertex in* $V_\Gamma \backslash V_\gamma$*, connected to it. For disconnected* γ*, this operation is defined by shrinking each connected component in this manner.*

The case where the subgraph γ is a forest occurs frequently in this text. In that case the number of loops does not change when shrinking γ, i.e. $|\Gamma| = |\Gamma_\gamma|$. If in particular γ is a spanning tree, then Γ_γ is a rose with $|\Gamma|$ petals.

2.2 Topological Notions

Since the main objective of this work is the calculation of homology groups, mathematical objects assigned to topological spaces, a few topological notions and notations are in order. This section provides a very short overview of general topology. Everything presented here is standard knowledge and the reader is referred to [vQ79] for details.

Definition 2.2.1. *Let* X *be a set,* $\tau \subseteq \mathcal{P}(X)$ *an element of the power set* $\mathcal{P}(X)$ *of* X*.* τ *is called a topology of* X *if*

- *$X, \emptyset \in \tau$*
- *For any collection $\{U_i\}_{i\in I}$ of sets $U_i \in \tau$ their union $\bigcup_{i\in I} U_i$ is an element of τ.*
- *For any finite collection $\{U_i\}_{i\in I}$ of sets $U_i \in \tau$ their intersection $\bigcap_{i\in I} U_i$ is an element of τ.*

The elements of τ are said to be the open sets in X.

A pair (X, τ) of a set X and a topology τ of X is called a topological space. If the specific topology used is obvious or of no relevance a topological space is often just denoted as the set X.
Let A be a set, (X, τ) a topological space, $i : A \hookrightarrow X$ an injective map. Then A can be endowed with a canonical topology τ' given by

$$\tau' = \{V \subseteq A \mid V = i^{-1}(U) \text{ for some open } U \subseteq X\}.$$

This topology is called the subspace topology and in particular for $A \subseteq X$ and $i : A \hookrightarrow X$ the natural inclusion map, the open sets are given by all sets of the form $U \cap A$ with $U \subseteq X$ open.
Given a topological space X and an equivalence relation $\sim$, the set of equivalence classes $X/_\sim := \{[x] \mid x \in X\}$ also comes with a canonical topology. Let $q : X \to X/_\sim$ be the natural quotient projection. A set $U \subseteq X/_\sim$ is defined to be open in this topology if its preimage under the quotient map $q^{-1}(U)$ is open in X. This is called the quotient topology.
One of the most elementary properties a topological spaces can have is connectedness.

Definition 2.2.2. *A topological space X is called connected if there are no two open non-empty sets $U, V \subset X$ such that $U \cup V = X$ and $U \cap V = \emptyset$.*

The most fundamental topological property a map between two topological spaces can have is continuity.

Definition 2.2.3. *Let X, Y be two topological spaces. A map $f : X \to Y$ is called continuous if for all open sets $U \subseteq Y$, the set $f^{-1}(U)$ is open in X.*

There is another notion of connectedness in topology referring to continuous maps.

Definition 2.2.4. *A topological space X is called path-connected if for any $x, y \in X$ there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.*

This is in fact the stronger condition, meaning that path-connectedness implies connectedness of a topological space (see [vQ79], page 69).
As with other mathematical structures like groups or vector spaces, two topological spaces may appear to be different but are essentially the same. In case of groups or vector spaces this is captured by the concept of group or vector space isomorphisms respectively, bijective maps preserving the mathematical structure of the objects under consideration. In topology continuous maps play the role of structure preserving maps and hence the following definition.

Definition 2.2.5. *Let X, Y be topological spaces. A homeomorphism between X and Y is a bijective continuous map $f : X \to Y$ such that f^{-1} is also continuous. If a homeomorphism $X \to Y$ exists, X and Y are said to be homeomorphic which is denoted by $X \cong Y$.*

A common example of a topological space is a metric space (X, d). It can be endowed with the topology of open disks $D_r(x) := \{y \in X \mid d(x, y) < r\}$ (with $x \in X$, $r \in \mathbb{R}$). A set $U \subseteq X$ in this topology is considered to be open if for all $x \in U$ there exists $\epsilon > 0$ such that $D_\epsilon(x) \subseteq U$. For the standard euclidean metric

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ (x, y) \mapsto \sqrt{x_1^2 + \ldots + x_n^2}$$

this is said to be the standard topology on $\mathbb{R}^n$.
Maps from one metric space to another can be required to preserve the distances between points up to a constant scaling factor. The construction of Outer space in chapter 4 will make use of this notion.

Definition 2.2.6. *Let $(X, d_X), (Y, d_Y)$ be two metric spaces. A map $h : X \to Y$ is called a homothety if there exists $\lambda \in \mathbb{R}^+$ such that $d_Y(h(x), h(y)) = \lambda d_X(x, y)$ for all $x, y \in X$. If $\lambda = 1$ then h is called an isometry.*

Given topological spaces X, Y and two continuous maps $f, g : X \to Y$, there is a notion of deforming one map continuously into the other.

Definition 2.2.7. *Let X, Y be two topological spaces, $f, g : X \to Y$ two continuous maps. A map $H : X \times [0, 1] \to Y$ is called a homotopy between f and g if it is continuous and $H(\bullet, 0) = f$, $H(\bullet, 1) = g$. In this case f and g are said to be homotopic.
Let in particular $X = [0, 1]$ (endowed with the subspace topology of $\mathbb{R}$), $f(0) = g(0)$ and $f(1) = g(1)$. Then H is called a homotopy with fixed end points if additionally $H(0, s) = f(0) = g(0)$ and $H(1, s) = f(1) = g(1)$ hold for all $s \in [0, 1]$.*

Homotopy is an equivalence relation since

- Any continuous map $f : X \to Y$ is homotopic to itself by $H : X \times [0, 1] \to Y$ defined by $H(t, s) = f(s)$ for all $t \in [0, 1]$.
- For two continuous maps $f, g : X \to Y$ with f homotopic to g by a homotopy $H : X \times [0, 1] \to Y$, g is homotopic to f by $H' : X \times [0, 1] \to Y$ defined by $H'(t, s) = H(t, -s)$.
- For three continuous maps $f, g, h : X \to Y$ with f homotopic to g by $H_1 : X \times [0, 1] \to Y$, g homotopic to h by $H_2 : X \times [0, 1] \to Y$, f is homotopic to h by $H : X \times [0, 1] \to Y$ defined by
$$H(x, t) := \begin{cases} H_1(x, 2t), & t \in [0, \frac{1}{2}] \\ H_2(x, 2t - 1), & t \in [\frac{1}{2}, 1] \end{cases}.$$

For a topological space X and two points $x, y \in X$ a continuous map $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$ is called a path from x to y. Two paths $f, g : [0, 1] \to X$ with $f(0) = x, f(1) = g(0) = y$, and $g(1) = z$ for $x, y, z \in X$ can be concatenated to a path $f \cdot g$ from x to z by

$$(f \cdot g)(t) := \begin{cases} g(2t), & t \in [0, \frac{1}{2}] \\ f(2t - 1), & t \in [\frac{1}{2}, 1] \end{cases}$$

Concatenation of paths is not associative but it is up to reparametrization of the path (see [Hat02], page 27).

Fixing a point $x_0 \in X$ one can consider all paths from x_0 to itself. Such a path is called a loop based at x_0. Every loop based at x_0 can be concatenated with every other. In fact, their homotopy classes (with respect to homotopy with fixed end points) form a group (again [Hat02], page 27) with the constant path $e_{x_0} : [0,1] \to X$, $t \mapsto x_0$ as a representative of the neutral element and the inverse of a class $[f]$ represented by a loop f is given by $[f^{-1}]$ with $f^{-1} : [0,1] \to X$, $t \mapsto f(1-t)$.

Definition 2.2.8. *Let X be a topological space, $x_0 \in X$ a distinguished point called the base point. The group of homotopy classes with fixed end points of all continuous maps $f : [0,1] \to X$ with $f(0) = f(1) = x_0$ is called the fundamental group of (X, x_0) denoted by $\pi_1(X, x_0)$.*

For any path-connected space X the fundamental group $\pi_1(X, x_0)$ does not depend on the base point x_0, in the sense that $\pi_1(X, x_0) \cong \pi_1(X, x_0')$ for all $x_0, x_0' \in X$(see [Hat02], page 28). Therefore the fundamental group is often simply denoted by $\pi_1(X)$ if the base point is of no relevance.
The fundamental group is a topological invariant in the sense that for two homeomorphic topological spaces X, Y with base points $x_0 \in X$ and a homeomorphism $f : X \to Y$ we have $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$. That does not mean that having isomorphic fundamental groups guarantees the existence of a homeomorhpism between the spaces. In fact two topological spaces can be similar in a weaker sense than being homeomorphic:

Definition 2.2.9. *Let X, Y be two topological spaces. X and Y are said to be homotopy equivalent if there exist continuous maps $f : X \to Y$, $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.*

Homotopy equivalent spaces also yield isomorphic fundamental groups (see [Hat02], page 37). Note that a homeomorphism is a special case of a homotopy equivalence.
There is a particular type of homotopy equivalence referring to subspaces that occurs often in topology:

Definition 2.2.10. *Let X be a topological space, A a subspace of X.*

- *A continuous map $r : X \to A$ is called a retraction if $r|_A = id_A$. In that case A is called a retract of X.*
- *A continuous map $r : X \times [0,1] \to A$ is called a deformation retract onto A if $r(\bullet, 0) = id_X$, $r(x, 1) \in A$ for all $x \in X$ and $r(\bullet, 1)|_A = id_A$.*

If a topological space admits a deformation retract to a single point, the space is called contractible. For example, any metric graph which is a tree can be regarded as a contractible space.

2.3 Group Theoretical Notions

In homology theory, abelian groups are assigned to topological spaces. The following chapters make use of various group theoretical notions that are shortly introduced in this section.

Let $(G, +)$ be an abelian group with neutral element 0. For $g \in G$ and $n \in \mathbb{Z}$ we denote

$$ng := \begin{cases} \underbrace{g + \ldots + g}_{n \text{ times}}, & \text{if } n \geq 1 \\ \underbrace{g^{-1} + \ldots + g^{-1}}_{n \text{ times}}, & \text{if } n \leq -1 \\ 0, & \text{if } n = 0. \end{cases}$$

Abelian groups can be combined by a tensor product to form a new abelian group.

Definition 2.3.1. *Let G, H be abelian groups. Their tensor product $G \otimes H$ is the group of all 2-tuples (g, h) with $g \in G$ and $h \in H$ where addition is defined as*

$$(g, h) + (g', h') := (g + g', h + h')$$

for all $g, g' \in G$ and $h, h' \in H$ subject to the equivalence relations $(g+g', h) \sim (g, h)+(g', h)$ and $(g, h + h') \sim (g, h) + (g, h')$.

Now let G be any group. An automorphism of G is an isomorphism $\phi : G \to G$. The automorphisms of G form a group themselves denoted by $\mathrm{Aut}(G)$. The group of automorphisms contains a subgroup consisting of all conjugations by an element $g \in G$.

Definition 2.3.2. *The group of inner automorphisms $Inn(G) \subseteq Aut(G)$ is the subgroup consisting of all automorphisms $\phi \in Aut(G)$ which act on all $x \in G$ as*

$$\phi(x) = g^{-1}xg$$

for some $g \in G$.

In particular the inner automorphism group of any abelian group is trivial. Taking the quotient by the inner automorphism group leaves the outer automorphisms of a group.

Definition 2.3.3. *The group of outer automorphisms $Out(G)$ of a group G is defined as*

$$Out(G) := Aut(G)/Inn(G).$$

This definition is important to understand the motivation of Outer space (see chapter 4). It is constructed such that the outer automorphism group $\mathrm{Out}(F_n)$ of the free group F_n has a particularly nice action on it.

2.3.1 Group Action

A common way to study groups is to interpret their elements as acting on a set.

Definition 2.3.4. *Let G be a group, X a non-empty set. A left action of G on X is a map $\sigma : G \times X \to X$ satisfying*

- $\sigma(e, x) = x$
- $\sigma(g_1 g_2, x) = \sigma(g_1 \sigma(g_2, x))$

for all $g_1, g_2 \in G$ and all $x \in X$. Analogously a right group action is defined for a map $\sigma : X \times G \to X$.

Typically one considers a set X with additional structure like a topological space or a vector space. A group action on a vector space is known as a representation of the group.
For a fixed $x \in X$, elements that lie in the image of $\sigma(\bullet, x) : G \to X$ are said to be in the orbit of x.

Definition 2.3.5. *Let G be a group, X a non-empty set and $\sigma : G \times X \to X$ a left group action. The orbit $G \cdot x$ of an element $x \in X$ is the set*

$$G \cdot x := \{\sigma(g, x) \mid g \in G\}.$$

Additionally, elements in X can have the property of being fixed by certain group elements acting on them.

Definition 2.3.6. *For $g \in G$ an element $x \in X$ satisfying $\sigma(g, x) = x$ is a fixed point of g. For a fixed $x \in X$ the group elements which have x as a fixed point form a subgroup G_x called the stabilizer of x.*

$$G_x := \{g \in G \mid \sigma(g, x) = x\}$$

Orbits and stabilizers are closely related: For any fixed $x \in X$, there is a natural bijection between G/G_x and $G \cdot x$ given by $g \cdot G_x \mapsto \sigma(g, x)$. This result is known as the orbit-stabilizer theorem (for a proof see [Lan93], pages 27-28).

2.3.2 Finite Symmetric Groups

The symmetric group $S(X)$ over a set X is defined as the group of bijections $X \to X$ with group multiplication given by the composition of these bijections. In particular the finite symmetric group S_n of order $n \in \mathbb{N}$ is defined as the group of bijections from a set of n elements to itself. Let $X := \{1, 2, \ldots, n\}$. An element $\sigma \in S_n$ can in full generality be taken to be a bijective map $\sigma : X \to X$ usually denoted by $\left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{smallmatrix}\right)$, where $p_i := \sigma(i)$ for all $1 \le i \le n$.
Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Elements of the form $\left(\begin{smallmatrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & n-1 & n \\ 1 & 2 & \cdots & j & \cdots & i & \cdots & n-1 & n \end{smallmatrix}\right)$ are called transpositions, denoted by $[i, j]$. Every element $\sigma \in S_n$ is the composition of transpositions. Such a decomposition is generally not unique, but the number of transpositions occurring in it modulo 2 depends only on σ. If this number is 0 then σ is called an even element, otherwise σ is called odd. The signum $|\sigma|$ of the element σ is defined by

$$|\sigma| := \begin{cases} 1, & \text{if } \sigma \text{ is even} \\ -1, & \text{otherwise} \end{cases}$$

For any $\left(\begin{smallmatrix} 1 & 2 & \cdots & n-1 & n \\ p_1 & p_2 & \cdots & p_{n-1} & p_n \end{smallmatrix}\right) \in S_n$ we denote a cyclic permutation of elements by

$$\tau_+ \left(\begin{smallmatrix} 1 & 2 & \cdots & n-1 & n \\ p_1 & p_2 & \cdots & p_{n-1} & p_n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & \cdots & n-1 & n \\ p_2 & p_3 & \cdots & p_n & p_1 \end{smallmatrix}\right).$$

The symmetric groups contain a lot of interesting subgroups.[2] A particular subgroup of S_n that is useful for representing the external leg structure of one-loop graphs in chapter 4 and 5 is the following. We let $n \geq 3$ and define

$$C_n \ltimes \mathbb{Z}_2 := \{\sigma \in S_n \mid \exists k \in \mathbb{N} \text{ s.t. } \sigma = \tau_+^k \left(\begin{smallmatrix} 1 & 2 & \dots & s-2 & s-1 \\ 1 & 2 & \dots & s-2 & s-1 \end{smallmatrix}\right) \text{ or } \sigma = \tau_+^k \left(\begin{smallmatrix} 1 & 2 & \dots & s-2 & s-1 \\ 1 & s-1 & \dots & 3 & 2 \end{smallmatrix}\right)\}.$$

This is even a normal subgroup as can be readily verified, hence $S_n/(C_n \ltimes \mathbb{Z}_2)$ is well-defined.[3]

2.3.3 Free Groups

Given any set X, a word w with letters in X is a finite formal sequence $w = x_1^{j_1} \dots x_n^{j_n}$ of elements $x_1, \dots, x_n \in X$. If $x_i \neq x_{i+1}$ for all $1 \leq i \leq n$ and $j_i \neq 0$ for all $1 \leq i \leq n$, w is called a reduced word. Every word can be reduced by a finite number of steps by subsequently replacing terms of the form $x_i^{j_i} x_{i+1}^{j_{i+1}}$ by $x_i^{j_i + j_{i+1}}$ if $x_i = x_{i+1}$.

Definition 2.3.7. *Let X be a set. The free group $F(X)$ generated by X is the group of all reduced words w with letters in X. Group multiplication is defined by concatenation and subsequent reduction and the neutral element is the empty word denoted by 1.*

If two sets X and X' have the same cardinality, the free groups $F(X)$ and $F(X')$ generated by these sets are isomorphic (see [Lan93], page 68). If a set X is finite and contains $n \in \mathbb{N}$ elements, $F(X)$ is called the free group on n generators and denoted by F_n. Any group isomorphic to some free group is called free.

Example 2.3.8. *The fundamental group $\pi_1(R_n)$ of the rose with n petals R_n is the free group on n generators. In particular $\pi_1(S^1 \cong R_1) \cong \mathbb{Z}$.*

Elements of the outer automorphism group of a free group are called cyclic words.
A free group with more than one generator is not abelian. There is, however, a notion of a free abelian group.

Definition 2.3.9. *Let X be a set. The free abelian group generated by X is the group of all formal sums $\sum_{x \in X} n_x x$ with $n_x \in \mathbb{Z}$ and only finitely many of the n_x non-zero. The addition of two elements $\sum_{x \in X} m_x x$ and $\sum_{x \in X} n_x x$ is defined by*

$$(\sum_{x \in X} m_x x) + (\sum_{x \in X} n_x x) := \sum_{x \in X} (m_x + n_x) x.$$

The other way around, if there exists a subset $B \subset G$ for a group G such that every element in G can be written as $\sum_{b \in B} n_b b$ with unique coefficients $n_b \in \mathbb{Z}$, then G is said to be a free abelian group.
The free abelian group generated by a set X is isomorphic to $\bigoplus_{x \in X} \mathbb{Z}$ (see [Lan93], pages 38-39).

[2] In fact any group G is isomorphic to a subgroup of $S(G)$. This statement is known as Cayley's theorem, for which a proof can be found for example in [Jac74], page 38. In particular, any finite group of order n is a subgroup of S_n.

[3] The mentioned subgroup is actually a semi-direct product of C_n and $\mathbb{Z}_2$. In this text only this particular semi-direct product will occur. To avoid the introduction of unnecessary theory, the subgroup is given by explicitly stating the elements it contains.

Chapter 3

Algebraic Topology

This section introduces all necessary notions of algebraic topology. In the first part a general definition of homology groups derived from chain complexes is given and their required properties are stated. The second part introduces three particular types of homology theory that are utilized in this work to make explicit calculations. Furthermore the last part provides a definition and some examples of exact sequences, one of the most important tools in algebraic homology.

3.1 Chain Complexes and Homology

A common method to examine a topological space is the calculation of topological invariants, mathematical objects associated to the space that are invariant under homeomorphisms. This can be a number like the famous Euler characteristic or the number of connected components of the space but also more complex objects like groups or modules. This thesis is mainly concerned with homology groups, a sequence of abelian groups that are such invariants. This section gives a short overview of the notions needed in this thesis.
Homology groups are defined via a chain complex, a sequence of groups and homomorphisms assigned to a topological space in a way that encodes the topological information.

Definition 3.1.1. *A collection of abelian groups $\{C_n\}_{n\in\mathbb{N}}$ together with a sequence of homomorphisms $\partial_n : C_n \longrightarrow C_{n-1}$ is called a chain complex $(C_\bullet, \partial_\bullet)$ if $\partial_n \circ \partial_{n+1} \equiv 0 \quad \forall n \in \mathbb{Z}$. The maps ∂_n are said to be the boundary morphisms and the chain complex is written as*

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots \tag{3.1}$$

The group C_n is called the chain group of dimension n, its elements are referred to as n-chains.

Elements in $\ker \partial_n$ are said to be n-cycles, while elements in $\operatorname{im} \partial_n$ are called n-boundaries. Heuristically speaking the general idea of homology is to look for cycles which are not boundaries. In this way, the nth homology group detects n-dimensional holes in a topological space.
Given any two groups A, B and a homomorphism $f : A \to B$, the kernel of f is a subgroup of A and the image of f is a subgroup of B. Hence for a chain complex $(C_\bullet, \partial_\bullet)$, the condition $\partial_n \circ \partial_{n+1} \equiv 0$ translates to $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$ for all $n \in \mathbb{Z}$, i.e. the image of ∂_{n+1} is a subgroup of $\ker \partial_n$. Since the chain groups C_n are all abelian and any subgroup of an abelian group is abelian itself, $\operatorname{im} \partial_{n+1}$ is in fact a normal subgroup of the kernel of ∂_n. This means $\ker \partial_n / \operatorname{im} \partial_{n+1}$ is a well-defined group and allows for the following definition:

Definition 3.1.2. *Let $(C_\bullet, \partial_\bullet)$ be a chain complex with $C_n = 0$ for all $n < 0$. The nth homology group H_n of $(C_\bullet, \partial_\bullet)$ is defined as*

$$H_n(C_\bullet, \partial_\bullet) := \ker \partial_n / \operatorname{im} \partial_{n+1} \tag{3.2}$$

In particular this means that cycles which are also boundaries are considered to be a trivial element in the corresponding homology group. Equation (3.2) can be restated as the homology group H_n being the group of all n-cycles subject to the equivalence relation $c_1 \sim c_2 :\iff c_1 - c_2 \in \operatorname{im} \partial_{n+1}$.
Chain maps are a particularly important class of maps between chain complexes.

Definition 3.1.3. *Let $(C_\bullet, \partial_\bullet)$, $(C'_\bullet, \partial'_\bullet)$ be two chain complexes, $f : C_\bullet \to C'_\bullet$ a map such that each $f_n := f|_{C_n}$ is a homomorphisms. f is called a chain map if the following diagram commutes for all n:*

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"'] \\ C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \end{tikzcd}$$

If each f_n is even an isomorphism, f is said to be a chain map isomorphism.

Chain maps have the property that they map elements in the kernel and image of $\partial_\bullet$ to elements in the kernel and image of $\partial'_\bullet$ respectively (see [PJH97], pages 24-25 or [Hat02], page 111). Thus a chain map $f : C_\bullet \to C'_\bullet$ descends to a homomorphism $f_* : H_\bullet(C_\bullet, \partial_\bullet) \to H_\bullet(C'_\bullet, \partial'_\bullet)$ defined by

$$f_*([x]) = [f(x)].$$

Furthermore there is a notion of homology with coefficients in a group. Let G be an abelian group, $(C_\bullet, \partial_\bullet)$ a chain complex. The corresponding chain complex $(C_\bullet(G), \partial_\bullet)$ with coefficients in G is defined by the chain groups $C_n(G) := C_n \otimes G$ for all $n \in \mathbb{Z}$ and the boundary morphism $\partial(x \otimes g) := \partial x \otimes g$ for all $x \in C_\bullet$ and $g \in G$.
A homology theory assigns a chain complex to a topological space and uses this complex to define the homology groups. We write $C_\bullet(X)$ to indicate the chain groups corresponding to a topological space X.
Sometimes it is useful to consider a chain complex of a topological space relative to a subspace.

Definition 3.1.4. *Let X be a topological space, $A \subseteq X$. The relative chain complex $(C_\bullet(X, A), \partial_\bullet)$ is defined by*

$$C_n(X, A) := C_n(X)/C_n(A).$$

The homology of the relative chain complex is called the relative homology, denoted by $H_\bullet(X, A)$.

A relative chain complex can also be endowed with coefficients in an abelian group G which is denoted by $C_\bullet(X, A; G)$, its homology groups by $H_\bullet(X, A; G)$.
In addition there is a notion of reduced homology. If

$$\dots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

is a chain complex with homology groups H_n, the chain complex can be slightly enlarged to

$$\dots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

with $\epsilon(\sum_i n_i\sigma_i) := \sum_i n_i$ for all $\sum_i n_i\sigma_i \in C_0$. Then the reduced homology groups $\tilde{H}_n$ are defined as

$$\tilde{H}_n := \begin{cases} \ker\partial_n / \operatorname{im}\partial_{n+1} = H_n, & \text{if } n \geq 1 \\ \ker\partial_0 / \operatorname{im}\epsilon, & \text{if } n = 0. \end{cases}$$

To also use reduced homology in the context of relative homology, the reduced relative homology $\tilde{H}_n(X, A)$ of a topological space X relative to a subspace $A \subset X$ is defined by

$$\tilde{H}_n(X, A) := \begin{cases} \tilde{H}_n(X), & \text{if } A = \emptyset \\ H_n(X, A), & \text{otherwise.} \end{cases}$$

There are various ways to assign a chain complex to a topological space thus obtaining a sequence of homology groups. The Eilenberg-Steenrod axioms (see [SE45]) set up a fairly general framework for homology theories from which many facts about homology groups can be derived.[1] They do not even define the homology via a chain complex, although most common homology theories arise in this form. For the purpose of this thesis however, such generality is not required and attention will be focused on a few particular homology theories.
Before getting into the details, there are some properties of homology this thesis makes use of that hold for any homology theory satisfying said axioms (as all homology theories considered here do). First of all, for any topological space that decomposes into a disjoint union of subspaces $X = \sqcup_{i\in I} X_i$, the relation $H_n(X) \cong \bigoplus_{i\in I} H_n(X_i)$ holds. This is in itself one of the Eilenberg-Steenrod axioms and is called additivity (again see [SE45]).
Let X, Y be two topological spaces, $f : X \to Y$ a continuous map. It is part of the definition of a homology theory that such a map induces a homomorphism $f_* : H_\bullet(X) \to H_\bullet(Y)$.[2] The next proposition states that sufficiently similar topological spaces have identical homology groups.

Proposition 3.1.5. *Let X, Y be topological spaces, $f : X \to Y$ a homotopy equivalence. Then $f_* : H_\bullet(X) \to H_\bullet(Y)$ is an isomorphism.*

A proof that this is a direct consequence of the axioms can be found in [Adh16], pages 433-434. In particular this holds for $f : X \to A$ a deformation retraction onto a subspace $A \subset X$. Since a homeomorphism is a homotopy equivalence, homeomorphic spaces have isomorphic homology groups. This justifies calling the homology a topological invariant.

3.2 Simplicial Homology

One of the simpler homology theories and one useful for practical calculations is called simplicial homology. It is a tool to study topological spaces that can be divided into simplices, which are very simple geometrical objects.

[1] There are in fact homology theories (in the sense that they are referred to as such in the common literature) that do not satisfy all of the cited axioms. They are sometimes said to be extraordinary homology theories.

[2] In the sense of category theory, a homology theory is a functor which assigns abelian groups to topological spaces and homomorphism to continuous maps.

Definition 3.2.1. *Let $k, n \in \mathbb{N}$ with $k \leq n$ and $v_0, \ldots, v_k \in \mathbb{R}^n$ such that $v_1 - v_0, \ldots, v_k - v_0$ are linearly independent. A k-simplex σ is the convex hull of such points, i.e.*

$$\sigma = \{\sum_{i=0}^{k} \lambda_i v_i \mid \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0 \quad \forall\, 1 \leq i \leq k\}.$$

It is denoted by $\sigma = [v_0, \ldots, v_k]$.

If $v_i = e_i$ are the canonical basis vectors in $\mathbb{R}^n$ for $0 \leq i \leq k$, the simplex is called the standard k-simplex and is denoted by Δ^k. The interior of a simplex σ is called an open simplex denoted by $\mathring{\sigma}$. The boundary of σ is defined as $\partial\sigma := \sigma - \mathring{\sigma}$. An example illustrating simplices in the first four dimensions can be found in figure 3.1.

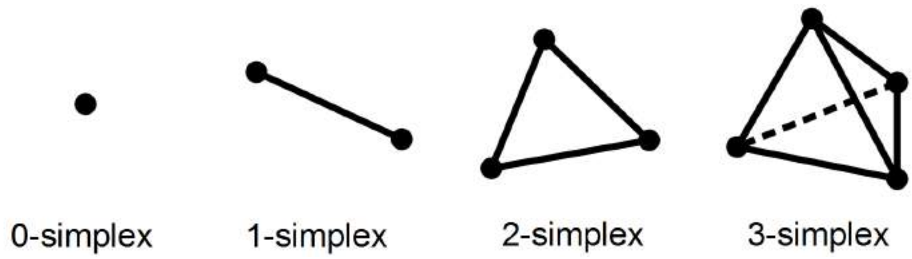


Figure 3.1: A symbolic illustration of simplices in dimensions zero to three.

Any k-simplex σ contains $k+1$ subsets

$$\sigma|\lambda_j := \{\sum_{i=0}^{k} \lambda_i v_i \in \sigma \mid \lambda_j = 0\},$$

which are themselves $(k-1)$-simplices and are called the faces of σ. Simplices can be glued together along their faces to form a complex.

Definition 3.2.2. *A collection of simplices K is called a simplicial complex if the following holds:*

- *For all $\sigma \in K$, the faces of σ are also elements in K.*
- *For all $\sigma, \sigma' \in K$, either $\sigma \cap \sigma' \in K$ or $\sigma \cap \sigma' = \emptyset$.*

It is often useful to relax these conditions a little. In a simplicial complex no two simplices can have the same vertex set. Take for example figure 3.2: The space depicted there is clearly a disjoint union of open simplices but not a simplicial complex, since the intersection of the two 1-simplices is the disjoint union of the two 0-simplices hence not a simplex itself.

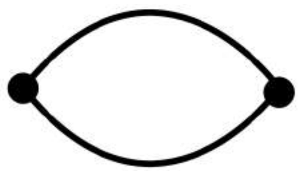


Figure 3.2: Not a simplicial complex

This problem can be fixed by the notion of a Δ-complex defined as follows (almost literally quoted from [Hat02]):

Definition 3.2.3. *A topological space X together with a collection of maps $\sigma_\alpha : \Delta^n \to X$ (with $n \in \mathbb{N}$ dependent on α) is called a Δ-complex if*

- *All restrictions of σ_α to the interior of Δ^n are injective such that each $x \in X$ is in the image of exactly one such restriction.*
- *For all $\sigma_\alpha : \Delta^n \to X$ the restriction to any face is a map $\sigma_\beta : \Delta^{n-1} \to X$.*
- *For any α, any $A \subset X$: $\sigma_\alpha^{-1}(A)$ is open $\iff$ A is open.*

This indeed generalizes the notion of a simplicial complex: If $K = \{\sigma_\alpha\}_{\alpha \in I}$ is a simplicial complex and $X \subset \mathbb{R}^n$ the union of all simplices in K, then the natural inclusion maps $\sigma_\alpha \hookrightarrow X$ give X the structure of a Δ-complex.
For a simplicial complex, its simplicial homology is defined as the homology of the chain complex $(C_\bullet, \partial_\bullet)$, where the chain groups C_n are the free abelian groups generated by the set of all n-simplices and the boundary operator $\partial_\bullet$ acts on n-chains $\langle\sigma\rangle$ (with $\sigma = [v_0, \ldots, v_n]$ an n-simplex) as

$$\partial_n(\langle\sigma\rangle) := \sum_{i=0}^{n} (-1)^i \langle [v_0, \ldots, \hat{v}_i, \ldots, v_n] \rangle. \tag{3.3}$$

As a subset of $\mathbb{R}^n$, a simplicial complex is naturally a topological space with subspace topology. Therefore any topological space homeomorphic to a simplicial complex can be assigned a sequence of homology groups in this way.
For a Δ-complex, the simplicial homology is defined in a similar manner: The chain groups are the free abelian groups generated by the set of all maps $\sigma_\alpha : \Delta^n \to X$ in the complex and the boundary operator ∂^Δ is defined via the restriction to the faces of $\Delta^n = [e_0, \ldots, e_n]$:

$$\partial_n^\Delta(\langle\sigma_\alpha\rangle) := \sum_{i=0}^{n} (-1)^i \langle \sigma_\alpha|_{[e_0, \ldots, \hat{e}_i, \ldots, e_n]} \rangle. \tag{3.4}$$

The simplicial homology groups of a topological space X calculated via a Δ-complex are denoted by $H_\bullet^\Delta(X)$.
To obtain a chain complex with a boundary operator that squares to 0, it is crucial to keep track of the order of vertices v_i. The reason is illustrated by the following example. Consider the two line segments in figure 3.3. They are topologically identical but they differ in the way they are divided into simplices. The left one consists of a single 1-simplex with a 0-simplex at each end, while the right consists of two 1-simplices with an additional 0-simplex v_2 in the middle.

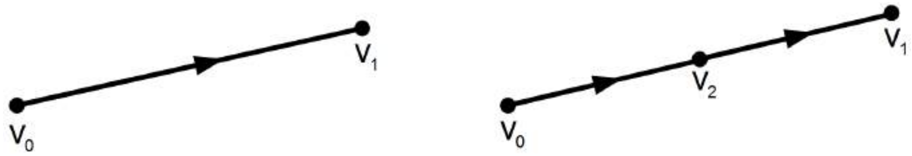


Figure 3.3: Two ways to divide a line segment into simplices.

Intuitively the boundary of each of these lines should consist of the outside vertices v_0 and v_1. However, if the boundary terms were just formally added, the result of the boundary morphism acting on the left line segment would be $v_0 + v_1$, while the right one would have boundary $(v_0 + v_2) + (v_2 + v_1) = v_0 + v_1 + 2v_2$. This problem can be fixed by keeping track of the order of vertices and assigning an appropriate sign. In figure 3.3, this is indicated by the arrows. Counting the 0-simplex with outgoing arrow as negative, the other as positive leads to the correct result: The boundary of the left line becomes $v_1 - v_0$, for the right one we get $(v_2 - v_0) + (v_1 - v_2) = v_1 - v_0$. Generalizing this reasoning to arbitrary dimensions leads to the necessity of defining

$$\langle[v_0, \ldots, v_i, \ldots, v_j, \ldots, v_n]\rangle = -\langle[v_0, \ldots, v_j, \ldots, v_i, \ldots, v_n]\rangle.$$

for any $\langle[v_0, \ldots, v_n]\rangle$ and any $1 \leq i, j \leq n$ with $i \neq j$.

3.3 Cubical Homology

A similar way to assign a chain complex and thus a sequence of homology groups to a topological space X is to subdivide the space into cubes instead of simplices. This method does not appear very often in the literature but it has certain computational advantages over simplices, some of which are utilized in chapters 4 and 5. The following establishes the basic definitions and facts for this homology theory.
With an interval $I \subset \mathbb{R}$ we mean a set of the form $I = [n, n+1]$ or $I = [n, n] = \{n\}$ for some $n \in \mathbb{Z}$. In the second case, the interval is called degenerate. A cube is obtained as the product of intervals.

Definition 3.3.1. *Let $d \in \mathbb{N}$. An elementary cube $w = I_1 \times \ldots \times I_d \subset \mathbb{R}^d$ is the product of intervals $I_i \subset \mathbb{R}$ $(1 \leq i \leq d)$. d is called the embedding number of w (written emb w), the number of non-degenerate intervals is called the dimension of w (written dim w).*

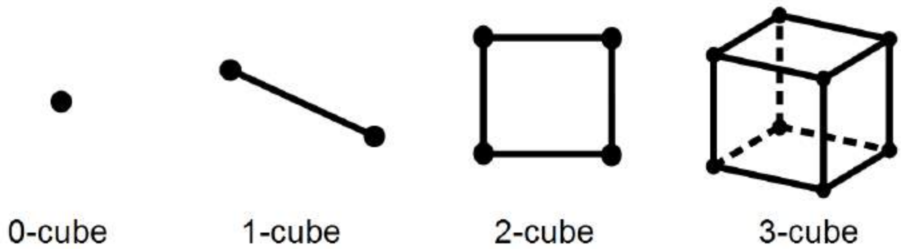


Figure 3.4: A symbolic illustration of non-degenerate cubes of dimensions zero to three.

We call $\square^n := [0,1] \times \ldots \times [0,1]$ with emb $\square^n$ = dim $\square^n = n$ the standard n-cube. For $n = 0$ we define $\square^0 := \{0\}$.
Analogous to simplices, elementary cubes contain faces, i.e. further cubes of lower dimension:

Definition 3.3.2. *Let w_1, w_2 be two elementary cubes. If $w_2 \subseteq w_1$, w_2 is said to be a face of w_1. It is called proper if $w_1 \neq w_2$ and called primary if dim w_2 = dim $w_1 - 1$.*

As with simplices, cubical homology works by writing a topological space as a certain union of cubes. The direct analog to a simplicial complex is to restrictive for the purpose of this work. Therefore we focus attention on the analog of a Δ-complex (compare definition 3.2.3).

Definition 3.3.3. *A topological space X together with a collection of maps $\sigma_\alpha : \square^n \to X$ (with $n \in \mathbb{N}$ dependent on α) is called a cubical complex if*

- *All restrictions of σ_α to the interior of $\square^n$ are injective such that each $x \in X$ is in the image of exactly one such restriction.*
- *For all $\sigma_\alpha : \square^n \to X$, the restriction to any face is a map $\sigma_\beta : \square^{n-1} \to X$.*
- *For any α, any $A \subset X$: $\sigma_\alpha^{-1}(A)$ is open $\iff$ A is open.*

The chain complex $(C_\bullet^\square(X), \partial_\bullet)$ associated to a cubical complex $(X, \{\sigma_\alpha\}_{\alpha\in A})$ is constructed by defining the chain groups $C_n^\square(X)$ to be the free abelian groups generated by all maps $\sigma_\alpha : \square^n \to X$. The boundary morphism $\partial_n^\square$ acts linearly on the n-chains and its action on a single chain group element $\langle\sigma_\alpha\rangle$ is defined by

$$\partial^\square(\langle w\rangle) := \partial_+^\square(\langle w\rangle) + \partial_-^\square(\langle w\rangle),$$

where

$$\partial_+^\square(\langle\sigma_\alpha\rangle) := \sum_{i=1}^{n}(-1)^{i-1}\langle\sigma_\alpha|_{[I_1,\dots,I_{i-1},\{0\},I_{i-1},\dots,I_n]}\rangle$$
$$\partial_-^\square(\langle\sigma_\alpha\rangle) := \sum_{i=1}^{n}(-1)^{i}\langle\sigma_\alpha|_{[I_1,\dots,I_{i-1},\{1\},I_{i-1},\dots,I_n]}\rangle.$$

This allows for a direct computation of the homology groups for a fair amount of topological spaces (in particular the ones encountered in this text). If there are only finitely many cubes involved, all cubes can in principal be listed and all boundary terms computed. The homology groups belonging to a cubical complex $(X, \{\sigma_\alpha\}_{\alpha\in A})$ is denoted by $H_\bullet^\square(X)$.
It must be remarked that caution is required when dealing with degenerate cubes, i.e. cubes $\sigma : \square^n \to X$ which are not injective. The treatment of this technicality is omitted here since this case does not occur in this text.

3.4 Singular Homology

One of the most important homology theories is called singular homology. It makes no reference to any particular decomposition of a topological space. Instead it uses continuous maps from a standard simplex to the space under consideration.

Definition 3.4.1. *Let X be a topological space, Δ^n the standard n-simplex. A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$.*

Note that this need not be an embedding of the standard n-simplex, i.e. σ is not required to be injective.
Given a topological space X, the singular chain complex $(C_\bullet(X), \partial_\bullet)$ of X is the chain complex where the chain groups $C_n(X)$ are the free abelian groups generated by all singular n-simplices and the boundary map acts on n-chains $\langle\sigma_n\rangle$ corresponding to a singular n-simplex σ_n as

$$\partial_n(\langle\sigma_n\rangle) := \sum_{i=1}^{n}(-1)^{i-1}\langle\sigma_n|_{\Delta^n|\lambda_i}\rangle. \tag{3.5}$$

The set of all continuous maps is usually uncountable and thus the chain groups are enormously large. It is not a priori clear that the homology groups of such a chain complex should be finitely generated even in reasonable cases. The following central theorem from algebraic topology however makes sure that this happens at least for finite cubical or Δ-complexes.

Theorem 3.4.2. *Let X be a topological space. If X can be given the structure of a simplicial complex then $H_\bullet(X) \cong H_\bullet^\Delta(X)$, if X is cubical then $H_\bullet(X) \cong H_\bullet^\square(X)$.*

For the proof of the first statement the reader is referred to [Hat02], where a whole subsection is devoted to the equivalence of simplicial and singular homology. The second statement follows from the fact that any cube can be decomposed into simplices, such that their union has the same boundary in the simplicial complex as the cube in the cubical complex.
An immediate corollary of this theorem is that the homology of any topological space that can be divided into simplices or cubes does not depend on the details of the division. Another very useful conclusion is that the reduced singular homology groups with coefficients in G of any one-point space $\{\text{pt.}\}$ are trivial since it is homeomorphic to a single 0-simplex. Thus $\operatorname{im}\partial_n = 0$ for all $n \in \mathbb{N}_0$ and $\ker\partial_n = 0$ for all $n \in \mathbb{N}$. In dimension zero the kernel has exactly one element so that $H_0(\{\text{pt.}\}) \cong G$ and hence $\tilde{H}_0(\{\text{pt.}\}) = 0$.

3.4.1 The Euler Characteristic

One of the most prominent topological invariants is the Euler characteristic, an integer assigned to a topological space. It was originally defined for the surfaces of polyhedra: Let V be the number of vertices of the polyhedron, E the number of edges, and F the number of faces. Then the Euler characteristic χ of the polyhedron is

$$\chi = V - E + F.$$

From a modern perspective, the Euler characteristic can be defined via the dimensions of the homology groups of a space.

Definition 3.4.3. *The Euler characteristic $\chi(X)$ of a topological space X is defined to be*

$$\chi(X) := \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(X) = \sum_{i=0}^{n} (-1)^i h_i(X), \tag{3.6}$$

where $h_i(X) := \operatorname{rank} H_i(X)$ is called the ith Betty number.

For cubical or Δ-complexes, the Euler characteristic can additionally be calculated in a slightly different way. For Δ-complexes it is given by

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \#(i - \text{simplices}), \tag{3.7}$$

for cubical complex by

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \#(i - \text{cubes}). \tag{3.8}$$

For a proof of these equations the reader is referred to [Adh16].

3.5 Exact Sequences

A useful algebraic tool to study homology groups is the notion of an exact sequence.

Definition 3.5.1. *A sequence of groups $\{G_n\}_{n\in\mathbb{Z}}$ together with homomorphisms $f_n : G_n \to G_{n-1}$ is called exact if $\ker f_n = \operatorname{im} f_{n+1}$ for all $n \in \mathbb{Z}$.*

The case where there is only one non-trivial group in an exact sequence does not occur, since if all groups except G_n would be trivial, $f_n : G_n \to G_{n-1} = 0$ would have $\ker f_n = G_n$, while $f_{n+1} : 0 = G_{n+1} \to G_n$ would have $\operatorname{im} f_{n+1} = 0$. Thus by exactness $G_n = \ker f_n = \operatorname{im} f_{n+1} = 0$.
The following two examples analyze the most simple non-trivial cases of exact sequences.

Example 3.5.2. *Let A and B be groups, $f : A \to B$ a homomorphism. If the sequence*

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

is exact, then $\ker f = 0$ by exactness of the left arrows and $\operatorname{im} f = B$ by exactness of the right arrows. Hence f is an isomorphism.

Example 3.5.3. *Let A,B, and C be groups, $f : A \to B$ and $g : B \to C$ two homomorphisms and consider the exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

where the out-most maps are the only available homomorphisms. The left-most map has trivial image, thus by exactness of the sequence f has trivial kernel and must be injective. The other way around, the right-most homomorphism maps all of C to 0, hence $\operatorname{im} g = C$ by exactness, i.e. g is surjective. Such a sequence is called a short exact sequence. Any exact sequence with more than three non-vanishing terms is called a long exact sequence.
The following three conditions are equivalent (see [PJH97] for the proof):

1. *f has a left inverse, i.e. there exists a map $\sigma : B \to A$ such that $\sigma \circ f = id_A$.*
2. *g has a right inverse, i.e. there exists a map $\tau : C \to B$ such that $g \circ \tau = id_C$.*
3. *$B \cong A \oplus C$*

In case any of these condition holds, the sequence is said to split. If A,B, and C are vector spaces, the short exact sequence always splits.

One of the huge theorems in algebraic topology is formulated via a split exact sequence.

Theorem 3.5.4. *Let X be a topological space, G an abelian group. There exist split exact sequences*

$$0 \longrightarrow H_n(X) \otimes G \longrightarrow H_n(X;G) \longrightarrow Tor(H_{n-1}(X);G) \longrightarrow 0,$$

and in particular $H_n(X;G) \cong (H_n(X) \otimes G) \oplus Tor(H_{n-1}(X);G)$.

This statement is called the universal coefficient theorem for homology for which a proof can be found in [Hat02], page 264. Here, $\mathrm{Tor}(H_{n-1}(X); G)$ is the torsion functor whose exact definition would lead to far for the purpose of this work. We note however that if G is a field of characteristic zero, the torsion group is trivial (again [Hat02], pages 265-266) and thus $H_n(X; G) \cong H_n(X) \otimes G$.
Another important tool in homology theory is the long exact sequence of pairs, connecting the homology of a space X, a subspace $A \subset X$, and their relative homology.

Theorem 3.5.5. *Let X be a topological space, $A \subset X$ a subspace. There exists a long exact sequence of the pair (X, A)*

$$\dots \longrightarrow H_{n+1}(X, A) \xrightarrow{\delta} H_n(A) \xrightarrow{i} H_n(X) \xrightarrow{j} H_n(X, A) \longrightarrow \dots,$$

where i and j are the natural inclusion maps. The long exact sequence of pairs also works for reduced homology:

$$\dots \longrightarrow \tilde{H}_{n+1}(X, A) \xrightarrow{\delta} \tilde{H}_n(A) \xrightarrow{i} \tilde{H}_n(X) \xrightarrow{j} \tilde{H}_n(X, A) \longrightarrow \dots$$

The map δ is called the connecting homomorphism. Its existence can be established by a purely algebraic statement called the algebraic snake lemma. This theorem is proven for example in [Adh16], pages 434-435 for the regular version, [Hat02], pages 117-118 for the reduced version.
Another long exact sequence relates the homology of a topological space X to the homology of two subspaces whose interior span the whole space.

Theorem 3.5.6. *Let X be a topological space, $A, B \subset X$ two subspaces such that $X = \mathring{A} \cup \mathring{B}$. Let $i : A \cap B \hookrightarrow A$, $j : A \cap B \hookrightarrow B$, $k : A \hookrightarrow X$, and $l : B \hookrightarrow X$ the natural inclusion maps. Then there exists a long exact sequence*

$$\dots \longrightarrow H_{n+1}(X) \xrightarrow{\partial_*} H_n(A \cap B) \xrightarrow{(i_*, j_*)} H_n(A) \oplus H_n(B) \xrightarrow{k_* - l_*} H_n(X) \longrightarrow \dots,$$

called the Mayer-Vietoris sequence. Additionally there is a reduced version

$$\dots \longrightarrow \tilde{H}_{n+1}(X) \xrightarrow{\partial_*} \tilde{H}_n(A \cap B) \xrightarrow{(i_*, j_*)} \tilde{H}_n(A) \oplus \tilde{H}_n(B) \xrightarrow{k_* - l_*} \tilde{H}_n(X) \longrightarrow \dots$$

A proof of these statements can be found in [Adh16], page 429. Both of the sequences from propositions 3.5.6 and 3.5.6 can also be used with coefficients in an abelian group G in the regular as well as in the reduced version.

Chapter 4

Outer Space and Its Generalizations

The evaluation of a Feynman graph is closely related to the evaluation of the graph with internal edges put on mass-shell and its reduced graphs. The combinatorial spirit of this is captured by Outer space, a topological space mathematicians use to study the automorphisms of free groups. This motivates physicists to take a closer look at Outer space, which was for example suggested in [DK15].
In this chapter Outer space is introduced, some of its most important properties mentioned, and the associated cubical chain complex described. The homology of the latter is discussed in detail for the simplest cases. A set of generators for the homology groups is given for up to five legs. For the highest non-trivial homology, the generators are given explicitly for arbitrary numbers of external edges.

4.1 Outer Space

Outer space X_n is introduced and formally constructed in [MC86] as a topological space on which $\mathrm{Out}(F_n)$ acts with finite stabilizer. We follow the construction from the review in [Vog02]:
First an identification of $\pi_1(R_n)$ with F_n is chosen such that a generator of F_n corresponds to a loop along a single edge of R_n. Points of X_n are equivalence classes of marked metric graphs (g, Γ), where Γ is an n-loop graph without external legs and the marking is a homotopy equivalence $g : R_n \to \Gamma$. The equivalence relation is defined to be $(g, \Gamma) \sim (g', \Gamma')$ if there exists a homothety $h : \Gamma \to \Gamma'$ such that $h \circ g$ is homotopic to g'. It is convenient to normalize the metric graphs in the sense that for all $(g, \Gamma) \in X_n$ we have $\sum_{e \in E_\Gamma^{\mathrm{int}}} \lambda(e) = 1$ and we will assume this has been done during the rest of this work. The equivalence relation in this case involves an isometry instead of a homothety. The relevance of the marking in a Feynman graph setting is not entirely clear yet. Thresholds of Feynman graphs are associated with a variation of the amplitude. In [DK15] it is mentioned the ambiguity related to this variation can be fixed by the markings.
A useful (but not unique) representation of a point (g, Γ) in Outer space X_n can be obtained by drawing the graph Γ, choosing a spanning tree T, and then assigning an orientation and an element of F_n to each edge not in T. This gives rise to a map $h : \Gamma \to R_n$ by collapsing T to the vertex of R_n and sending the remaining edges to the loops in R_n given by the labels. This is done such that h is a homotopy inverse to the marking g. Figure 4.1 shows an example of four representations of the same point in X_3, taken from [MC86].
There is a right group action of $\mathrm{Out}(F_n)$ on X_n. The identification of F_n with $\pi_1(R_n)$ allows outer automorphisms $\phi \in \mathrm{Out}(F_n)$ to be represented by homotopy equivalences of R_n. So any $\phi \in \mathrm{Out}(F_n)$ has a representative $f : R_n \to R_n$ that allows for the definition of a group

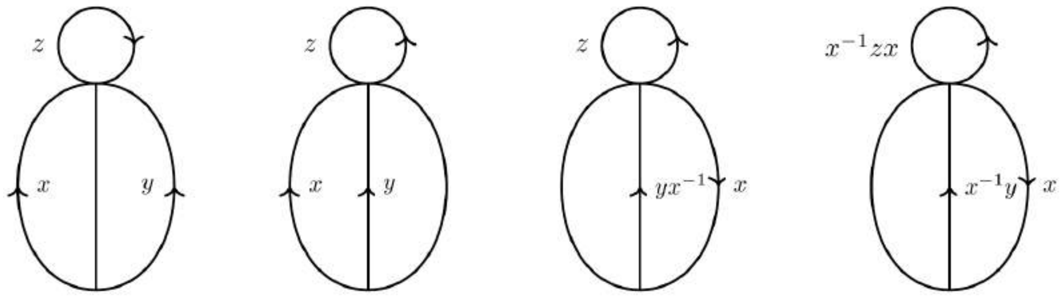


Figure 4.1: Four representations of the same point in X_3

action $\sigma : X_n \times \mathrm{Out}(F_n) \to X_n$ by $\sigma((g, \Gamma), \phi) = (g \circ f, \Gamma)$.
The topology on X_n is defined by a map $\phi : X_n \to \mathbf{RP}^{\mathcal{C}}$, where $\mathcal{C}$ is the set of all cyclic words in F_n. ϕ assigns each reduced word w the length of the (uniquely determined) loop in Γ that is homotopic to $g(w)$. As shown in [MC87], ϕ is injective and thus induces a subspace topology on X_n.
Outer space X_n can be described as the disjoint union of open simplices in this topology. Any marked metric graph $(g, \Gamma) \in X_n$ corresponds to a family of graphs by varying the length of the edges. Such a family can be parametrized in Euclidean space by choosing an ordering of edges in Γ and then taking the ith coordinate to be the length of the ith edge. Since the volume of each graph is normalized to 1, this parametrizes an open simplex of dimension $|E_\Gamma^{\text{int}}| - 1$.
It is often useful to work with a slightly simpler space. There is a deformation retraction of X_n constructed by shrinking all separating edges in non-1PI graphs to zero length (this is described for example in [Vog02]). The upshot of this is that, since a deformation retraction induces an isomorphism on the homology groups, only 1PI graphs have to be considered in any calculation. From this point on we will work exclusively with these reduced spaces, and Outer space X_n will refer to them instead as is often done in the literature.

4.1.1 The Spine of X_n and the Cubical Chain Complex

The mathematical motivation for Outer space X_n is to study the group action of $\mathrm{Out}(F_n)$ on this space. Unfortunately the quotient of X_n by the outer automorphism group of F_n is not a compact space (see [Vog02]). Additionally, the description of Outer space as a disjoint union of open simplices does not carry over to the quotient space. Informally speaking the problem is that some of the simplices get folded onto themselves under the quotient as is illustrated in figure 4.2. Without the markings, each point on a boundary 1-simplex is identified with its mirror image,[1] and in particular all corners become the same point. There is however a subspace K_n, called the spine of Outer space, which is compact under this quotient (again [Vog02]). K_n is a deformation retract of Outer space and can be grouped into cubes (Γ, F) specified by a graph Γ and a spanning forest F of Γ. Such a cube (Γ, F) contains all marked graphs that can be obtained by shrinking a subforest of F to zero length. The dimension of any cube $(\Gamma, F) \in K_n$ is given by the number of edges contained in the forest F. For a forest F with k edges the cube contains 2^k simplices.

[1] A 1-simplex is a line segment and can be parametrized by a single number $t \in [0, 1]$. Using this parametrization, identification with the mirror image means quotient by the equivalence relation given by $t \sim -t$ for all $t \in [0, 1]$.

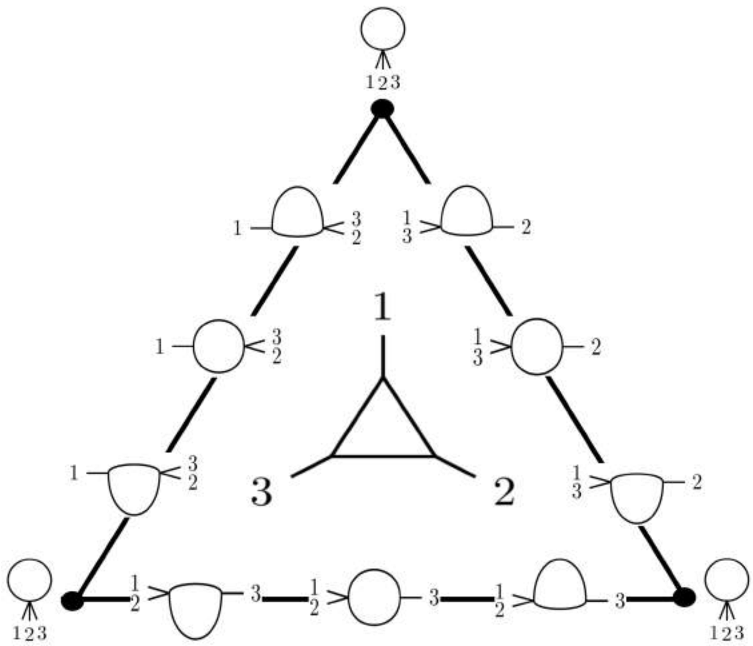


Figure 4.2: Under the quotient by the group action, some simplices get folded on themselves.

To define an action of the boundary operator on the cubes, they have to be assigned an orientation. This can be done by choosing an ordering of the edges in the spanning-forest. For F a spanning forest of a graph with n edges, we denote its set of edges by $E_F = \{e_1, \dots, e_n\}$. The k-chain corresponding to a k-cube (Γ, F) is denoted by $\langle \Gamma, F \rangle$. The boundary morphism $\partial : C_\bullet(K_n) \to C_\bullet(K_n)$ is defined by acting on k-chains $\langle \Gamma, F \rangle$ as

$$\partial_k : C_k(K_n) \to C_{k-1}(K_n)\ ,\ \langle \Gamma, F \rangle \mapsto \sum_{i=1}^{|E_F|} (-1)^{i-1} (\langle \Gamma, F \backslash \{e_i\} \rangle - \langle \Gamma_{e_i}, F_{e_i} \rangle). \tag{4.1}$$

This constitutes chain complexes $(C_\bullet(K_n), \partial_\bullet)$ which gives rise to homology groups $H_\bullet(K_n)$.

4.1.2 Generalizations of Outer Space

The construction of Outer space can easily be generalized to graphs with s external edges. The group action on the resulting space will be required to fix the external legs. To make this precise, one can think of the external edges as being full edges (instead of half-edges) with 1-valent vertices $x_1, \dots, x_s$ at there non-connected ends. These peripheral vertices are called leaves.

Definition 4.1.1. *Let Γ be an admissible n-loop graph with s leaves $x_1, \dots, x_s$. The group $\Gamma_{n,s}$ is the group of homotopy classes of homotopy equivalences of Γ, where both homotopies are required to fix the x_i.*

This definition is taken from [JC15] which also contains the proof that these are indeed groups. In particular the case without any external legs retrieves the outer automorphism group, i.e. $\Gamma_{n,0} \cong \mathrm{Out}(F_n)$. The case with only one external leg (or equivalently the case of graphs with a distinguished vertex called the base point) is the full automorphism group $\mathrm{Aut}(F_n)$ (see [AH98]).

Just as before, non-1PI graphs do not have to be considered. This construction results again in cubical complexes, where the set of all cubes is

$$X_{n,s} := \{(\Gamma, F) \mid \Gamma \text{ an admissible graph with } |\Gamma| = n, |E_\Gamma^{ext}| = s, F \subset \Gamma \text{ a spanning forest}\}$$

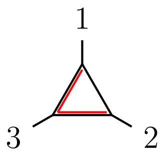
and the k-skeleta

$$X_{n,s}^k := \{(\Gamma, F) \in X_{n,s} \mid |E_F| = k\}.$$

These sets become cubical chain complexes when endowing them with an appropriate boundary operator. The operator from equation (4.1) makes no reference to external legs and can be generalized to a map defined to act on k-cubes $\langle \Gamma, F\rangle \in C_k(X_{n,s})$ as

$$\partial_k : C_k(X_{n,s}) \to C_{k-1}(X_{n,s}) \ , \ \langle \Gamma, F\rangle \mapsto \sum_{i=1}^{k} (-1)^{i-1}(\langle \Gamma, F \backslash \{e_i\}\rangle - \langle \Gamma_{e_i}, F_{e_i}\rangle). \tag{4.2}$$

This map still fulfills $\partial \circ \partial = 0$ (see chapter 5, where it is shown that this holds for an even more general boundary operator).
In this section, a pair (Γ, F) of a graph Γ and a spanning forest F of Γ or its corresponding chain group element $\langle \Gamma, F\rangle$ are depicted by drawing the graph as described in chapter 2 and marking the edges of F in red. Since we are dealing exclusively with spanning forests, the vertices belonging to F are not explicitly drawn. Instead it is always assumed that $V_F = V_\Gamma$ unless otherwise stated. For example



represents the pair (Γ, F) with $V_\Gamma = \{v_1, v_2, v_3\}$, $H_\Gamma = \{h_1, h_2, \ldots, h_9\}$, s_Γ defined by

$$s(h_i) = \begin{cases} v_1, & \text{if } 1 \le i \le 3 \\ v_2, & \text{if } 4 \le i \le 6 \\ v_3, & \text{if } 7 \le i \le 9, \end{cases}$$

c_Γ given by

i	1	2	3	4	5	6	7	8	9
$c(h_i)$	h_1	h_3	h_2	h_4	h_6	h_5	h_7	h_9	h_8

and the edges of the spanning tree F by $E_F = \{\{h_5, h_6\}, \{h_8, h_9\}\}$.
The following example illustrates how these pairs form cubes in the spine of Outer space.

Example 4.1.2. *Consider the cubical complex associated with $X_{1,4}$. Figure 4.3 contains a geometric representation of two neighboring 2-cubes in this complex.*

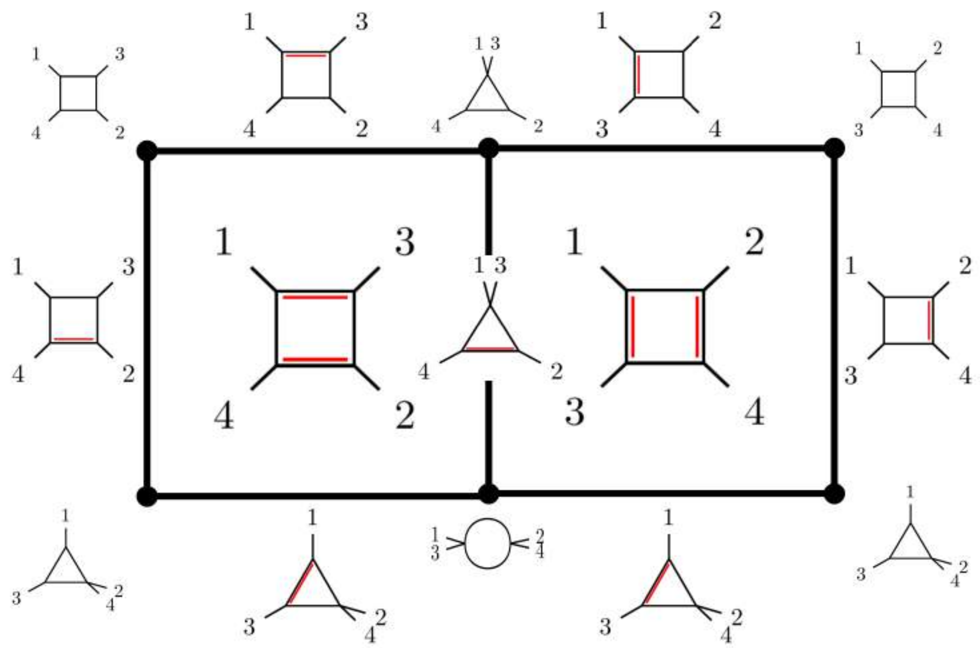


Figure 4.3: Two neighboring cubes in the cubical chain complex

Shrinking an edge to zero length in general yields several possibilities to blow the edge up again. Note that the 1-cubes at the bottom are also identified with each other.

To simplify the notation, elements of the homology groups, which are equivalence classes of chains, will simply be denoted by any of their representatives when no confusion can arise.[2]

4.2 The Homology of $X_{n,s}$

Much is already known about the homology of $X_{n,s}$. The homology of $X_{1,s}$ with coefficients in the field $\mathbb{Q}$ is fully determined in [JC15].[3]

$$H_k(X_{1,s};\mathbb{Q}) = \begin{cases} \mathbb{Q}^{\binom{s}{k}}, & \text{if k is even} \\ 0, & \text{otherwise} \end{cases} \tag{4.3}$$

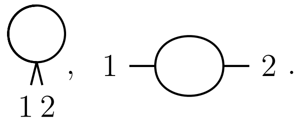
The primary focus of this thesis is the computation of the rational homology of moduli spaces of colored graphs in the one-loop case, i.e. (in the non-colored case of this chapter) the calculation of $H_\bullet(X_{1,s};\mathbb{Q})$. The case without colors, or equivalently a single color, serves as a good starting point to understand the computation and compare results with already established knowledge like equation (4.3). The next examples will provide a detailed analysis of the homology in the most simple cases.

Example 4.2.1. *We consider the cubical chain complex associated to $X_{1,2}$. In dimension zero there is a cube for each one-loop graph with two external edges (valency of each vertex*

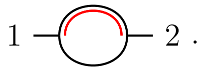
[2] The usual notation for an equivalence class of an element x is the notation with square-brackets $[x]$. Since in this work very lengthy elements of the homology group will be written down as formal series of graphs, drawn as in the previous example, adding these brackets would unnecessarily blow up these passages.

[3] In the mentioned paper, this result is stated in terms of the cohomology of the groups $\Gamma_{1,s}$.

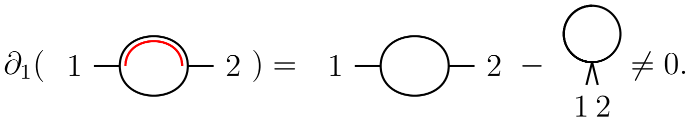
at least three) together with an edgeless spanning forest. There are two such graphs, namely



In one dimension, there is a cube for each such graph together with a forest containing one edge. The self-loop graph does not admit such a forest, so the only graph in dimension one is



The boundary operator applied to zero-dimensional cubes yields zero per definition. Application of the boundary morphism in dimension one yields



This means that ∂_1 can not vanish on any non-trivial linear combination of graphs, i.e. $\ker\partial_1 = 0$ *which implies $H_1(X_{1,2};\mathbb{Q}) = 0$. Furthermore this means that the two graphs in dimension zero are equivalent with respect to $x \sim y :\Leftrightarrow x - y \in \operatorname{im}\partial_1$, thus the same element in the quotient space $H_0(X_{1,2};\mathbb{Q}) = \ker\partial_0/\operatorname{im}\partial_1$. Hence*

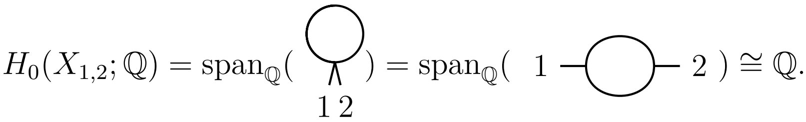


Figure 4.4 depicts a geometric realization of $X_{1,2}$ as a cubical complex.

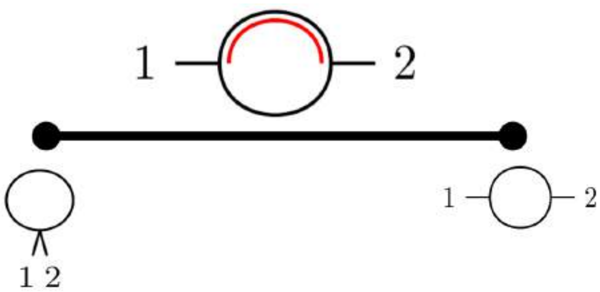
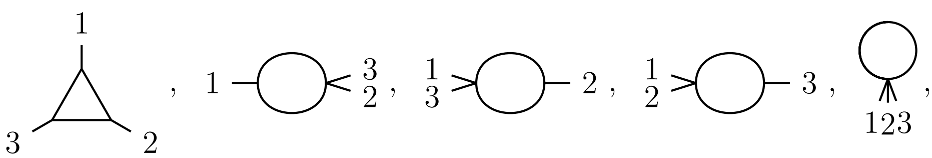
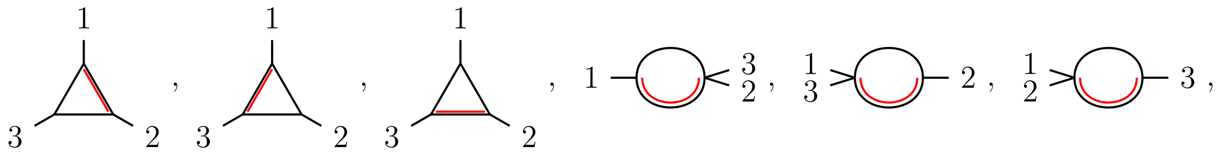


Figure 4.4: A geometric representation of $X_{1,2}$ as a cubical complex.

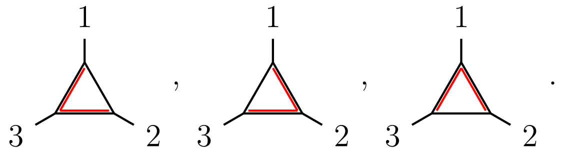
Example 4.2.2. *The simplest example with any non-trivial generators occurs in $X_{1,3}$. The space is made up from five 0-cubes*



six 1-cubes



and three 2-cubes



The straight-forward calculation of the boundary of these cubes yields

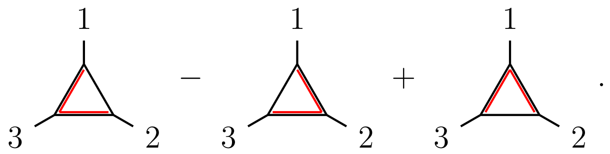
$$\dim \ker \partial_1 = 2 \qquad \dim \operatorname{im} \partial_1 = 4$$

$$\dim \ker \partial_2 = 1 \qquad \dim \operatorname{im} \partial_2 = 2$$

and hence

$$H_k(X_{1,3}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } k = 0, 2 \\ 0, & \text{otherwise.} \end{cases}$$

The generator of $H_2(X_{1,3}; \mathbb{Q})$ can be chosen to be

$g =$  (4.4)

Note that the signs result from an arbitrary orientation convention, assigning an order to the edges of each spanning forest. In the above calculation a lexigraphical ordering described in the following subsection 4.2.1 is employed. Figure 4.5 contains a graphical representation of the complex in which cubes labeled by identical pairs of a graph and a spanning forest are understood to be identified. That the generator g in the highest dimension is a sum of all cubes (with appropriate signs) is an instance of the more general result from proposition 4.2.5 which is proven at the end of this chapter.

Example 4.2.3. *The space $X_{1,4}$ consists of 17 0-cubes, 37 1-cubes, 36 2-cubes, and 12 3-cubes. The computation of the boundaries reveals the ranks of the maps ∂_k to be*

$$\dim \ker \partial_1 = 21 \qquad \dim \operatorname{im} \partial_1 = 16$$

$$\dim \ker \partial_2 = 15 \qquad \dim \operatorname{im} \partial_2 = 21$$

$$\dim \ker \partial_3 = 0 \qquad \dim \operatorname{im} \partial_3 = 12$$

and thus

$$H_k(X_{1,3}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } k = 0 \\ \mathbb{Q}^3, & \text{if } k = 2 \\ 0, & \text{otherwise.} \end{cases}$$

We consider the computation of the second homology group $H_2(X_{1,4}; \mathbb{Q})$ in detail. The first step is the calculation of $\ker \partial_2$ which is generated by 15 elements. Note that there is a sort

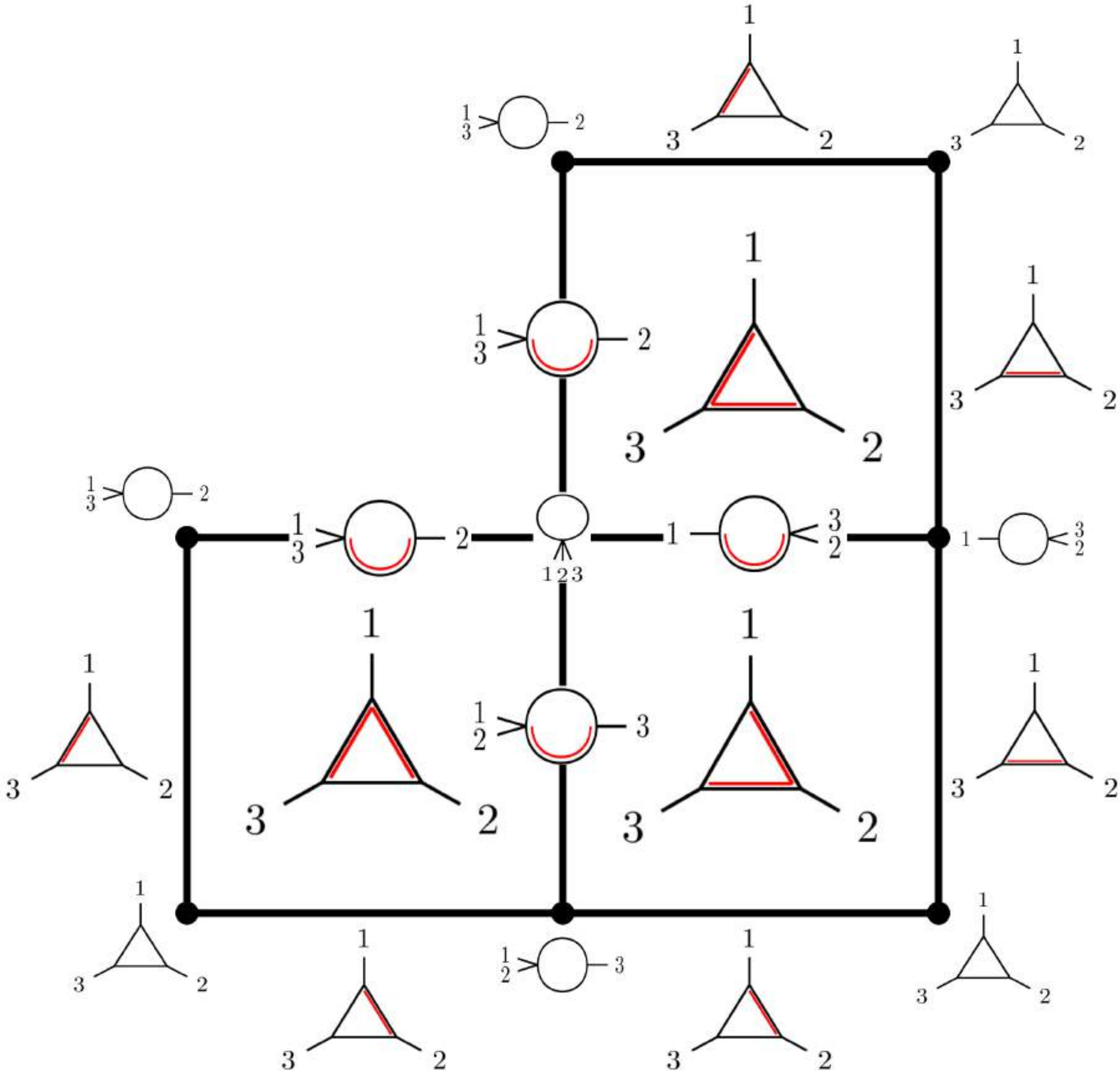
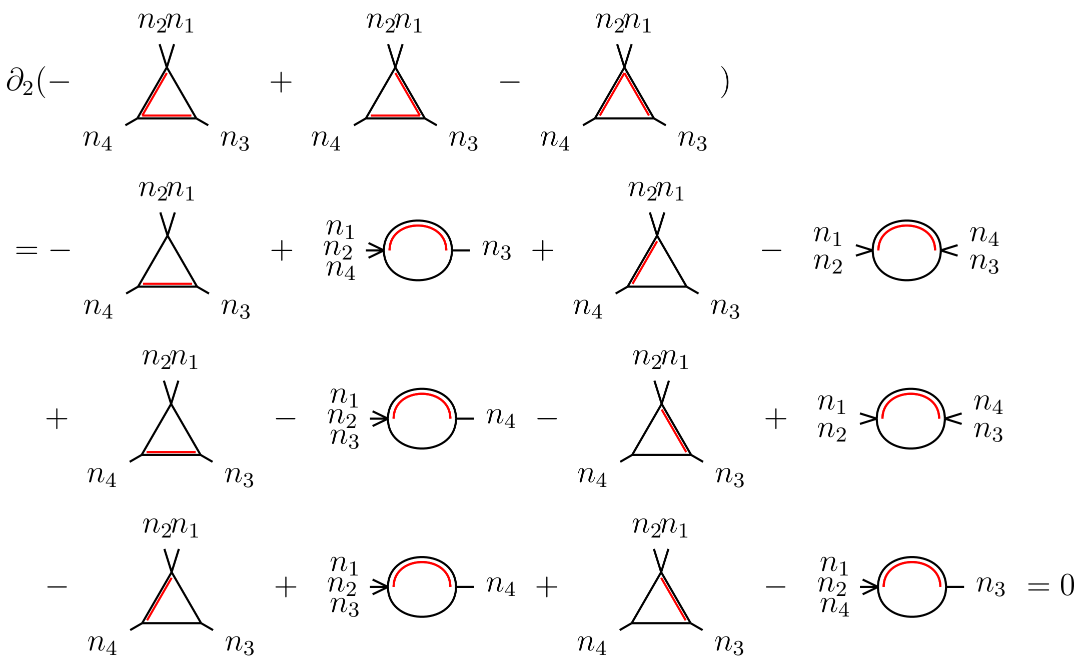
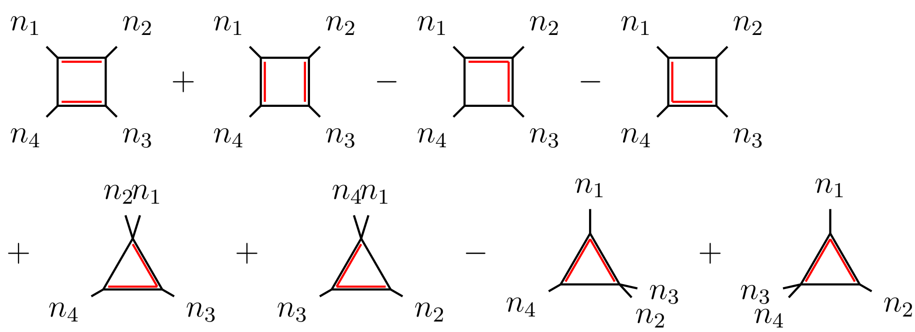


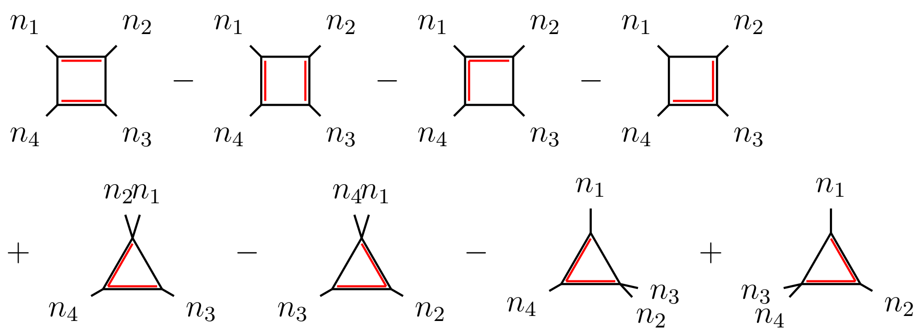
Figure 4.5: A geometric representation of $X_{1,3}$ as a cubical complex.

of inclusion of the elements of $\ker \partial_2 = H_2(X_{1,3})$ *from equation 4.4, since adding an external leg or re-labeling the external edges of every graph in the same way for every graph does not change the fact, that the boundary operator vanishes on this element. That is*

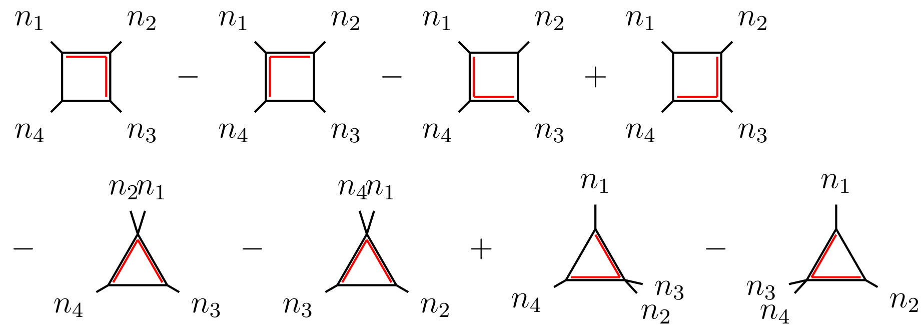


for all $n_1, n_2, n_3, n_4 \in \{1, 2, 3, 4\}$.[4] *Since there are* $\binom{4}{2} = 6$ *possibilities to group two legs together, this accounts for 6 elements of the kernel. The remaining elements are of the form*



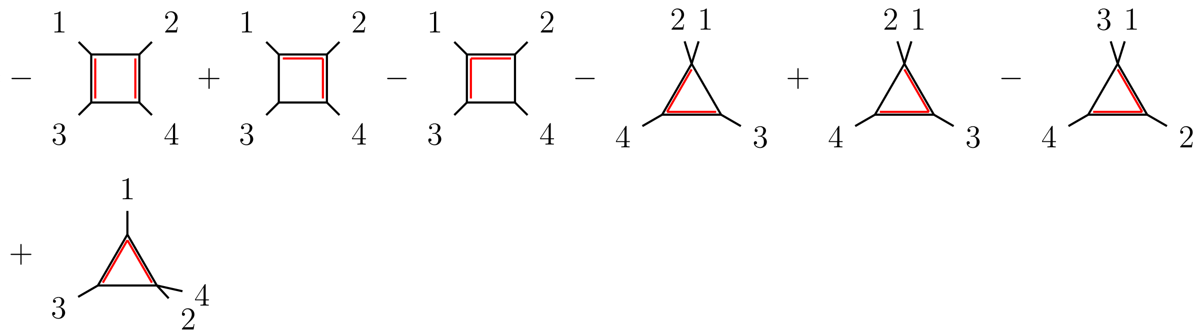


[4]This is unnecessarily general, the legs in the complexes under consideration have uniquely labeled external legs.

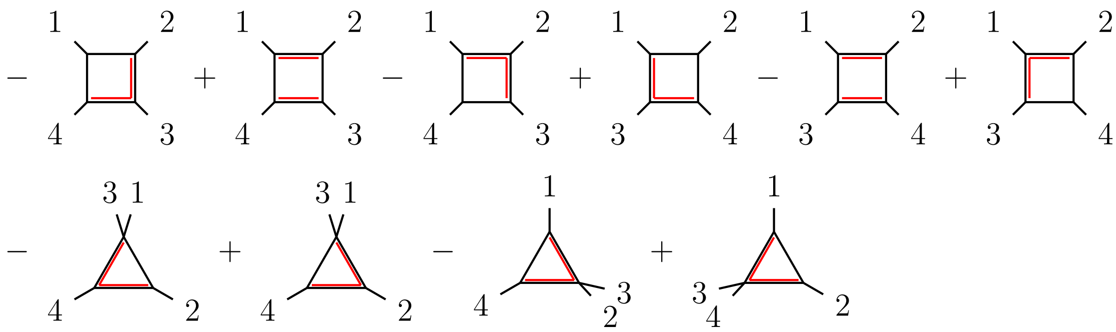


as can be easily checked by explicit calculation of the boundary and comparing the terms containing the box graphs to verify linear independence. This accounts for $3 \cdot 3 = 9$ *elements, since there are three distinct external leg structures for the box graph. Together this amounts for the whole kernel, which contains 15 elements in total.*
After applying the equivalence relation enforced by dividing out $\operatorname{im} \partial_3$*, a set of generators* g_1, g_2, g_3 *of* $H_2(X_{1,4}; \mathbb{Q})$ *can be chosen to be*

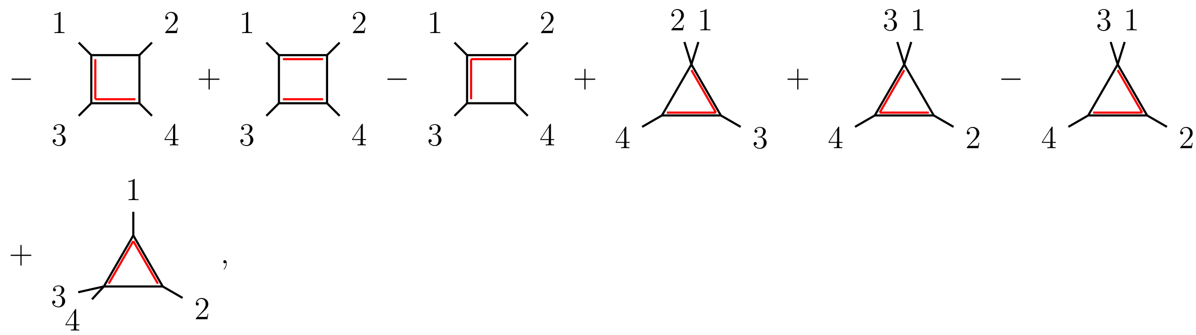
$g_1 =$



$g_2 =$



$g_3 =$



a result obtained with computer assistance by a method described in the following subsection 4.2.1.

4.2.1 Calculating the Homology

In principle the task of calculating the homology groups $H_\bullet(X_{n,s};\mathbb{Q})$ is straightforward: The cubical complexes under consideration contain finitely many cubes, so it is possible to list all of them, apply the boundary operator ∂ to each cube, and write the results into a big matrix. Each k-dimensional cube contributes a basis vector to $C_k(X_{n,s};\mathbb{Q})$. Let A_{∂_k} be the matrix representing ∂_k with respect to this basis. Then the dimension of the kth homology group is[5]

$$\begin{aligned}\dim H_k(X_{n,s};\mathbb{Q}) &= \dim\ker\partial_k - \dim\operatorname{im}\partial_{k+1}\\ &= \dim C_k(X_{n,s};\mathbb{Q}) - \operatorname{rank}(A_{\partial_k}) - \operatorname{rank}(A_{\partial_{k+1}}).\end{aligned} \tag{4.5}$$

Unfortunately the number of cubes in $X_{n,s}$ growth extremely fast with increasing loop number n and number of external legs s. Direct calculations can only be performed by hand for very small numbers of n and s, so further computations have to be done with computer assistance. For this purpose, a Matlab program was written for this thesis, which is shortly described in subsubsection 4.2.1.2. A detailed list of its content can be found in appendix A.

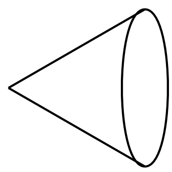
In this thesis, the main concern will be the one-loop case with rational coefficients, i.e. the homology of $(C_\bullet(X_{1,s};\mathbb{Q}),\partial_\bullet)$.

4.2.1.1 Representation of Graphs

In order to perform the necessary calculations on a computer, a way to represent a graph in the program must be chosen. We will follow [AH98], which sets up such a representation for graphs with a base point (a distinguished vertex):[6]

For an admissible graph Γ with r vertices, no external edges, and basepoint $v_0\in V_\Gamma$, it suffices to specify an ordering of the vertices with v_0 first (to simplify notation we denote $V_\Gamma=\{1,2,\ldots,r\}$, in particular $v_0=1$) and a list of pairs $\{i,j\}$ with $1\le i\le j\le r$ for each edge connecting the ith and the jth vertex.

Example 4.2.4. *Consider the Dunce's cap graph*



Up to reordering of the pairs the graph accepts three representations in form of a list as described above:

$$\{\{1,2\},\{1,2\},\{1,3\},\{2,3\}\},$$

[5] This is simply a consequence of the two basic statements from linear algebra that for any finite dimensional vector space V and any subspace $U\subseteq V$, the formula $\dim V/U=\dim V-\dim U$ holds, and for any linear map $f: X\to Y$ between finite dimensional vector spaces X and Y, the formula $\dim V=\dim\ker f+\dim\operatorname{im} f$ holds.

[6] The case without any fixed vertex, i.e. without external legs does not occur in this work. Explicit computations are performed for the one-loop case only and the rational homology of $X_{1,0}$ is know to be trivial.

$$\{\{1,2\},\{1,3\},\{1,3\},\{2,3\}\},$$
$$\{\{1,2\},\{1,3\},\{2,3\},\{2,3\}\}.$$

As the example illustrates, this representation is not unique. The ordering of the vertices is an arbitrary choice resulting in different such lists. To obtain a unique representation, the graphs are put into a normal form: Starting with an arbitrary representation, we put all pairs in the list into lexicographical ordering. To the result, we apply all permutations of the vertices labeled by $2, \dots, r$, make sure each resulting pair $\{i, j\}$ has $i \leq j$ by switching i and j if necessary, and sort again lexicographically. Of all obtained lists, the first one with respect to the lexicographical ordering is called the normal form of the graph. As proven in [AH98], two graphs have the same normal form if and only if they are isomorphic to each other.
Let $s \geq 3$ and consider the one-loop graph Γ with k edges and s leaves (see figure 4.6).

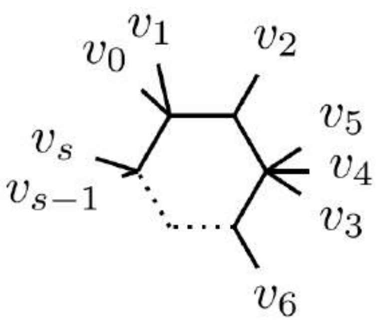


Figure 4.6: The one-loop graph with r edges and s external legs.

The normal form of Γ is

$$\{\{1,2\},\{1,r\},\{2,3\},\dots,\{r-1,r\}\}. \tag{4.6}$$

The one-loop graph with one internal edge has the normal form $\{\{1,1\}\}$, the one-loop graph with two internal edges the normal form $\{\{1,2\},\{1,2\}\}$.
The external leg structure of Γ can be uniquely specified by a partition of $\{1,\dots,s\}$ into r groups to indicate which legs share the same source vertex, and an element of $S_r/(C_r \ltimes \mathbb{Z}_2)$. The latter comes from the fact that an element of S_r characterizes an arbitrary permutation of the vertices the legs are attached to. But merely reversing the order of the vertices (in the sense that the rth group gets replaced by 2nd one, the $(r-1)$th by 3rd, etc.) results in the same graph. Additionally any cyclic permutation of the vertices also yields an identical graph. Therefore exactly the elements of $C_r \ltimes \mathbb{Z}_2$ have to be divided out to obtain a unique representation for the structure of the external edges.[7]

4.2.1.2 The Program

In the program, a chain element $\langle \Gamma, F \rangle \in C_\bullet(X_{n,s})$ corresponding to a graph Γ and a spanning forest F consists of the following data:

[7]It is well-known that the symmetry factor $S = \frac{1}{|Aut(\Gamma)|}$ for the general one-loop graph with r vertices is $\frac{1}{2r}$. Since $C_r \ltimes \mathbb{Z}_2$ has $2r$ elements, we know that this is enough and no further elements have to divided out.

List of internal edges The normal form representation of Γ as described above. Since the main objective is to calculate the homology groups for the one-loop case, this is given by the expression (4.6).

List of edges in F A list of edges in F given as a subset of the list of internal edges in the inherited ordering.

External leg structure For a graph with r vertices and s external edges, the external leg structure is stored in a $s \times r$-matrix $(a_{ij})_{ij}$, where $a_{ij} = 1$ if the ith external leg is connected to the jth vertex, $a_{ij} = 0$ otherwise. To fix the ambiguity arising from cyclic permutations, we always set $a_{11} = 1$. This representation is not entirely unique: There are two such representations for each distinct leg structure since reversing the order of the vertices yields a different matrix. This matrix might seem to big an object to encode this information. But it contains only s non-zero entries and can be effectively stored in a Matlab program using the sparse-command. The matrix form makes the calculation of the graph with an edge shrunken to zero length particularly easy.

The boundary operator requires the cubes to be oriented which corresponds to an ordering of the edges in the spanning forest. The orientation can be freely chosen (as long as it is used consistently) and since the structure set up above already provides the edges of the forest with an ordering inherited from the normal form of a graph, we will use this ordering to orient the cubes.
Given a number of external legs $s \in \mathbb{N}$, the program performs the following steps:

List all cubes: As first step, all cubes corresponding to graphs with s external edges are listed in order of increasing dimension. For a one-loop graph this is achieved in the following way: For each $0 \leq k \leq s-1$ let $l \in \mathbb{N}$ (the number of internal edges of a graph) run from $k+1$ to s. For each l the normal form representation (4.6) for a graph with l internal edges is set up, all k element subsets of this list are computed to get all spanning k-forests. Then for each such pair, all ways to group the s external edges are determined. In each dimension the boundary of each cube is calculated and listed next to corresponding cube.

Translate to matrices: Each k-cube represents a basis vector in the chain group $C_k(X_{n,s};\mathbb{Q})$. The order in which the cubes are listed is determined by the first step and defines an ordering for these basis vectors. The matrices $A_{\partial_k} = (a^k_{ij})_{ij}$ representing ∂_k with respect to these bases are obtained in the following way: Start with a zero-matrix with appropriate dimensions. Compare the boundary of each cube (consisting of $(k-1)$-cubes with rational coefficients) with the complete list of $(k-1)$-cubes. If a term in the boundary of the ith k-cube is in the jth place of the complete list of $(k-1)$-cubes, the coefficient of that term is added to the entry a^k_{ij}. Having obtained all matrices, the dimension of the homology groups is then calculated by equation (4.5).

Calculate generators: The kernel of ∂_k can be obtained by performing a row reduction on the matrix A_{∂_k}. This corresponds to a change of basis in the domain of ∂_k. To quotient out the image of ∂_{k+1}, the same change of basis can be applied to the target space of ∂_{k+1}. The resulting matrix representation $A_{\partial_{k+1}}$ is not necessarily in row echelon form, but completing the row reduction and applying the same basis change to the domain of ∂_k leaves A_{∂_k} unchanged, since $\operatorname{im}\partial_{k+1} \subseteq \ker \partial_k$. The resulting two matrices in row

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7
$X_{1,1}$	1	-	-	-	-	-	-	-
$X_{1,2}$	1	0	-	-	-	-	-	-
$X_{1,3}$	1	0	1	-	-	-	-	-
$X_{1,4}$	1	0	3	0	-	-	-	-
$X_{1,5}$	1	0	6	0	1	-	-	-
$X_{1,6}$	1	0	10	0	5	0	-	-
$X_{1,7}$	1	0	15	0	15	0	1	-
$X_{1,8}$	1	0	21	0	35	0	7	0

Table 4.1: The dimension of the homology groups $H_k(X_{1,s};\mathbb{Q})$ for $1 \leq s \leq 8$ and $0 \leq k \leq 7$.

	$\ker\partial_0$	$\ker\partial_1$	$\ker\partial_2$	$\ker\partial_3$	$\ker\partial_4$	$\ker\partial_5$
$X_{1,1}$	1	-	-	-	-	-
$X_{1,2}$	2	0	-	-	-	-
$X_{1,3}$	5	2	1	-	-	-
$X_{1,4}$	17	21	15	0	-	-
$X_{1,5}$	83	188	187	59	1	-
$X_{1,6}$	557	1785	2355	1435	365	0

	$\operatorname{im}\partial_0$	$\operatorname{im}\partial_1$	$\operatorname{im}\partial_2$	$\operatorname{im}\partial_3$	$\operatorname{im}\partial_4$	$\operatorname{im}\partial_5$
$X_{1,1}$	0	-	-	-	-	-
$X_{1,2}$	1	0	-	-	-	-
$X_{1,3}$	4	2	0	-	-	-
$X_{1,4}$	16	21	12	0	-	-
$X_{1,5}$	82	188	181	59	0	-
$X_{1,6}$	556	1785	2345	1435	360	0

Table 4.2: The dimension of the kernel and image of ∂_k in the chain complex $(C_\bullet(X_{1,s};\mathbb{Q}),\partial_\bullet)$ for $1 \leq s \leq 6$ and $0 \leq k \leq 5$.

echelon form are used to obtain a basis $b_1,\ldots,b_M$ of $\operatorname{im}\partial_{k+1}$ and a basis $b_1,\ldots,b_N$ of $\ker\partial_k$ (with $N \geq M$) such that $[b_{M+1}],\ldots,[b_N]$ is a basis of $\ker\partial_k/\operatorname{im}\partial_{k+1}$. These vectors can than be translated to cubes again by looking them up in the lists obtained in the first step.

4.2.1.3 Results

As mentioned above, the dimension of the homology groups $H_k(X_{1,s};\mathbb{Q})$ is determined in [JC15] for all $k,s \in \mathbb{N}_0$ (see equation (4.3)). While calculating the generators of these groups, these results where reproduced up to six external legs. Table 4.1 depicts the dimension of the homology groups for s up to 8. Rows colored in gray represent results not reproduced here. Additionally, in tables 4.2, the dimensions of the kernels and images are listed separately.

A choice of generators for up to four external edges is given in the previous examples 4.2.1, 4.2.2, and 4.2.3. For five external legs, such a choice can be found in appendix B.[8]

[8] While a set of generators for six external legs was also computed, they are not contained in this work due to their enormous length and since these particular generators did not yield any further insight into the matter yet.

For $H_{s-1}(X_{1,s};\mathbb{Q})$ there is either no generator (if s is even) or exactly one generator (if s is odd). In the second case, this generator is the sum of all possible cubes with appropriately chosen signs. Since there has to be one external leg per vertex in this case, the structure of the external legs is uniquely specified by an element in $S_s/(C_s \ltimes \mathbb{Z}_2)$ because there is only one available partition.

Proposition 4.2.5. *For $s \geq 3$ odd* [9] *the homology group $H_{s-1}(X_{1,s};\mathbb{Q})$ is generated by*

$$\sum_{[\sigma]\in S_s/(C_s\ltimes\mathbb{Z}_2)} \sum_{k=1}^{s} (-1)^{k-1}|\sigma|\langle\Gamma_{[\sigma]}, \{e_1,\dots,\hat{e}_k,\dots,e_s\}\rangle.$$

Proof. First we note that the signum $|\sigma|$ is well-defined, since the subgroup $C_s \ltimes \mathbb{Z}_2$ contains only even permutations for odd s. Hence two representatives σ, σ' of $[\sigma] = [\sigma']$ have the same signum, i.e. $|\sigma| = |\sigma'|$.
We consider the ∂^+- and ∂^--parts of the boundary operator separately for the proof. For the ∂^+-part we have

$$\begin{aligned}
&\partial^+_{s-1}\Big(\sum_{k=1}^{s}(-1)^{k-1}\langle\Gamma_{[\sigma]}, \{e_1,\dots,\hat{e}_k,\dots,e_s\}\rangle\Big)\\
&= \sum_{\substack{i,k=1\\ i<k}}^{s} (-1)^{k+i}\langle\Gamma_{[\sigma]}, \{e_1,\dots,\hat{e}_i,\dots,\hat{e}_k,\dots,e_s\}\rangle\\
&+ \sum_{\substack{i,k=1\\ i>k}}^{s} (-1)^{k+i-1}\langle\Gamma_{[\sigma]}, \{e_1,\dots,\hat{e}_k,\dots,\hat{e}_i,\dots,e_s\}\rangle\\
&= \sum_{\substack{i,k=1\\ i<k}}^{s} (-1)^{k+i}\langle\Gamma_{[\sigma]}, \{e_1,\dots,\hat{e}_i,\dots,\hat{e}_k,\dots,e_s\}\rangle\\
&- \sum_{\substack{i,k=1\\ i<k}}^{s} (-1)^{k+i}\langle\Gamma_{[\sigma]}, \{e_1,\dots,\hat{e}_i,\dots,\hat{e}_k,\dots,e_s\}\rangle = 0,
\end{aligned} \tag{4.7}$$

where in the last step the names of the summation indices were exchanged.
For the ∂^--part, we fix a $k \in \mathbb{N}$ with $1 \leq k \leq s$ and write $F = \{\tilde{e}_1, \cdots, \tilde{e}_{s-1}\}$ with $\tilde{e}_i = e_i$ if $i < k$ and $\tilde{e}_i = e_{i+1}$ otherwise. Then

$$\begin{aligned}
&\partial^-_{s-1}\Big(\sum_{[\sigma]\in S_s/(C_s\ltimes\mathbb{Z}_2)} |\sigma|\langle\Gamma_{[\sigma]}, F\rangle\Big)\\
&= \sum_{[\sigma]\in S_s/(C_s\ltimes\mathbb{Z}_2)} \sum_{i=1}^{s-1}(-1)^{i-1}|\sigma|\langle\Gamma_{[\sigma],\tilde{e}_i}, F_{\tilde{e}_i}\rangle\\
&= \sum_{\substack{[\sigma]\in S_s/(C_s\ltimes\mathbb{Z}_2)\\ \sigma \text{ even}}} \sum_{i=1}^{s-1}(-1)^{i-1}\langle\Gamma_{[\sigma],\tilde{e}_i}, F_{\tilde{e}_i}\rangle - \sum_{\substack{[\sigma']\in S_s/(C_s\ltimes\mathbb{Z}_2)\\ \sigma' \text{ odd}}} \sum_{i=1}^{s-1}(-1)^{i-1}\langle\Gamma_{[\sigma'],\tilde{e}_i}, F_{\tilde{e}_i}\rangle\\
&= \sum_{\substack{[\sigma]\in S_s/(C_s\ltimes\mathbb{Z}_2)\\ \sigma \text{ even}}} \sum_{i=1}^{s-1}(-1)^{i-1}\langle\Gamma_{[\sigma],\tilde{e}_i}, F_{\tilde{e}_i}\rangle - \sum_{\substack{[\sigma]\in S_s/(C_s\ltimes\mathbb{Z}_2)\\ \sigma \text{ even}}} \sum_{i=1}^{s-1}(-1)^{i-1}\langle\Gamma_{[\sigma],\tilde{e}_i}, F_{\tilde{e}_i}\rangle = 0
\end{aligned} \tag{4.8}$$

[9] For $s = 1$ the formula is only morally valid, since we defined $S_n/(C_n \ltimes \mathbb{Z}_2)$ only for $n \geq 3$. Nevertheless the generator of $H_0(X_{1,1};\mathbb{Q})$ is the sum of all cubes, since $X_{1,1}$ consists of a single 0-cubes which generates the group.

where the last line is due to the fact that exchanging the two legs where the edge $\tilde{e}_i$ was shrunken yields an identical graph but changes the sign of the representative σ. Putting the results together gives

$$\begin{aligned}
&\partial_{s-1}\Big(\sum_{[\sigma]\in S_s/(C_s\ltimes\mathbb{Z}_2)}\sum_{k=1}^{s}(-1)^{k-1}|\sigma|\langle\Gamma_{[\sigma]},\{e_1,\dots,\hat{e}_k,\dots,e_s\}\rangle\Big)\\
&=(\partial_{s-1}^{+}+\partial_{s-1}^{-})\Big(\sum_{[\sigma]\in S_s/(C_s\ltimes\mathbb{Z}_2)}\sum_{k=1}^{s}(-1)^{k-1}|\sigma|\langle\Gamma_{[\sigma]},\{e_1,\dots,\hat{e}_k,\dots,e_s\}\rangle\Big)\\
&=\sum_{[\sigma]\in S_s/(C_s\ltimes\mathbb{Z}_2)}|\sigma|\partial_{s-1}^{+}\Big(\sum_{k=1}^{s}(-1)^{k-1}\langle\Gamma_{[\sigma]},\{e_1,\dots,\hat{e}_k,\dots,e_s\}\rangle\Big)\\
&+\sum_{k=1}^{s}(-1)^{k-1}\partial_{s-1}^{-}\Big(\sum_{[\sigma]\in S_s/(C_s\ltimes\mathbb{Z}_2)}|\sigma|\langle\Gamma_{[\sigma]},\{e_1,\dots,\hat{e}_k,\dots,e_s\}\rangle\Big)=0,
\end{aligned}$$

where equations (4.7) and (4.8) where used in the last step. Since $H_{s-1}(X_{1,s};\mathbb{Q})=\ker\partial_{s-1}$, this completes the proof. □

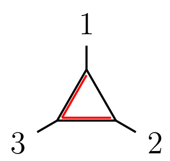
Chapter 5

Moduli Spaces of Colored Graphs

In the above chapter moduli spaces of graphs are described which give rise to cubical chain complexes in which cubes are represented by pairs of a metric graph and a spanning-forest. In these considerations, the only data assigned to the edges of a graph is their length. When physicists use Feynman graphs, they want to distinguish between different kinds of particles or masses represented by graph edges. For this purpose we introduce a coloring of graphs:

Definition 5.0.1. *Let Γ be a graph and $m \in \mathbb{N}$. An m-coloring of Γ is a map $c : E_\Gamma^{int} \to \{1, 2, ..., m\}$.*

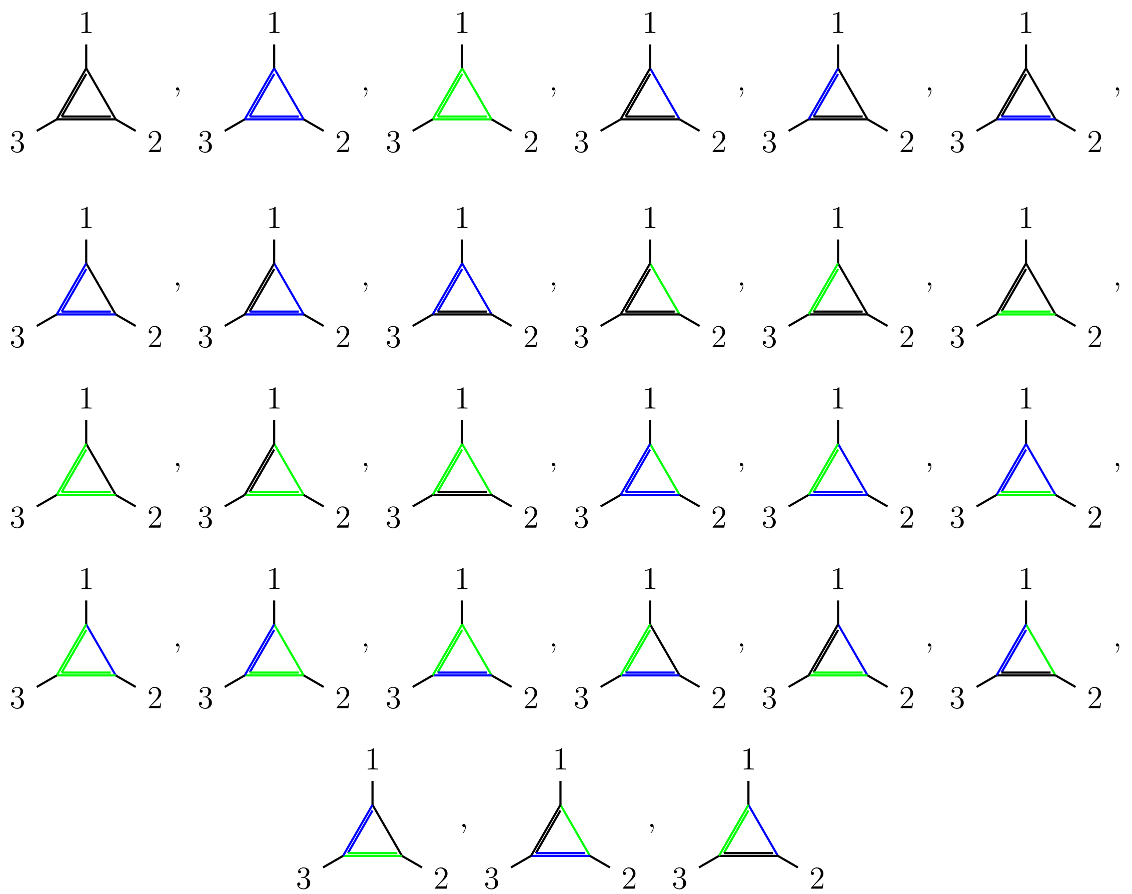
An m-coloring of a graph represents some (physical) property of the edges which can take m different values. For example one might think of a Feynman graph in a scalar field theory with three different masses as a graph endowed with a 3-coloring, one color for each mass in the theory.[1]
Analogous to chapter 4 metric graphs endowed with a coloring can be thought of as points in a moduli space of colored graphs. Three different moduli spaces of colored graphs are considered here: Section 5.1 is concerned with spaces $X_{n,s,m}$ in which any m-coloring is allowed. The following section 5.2 examines spaces of holo-colored graphs, in which graphs only admit colorings where each edge is assigned a different color. Two cases are distinguished: Subsection 5.2.1 uses colored graphs as before, while subsection 5.2.2 deals with graphs which retain the information of the coloring upon shrinking edges.
In this section we represent a colored graph with a spanning forest the following way: Graphs are drawn as in chapter 4 but with appropriately colored edges to represent the coloring. To avoid confusion, spanning forests are no longer depicted in red. Instead, edges that belong to the forest are drawn twice in the correct color. Consider for example the triangle graph with three external legs and a fixed spanning tree, which is depicted as



in the last chapter. It accepts 27 different 3-colorings (represented with the colors black,

[1] In many actual quantum field theoretical calculations, the spin of particles adds an additional feature: Half-integer spin particles come as oriented edges. The orientation of edges is a further complication that is not considered in this work.

blue, and green), which yields colored graphs depicted as



in the current chapter. As before, the vertices in the spanning forest (which are all vertices of the graph it spans) are not depicted for simpler notation.

5.1 Moduli Spaces of Graphs with Arbitrary Coloring

The first case that is considered here allows for any possible coloring of graphs, so we do not assume any restriction on the maps $c : E_\Gamma^{int} \to \{1, 2, ..., m\}$. Analogous to the previous chapter, one can consider the space of all triples (Γ, F, c), where Γ is an admissible metric graph with n loops and s external legs, F a spanning forest of Γ, and c an m-coloring of Γ. This space is denoted by $X_{n,s,m}$.
The construction of the cubical chain complex is compatible with the coloring of graphs. The chain groups $C_k(X_{n,s,m})$ in this case are the free groups generated by all k-cubes in $X_{n,s,m}$. The boundary operator ∂ from equation (4.2) can easily be generalized to act on triples (Γ, F, c). We define the action of the boundary homomorphism $\partial^m : C_\bullet(X_{n,s,m}) \to C_\bullet(X_{n,s,m})$ on a cube $\langle \Gamma, F, c\rangle \in C_k(X_{n,s,m})$ by

$$\partial_k^m(\langle \Gamma, F, c\rangle) = \sum_{i=1}^{|E_F|} (-1)^{i-1} (\langle \Gamma, F \backslash \{e_i\}, c\rangle - \langle \Gamma_{e_i}, F_{e_i}, c_{e_i}\rangle), \tag{5.1}$$

where $c_{e_i} := c|_{\{e_1,...,\hat{e}_i,...,e_{|E_F|}\}}$ is the coloring of the graph with edge e_i shrunken to zero length. This operator still squares to zero as is required for a chain complex: Let $\langle \Gamma, F, c\rangle \in$

$C_k(X_{n,s,m})$. Then

$$
\begin{aligned}
\partial_k^m(\partial_{k+1}^m(\langle\Gamma,F,c\rangle)) &= \partial_k^m(\sum_{i=1}^{k+1}(-1)^{i-1}(\langle\Gamma,F\backslash\{e_i\},c\rangle - \langle\Gamma_{e_i},F_{e_i},c_{e_i}\rangle))\\
&= \sum_{i=1}^{k+1}(-1)^{i-1}(\partial_k^m(\langle\Gamma,F\backslash\{e_i\},c\rangle) - \partial_k^m(\langle\Gamma_{e_i},F_{e_i},c_{e_i}\rangle))\\
&= \sum_{i=1}^{k+1}(-1)^{i-1}(\sum_{j=1}^{i-1}(-1)^{j-1}(\langle\Gamma,F\backslash\{e_i,e_j\},c\rangle - \langle\Gamma_{e_j},(F\backslash\{e_i\})_{e_j},c_{e_j}\rangle)\\
&\qquad - \sum_{j=i+1}^{k+1}(-1)^{j-1}(\langle\Gamma,F\backslash\{e_i,e_j\},c\rangle - \langle\Gamma_{e_j},(F\backslash\{e_i\})_{e_j},c_{e_j}\rangle)\\
&\qquad - \sum_{j=1}^{i-1}(-1)^{j-1}(\langle\Gamma_{e_i},F_{e_i}\backslash\{e_j\},c_{e_i}\rangle - \langle\Gamma_{e_i,e_j},F_{e_i,e_j},c_{e_i,e_j}\rangle)\\
&\qquad + \sum_{j=i+1}^{k+1}(-1)^{j-1}(\langle\Gamma_{e_i},F_{e_i}\backslash\{e_j\},c_{e_i}\rangle - \langle\Gamma_{e_i,e_j},F_{e_i,e_j},c_{e_i,e_j}\rangle))\\
&= \sum_{i=1}^{k+1}(-1)^{i}(\sum_{j=1}^{i-1}(-1)^{j}(\langle\Gamma,F\backslash\{e_i,e_j\},c\rangle - \langle\Gamma_{e_j},(F\backslash\{e_i\})_{e_j},c_{e_j}\rangle\\
&\qquad\qquad - \langle\Gamma_{e_i},F_{e_i}\backslash\{e_j\},c_{e_i}\rangle + \langle\Gamma_{e_i,e_j},F_{e_i,e_j},c_{e_i,e_j}\rangle)\\
&\qquad - \sum_{j=i+1}^{k+1}(-1)^{j}(\langle\Gamma,F\backslash\{e_i,e_j\},c\rangle - \langle\Gamma_{e_j},(F\backslash\{e_i\})_{e_j},c_{e_j}\rangle\\
&\qquad\qquad - \langle\Gamma_{e_i},F_{e_i}\backslash\{e_j\},c_{e_i}\rangle + \langle\Gamma_{e_i,e_j},F_{e_i,e_j},c_{e_i,e_j}\rangle))).
\end{aligned}
$$

The above sum consists of terms invariant under exchange of the indices i and j, namely

$$g_{ij} := \langle\Gamma,F\backslash\{e_i,e_j\},c\rangle - \langle\Gamma_{e_j},(F\backslash\{e_i\})_{e_j},c_{e_j}\rangle - \langle\Gamma_{e_i},F_{e_i}\backslash\{e_j\},c_{e_i}\rangle + \langle\Gamma_{e_i,e_j},F_{e_i,e_j},c_{e_i,e_j}\rangle.$$

With this we obtain

$$
\begin{aligned}
\partial_k^m(\partial_{k+1}^m(\langle\Gamma,F,c\rangle)) &= \sum_{i=1}^{k+1}\sum_{j=1}^{i-1}(-1)^{i+j}g_{ij} - \sum_{i=1}^{k+1}\sum_{j=i+1}^{k+1}(-1)^{i+j}g_{ij}\\
&= \sum_{i=1}^{k+1}\sum_{j=1}^{i-1}(-1)^{i+j}g_{ij} - \sum_{j=1}^{k+1}\sum_{i=j+1}^{k+1}(-1)^{j+i}g_{ij}\\
&= \sum_{i=1}^{k+1}\sum_{j=1}^{i-1}(-1)^{i+j}g_{ij} - \sum_{i=1}^{k+1}\sum_{j=1}^{i-1}(-1)^{j+i}g_{ij} = 0.
\end{aligned}
$$

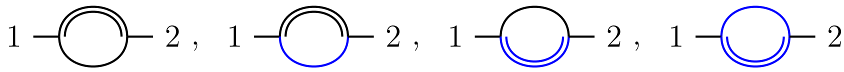
Hence by linearity of ∂^m this means $\partial^m \circ \partial^m = 0$. The cubical chain complex constructed in this way will be denoted by $(C_\bullet(X_{n,s,m}), \partial_\bullet^m)$ where again n is the number of loops, s the number of external legs, and m the number of colors. Note that this is indeed a generalization of $(C_\bullet(X_{n,s}), \partial_\bullet)$ since $C_k(X_{n,s,1}) \cong C_k(X_{n,s})$ for all $k \in \mathbb{N}_0$ by the chain map isomorphism $\langle\Gamma,F,c\rangle \mapsto \langle\Gamma,F\rangle$.

5.1.1 Computation of $H_\bullet(X_{1,s,m};\mathbb{Q})$

Just like the $C_k(X_{n,s};\mathbb{Q})$, the chain groups $C_k(X_{n,s,m};\mathbb{Q})$ are finite and have the structure of vector spaces due to the coefficients in $\mathbb{Q}$ (a field of characteristic zero). Thus the boundary

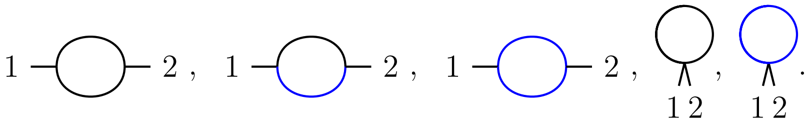
map ∂^m is a linear map between vector spaces and can be represented by a matrix A_{∂^m}. The dimension of the homology groups can obtained from equation (4.5) as before.
The next examples provide an analysis of the most simple cases.

Example 5.1.1. *Consider the space $X_{1,2,2}$ of one-loop graphs with two external legs and two colors. It consists of four one-dimensional cubes*



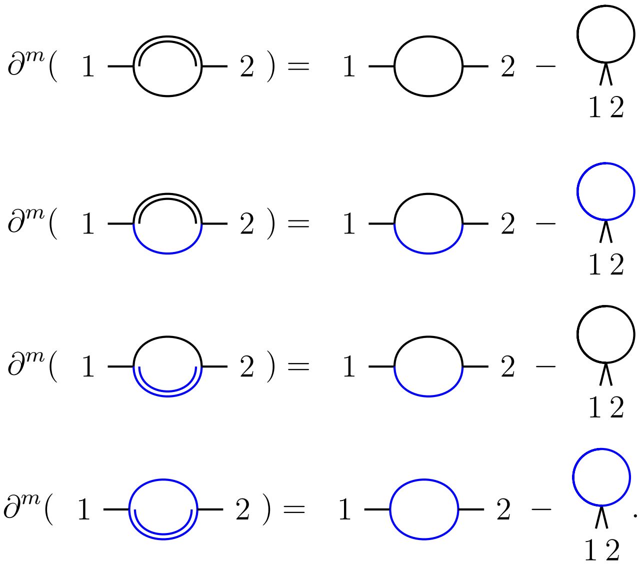
(5.2)

and five zero-dimensional cubes



(5.3)

The boundaries of the 0-cubes are trivial by definition. A straight-forward calculation of the boundaries of the 1-cubes yields



The matrix $A_{\partial_1^m}$ representing ∂_1^m with respect to the bases (5.2) of $C_1(X_{1,2,2};\mathbb{Q})$ and (5.3) of $C_0(X_{1,2,2};\mathbb{Q})$ reads

$$A_{\partial_1^m} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

and we have $\operatorname{rank} A_{\partial_1^m} = 4$*. Hence by equation (4.5)* $\dim H_1(X_{1,2,2};\mathbb{Q}) = 4 - 4 - 0 = 0$ *and* $\dim H_0(X_{1,2,2};\mathbb{Q}) = 5 - 0 - 4 = 1$ *or equivalently*

$$H_k(X_{1,2,2};\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } k = 0 \\ 0, & \text{otherwise.} \end{cases}$$

A geometric representation of the associated cubical complex, illustrating the face relations of the above cubes, can be found in figure 5.1.

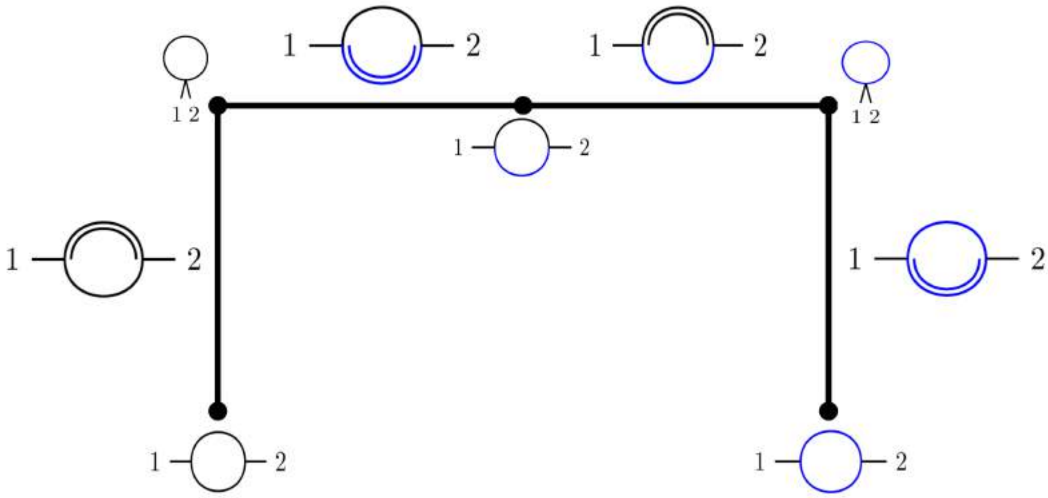
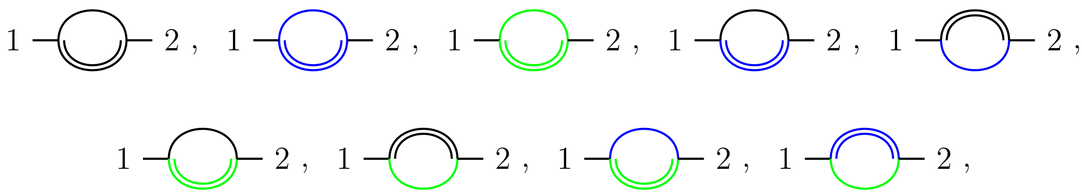


Figure 5.1: A geometric representation of $X_{1,2,2}$ as a cubical complex.

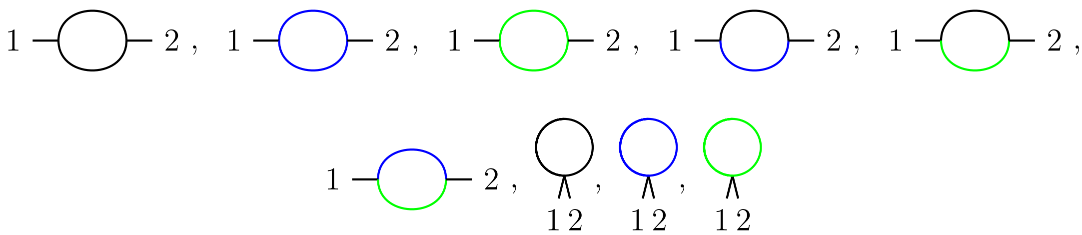
The space $X_{1,2,2}$ is evidently contractible in accordance with the obtained homology groups. Any of the four 0-cubes can be chosen as a representative for the generator of the zeroth homology group $H_0(X_{1,2,2})$.

This is the same result as for the homology of $X_{1,2}$ (in the sense that $H_k(X_{1,2,2}) \cong H_k(X_{1,2})$ for all $k \in \mathbb{N}_0$), so the coloring does not provide additional topological structure in this case. Adding a third color to the space of one-loop graphs with two external legs changes the properties of the space as the next example illustrates.

Example 5.1.2. *Consider the space $X_{1,2,3}$ of one-loop graphs with two external legs and three colors. There are nine 1-cubes*



and nine 0-cubes



in this cubical complex. We remark that the Euler characteristic is thus $\chi(X_{1,2,3}) = 9-9 = 0$. The straight-forward calculation of the boundaries of all cubes reveals $\operatorname{rank} \partial_1 = 8$ *and hence* $\dim H_0(X_{1,2,3};\mathbb{Q}) = \dim H_1(X_{1,2,3};\mathbb{Q}) = 1$, *i.e.*

$$H_k(X_{1,2,3};\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } k = 0,1 \\ 0, & \text{otherwise.} \end{cases}$$

Figure 5.2 depicts $X_{1,2,3}$ as a cubical complex.

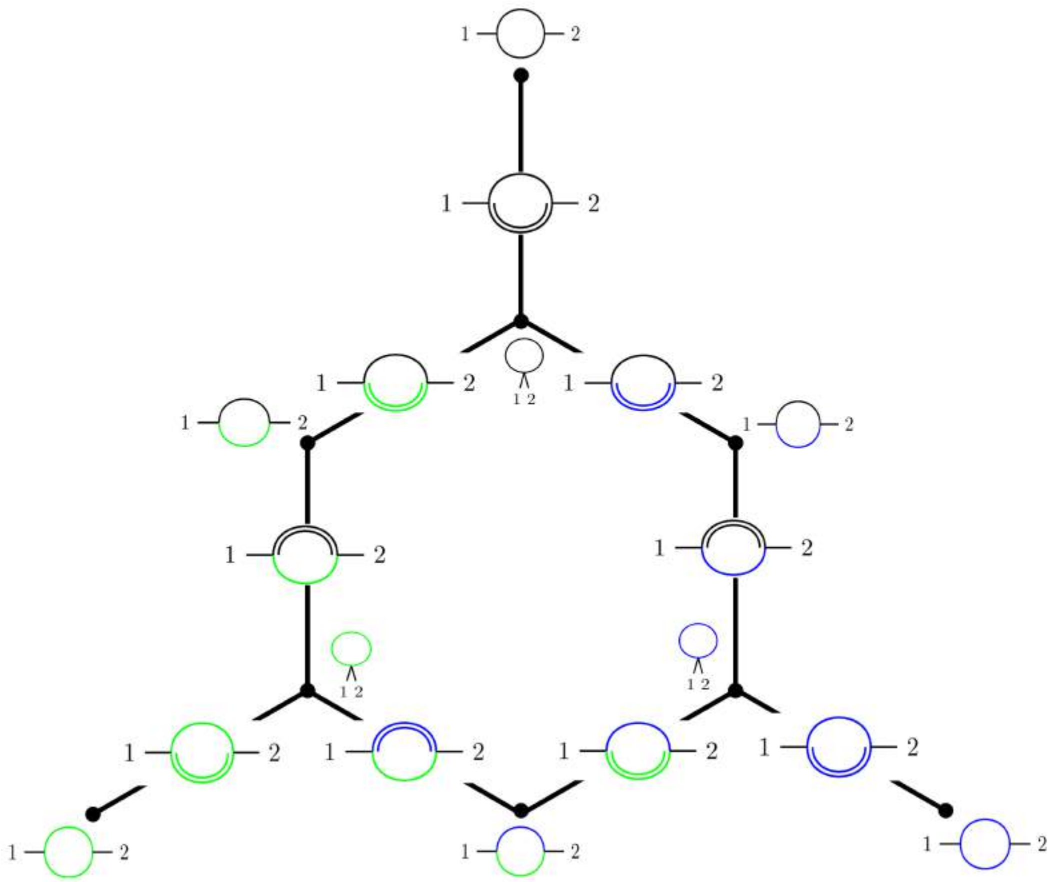
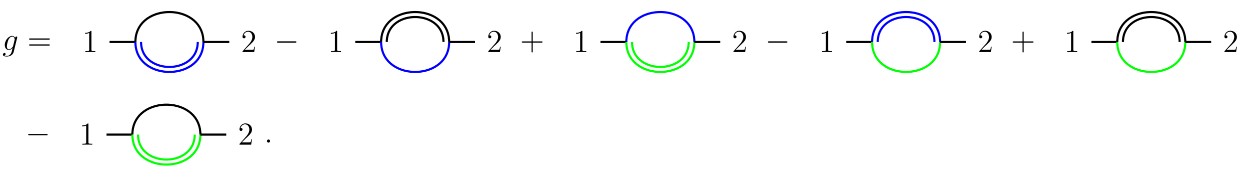


Figure 5.2: A geometric representation of $X_{1,2,3}$ as a cubical complex.

The space $X_{1,2,3}$ is homotopy equivalent to the circle S^1. The generator g of the first homology group can be chosen to be



Each of the $X_{1,2,m}$ is homotopy equivalent to the rose R_n for some $n \in \mathbb{N}_0$.[2] The dimension of their homology groups (and hence the number of petals n) is determined in subsection 5.1.2 for all $m \in \mathbb{N}$.

Example 5.1.3. *Consider the space of two-colored one-loop graphs with three external legs $X_{1,3,2}$. It consists of 24 2-cubes, 36 1-cubes, and 19 0-cubes. Computing the boundary of each cube and translating the result to matrix representatives of ∂_k^2 yields*

$$\dim\ker\partial_1^2 = 18 \qquad \dim\operatorname{im}\partial_1^2 = 18$$

[2] Any cubical or simplicial complex of dimension one consists of points and line segments only. It can be regarded as a topological graph and admits a deformation retraction to a rose by choosing any spanning tree and continuously shrinking it to a point.

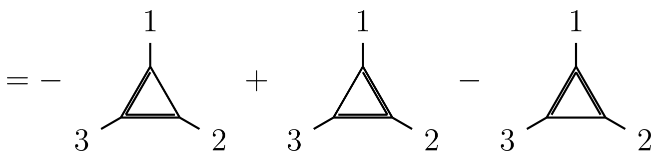
$$\dim\ker\partial_2^2 = 6 \qquad \dim\operatorname{im}\partial_2^2 = 18,$$

and hence

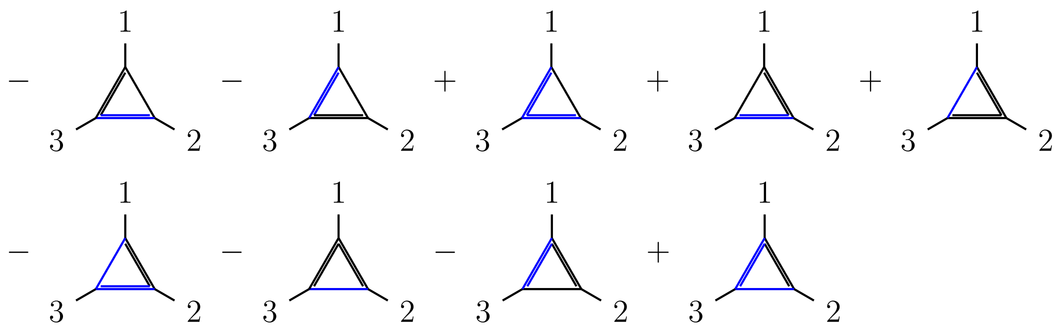
$$H_k(X_{1,3,2};\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \textit{if } k=0 \\ \mathbb{Q}^6, & \textit{if } k=2 \\ 0, & \textit{otherwise} \end{cases}$$

where $H_0(X_{1,3,2};\mathbb{Q})$ is generated by the equivalence class of any 0-cube and a particular choice of generators $g_1,\ldots,g_6$ for the second homology group $H_2(X_{1,3,2};\mathbb{Q})$ is

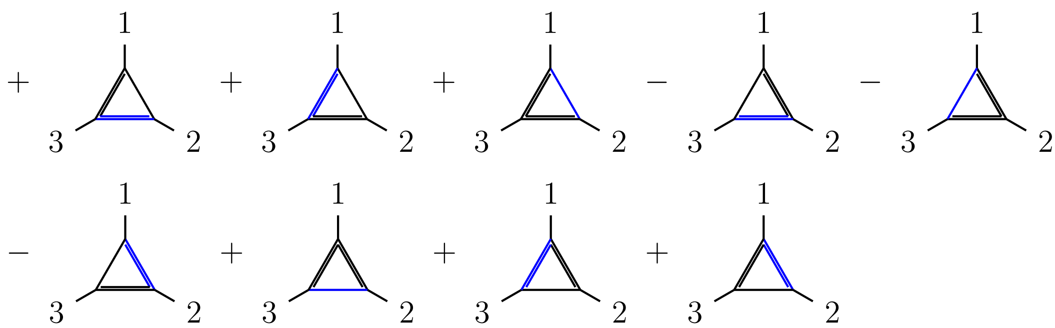
$g_1 =$



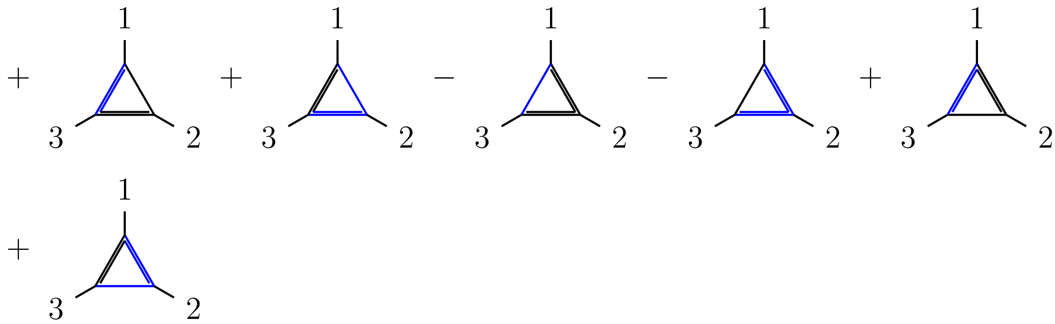
$g_2 =$

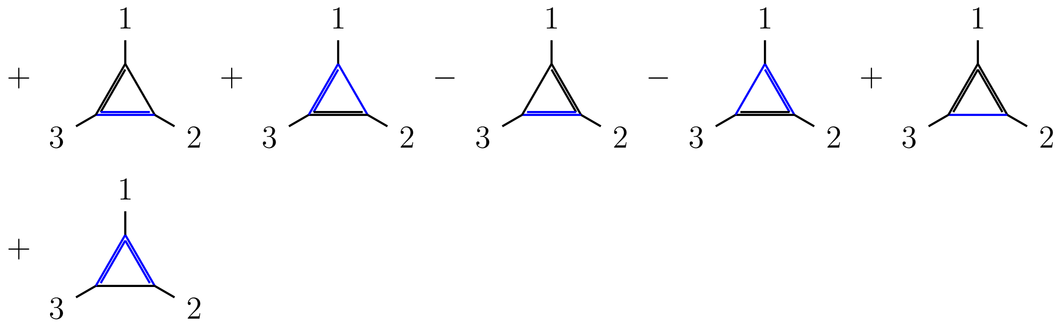


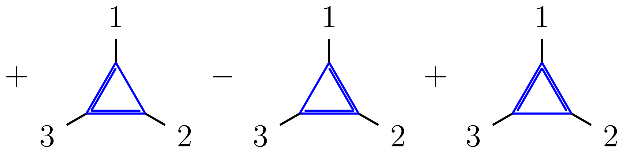
$g_3 =$



$g_4 =$



$g_5 =$ 

$g_6 =$ 

Note that g_1 and g_6 have the same structure as the single generator of $H_2(X_{1,3};\mathbb{Q})$ (see equation 4.4) in the sense that they consist of the same pairs of graphs and spanning forests and their coloring is a constant map and the same for all terms in the sum.

As in the colorless case from the previous chapter, the homology groups have been calculated with a Matlab program, in this case for $n = 1$, $1 \leq m \leq 7$, and small s. Endowing graphs with the additional data of color leads to an even greater growth of the number of cubes with increasing s. Computing the homology of the $X_{1,s,m}$ by explicit calculation gets increasingly more difficult when increasing the number of colors m so that the maximal number of external legs s to which the calculations could be performed decreases.
There are only minor changes necessary to apply the algorithm described in chapter 4 to the case with colored edges. Each cube is set up with an additional array containing a number from 1 to m, indicating the color of an edge. The order of this array is taken with respect to the lexicographical ordering of the list of internal edges. The action of the boundary operator also has to be slightly altered in accordance with equation (5.1).
The results for the homology dimensions for different numbers of colors are listed in table 5.1, a specific choice of generators for each group can be found in appendix B.

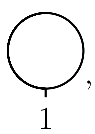
5.1.2 Proofs of Special Cases

The homology groups of $X_{1,s,m}$ can be partially determined without explicit calculation. For any $s \geq 2$ for example, $X_{1,s,m}$ is path-connected by construction and hence $\dim H_0(X_{1,s,m};\mathbb{Q}) = 1$. Furthermore, the following two propositions hold:

Proposition 5.1.4. *For all $m \in \mathbb{N}$ the dimension of the zeroth homology group is given by*

$$\dim H_0(X_{1,1,m};\mathbb{Q}) = m. \tag{5.4}$$

Proof. The cubical chain complex associated to $X_{1,1,m}$ consists only of zero dimensional cubes associated to graphs of the form



	H_0	H_1	H_2	H_3	H_4
$X_{1,1,2}$	2	-	-	-	-
$X_{1,2,2}$	1	0	-	-	-
$X_{1,3,2}$	1	0	6	-	-
$X_{1,4,2}$	1	0	3	9	-
$X_{1,5,2}$	1	0	6	0	84

	H_0	H_1	H_2	H_3
$X_{1,1,3}$	3	-	-	-
$X_{1,2,3}$	1	1	-	-
$X_{1,3,3}$	1	0	20	-
$X_{1,4,3}$	1	0	3	103

	H_0	H_1	H_2	H_3
$X_{1,1,4}$	4	-	-	-
$X_{1,2,4}$	1	3	-	-
$X_{1,3,4}$	1	0	49	-
$X_{1,4,4}$	1	0	3	426

	H_0	H_1	H_2	H_3
$X_{1,1,5}$	5	-	-	-
$X_{1,2,5}$	1	6	-	-
$X_{1,3,5}$	1	0	99	-

	H_0	H_1	H_2
$X_{1,1,6}$	6	-	-
$X_{1,2,6}$	1	10	-
$X_{1,3,6}$	1	0	176

	H_0	H_1	H_2
$X_{1,1,7}$	7	-	-
$X_{1,2,7}$	1	15	-
$X_{1,3,7}$	1	0	286

Table 5.1: The dimension of the homology groups $H_k(X_{1,s,m};\mathbb{Q})$ for $1 \leq m \leq 7$ and various s.

where the one edge may be colored with any of the m available colors. The only available spanning forest is the one that contains the vertex only. The boundary operator vanishes on any 0-cube by definition, so all of these m cubes lie in the kernel of ∂_0^m. Since there are no 1-cubes, the zeroth homology group is just the kernel of ∂_0^m. Thus the set of all m-cubes generates the homology group $H_0(X_{1,1,m};\mathbb{Q})$. □

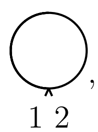
Proposition 5.1.5.

$$\dim H_1(X_{1,2,m};\mathbb{Q}) = \frac{m^2}{2} - \frac{3m}{2} + 1 \tag{5.5}$$

Proof. The space $X_{1,2,m}$ consist of zero-dimensional and one-dimensional cubes. Its Euler characteristic χ can be calculated as the number of 0-cubes minus the number of 1-cubes (see equation (3.8)). On the other hand, χ can also be calculated as $\chi = \dim H_0(X_{1,2,m};\mathbb{Q}) - \dim H_1(X_{1,2,m};\mathbb{Q})$ by definition (see equation (3.6)). By the path-connectedness of any $X_{1,s,m}$ with $s \geq 2$, we have $\dim H_0(X_{1,2,m};\mathbb{Q}) = 1$ and hence

$$\dim H_1(X_{1,2,m};\mathbb{Q}) = \dim H_0(X_{1,2,m};\mathbb{Q}) - \chi = 1 - \chi.$$

In dimension zero there are m graphs of the form



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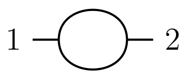
one for every color the edge admits, and $\sum_{i=1}^{m}\sum_{j\leq i} 1 = \sum_{i=1}^{m} i = \frac{m(m+1)}{2}$ graphs of the form



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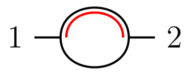
for each pair of colors the edges admit. Hence there are $\frac{m(m+1)}{2} + m = \frac{m^2+3m}{2}$ elements in the lowest dimension. In dimension one, there are m^2 graphs of the form



so that the Euler characteristic is $\chi = \frac{m^2+3m}{2} - m^2 = \frac{3m}{2} - \frac{m^2}{2}$ and one obtains

$$\dim H_1(X_{1,2,m};\mathbb{Q}) = 1 - (\frac{3m}{2} - \frac{m^2}{2}) = \frac{m^2}{2} - \frac{3m}{2} + 1.$$

$\square$

Proof. An alternative proof of the proposition makes use of the Mayer-Vietoris sequence and might provide more insight into the topology at work.
We divide the space $X_{1,2,m}$ into

$$A := \{(\Gamma, F, c) \mid m \in \operatorname{im} c\}$$

and

$$B := \{(\Gamma, F, c) \mid c(e) \neq m \text{ for at least one } e \in E_\Gamma^{\text{int}}\},$$

so that $X_{1,2,m} = \mathring{A} \cup \mathring{B}$. The subspaces A and B are in figure 5.3 and figure 5.4 respectively. In these figures, black labels represent the labels of external legs, while red labels represent the coloring.

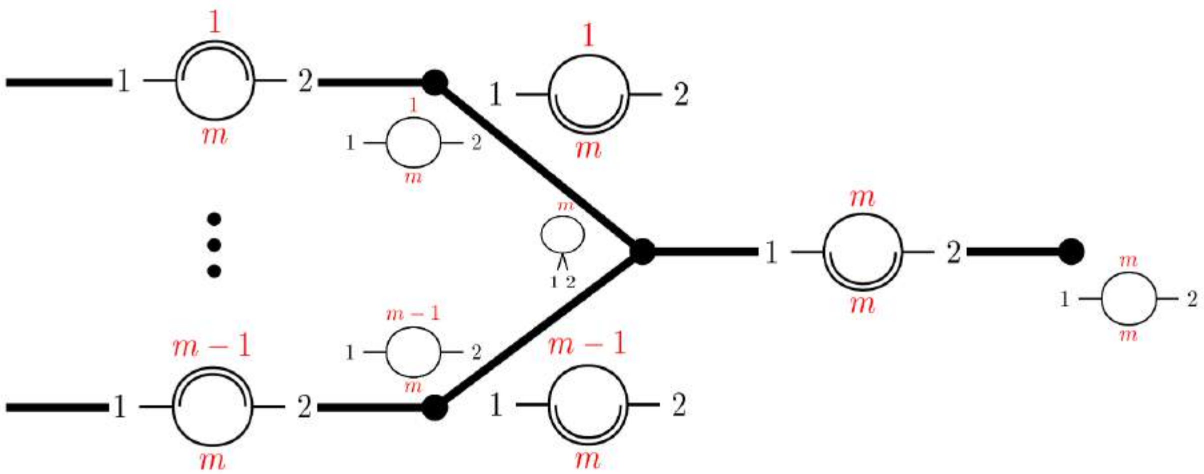


Figure 5.3: The subspace $A \subset X_{1,2,m}$ containing all cubes corresponding to graphs with at least one edge colored in color m.

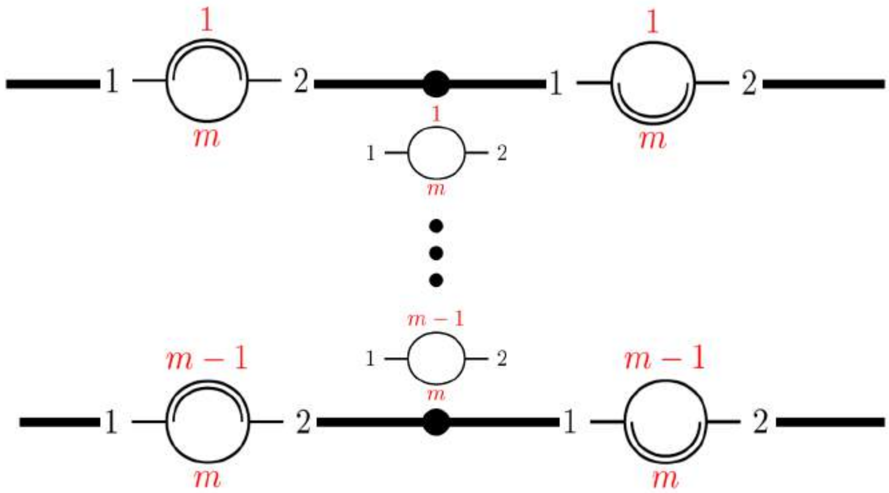


Figure 5.5: The intersection $A \cap B$ of the involved subspaces is a disjoint union of line segments.

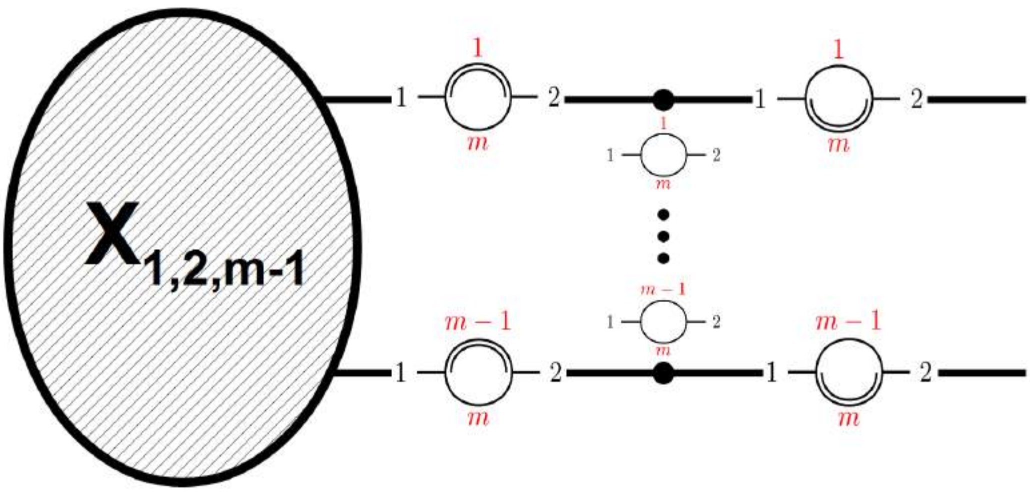


Figure 5.4: The subspace $B \subset X_{1,2,m}$ containing all cubes corresponding to graphs with at least one edge not colored in color m.

The intersection $A \cap B$ consists of all cubes corresponding to graphs that contain an edge colored with m but with the other edge colored differently. Figure 5.5 illustrates this intersection. A is contractible and B admits a deformation retract to a subspace homeomorphic to $X_{1,2,m-1}$ that shrinks all cubes containing an edge colored with m to zero length. Their intersection $A \cap B$ consists of $m-1$ disjoint lines and can be deformation retracted to $m-1$ disjoint points.
The reduced Mayer-Vietoris sequence for $X_{1,2,m}$ and subspaces A and B reads

$$\begin{aligned} 0 \longrightarrow \tilde{H}_1(A \cap B; \mathbb{Q}) &\longrightarrow \tilde{H}_1(A; \mathbb{Q}) \oplus \tilde{H}_1(B; \mathbb{Q}) \longrightarrow \tilde{H}_1(X_{1,2,m}; \mathbb{Q}) \\ \longrightarrow \tilde{H}_0(A \cap B; \mathbb{Q}) &\longrightarrow \tilde{H}_0(A; \mathbb{Q}) \oplus \tilde{H}_0(B; \mathbb{Q}) \longrightarrow \ldots \end{aligned}$$

Now $\tilde{H}_n(A; \mathbb{Q}) = 0$ since A is homotopy equivalent to a point. By the additivity axiom

$$\tilde{H}_1(A \cap B; \mathbb{Q}) \cong \bigoplus_{i=1}^{m-1} \tilde{H}_1(\{pt.\}; \mathbb{Q}) = 0$$

and

$$H_0(A \cap B; \mathbb{Q}) \cong H_0(\bigsqcup_{i=1}^{m-1} \{pt.\}; \mathbb{Q}) \cong \mathbb{Q}^{m-1},$$

hence $\tilde{H}_0(A \cap B; \mathbb{Q}) \cong \mathbb{Q}^{m-2}$. Furthermore $\tilde{H}_1(B; \mathbb{Q}) \cong \tilde{H}_1(X_{1,2,m-1}; \mathbb{Q})$. Therefore the above long exact sequence contains a short exact sequence which under application of the above isomorphisms reads

$$0 \longrightarrow \tilde{H}_1(X_{1,2,m-1}; \mathbb{Q}) \longrightarrow \tilde{H}_1(X_{1,2,m}; \mathbb{Q}) \longrightarrow \mathbb{Q}^{m-2} \longrightarrow 0,$$

where the first zero is $\tilde{H}_1(A \cap B; \mathbb{Q})$. This is a split exact sequence, so we immediately obtain

$$\tilde{H}_1(X_{1,2,m}; \mathbb{Q}) \cong \tilde{H}_1(X_{1,2,m-1}; \mathbb{Q}) \oplus \mathbb{Q}^{m-2}$$

and in particular $\dim H_1(X_{1,2,m}; \mathbb{Q}) = \dim H_1(X_{1,2,m-1}; \mathbb{Q}) + m - 2$. The case $m = 1$ is clear since $X_{1,2,1} \cong X_{1,2}$, hence $\dim H_1(X_{1,2,1}; \mathbb{Q}) = 0$. With this the proposition follows by induction. □

5.1.3 Algebraic Considerations

For any given $m \in \mathbb{N}$, there is a canonical linear map $f^{(m)} : C_\bullet(X_{n,s,m}) \to C_\bullet(X_{n,s})$ defined by forgetting the coloring on each graph. This means for any $\langle \Gamma, F, c \rangle \in C_\bullet(X_{n,s,m})$, the map $f^{(m)}$ is defined by

$$f^{(m)}(\langle \Gamma, F, c \rangle) := \langle \Gamma, F \rangle \tag{5.6}$$

and extended by linearity. This map induces a canonical map on the corresponding homology groups due to the following proposition.

Proposition 5.1.6. *The map $f^{(m)}$ is a chain map between the chain complexes $(C_\bullet(X_{n,s,m}), \partial_\bullet^m)$ and $(C_\bullet(X_{n,s}), \partial_\bullet)$, i.e. $f^{(m)} \circ \partial^m = \partial \circ f^{(m)}$.*

Proof. Let $x \in C_k(X_{n,s,m}; \mathbb{Q})$ and write $x = \sum_i q_i \langle \Gamma_i, F_i, c_i \rangle$ with $q_i \in \mathbb{Q}$ for all i. Then

$$\begin{aligned}
(f^{(m)} \circ \partial_k^m)(x) &= f^{(m)}(\sum_i q_i \partial_k^m(\langle \Gamma_i, F_i, c_i \rangle)) \\
&= f^{(m)}(\sum_i q_i \sum_{j=1}^k (-1)^{j-1}(\langle \Gamma_i, F_i \backslash \{e_j\}, c_i \rangle - \langle \Gamma_{i,e_j}, F_{i,e_j}, c_{i,e_j} \rangle)) \\
&= \sum_i q_i \sum_{j=1}^k (-1)^{j-1}(f^{(m)}(\langle \Gamma_i, F_i \backslash \{e_j\}, c_i \rangle) - f^{(m)}(\langle \Gamma_{i,e_j}, F_{i,e_j}, c_{i,e_j} \rangle)) \\
&= \sum_i q_i \sum_{j=1}^k (-1)^{k-1}(\langle \Gamma_i, F_i \backslash \{e_j\} \rangle - \langle \Gamma_{i,e_j}, F_{i,e_j} \rangle) \\
&= \sum_i q_i \partial_k(\langle \Gamma_i, F_i \rangle) \\
&= \partial_k(\sum_i q_i f^{(m)}(\langle \Gamma_i, F_i, c_i \rangle)) \\
&= (\partial_k \circ f^{(m)})(x).
\end{aligned} \tag{5.7}$$

□

As a chain map $f^{(m)}$ descends to a linear map $f_*^{(m)} : H_\bullet(X_{n,s,m}) \to H_\bullet(X_{n,s})$ as described in the paragraph following definition 3.1.3.
On the other hand there is a linear map in the other direction. Let $\langle \Gamma, F\rangle \in C_\bullet(X_{n,s})$ and consider the map $c^{(m)} : C_\bullet(X_{n,s}) \to C_\bullet(X_{n,s,m})$ defined by

$$\begin{aligned} c^{(m)}(\langle \Gamma, F\rangle) :&= \sum_{\text{all colorings c}} \frac{1}{m^{|E_\Gamma^{int}|}} \langle \Gamma, F, c\rangle \\ &= \sum_{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}=1}^{m} \frac{1}{m^{|E_\Gamma^{int}|}} \langle \Gamma, F, \{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}\}\rangle \end{aligned} \tag{5.8}$$

and extended by linearity. Here $\{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}\}$ denotes the m-coloring c that assigns an edge $e_j \in E_\Gamma^{\text{int}} = \{e_1,\ldots,e_{|E_\Gamma^{\text{int}}|}\}$ the value $c(e_j) = i_j$ for all $1 \le j \le |E_\Gamma^{\text{int}}|$. This is certainly not unique since it involves an arbitrary ordering of the edges, but this is no problem for the definition of $c^{(m)}$ since it involves only the sum over all possible colorings. What follows will assume that some ordering of edges was fixed.

Proposition 5.1.7. *The map $c^{(m)}$ is a chain map between the chain complexes $(C_\bullet(X_{n,s}), \partial)$ and $(C_\bullet(X_{n,s,m}), \partial^m)$, i.e. $c^{(m)} \circ \partial = \partial^m \circ c^{(m)}$.*

Proof. Let $\langle \Gamma, F\rangle \in C_k(X_{n,s})$. For $k = 0$ the equation $c^{(m)} \circ \partial_0 = \partial_0^m \circ c^{(m)}$ is trivially true. Note that for any $e \in F$ the equation $|E_{\Gamma_e}^{int}| = |E_\Gamma^{int}| - 1$ holds. Then for all $k \in \mathbb{N}$ we have

$$\begin{aligned} (c^{(m)} \circ \partial_k)(\langle \Gamma, F\rangle) &= c^{(m)}(\sum_{i=1}^{k}(-1)^{i-1}(\langle \Gamma, F\backslash\{e_i\}\rangle - \langle \Gamma_{e_i}, F_{e_i}\rangle)) \\ &= \sum_{i=1}^{k}(-1)^{i-1}(c^{(m)}(\langle \Gamma, F\backslash\{e_i\}\rangle) - c^{(m)}(\langle \Gamma_{e_i}, F_{e_i}\rangle)) \\ &= \sum_{i=1}^{k}(-1)^{i-1}(\sum_{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}=1}^{m} \frac{1}{m^{|E_\Gamma^{int}|}} \langle \Gamma, F\backslash\{e_i\}, \{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}\}\rangle \\ &\qquad - \sum_{j_1,\ldots,j_{|E_\Gamma^{\text{int}}|-1}=1}^{m} \frac{1}{m^{|E_\Gamma^{int}|-1}} \langle \Gamma_{e_i}, F_{e_i}, \{j_1,\ldots,j_{|E_\Gamma^{\text{int}}|-1}\}\rangle) \\ &= \sum_{i=1}^{k}(-1)^{i-1}(\sum_{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}=1}^{m} \frac{1}{m^{|E_\Gamma^{int}|}} \langle \Gamma, F\backslash\{e_i\}, \{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}\}\rangle \\ &\qquad - \sum_{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}=1}^{m} \frac{1}{m^{|E_\Gamma^{int}|}} \langle \Gamma_{e_i}, F_{e_i}, \{i_1,\ldots,\hat{i_i},\ldots,i_{|E_\Gamma^{\text{int}}|}\}\rangle) \\ &= \sum_{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}=1}^{m} \frac{1}{m^{|E_\Gamma^{int}|}} \sum_{i=1}^{k}(-1)^{i-1}(\langle \Gamma, F\backslash\{e_i\}, \{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}\}\rangle \\ &\qquad - \langle \Gamma_{e_i}, F_{e_i}, \{i_1,\ldots,\hat{i_i},\ldots,i_{|E_\Gamma^{\text{int}}|}\}\rangle) \\ &= \sum_{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}=1}^{m} \frac{1}{m^{|E_\Gamma^{int}|}} \partial_k^m(\langle \Gamma, F, \{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}\}\rangle) \\ &= \partial_k^m(\sum_{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}=1}^{m} \frac{1}{m^{|E_\Gamma^{int}|}} \langle \Gamma, F, \{i_1,\ldots,i_{|E_\Gamma^{\text{int}}|}\}\rangle) \\ &= (\partial_k^m \circ c^{(m)})(\langle \Gamma, F\rangle). \end{aligned} \tag{5.9}$$

Hence by linearity of all involved maps we have $(c^{(m)} \circ \partial_k)(x) = (\partial_k^m \circ c^{(m)})(x)$ for all $x = \sum_i q_i \langle \Gamma_i, F_i \rangle \in C_\bullet(X_{n,s}; \mathbb{Q})$ (with $q_i \in \mathbb{Q}$ for all i). $\square$

This means the map $c^{(m)}$ also descends to a homomorphism $c_*^{(m)} : H_\bullet(X_{n,s}; \mathbb{Q}) \to H_\bullet(X_{n,s,m}; \mathbb{Q})$ on the homology groups. The situation can be summarized by the following two commutative diagrams.

$$\begin{tikzcd}
C_k(X_{n,s,m};\mathbb{Q}) \arrow[r, "\partial_k^m"] \arrow[d, "f^{(m)}"'] & C_{k-1}(X_{n,s,m};\mathbb{Q}) \arrow[d, "f^{(m)}"'] \\
C_k(X_{n,s};\mathbb{Q}) \arrow[r, "\partial_k"] & C_{k-1}(X_{n,s};\mathbb{Q})
\end{tikzcd}
\qquad
\begin{tikzcd}
C_k(X_{n,s,m};\mathbb{Q}) \arrow[r, "\partial_k^m"] & C_{k-1}(X_{n,s,m};\mathbb{Q}) \\
C_k(X_{n,s};\mathbb{Q}) \arrow[r, "\partial_k"] \arrow[u, "c^{(m)}"] & C_{k-1}(X_{n,s};\mathbb{Q}) \arrow[u, "c^{(m)}"]
\end{tikzcd}$$

For the two homomorphisms $f_*^{(m)}$ and $c_*^{(m)}$, the following relation holds:

Proposition 5.1.8. $f_*^{(m)} \circ c_*^{(m)} = id_{H_\bullet(X_{n,s};\mathbb{Q})}$. *In particular* $f_*^{(m)}$ *is surjective and* $c_*^{(m)}$ *is injective.*

Proof. Let $k \in \mathbb{N}_0$. For all $[x] = [\sum_i q_i \langle \Gamma_i, F_i \rangle] \in H_k(X_{n,s}; \mathbb{Q})$ with $q_i \in \mathbb{Q}$ for all i we have

$$(f_*^{(m)} \circ c_*^{(m)})([x]) = [\sum_i q_i (f^{(m)} \circ c^{(m)})(\langle \Gamma_i, F_i \rangle)]$$

and

$$\begin{aligned}
(f^{(m)} \circ c^{(m)})(\langle \Gamma_i, F_i \rangle) &= f^{(m)}(\sum_{\text{all colorings c}} \frac{1}{m^{|E_\Gamma^{int}|}} \langle \Gamma_i, F_i, c \rangle) \\
&= \sum_{\text{all colorings c}} \frac{1}{m^{|E_\Gamma^{int}|}} f(\langle \Gamma_i, F_i, c \rangle) \\
&= \sum_{\text{all colorings c}} \frac{1}{m^{|E_\Gamma^{int}|}} \langle \Gamma_i, F_i \rangle \\
&= \langle \Gamma_i, F_i \rangle.
\end{aligned}$$

for all i. Hence $(f_*^{(m)} \circ c_*^{(m)})(x) = x$ which completes the proof. $\square$

This means in particular that $\dim H_k(X_{1,s}; \mathbb{Q}) \leq \dim H_k(X_{1,s,m}; \mathbb{Q})$ for all k, s and m. Taking a closer look at the tables 5.1 suggest that there might even be an isomorphism $H_k(X_{1,s,m}; \mathbb{Q}) \to H_k(X_{1,s}; \mathbb{Q})$ for $k < s - 1$. The two maps considered in this subsection are plausible candidates for such a potential isomorphism.

5.1.4 Homology in the Highest Non-Trivial Dimension

The spaces $X_{n,s,m}$ naturally contain m copies of $X_{n,s}$, containing all graphs whose edges are colored identically. We use this fact by considering the subspace

$$A := \{(\Gamma, F, c) \in X_{n,s,m} \mid c \text{ is a constant map}\}$$

and the long exact sequence of the pair $(X_{n,s,m}, A)$. To simplify notation we set $X := X_{n,s,m}$. The exact sequence then reads

$$\ldots \to H_{k+1}(X, A; \mathbb{Q}) \to H_k(A; \mathbb{Q}) \to H_k(X; \mathbb{Q}) \to H_k(X, A; \mathbb{Q}) \to \ldots$$

We have $A \cong \sqcup_{i=1}^{m} X_{n,s,1} \cong \sqcup_{i=1}^{m} X_{n,s}$ and hence $H_k(A; \mathbb{Q}) \cong (H_k(X_{n,s}; \mathbb{Q}))^m$ for all $k \in \mathbb{N}_0$. In the one-loop case this means in particular that $H_k(A; \mathbb{Q}) = 0$ for all odd k (see equation (4.3)). Thus, in case of an odd number of external legs s, this yields a short exact sequence

$$0 \to H_{s-1}(A; \mathbb{Q}) \to H_{s-1}(X; \mathbb{Q}) \to H_{s-1}(X, A; \mathbb{Q}) \to 0.$$

This sequence splits since we consider coefficients in $\mathbb{Q}$, which makes every homology group under consideration a vector space. Thus

$$H_{s-1}(X; \mathbb{Q}) \cong (H_{s-1}(X_{1,s}; \mathbb{Q}))^m \oplus H_{s-1}(X, A; \mathbb{Q}),$$

where the isomorphism is

$$(H_{s-1}(X_{1,s}; \mathbb{Q}))^m \oplus H_{s-1}(X, A; \mathbb{Q}) \to H_{s-1}(X; \mathbb{Q}),$$

$$(x_1, \ldots, x_m, y) \mapsto \sum_{a=1}^{m} (j_a)_*(x_a) + i_*^{-1}(y)$$

with i_*^{-1} the left inverse of the homomorphism $i_* : H_{s-1}(X; \mathbb{Q}) \to H_{s-1}(X, A; \mathbb{Q})$ induced by the natural inclusion and the $(j_a)_* : H_{s-1}(X_{1,s}; \mathbb{Q}) \to H_{s-1}(X; \mathbb{Q})$ are homomorphisms induced by the inclusion maps $j_a : X_{1,s} \to X$ defined by

$$j_a(\Gamma, F) := (\Gamma, F, c_a),$$

where the c_a are constant m-colorings with $c_i(e) = a$ for all $e \in E_\Gamma^{\text{int}}$. This accounts for what was already encountered in example 5.1.3: A generator of $H_{s-1}(X_{1,s}; \mathbb{Q})$ endowed with a constant coloring is a generator of $H_{s-1}(X_{1,s,m}; \mathbb{Q})$ and moreover these generators are independent for different colors. Therefore the full homology can in principle be understood by studying the relative homology groups $H_{s-1}(X_{1,s,m}, A; \mathbb{Q})$ and thus adding the remaining generators.

5.2 Moduli Space of Holo-Colored Graphs

As mentioned in chapter 4, some of the simplices get folded onto themselves under the quotient by the group action, which leads to the necessity of a subdivision into cubes to calculate the homology. This problem can be avoided by assigning each edge of a graph a different color which allows for a computation of the homology groups as a Δ-complex. Such graphs are referred to as holo-colored graphs in this text.

5.2.1 Holo-Colored Graphs with Forgetful Edges

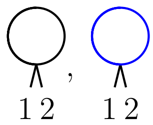
The first case that is considered here is very similar to section 5.1 in its construction. The only difference is that we restrict each complex $X_{n,s,m}$ to the cubes associated to holo-colored graphs. The resulting cubical complex will be denoted by $\tilde{X}_{n,s}$, the corresponding chain complex by $C_\bullet(\tilde{X}_{n,s}, \tilde{\partial}_\bullet)$, where again n is the number of loops and s the number of external legs. The boundary operator of the Δ-complex contains the terms with shrunken edges only and reads[3]

$$\tilde{\partial}_k(\langle \Gamma, c\rangle) := \sum_{i=1}^{k} (-1)^{i-1} \langle \Gamma_{e_i}, c_{e_i}\rangle,$$

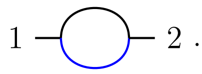
where the dimension k is the number of internal edges in Γ. Spanning forests do not appear in this case, since their appearance is a result of the cubical subdivision. The number of colors used will always be chosen to be the maximal number of internal edges occurring in the complex. For the one-loop case $n = 1$ this is s.

$\tilde{X}_{1,1}$ contains only a single 0-simplex and is identical to $X_{1,1,1} \cong X_{1,1}$. The next examples analyze the cases $n = 1, s = 2$ and $n = 1, s = 3$ in detail.

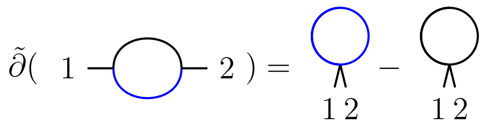
Example 5.2.1. *The Δ-complex associated to $\tilde{X}_{1,2}$ contains two 0-simplices*



and one 1-simplex



Applying $\tilde{\partial}$ to the latter yields



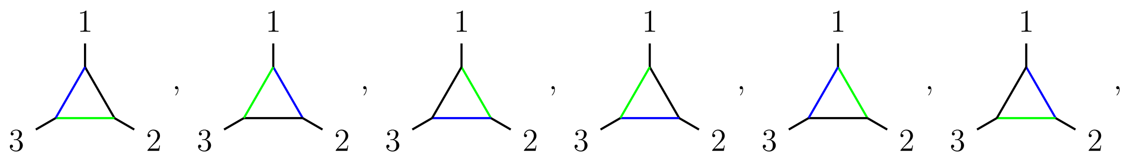
and thus

$$\ker \tilde{\partial}_1 = 0 \qquad \operatorname{im} \tilde{\partial}_1 = 1.$$

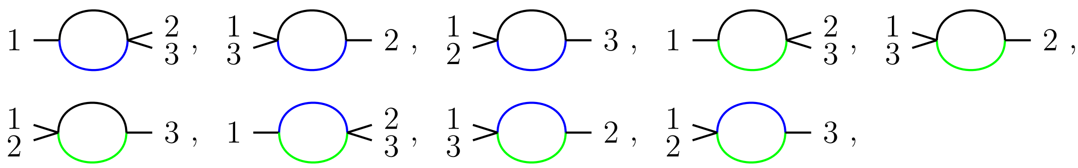
The homology groups are hence

$$H_k(\tilde{X}_{1,2}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \textit{if } k = 0 \\ 0, & \textit{otherwise.} \end{cases}$$

Example 5.2.2. *We consider the space of holo-colored graphs with three external legs $\tilde{X}_{1,3}$. It consists of six 2-simplices*

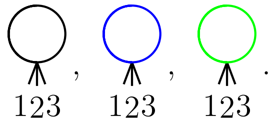


nine 1-simplices



[3]This simple definition works only for the one-loop space $\tilde{X}_{1,s}$. Otherwise it might not be possible to shrink an edge to zero length, i.e. some faces are missing in the Δ-complex.

and three 0-simplices



As before, the computation of the boundaries for each simplex is straightforward and reveals

$$\dim\ker\tilde{\partial}_1 = 7 \qquad \dim\operatorname{im}\tilde{\partial}_1 = 2$$

$$\dim\ker\tilde{\partial}_2 = 1 \qquad \dim\operatorname{im}\tilde{\partial}_2 = 5.$$

A geometric representation of the associated Δ*-complex can be found in figure 5.6.*

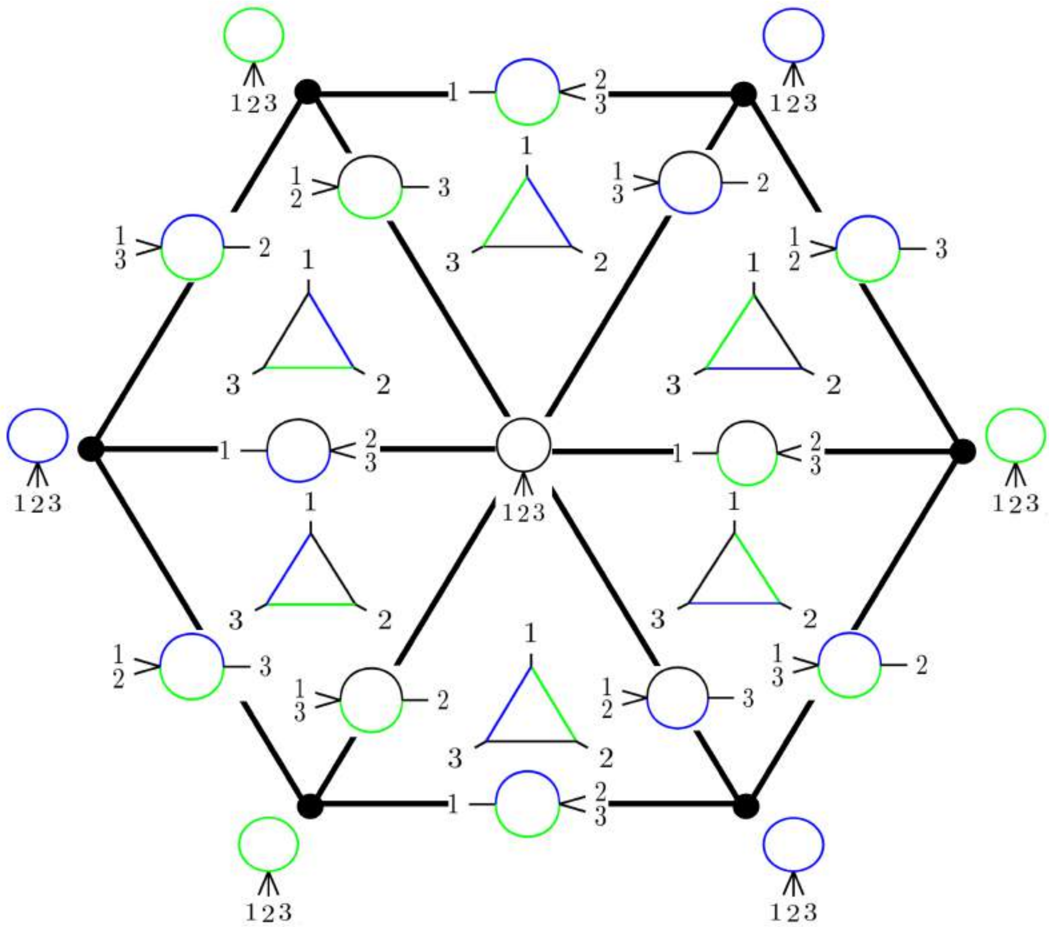
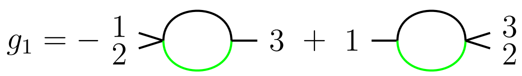
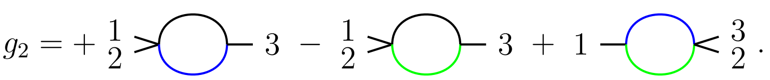


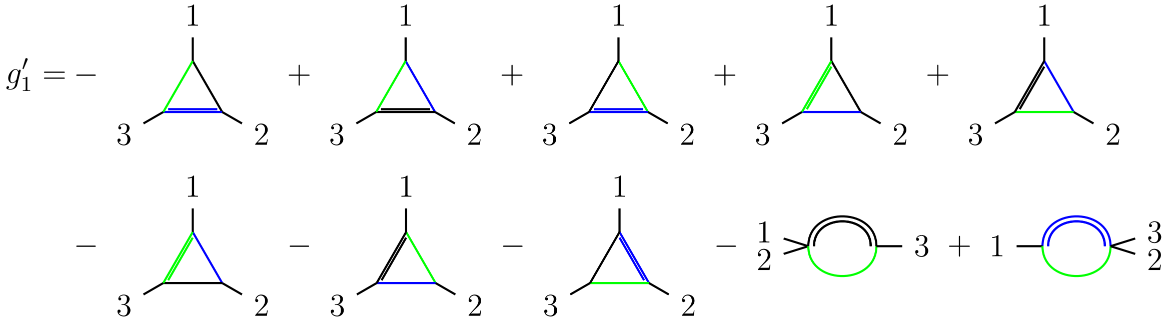
Figure 5.6: A geometric representation of $\tilde{X}_{1,3}$ as a Δ-complex.

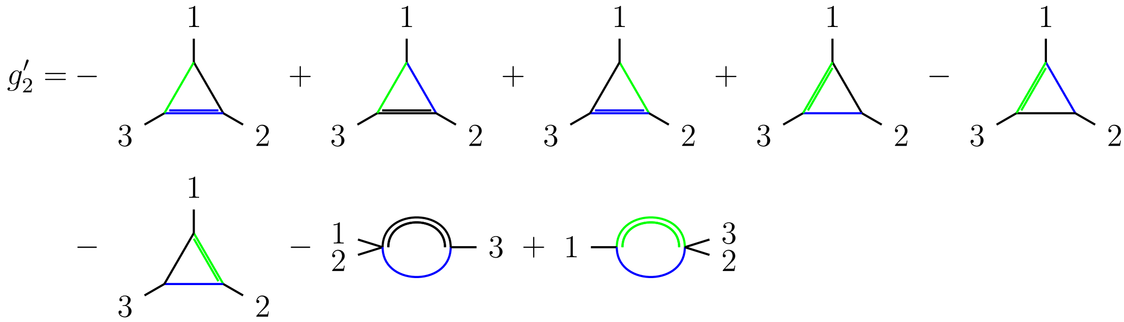
In this picture identification of lines and vertices labeled by the same graphs is implied. A closer inspection reveals this space to be a 2-torus. A choice of generators g_1 *and* g_2 *for* $H_1(\tilde{X}_{1,3};\mathbb{Q})$ *is*



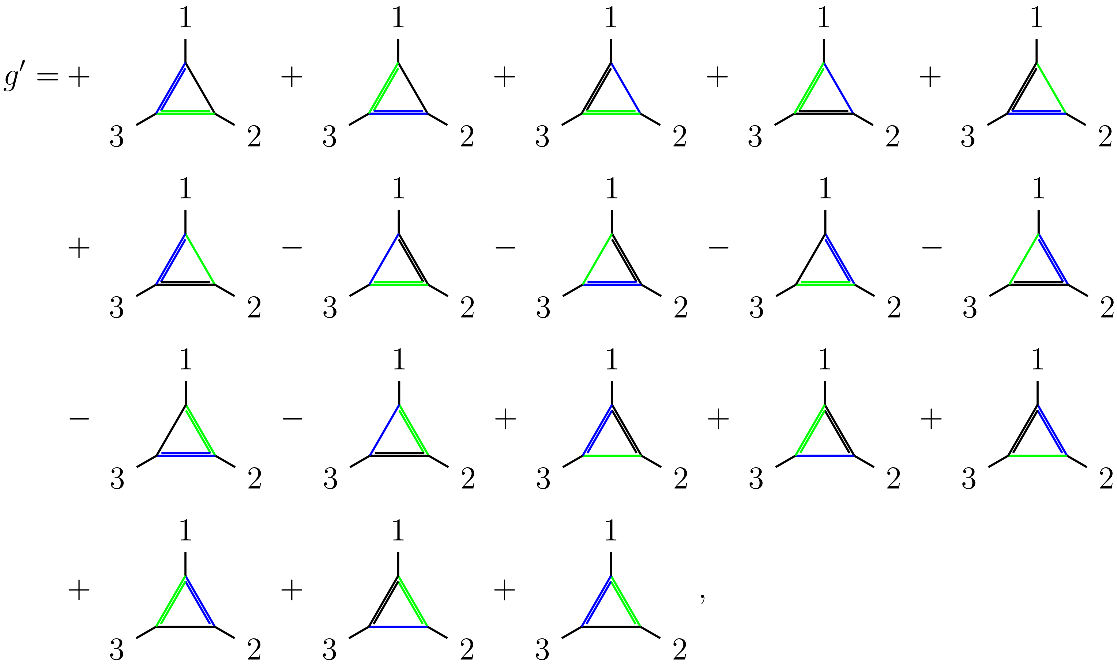


Alternatively, when performing the calculation in cubical homology, the set of generators can be chosen to be





for $H_1(\tilde{X}_{1,3};\mathbb{Q})$ *and*



which contains all possible 2-cubes, for $H_2(\tilde{X}_{1,3};\mathbb{Q})$*.*[4]

As in the previous cases the homology of the moduli space of holo-colored graphs was calculated with the help of a computer program. The only necessary change is a routine that sorts

[4]This is slight abuse of notation: The homology groups derived from the Δ-complex should in principal be distinguished from those derived from the cubical complex, although they are isomorphic.

	H_0	H_1	H_2	H_3	H_4
$\tilde{X}_{1,1}$	1	-	-	-	-
$\tilde{X}_{1,2}$	1	0	-	-	-
$\tilde{X}_{1,3}$	1	2	1	-	-
$\tilde{X}_{1,4}$	1	0	36	3	-
$\tilde{X}_{1,5}$	1	0	6	824	12

Table 5.2: The dimension of the homology groups $H_k(\tilde{X}_{1,s};\mathbb{Q})$ for $1 \leq s \leq 5$ and $0 \leq k \leq 4$.

out all graphs which are not holo-colored after listing all cubes. In this case, the calculation via a Δ-complex is also available. This simplifies the computations substantially, since there are fewer simplices than cubes and a k-simplex has less faces than a k-cube.
The dimensions of the homology groups belonging to the spaces $\tilde{X}_{1,s}$ that could be obtained are listed in table 5.2 for $1 \leq s \leq 5$. A particular choice of generators for these groups can be found in appendix B.

5.2.1.1 Combinatorics

The number of k-simplices in the space $\tilde{X}_{1,s}$ of metric holo-colored graphs can be determined by combinatorial means. For an admissible graph Γ without external legs and k internal edges, denote the set of all holo-colorings (with s available colors) of Γ by $\mathcal{C}_{k,s}(\Gamma)$, the set of non-equivalent external leg structures with s legs the graph Γ admits by $\mathcal{S}_{k,s}(\Gamma)$.
For a one-loop holo-colored graph Γ with k internal edges there are $\binom{s}{k}$ ways to choose which colors the edges of Γ can have, i.e. ways to choose the image of the coloring. Furthermore there are $k!$ ways to permute these edges to get different graphs, except in the case $k = 2$ where both permutations of edges yield the same graph. Thus

$$|\mathcal{C}_{k,s}(\Gamma)| = \binom{s}{k}\frac{k!}{1+\delta_{k,2}}. \tag{5.10}$$

The external leg structures is uniquely determined by a partition of the s legs into k non-empty groups and an element of $S_k/(C_k \ltimes \mathbb{Z}_2)$. There are $\left\{ {s \atop k} \right\}$ number of ways to choose such a partition, where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind (see for example [RG94], first section of chapter 6) given by

$$\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

The group $S_k/(C_k \ltimes \mathbb{Z}_2)$ has $\frac{k!}{2}\frac{1}{k} = \frac{(k-1)!}{2}$ elements, thus there are exactly this many ways to order the groups of legs to yield distinct graphs. The only exceptions are $k = 1$ and $k = 2$: In both cases there is clearly only one way to organize the groups of legs (and the above consideration is not valid since we defined $S_k/(C_k \ltimes \mathbb{Z}_2)$ only for $k \geq 3$). Hence

$$|\mathcal{S}_{k,s}(\Gamma)| = \left\{ {s \atop k} \right\}\frac{(k-1)!}{2}(1+\delta_{k,1}+\delta_{k,2}). \tag{5.11}$$

With this, the number of $(k-1)$-simplices $N_{k,s}$ in $\tilde{X}_{1,s}$ can be calculated. Each such $(k-1)$-simplex belongs to a one-loop holo-colored graph with k internal edges and s external edges.

There are $|\mathcal{C}_{k,s}(\Gamma)| \cdot |\mathcal{S}_{k,s}(\Gamma)|$ such graphs since the choices of the coloring and the external legs structure are independent, so by the above equations (5.10) and (5.11)

$$\begin{aligned} N_{k,s} &= (\binom{s}{k} k!)(\left\{ {s \atop k} \right\} \frac{(k-1)!}{2}(1+\delta_{k,1})) \\ &= \frac{(k-1)!}{2-\delta_{k,1}} \binom{s}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^s. \end{aligned}$$

This in turn can be used to calculate the Euler characteristic χ of $\tilde{X}_{1,s}$:

$$\begin{aligned} \chi(\tilde{X}_{1,s}) &= \sum_{k=1}^{s} (-1)^{k-1} N_{k,s} \\ &= \sum_{k=1}^{s} (-1)^{k-1} \frac{(k-1)!}{2-\delta_{k,1}} \binom{s}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^s \\ &= \sum_{k=1}^{s} \sum_{j=0}^{k} (-1)^{j+1} \frac{j^s (k-1)!}{2-\delta_{k,1}} \binom{s}{k} \binom{k}{j} \end{aligned} \tag{5.12}$$

For $1 \le s \le 8$, the Euler characteristic obtained by this formula can be found in table 5.3.

s	1	2	3	4	5	6	7	8
$\chi(\tilde{X}_{1,s})$	1	1	0	34	-805	26541	-1122506	59485588

Table 5.3: The Euler characteristic of $\tilde{X}_{1,s}$ for $1 \le s \le 8$

5.2.1.2 Homology in the Highest Non-Trivial Dimensions

A one-loop holo-colored graph in the highest non-trivial dimension can be parametrized in the following way: Let $(\Gamma, F, c) \in \tilde{X}_{1,s}$. As a metric graph, Γ can be thought of as embedded in $\mathbb{R}^2$ by drawing a circle of finite radius (which is $\frac{1}{2\pi}$ if working with normalized graphs) to represent the union of all edges. Fix an arbitrary point on the circle and identify it with a vertex $v_0 \in V_\Gamma$. The length of each edge is uniquely determined by fixing $s-1$ angles $\theta_1, \dots, \theta_{s-1}$ representing the position of the remaining vertices $v_1, \dots, v_{s-1} \in V_\Gamma$ on the circle with respect to the position of the distinguished vertex v_0. For convenience we choose a parametrization such that angles θ_i reach from 0 to 1.
In this way (Γ, F, c) can be uniquely represented as $(\theta_1, \dots, \theta_{s-1}, \sigma)$, where $\sigma \in S_s/_\sim$ with $\left(\begin{smallmatrix} 1 & 2 & \dots & s \\ p_1 & p_2 & \dots & p_s \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 2 & \dots & s \\ p_1 & p_s & \dots & p_2 \end{smallmatrix}\right)$ signifies a permutation of the colored edges with respect to some fixed holo-coloring.[5] This can be used to set up a coarser cubical complex in which each σ designates an $(s-1)$-cube $C_\sigma := \langle \theta_1, \dots, \theta_{s-1}, \sigma \rangle$ with the boundary operator ∂ defined by

$$\partial^+(C_\sigma) := \sum_{i=1}^{s-1} (-1)^{i-1} \langle \theta_1, \dots, \theta_{i-1}, 0, \theta_{i+1}, \dots, \theta_{s-1}, \sigma \rangle$$

and

$$\partial^-(C_\sigma) := \sum_{i=1}^{s-1} (-1)^{i} \langle \theta_1, \dots, \theta_{i-1}, 1, \theta_{i+1}, \dots, \theta_{s-1}, \sigma \rangle.$$

[5] The quotient by the equivalence relation is necessary since $(\theta_1, \dots, \theta_{s-1}, \left(\begin{smallmatrix} 1 & 2 & \dots & s \\ p_1 & p_2 & \dots & p_s \end{smallmatrix}\right)) = (\theta_{s-1}, \dots, \theta_1, \left(\begin{smallmatrix} 1 & 2 & \dots & s \\ p_1 & p_s & \dots & p_2 \end{smallmatrix}\right))$.

Letting any external leg rotate once around the graph leads to a cyclic permutation of the edges. More specifically for any $\sigma \in S_s/_\sim$ we have

$$(\theta_1, \dots, \theta_{i-1}, 0, \theta_{i+1}, \dots, \theta_{s-1}, \sigma) = (\theta_1, \dots, \theta_{i-1}, 1, \theta_{i+1}, \dots, \theta_{s-1}, \tau_+(\sigma)).$$

This immediately yields

$$\partial_{s-1}(C_\sigma) = \partial^+_{s-1}(C_\sigma) + \partial^-_{s-1}(C_\sigma) = \partial^+_{s-1}(C_\sigma) - \partial^+_{s-1}(C_{\tau_+(\sigma)}) \tag{5.13}$$

for the boundary of any cube C_σ.
The highest non-trivial homology group $H_{s-1}(\tilde{X}_{1,s};\mathbb{Q})$ is just $\ker \partial_{s-1}$. Equation (5.13) sets up a linear system of equations to determine this kernel. To make this explicit, we quotient out the cyclic permutations generated by τ_+. The quotient group is $(S_s/_\sim)/C_s \cong S_s/(C_s \ltimes \mathbb{Z}_2)$ consisting of $\frac{s!}{2s} = \frac{(s-1)!}{2}$ equivalence classes and we choose a representatives $\sigma_1, \dots, \sigma_N$ (with $N = \frac{(s-1)!}{2}$) for each one. The matrix representation of ∂_{s-1} with respect to the basis

$$\partial^+_{s-1}(C_{\sigma_1}), \partial^+_{s-1}(C_{\tau_+(\sigma_1)}), \dots, \partial^+_{s-1}(C_{\tau_+^{s-1}(\sigma_1)}), \partial^+_{s-1}(C_{\sigma_2}), \partial^+_{s-1}(C_{\tau_+(\sigma_2)}), \dots, \partial^+_{s-1}(C_{\tau_+^{s-1}(\sigma_N)})$$

of the target space then reads

$$A_{\partial_{s-1}} = \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix},$$

containing N copies of

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}.$$

The matrix A can easily be brought into row-echolon form and reveal its rank to be rank $A = s-1$. Thus there are $\frac{(s-1)!}{2}$ solutions to the homogeneous system of equations defined by $A_{\partial_{s-1}}$ and we obtain

$$H_{s-1}(\tilde{X}_{1,s}) \cong \mathbb{Q}^{\frac{(s-1)!}{2}}. \tag{5.14}$$

5.2.2 Holo-Colored Graphs with Remembering Edges

When considering Feynman graphs in the operator product expansion, edges shrunken to zero length still carry the physical information (like mass or particle type) of the edge. So from a physicists perspective, it might be more interesting to consider an alternative complex, in which the face relations of cubes respect this restriction. This complex will be denoted by $\bar{X}_{n,s}$ where again n is the number of loops, s the number of external legs and the number of colors is taken to be the maximal number of internal edges that can occur.
Up to and including the triangle graph, this is no different than the complex of holo-colored graphs with forgetful edges. The examples from subsection 5.2.1 serve equally well as examples for the alternative space with remembering edges. In general a difference to the complex

from subsection 5.2.1 can only occur if there are at least two vertices to which more than one external edge is connected. In particular $H_{s-1}(\bar{X}_{1,s}) \cong H_{s-1}(\tilde{X}_{1,s})$ which means that the results from equation (5.14) is still valid, hence

$$H_{s-1}(\bar{X}_{1,s}) \cong \mathbb{Q}^{\frac{(s-1)!}{2}}. \tag{5.15}$$

For explicit calculations of the homology of $\bar{X}_{1,s}$ the representation of a simplex has to be slightly modified. Since the color of shrunken edges is now relevant when identifying faces, for each vertex with more than one external leg connected to it we add a set of colors associated to the edges shrunken to that vertex.
The homology dimensions of the one-loop case for $1 \leq s \leq 5$ that were again calculated with computer assistance can be found in table 5.4. An explicit choice of generators can be found in appendix B.

	H_0	H_1	H_2	H_3	H_4
$\bar{X}_{1,1}$	1	-	-	-	-
$\bar{X}_{1,2}$	1	0	-	-	-
$\bar{X}_{1,3}$	1	2	1	-	-
$\bar{X}_{1,4}$	1	0	18	3	-
$\bar{X}_{1,5}$	1	0	48	166	12

Table 5.4: The dimension of the homology groups $H_k(\bar{X}_{1,s}; \mathbb{Q})$ for $1 \leq s \leq 5$ and $0 \leq k \leq 4$.

Chapter 6

Conclusions

In this thesis the homology groups of moduli spaces of colored graphs were studied with primary focus on one-loop graphs. A set of generators for the homology groups of a non-colored space derived from generalized one-loop Outer space is determined for up to five external legs. For the highest dimensional non-trivial homology groups, which are 0 for even numbers of external legs and one-dimensional otherwise, the generator of the latter case is given by the sum of all cubes with proper signs for any number of external edges.
Furthermore a set of generators for three versions of a colored moduli spaces of one-loop graphs is obtained for small numbers of external legs and colors. Additional to the directly obtained generators calculated with computer assistance, the following results for arbitrary numbers of colors m or numbers of external legs s are established:

$$H_0(X_{1,1,m};\mathbb{Q}) \cong \mathbb{Q}^m,$$

$$H_1(X_{1,2,m};) \cong \mathbb{Q}^{\frac{(m-1)(m-2)}{2}},$$

$$H_{s-1}(\tilde{X}_{1,s};\mathbb{Q}) \cong H_{s-1}(\bar{X}_{1,s};\mathbb{Q}) \cong \mathbb{Q}^{\frac{(s-1)!}{2}}$$

An explicit formula for the Euler characteristic is $\tilde{X}_{1,s}$ also obtained by combinatorial means.
A study of maps between the arbitrarily colored spaces and the non-colored spaces yields some connections between the two, in particular that $\dim H_k(X_{1,s};\mathbb{Q}) \leq \dim H_k(X_{1,s,m};\mathbb{Q})$ for all k, s, and m.
In typical quantum field theories like QED, the half-integer spin of particles result in oriented lines. This is a feature that was not considered in this work and deserves further study. Additionally, results beyond the one-loop case would be desirable, since its combinatorial simplicity is likely to obscure more general features of the spaces.
In [JC15] the authors are able to calculate the dimensions of the homology groups for the two-loop non-colored moduli space by using the one-loop results and an assembly map. These techniques might also be applicable to the colored versions. Further study is required.
This work presents a first possible step to understand the connection between the topological structure of moduli spaces of graphs, capturing the combinatorics of Cutskosky cut and reduced graphs, and the analytical structure of Feynman amplitudes. Homology generators encode significant features of the topological spaces they belong to and should contain some interesting physics. The next step is to translate these generators via Feynman rules to a physical result, as far this turns out to be possible.

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Appendix A

The Program

The computer assisted computations were performed with a Matlab script. Section A.1 contains a detailed description of the class used to represent cubes and simplices in the computations. The following section A.2 explains the central functions used to list all cubes or simplices, generate the matrices representing the boundary operator, and computing the generators.

A.1 Graphs

Let $\langle x\rangle = \sum_{i=1}^{N} q_i\langle\Gamma_i, F_i, c_i\rangle$ be a chain element from any of the complexes under consideration. A term $q_i\langle\Gamma_i, F_i, c_i\rangle$ is represented by a class *graphPhysical* containing the following data:

Properties of the class graphPhysical

edges A $2\times|E^{\text{int}}_{\Gamma_i}|$-matrix A with entries in the natural numbers where ith column contains the ith edge (with respect to the normal ordering).

forest A $2\times|E_{F_i}|$-matrix where the columns are a subset of the columns in **edges** in the inherited ordering.

leaves A $|E^{\text{ext}}_{\Gamma_i}|\times|V_{\Gamma_i}|$-matrix containing entries a_{ij} such that $a_{ij}=1$ if and only if the ith external leg is connected to the jth vertex, $a_{ij}=0$ otherwise.

colors An array of length n containing numbers in the range of $1,\ldots,m$. The ith entry stands for the color of the ith edge with respect to the normal form.

missingColors A cell array containing the colors associated to edges of zero length. The ith entry is an array of natural numbers (in ascending order) representing the colors associated to the ith vertex of Γ.

coeff The coefficient $q_i\in\mathbb{Q}$.

Furthermore the class contains the following functions:

Methods of the class graphPhysical

graphPhysical(n) Constructor setting up a cube corresponding to a one-loop graph with n edges. The normal form list of edges is given by equation (4.6), **forest**, **leaves**, and **missingColors** are initialized as empty. The **colors** are initialized as 1 for every edge and **coeff** is also set to 1.

normalOrder() Puts the representation of a graph into normal order by the procedure described in subsubsection 4.2.1.1.

parity() Let $r = |V_{\Gamma_i}|$ and $V_{\Gamma_i} = \{v_1, v_2, \dots, v_r\}$. This function returns the representation of the graph under an exchange of vertices $v_i \leftrightarrow v_{r-i+2}$ for all $2 \leq i \leq r$.

shrinkEdge(e) Returns the graph with edge e shrunken to zero length.

removeEdge(e) Returns the graph with edge e removed from its spanning forest.

boundaryAH() Calculates the result of ∂ or ∂^m on a cube by calculating **shrinkEdge(e)** and **removeEdge(e)** for all edges **e** in the spanning forest and writing them in a cell array with appropriate coefficients according to equation (4.2) or (5.1) respectively.

A.2 Functions

The task of calculating the homology groups is split into several functions given by the following list.

getAllCubes(s, k, m) Returns a cell array containing all k-dimensional cubes in either $X_{1,s,m}$,$\tilde{X}_{1,s}$, or $\bar{X}_{1,s}$. This is achieved by calling the constructor of **graphPhysical** and the functions **getForests**, **getLeaves**, and **getColors**.

getAllSimplices(s, k, m) Returns a cell array containing all k-dimensional simplices in either $\tilde{X}_{1,s}$ or $\bar{X}_{1,s}$.

getForests(k, g) Takes an integer $k \geq 0$ and a graph g and returns a cell array containing all spanning forests of g with k internal edges.

getLeaves(s, n) Takes two natural numbers s, n and returns a cell array containing all matrices representing distinct external leg structures for a one-loop graph with s external edges and n vertices (or equivalently n internal edges).

getColors(m, n) Takes two natural numbers m, n and returns a cell array containing all m-colorings of a graph with n internal edges.

getVector(graphs, graphsComp) Converts *graphs* to a vector by comparing each graph in *graphs* with the list *graphsComp* to find the proper entry and then entering the coefficient of the graph in question.

getMatrix(graphs, graphsComp) Obtains a matrix representation of the boundary operators by successively calling **getVector** with all entries of *graphs*. The result is stored in a huge sparse-matrix.

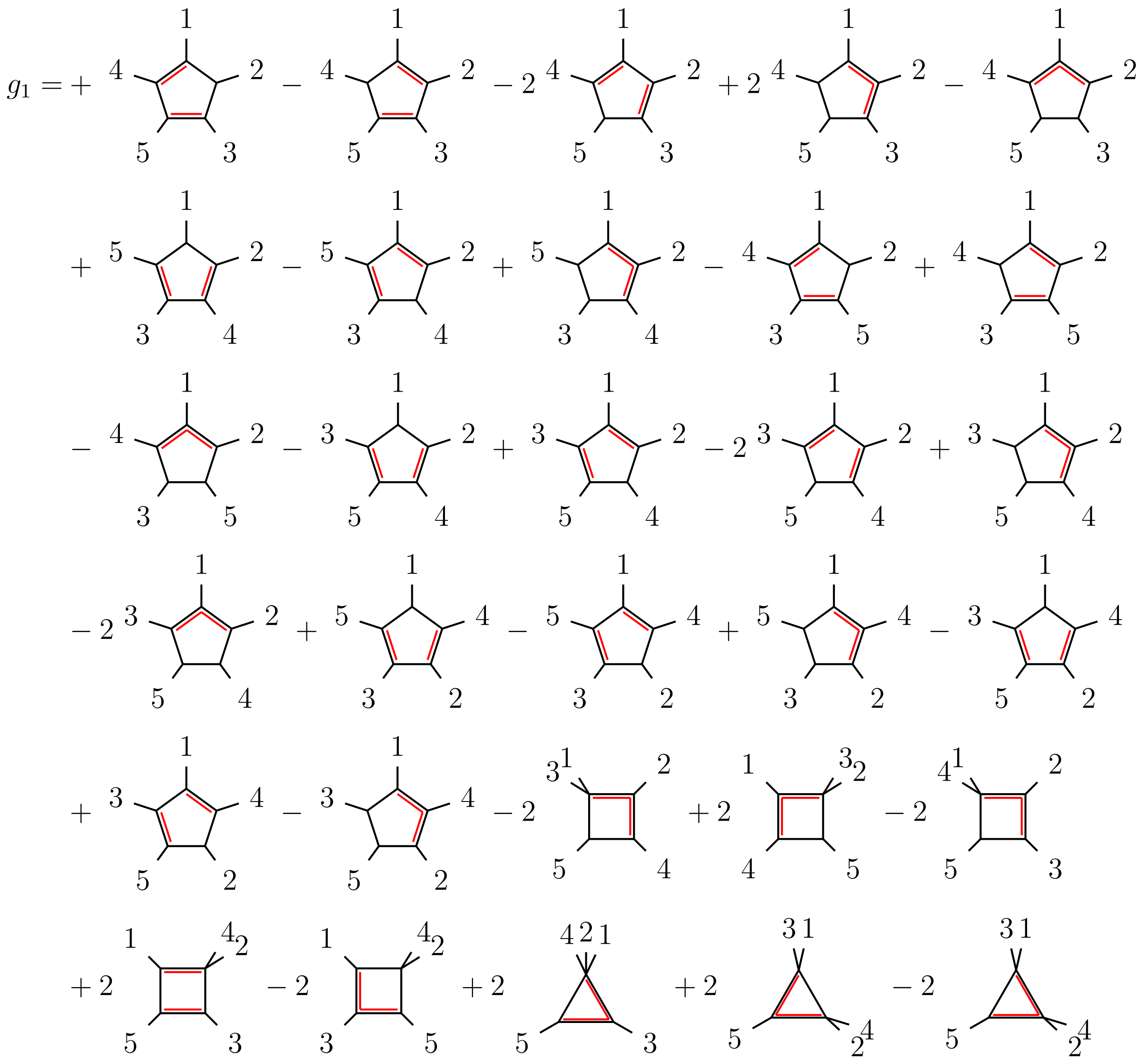
getGenerators(mat, graphs) Calculates a set of generators by simultaneously putting the matrix representations of ∂_n and ∂_{n+1} in row-echolon form for all n and then translating the results back to graphs. The row and column reductions are performed with a method to find exact solutions to linear systems of equations with rational coefficients described in [IB66] and [ASF71].

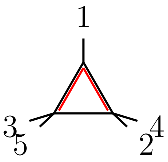
Appendix B

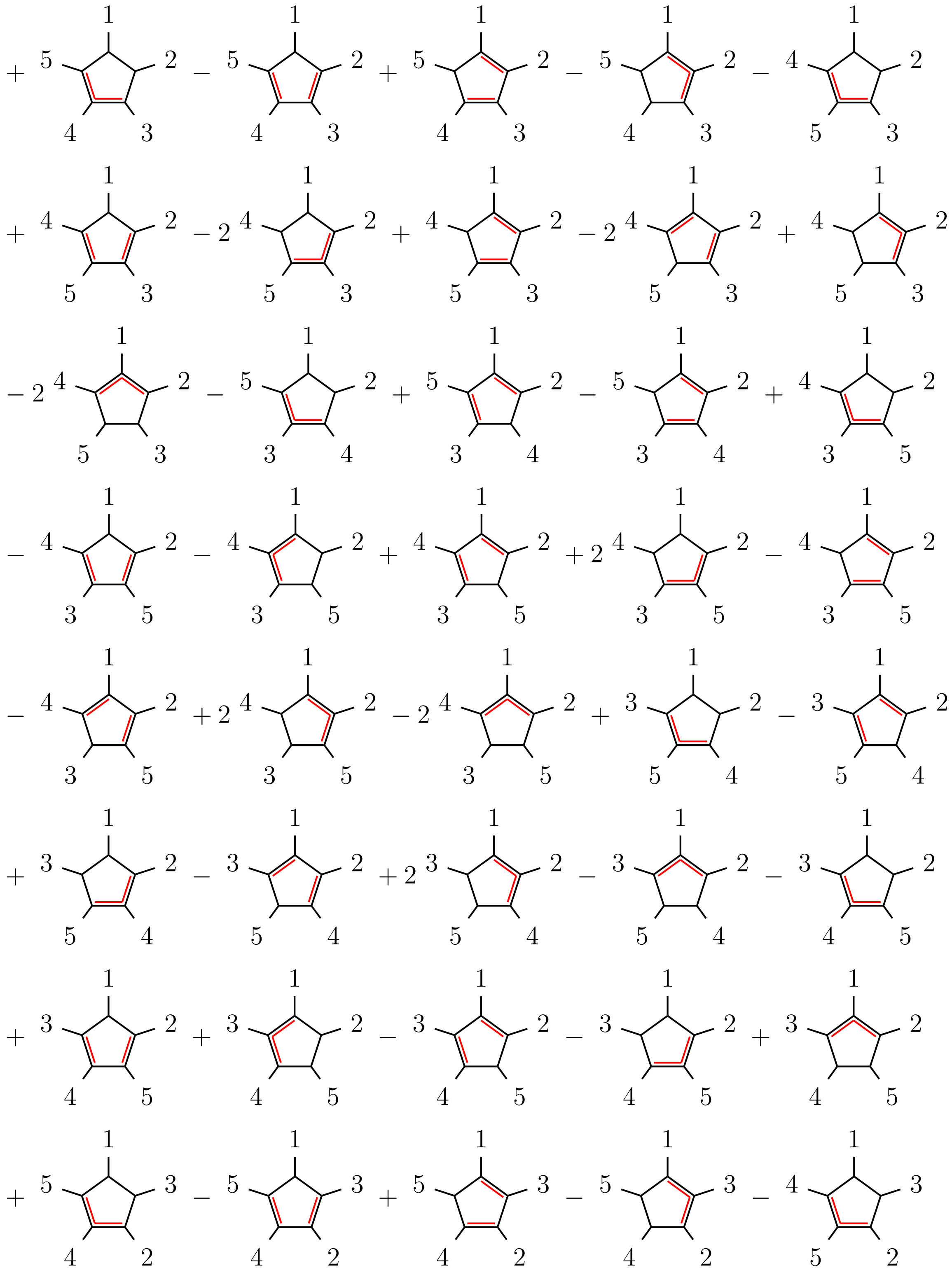
Homology Generators

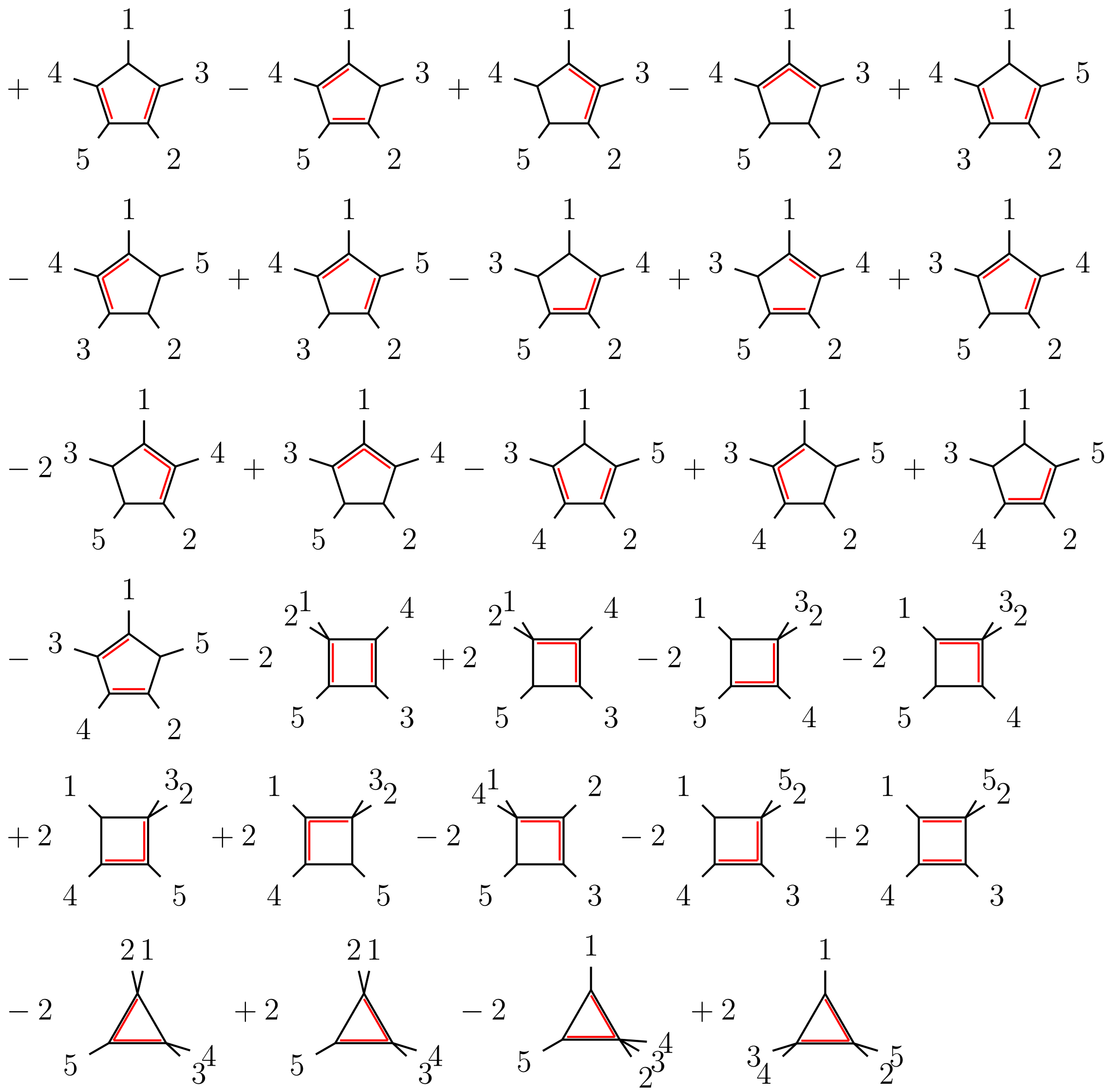
This chapter contains all sets of generators of the homology groups under consideration obtained, which were not explicitly given in chapters 4 and 5.

B.1 Generators of $H_2(X_{1,5};\mathbb{Q})$



$+\,2$ 

$g_2 =$ 



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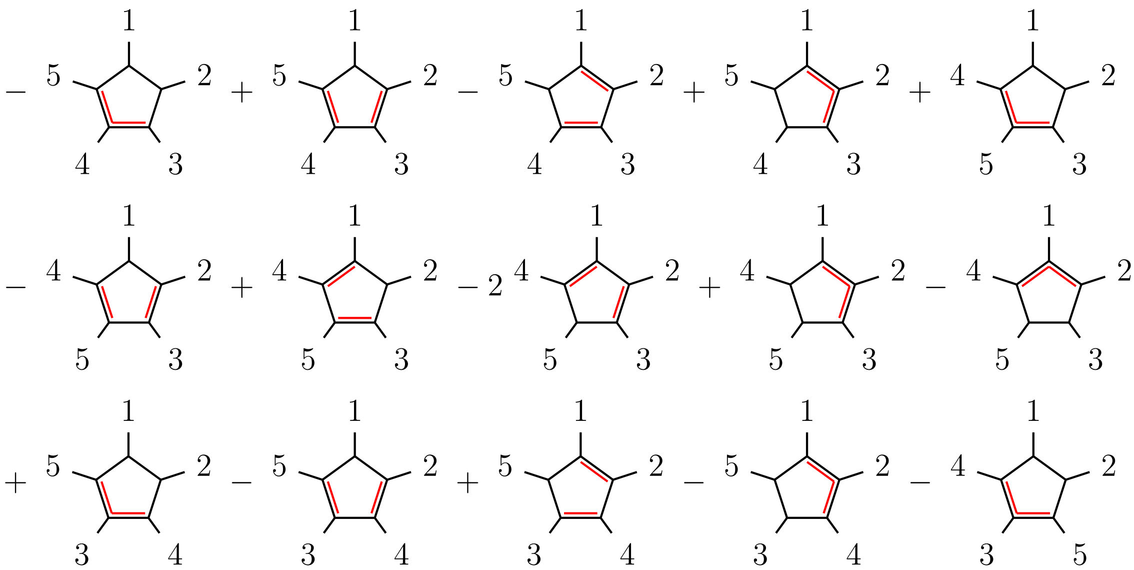
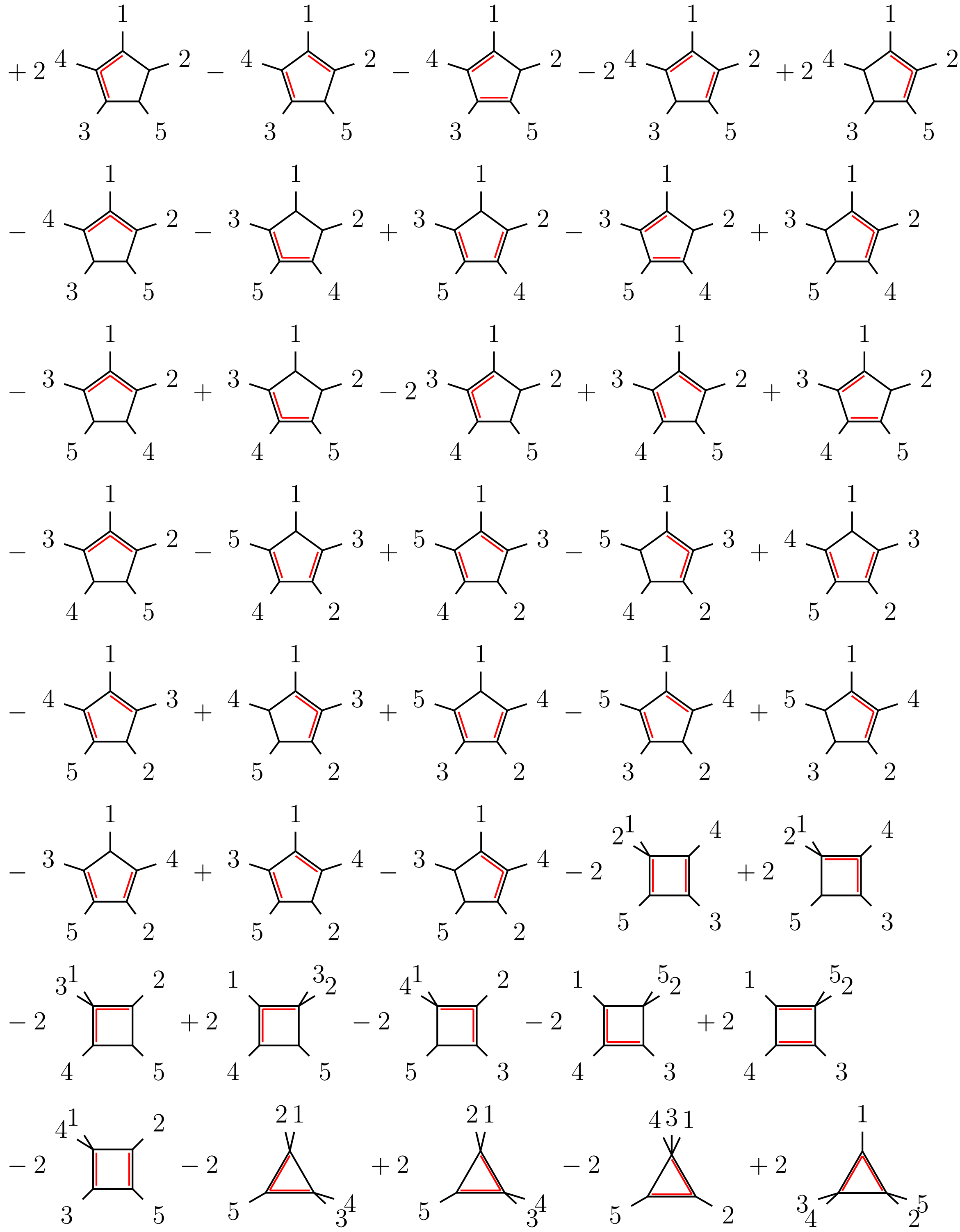
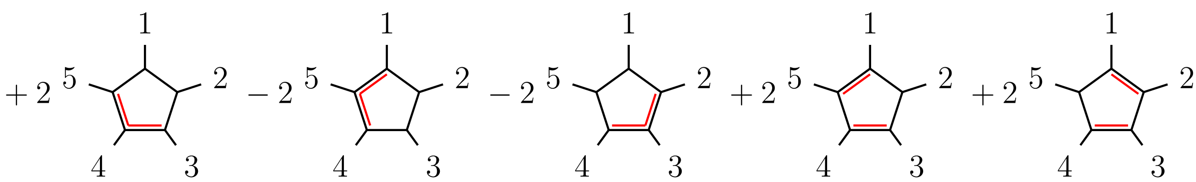
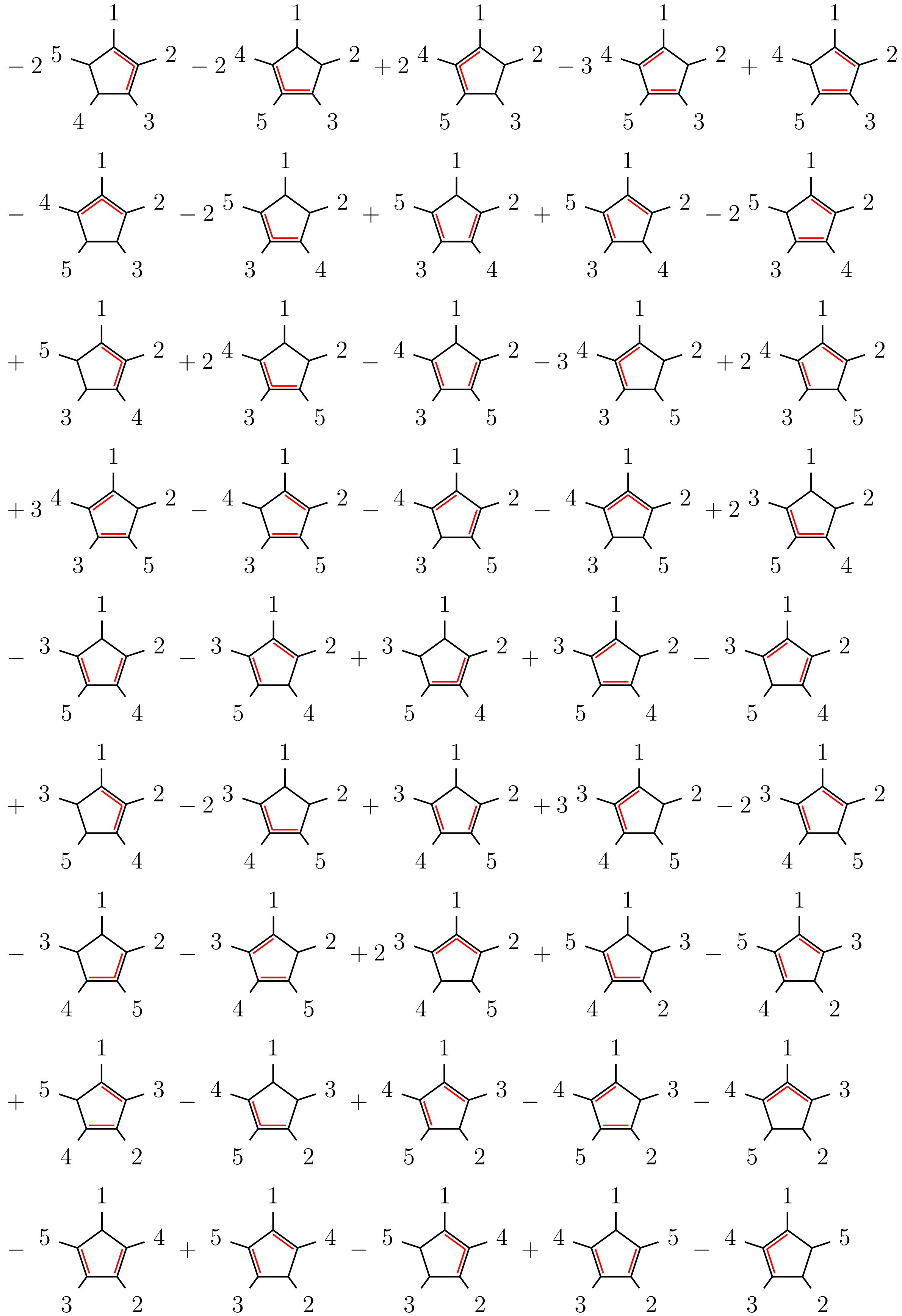
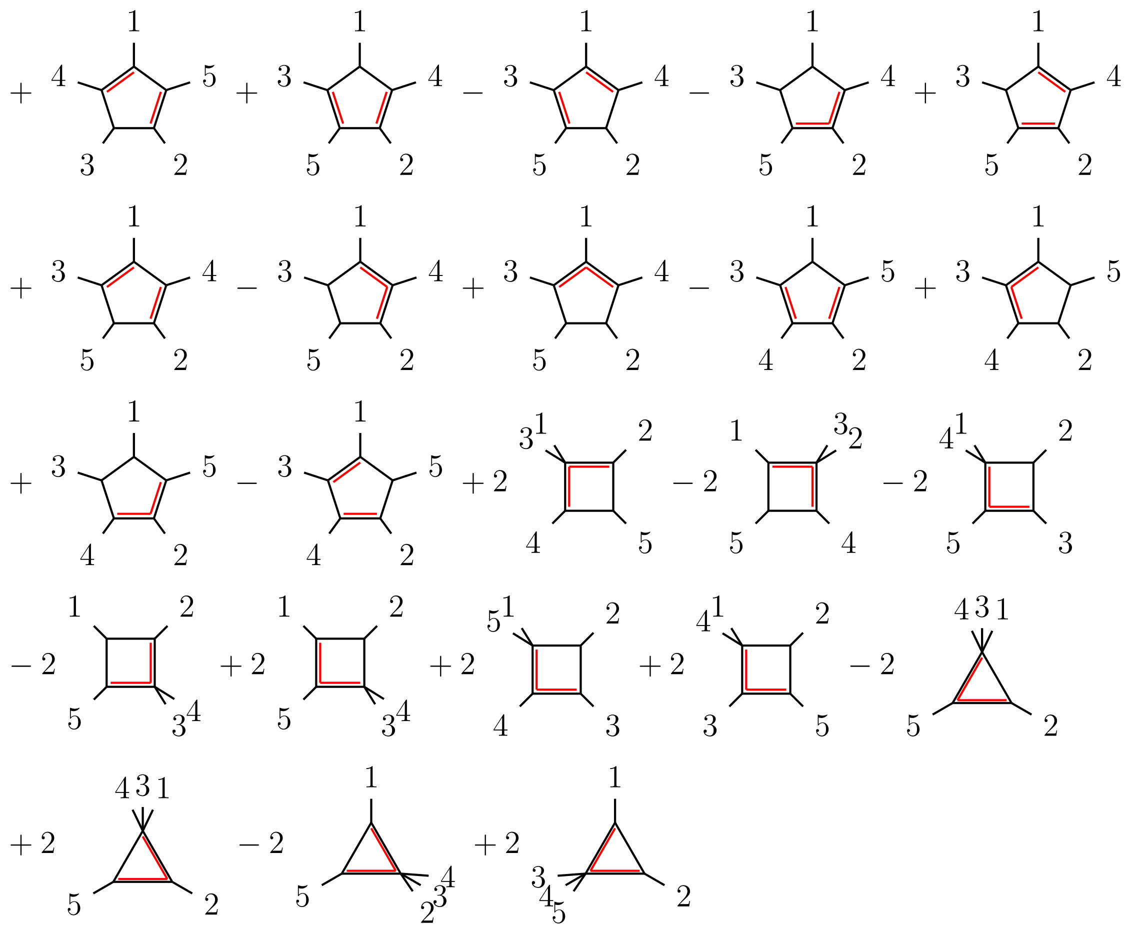
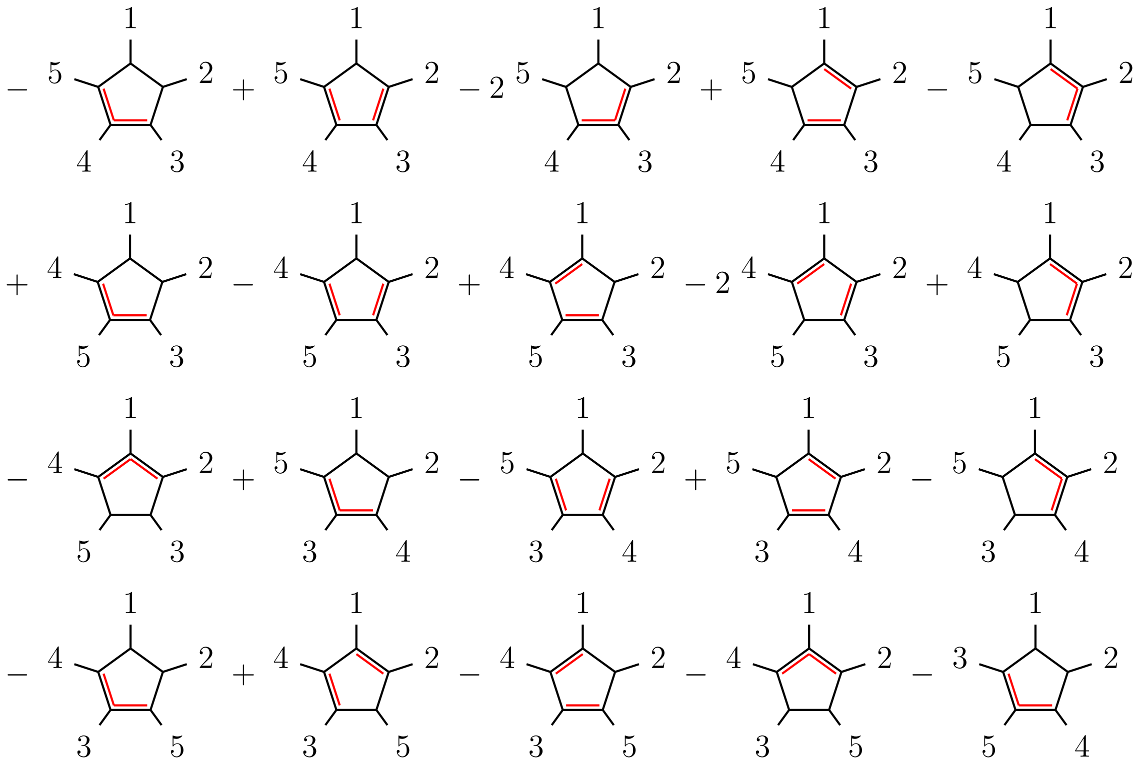


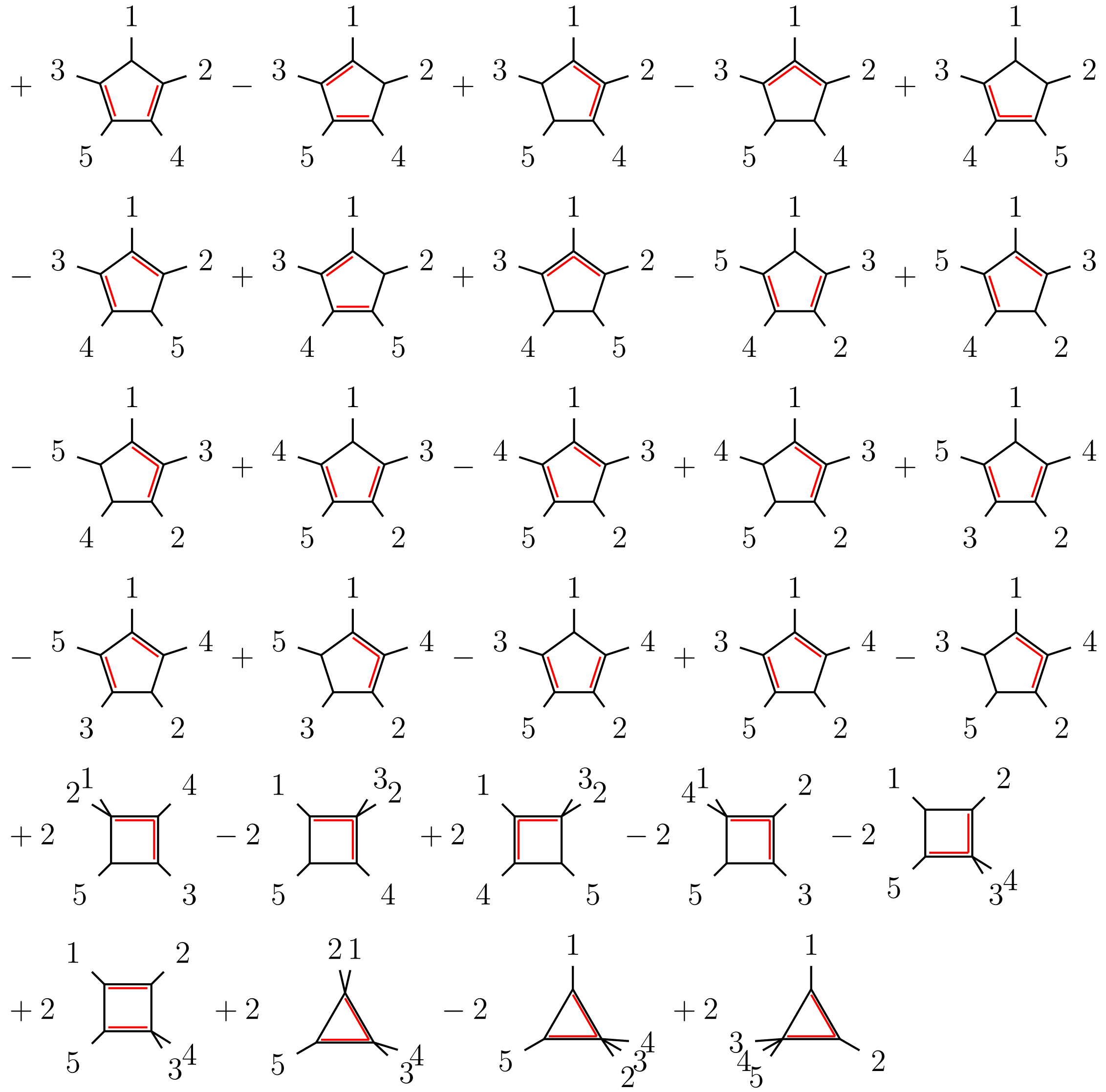


$g_4 =$ 





$g_5 =$ 



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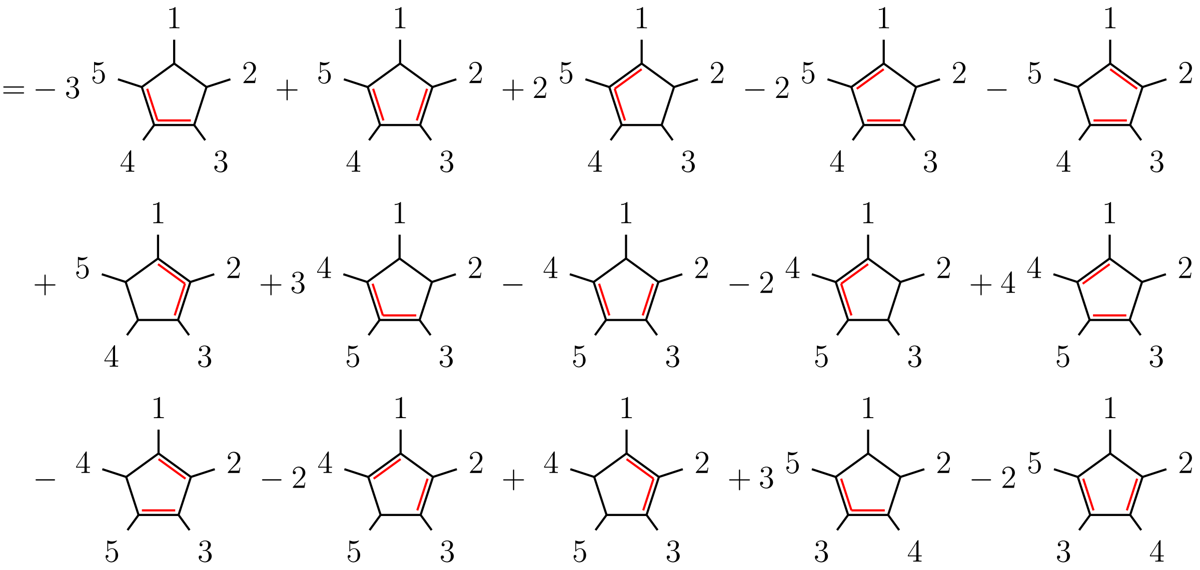

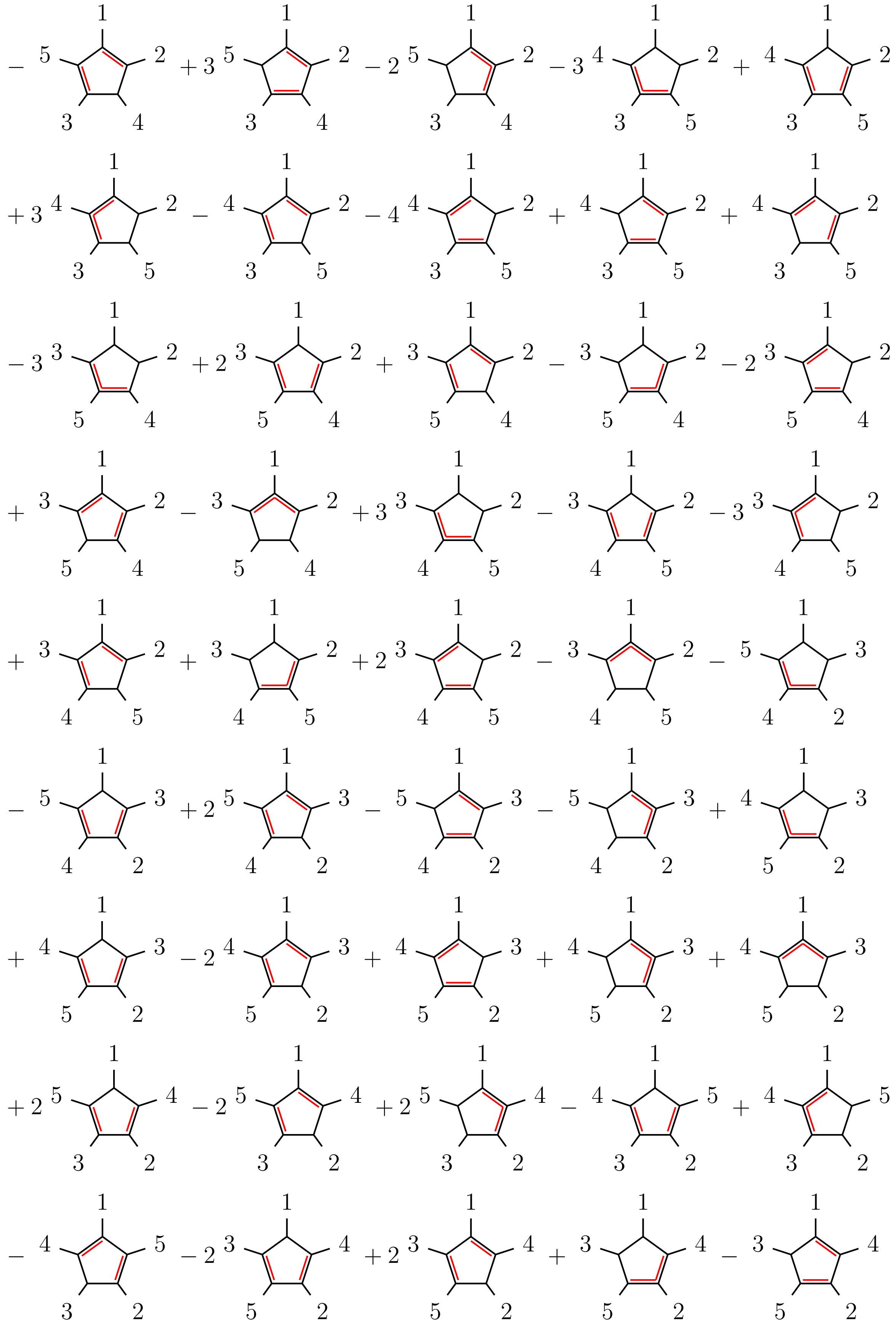
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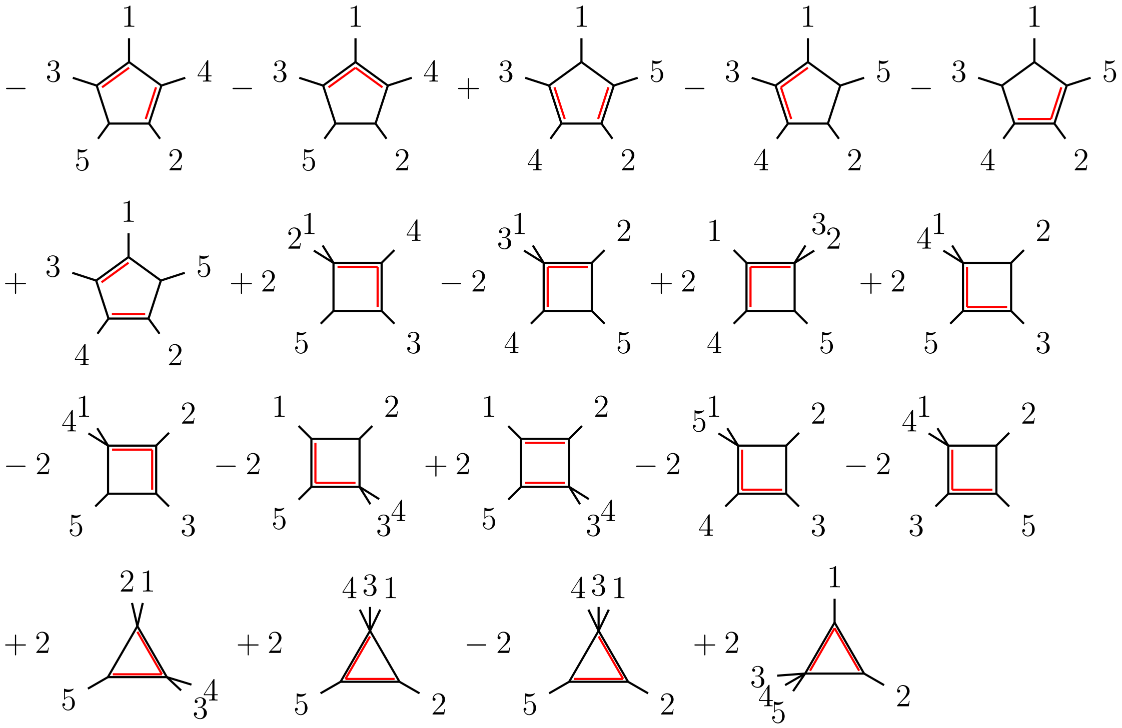

B.2 Generators of $H_2(X_{1,4,2};\mathbb{Q})$

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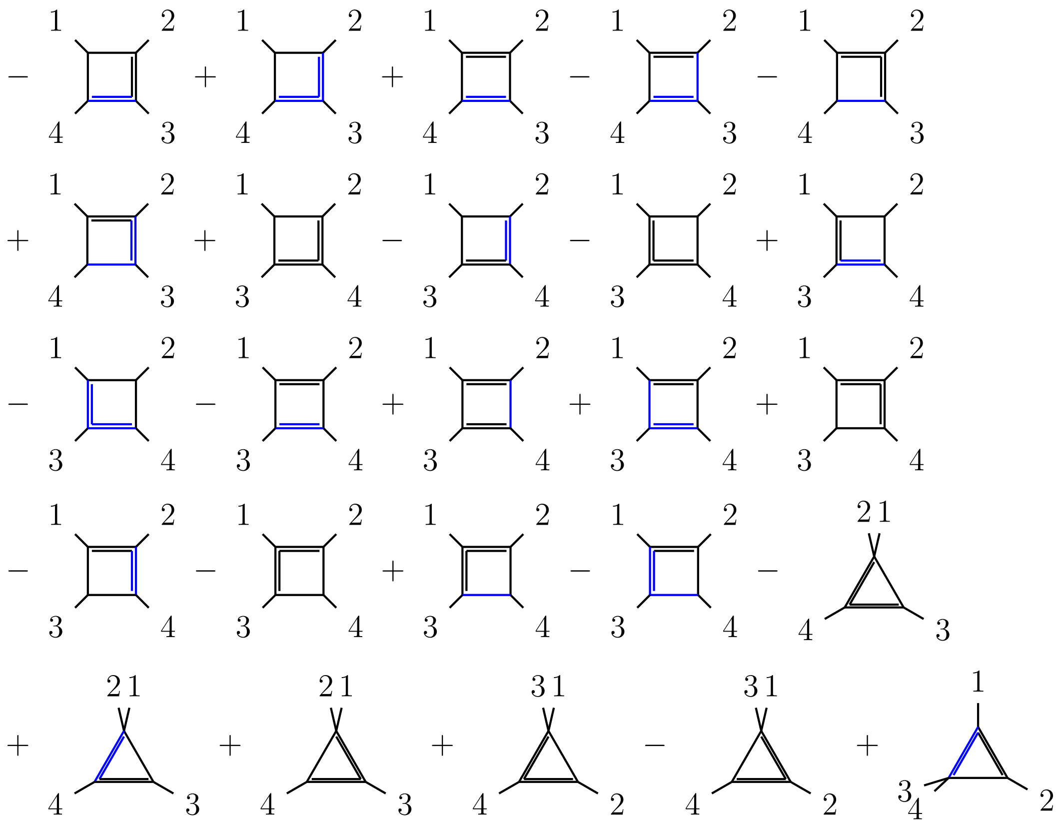

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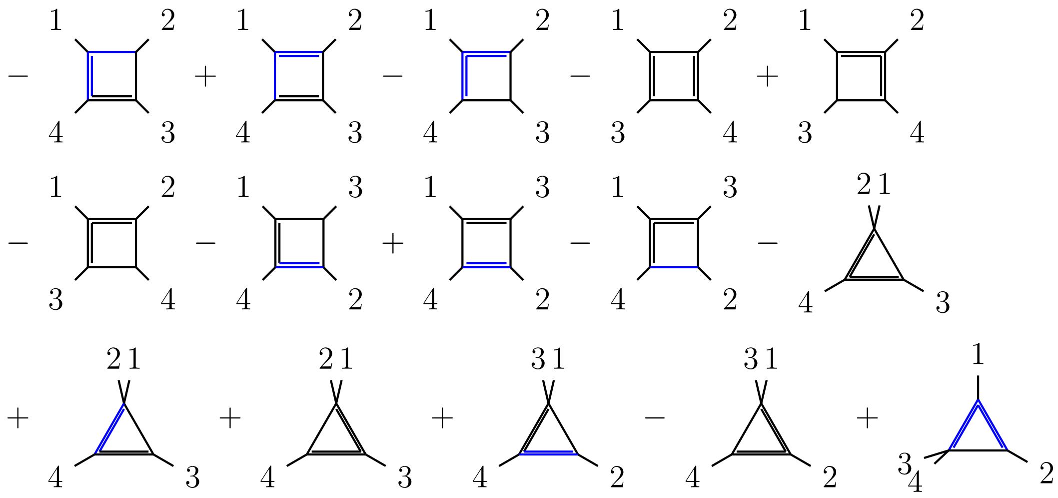

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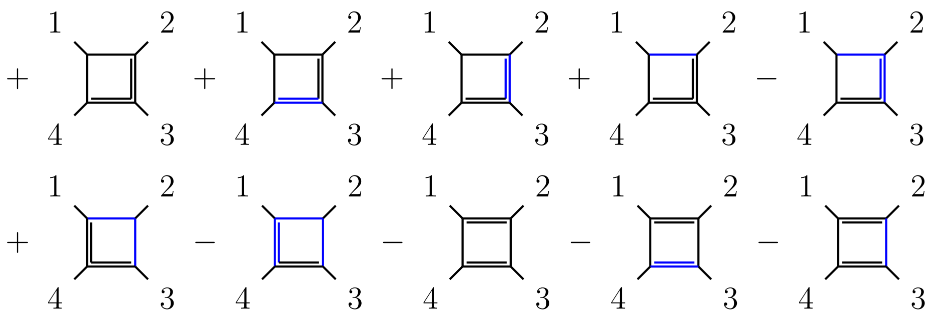

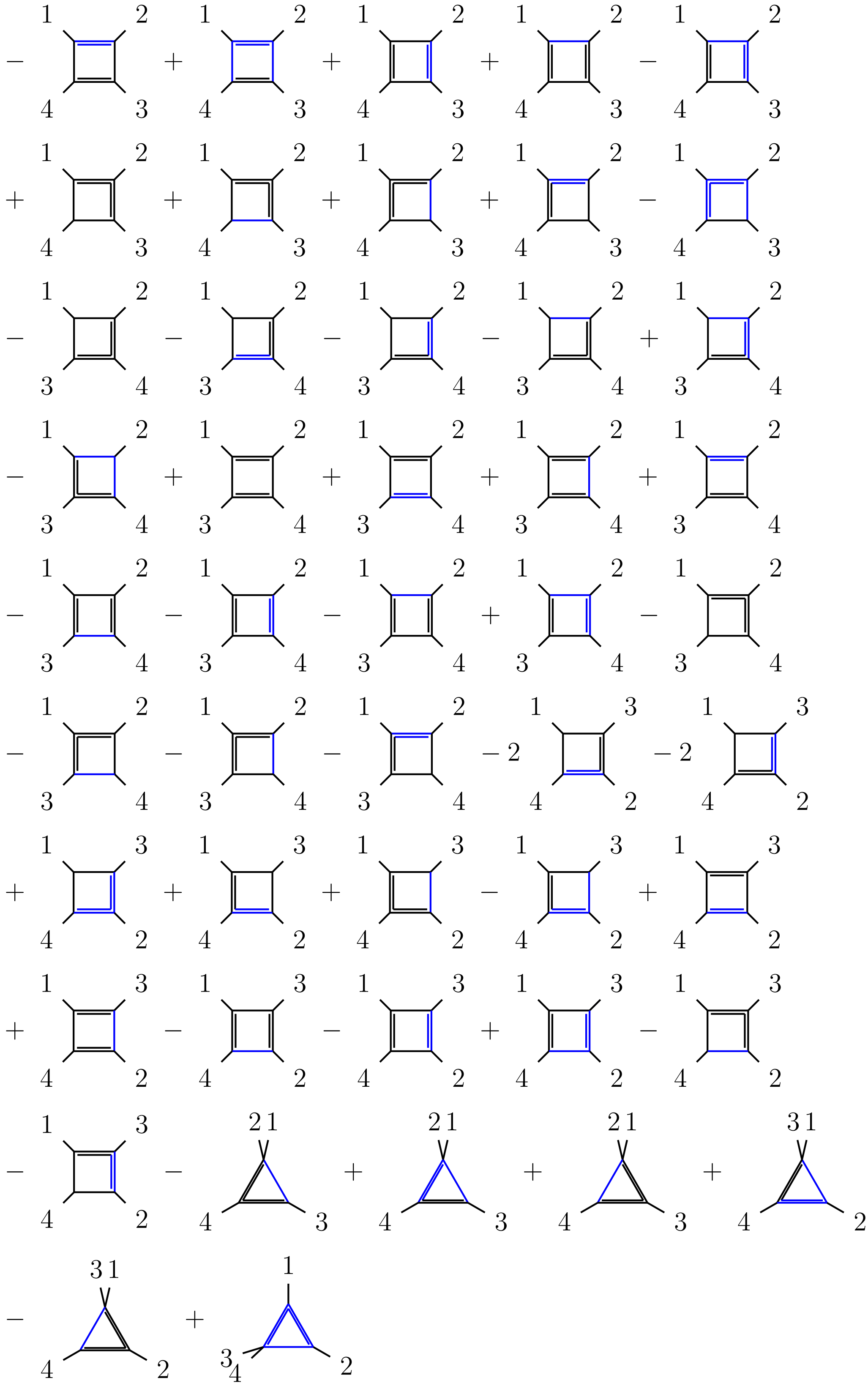

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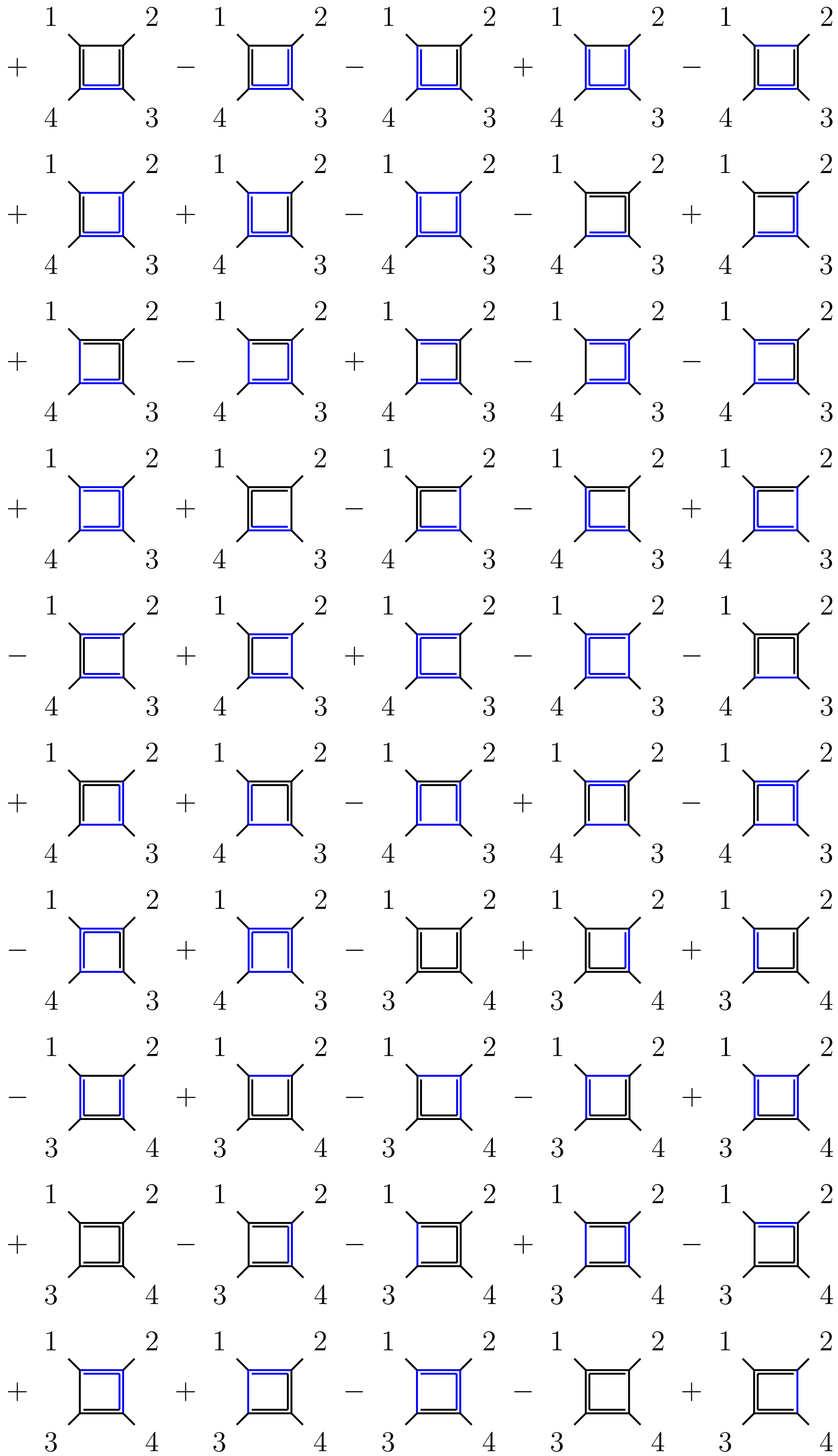

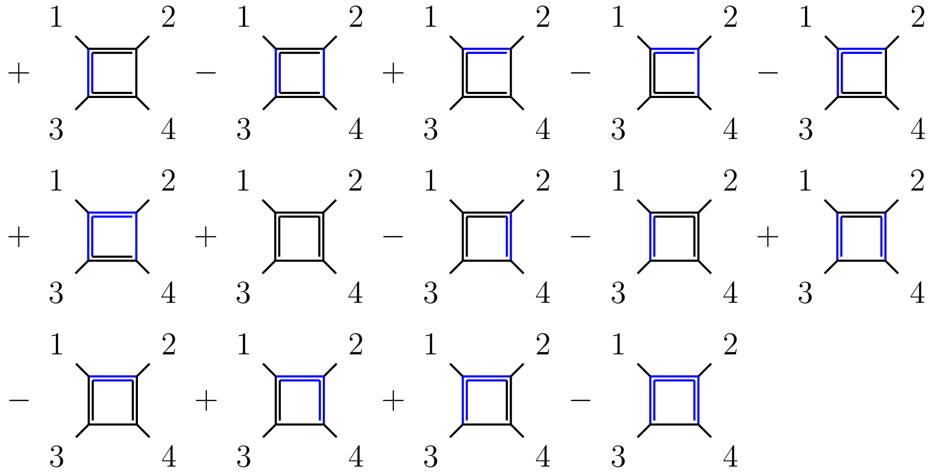

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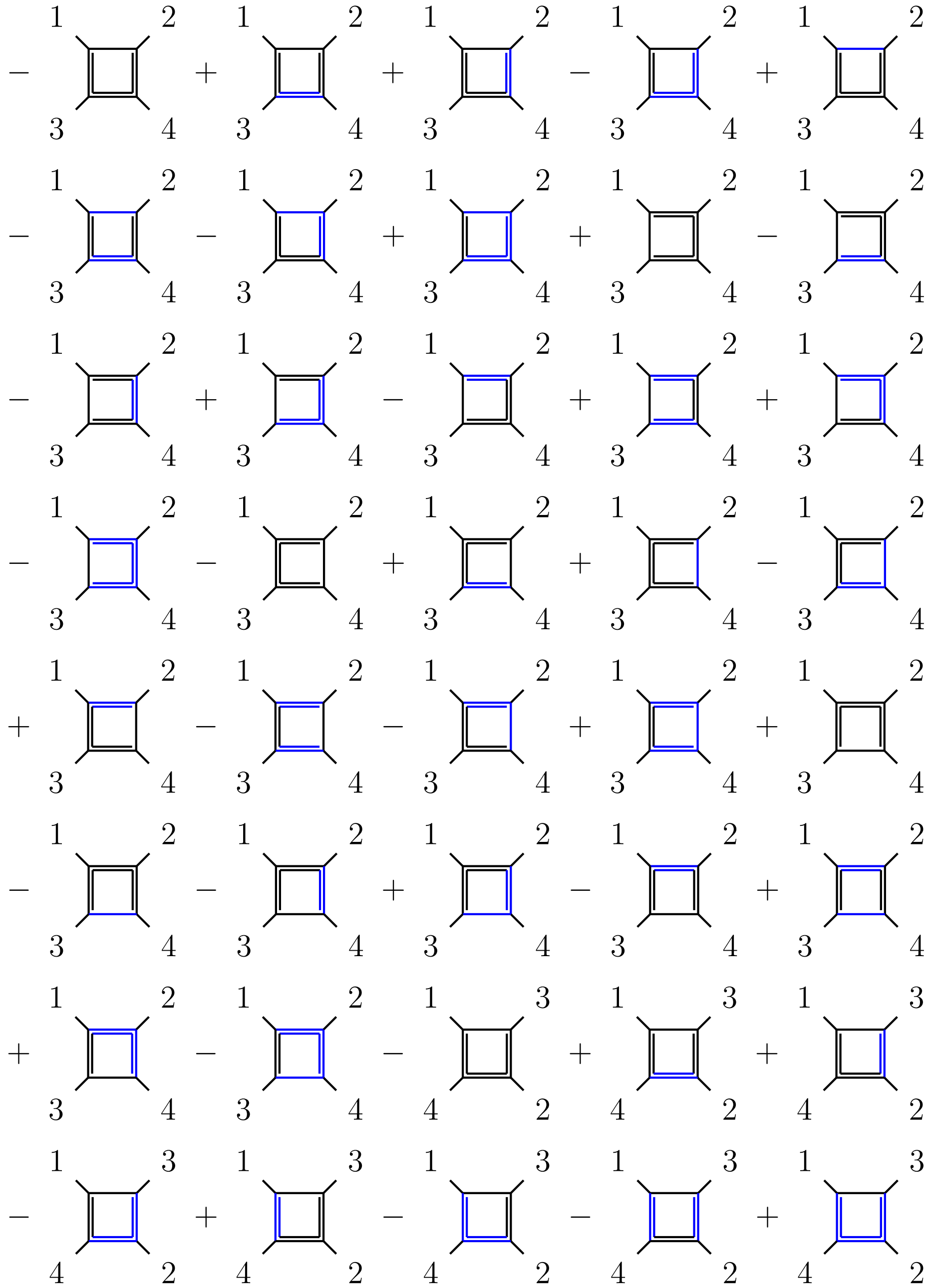

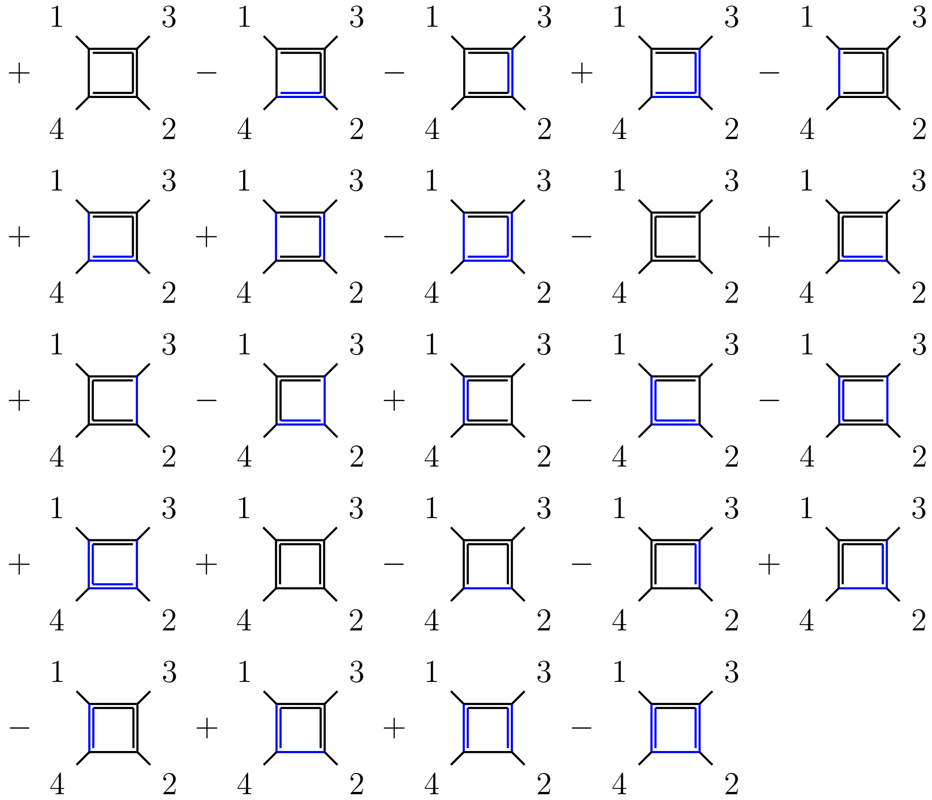

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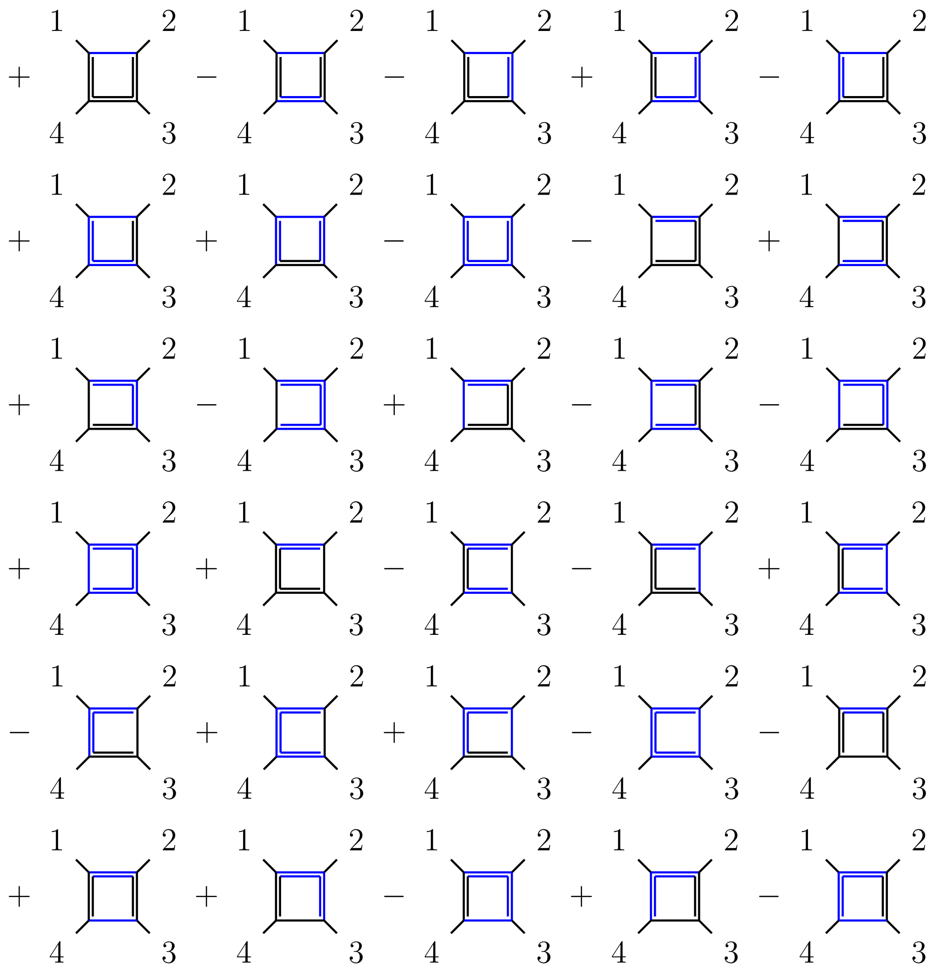

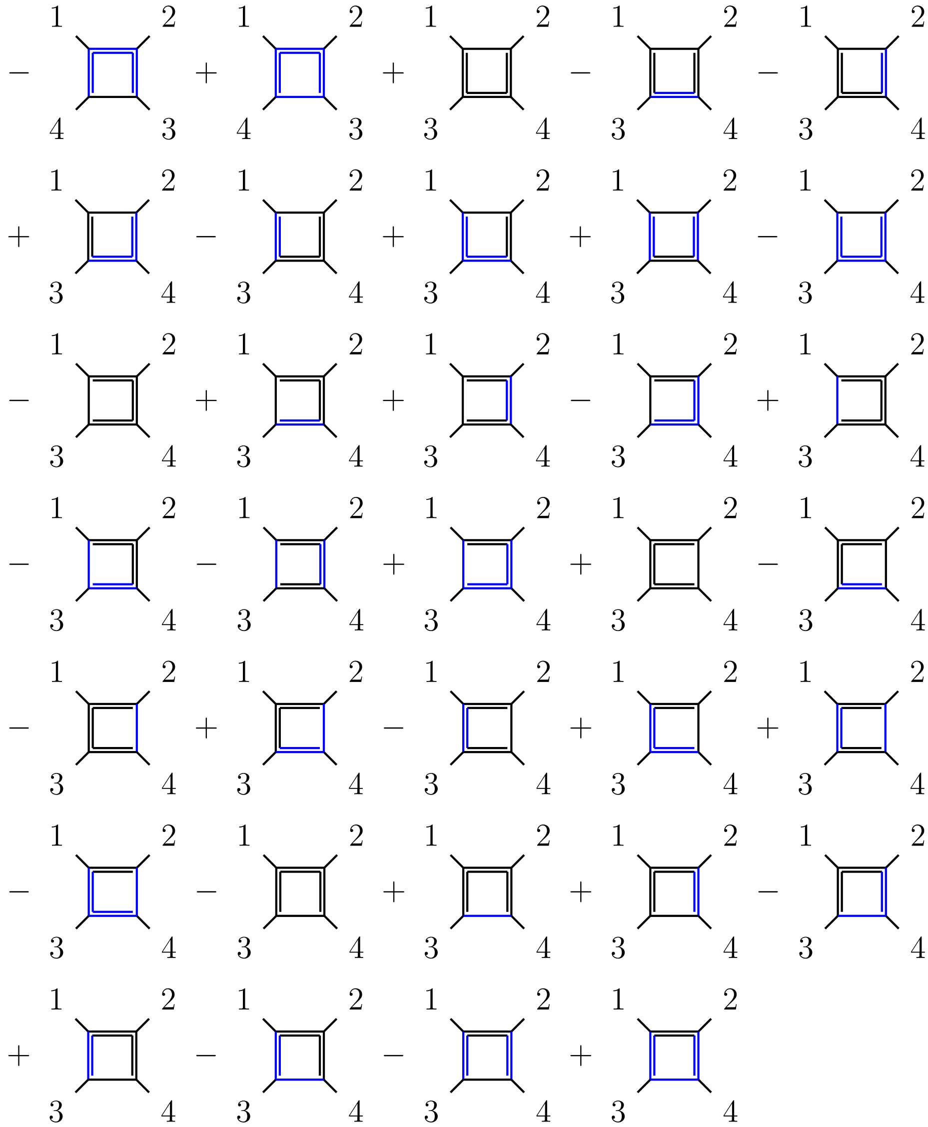

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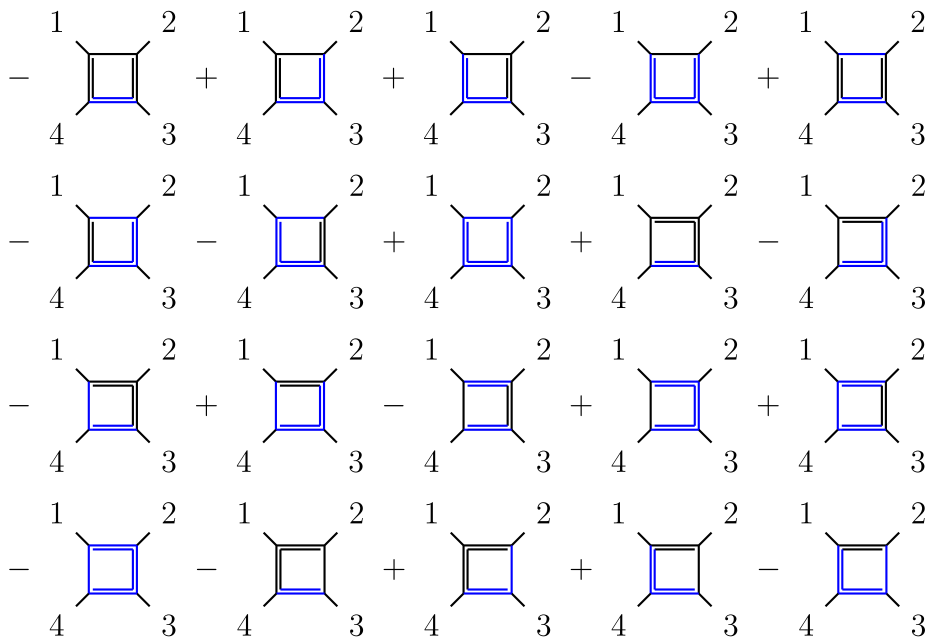

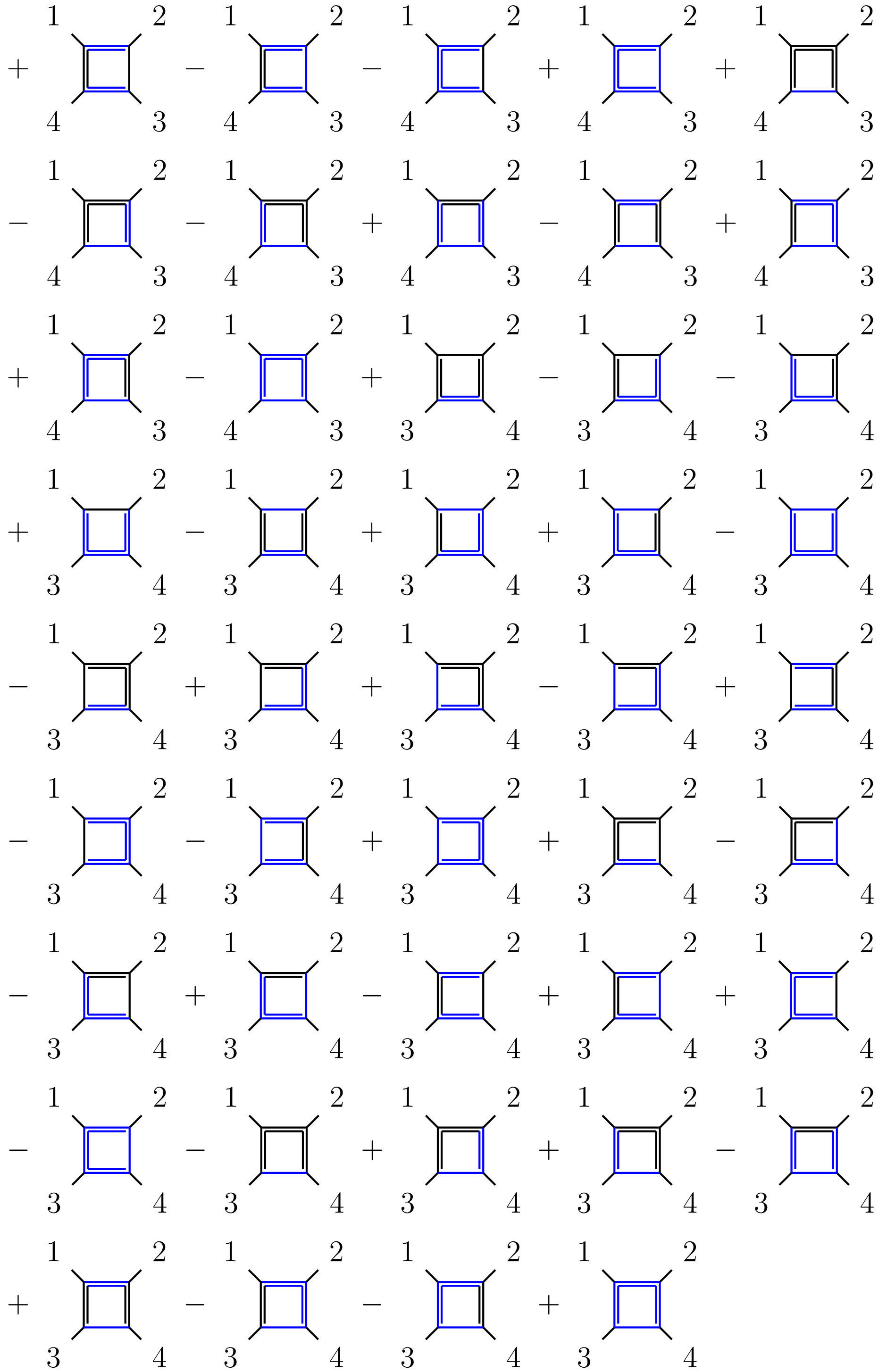

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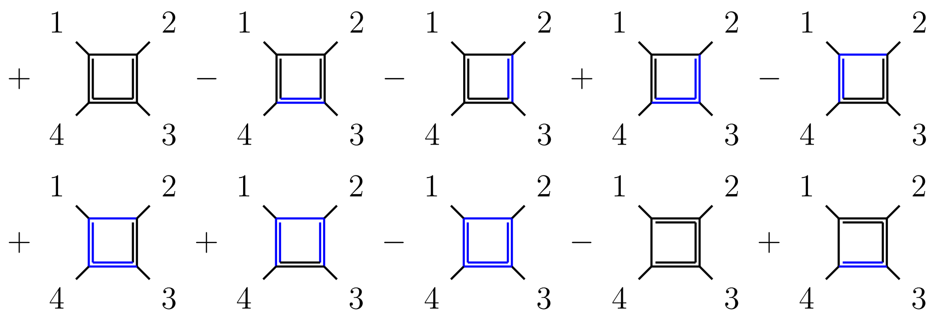

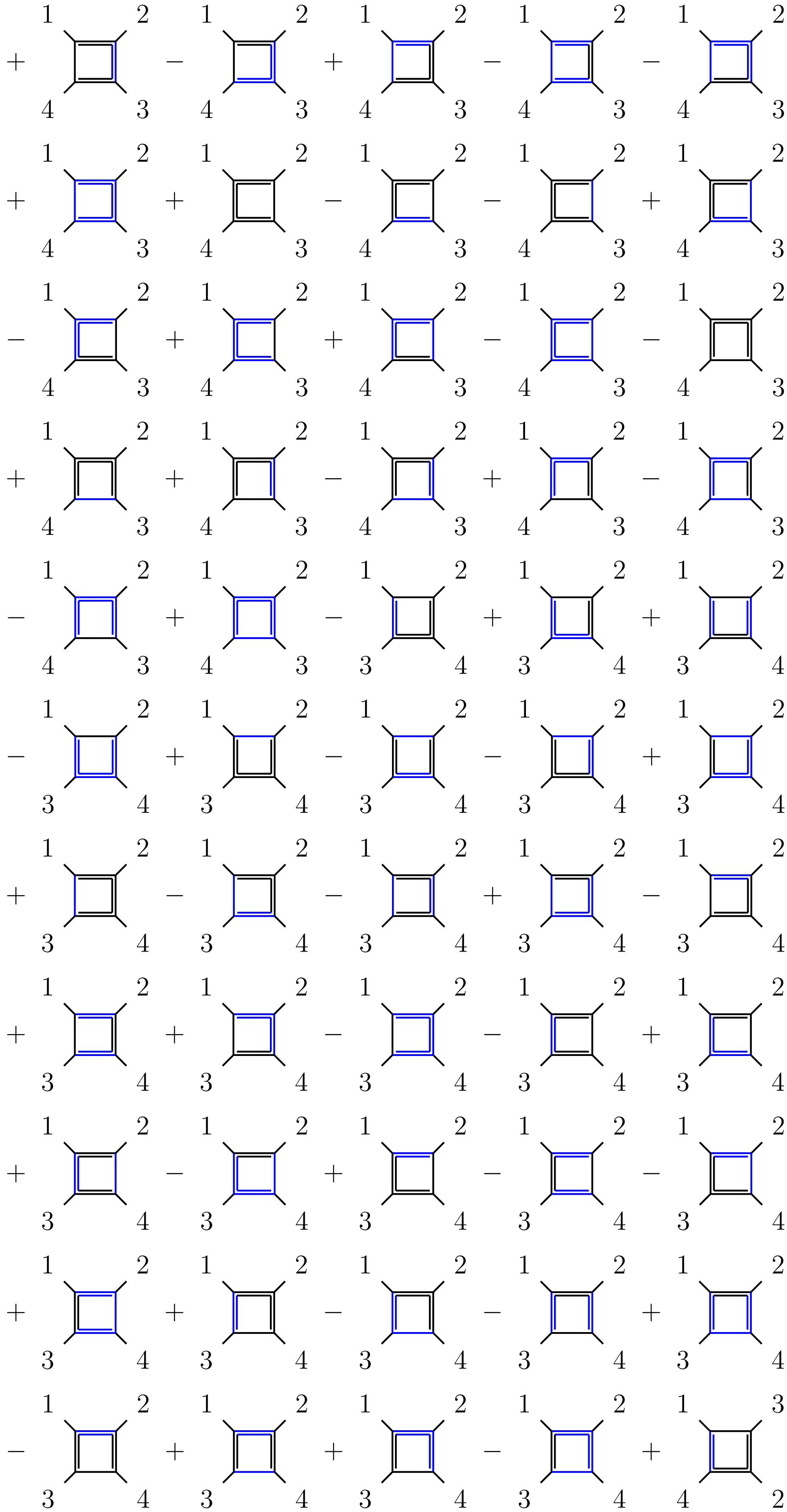

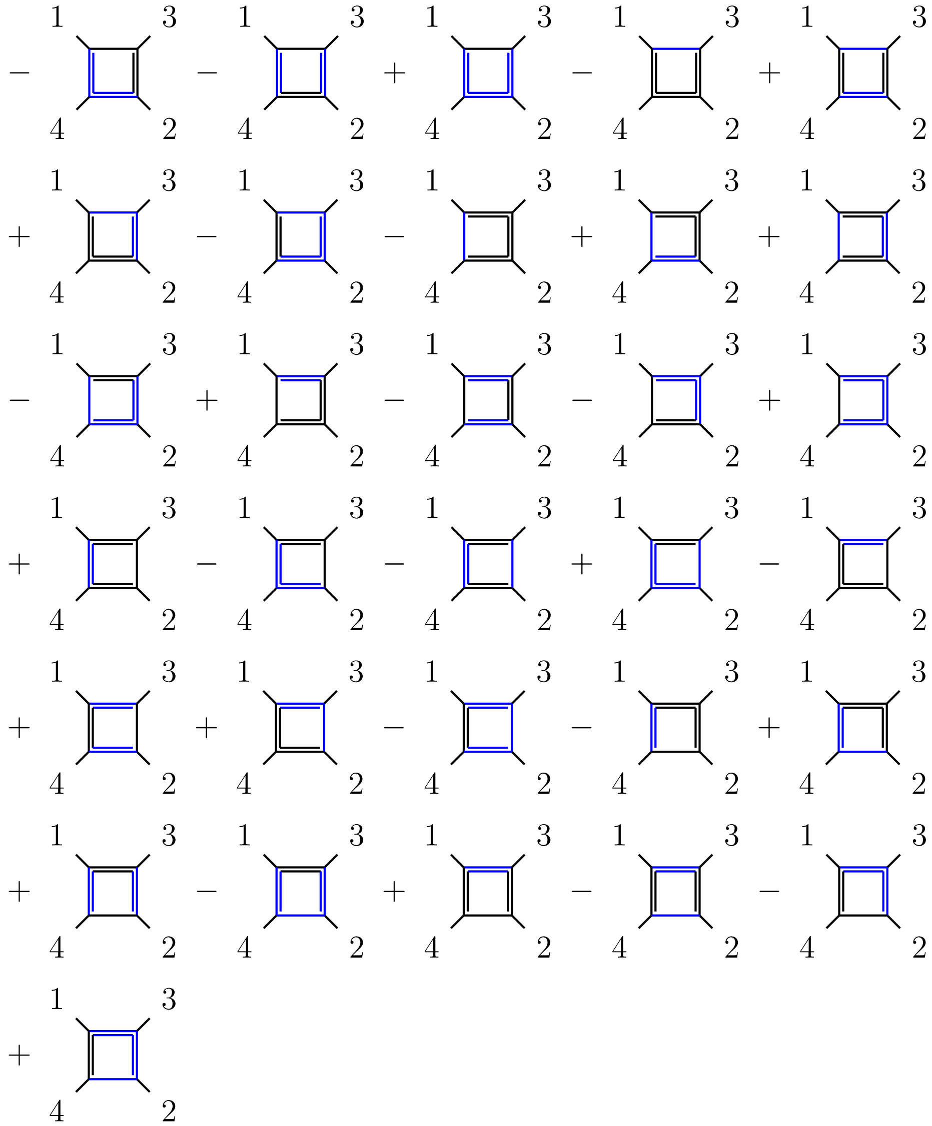

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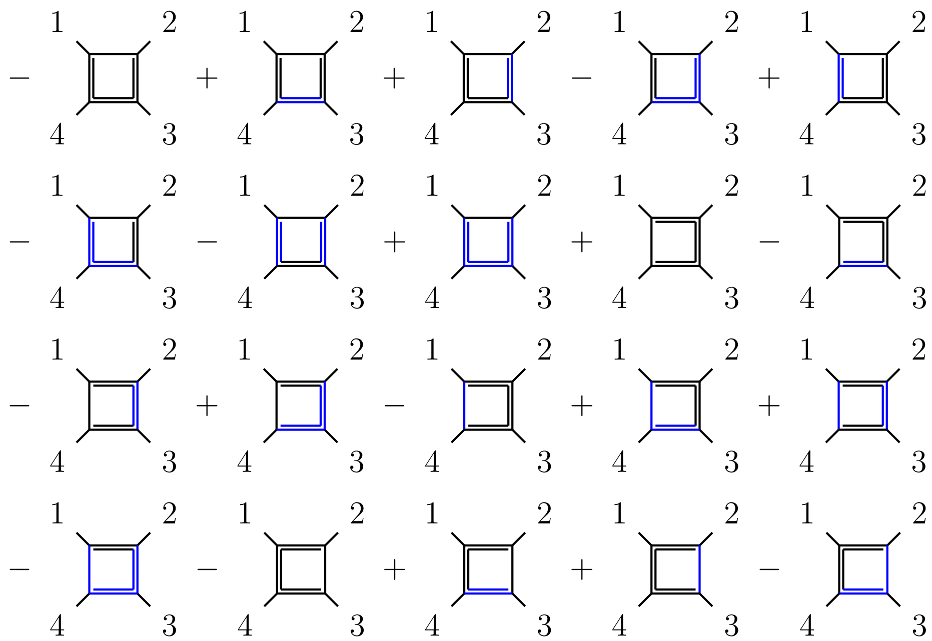

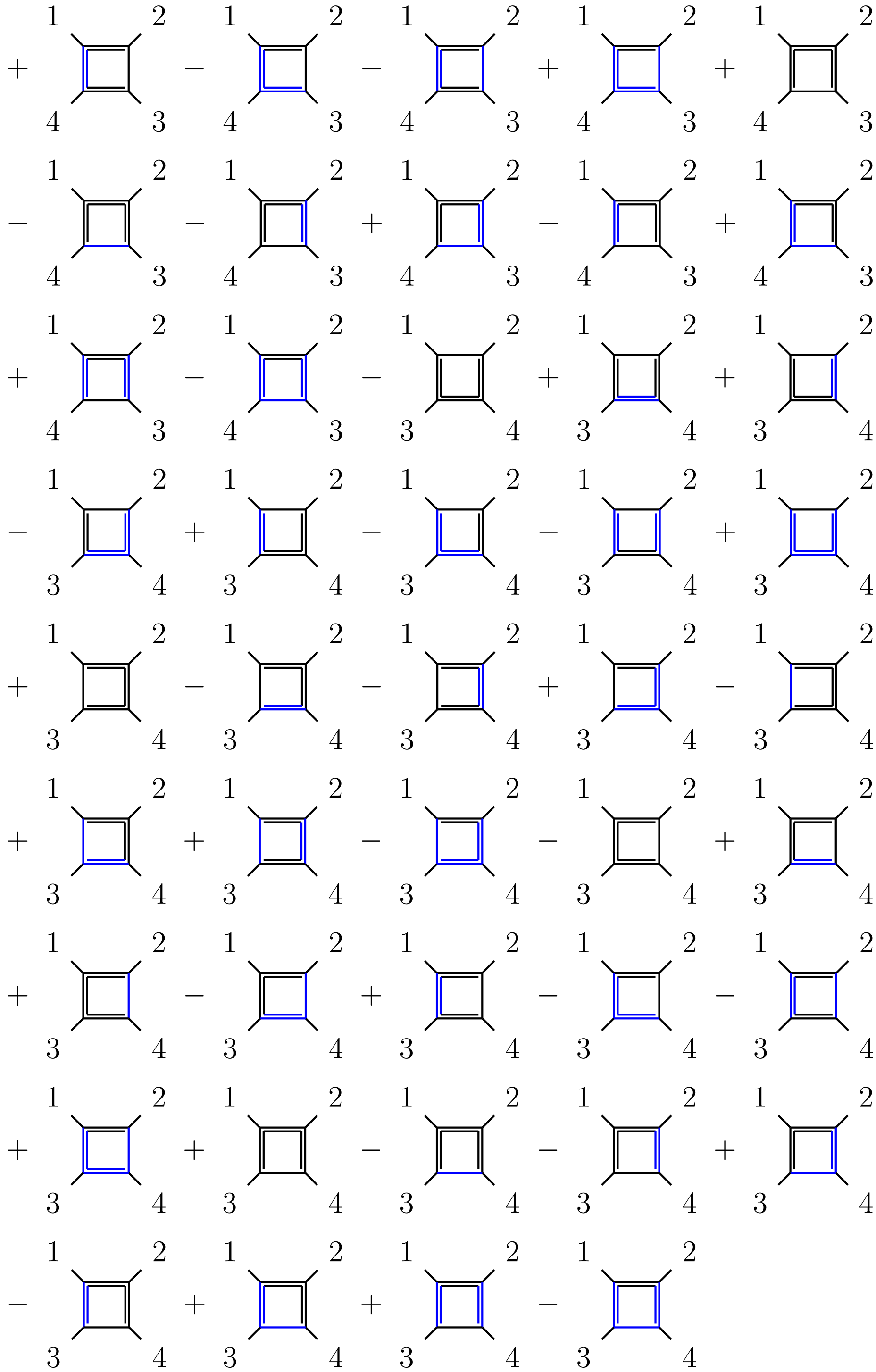

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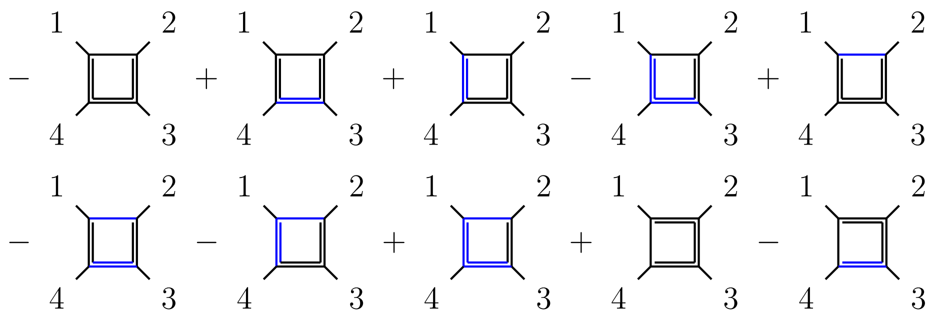

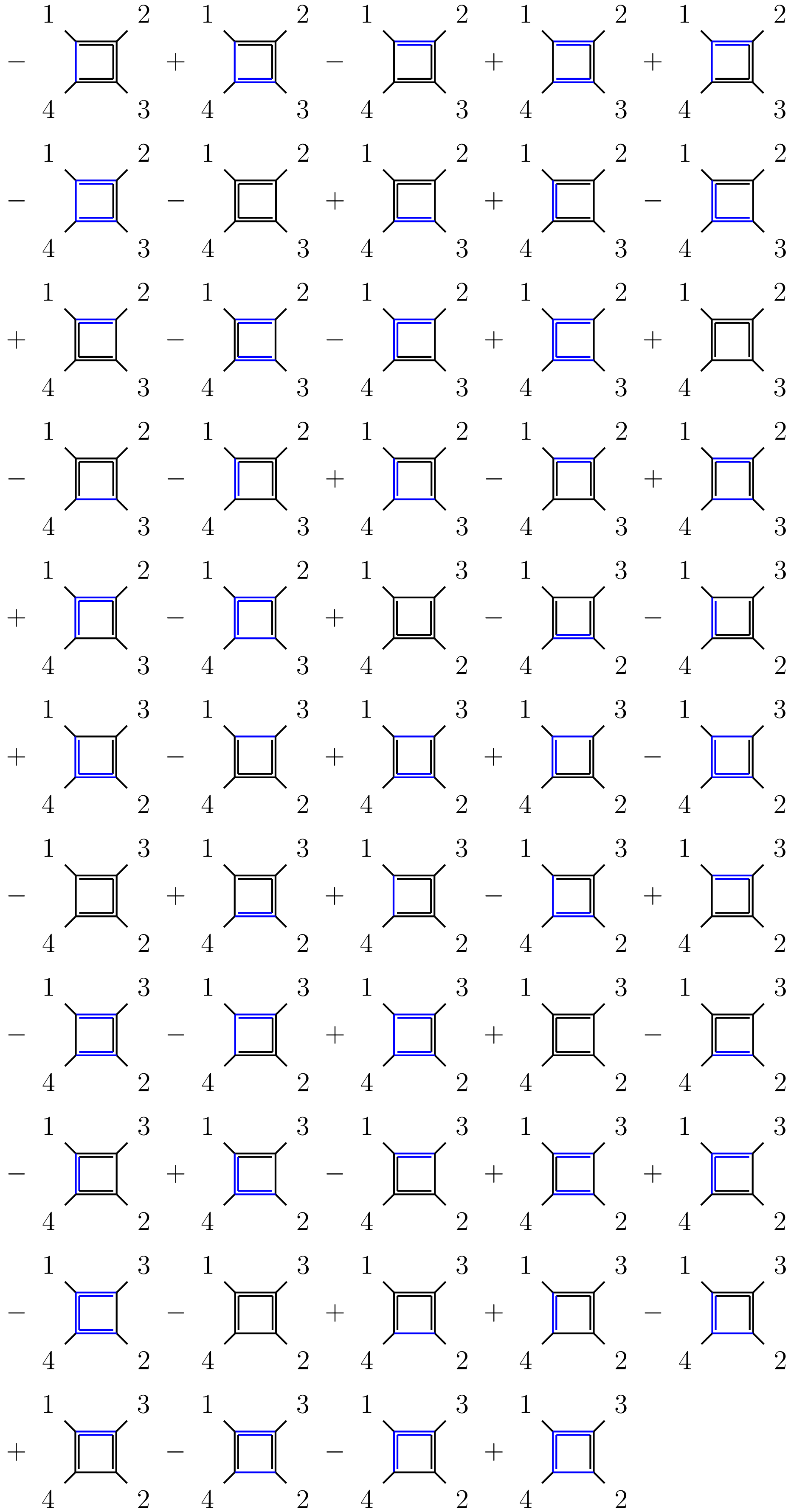

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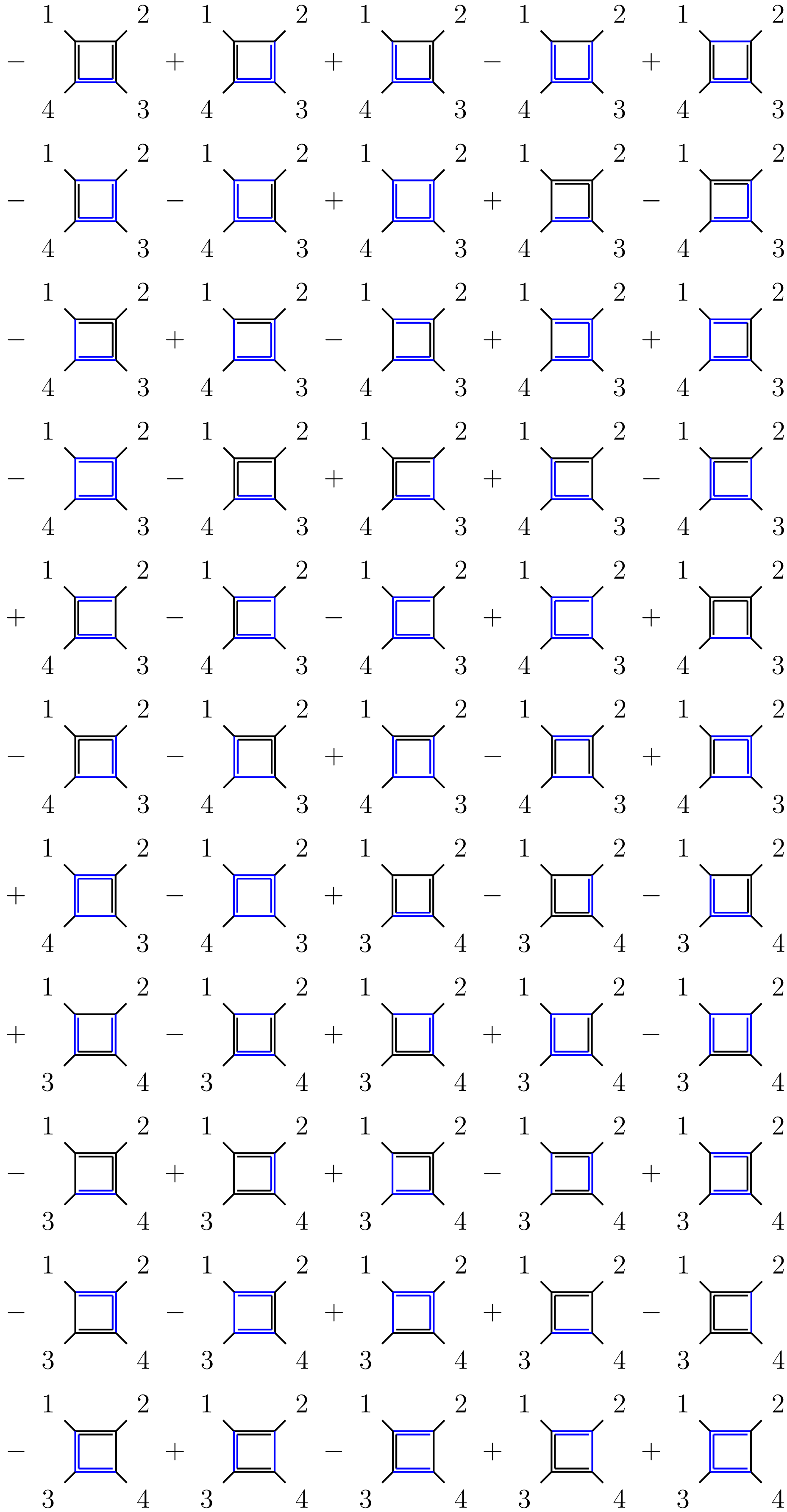
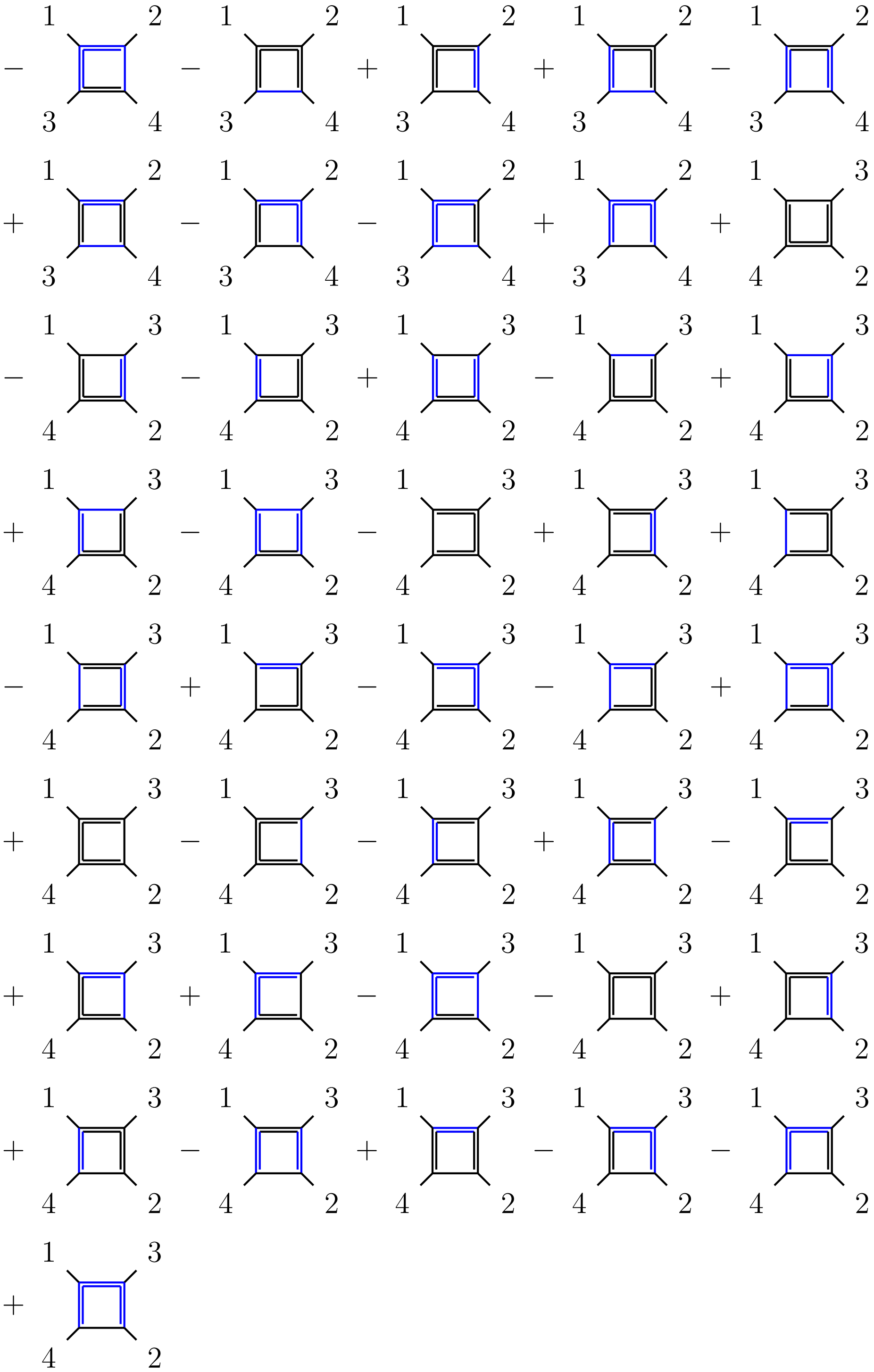
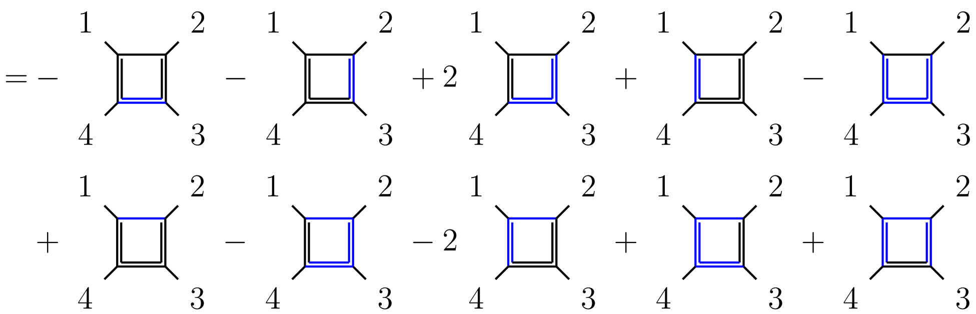
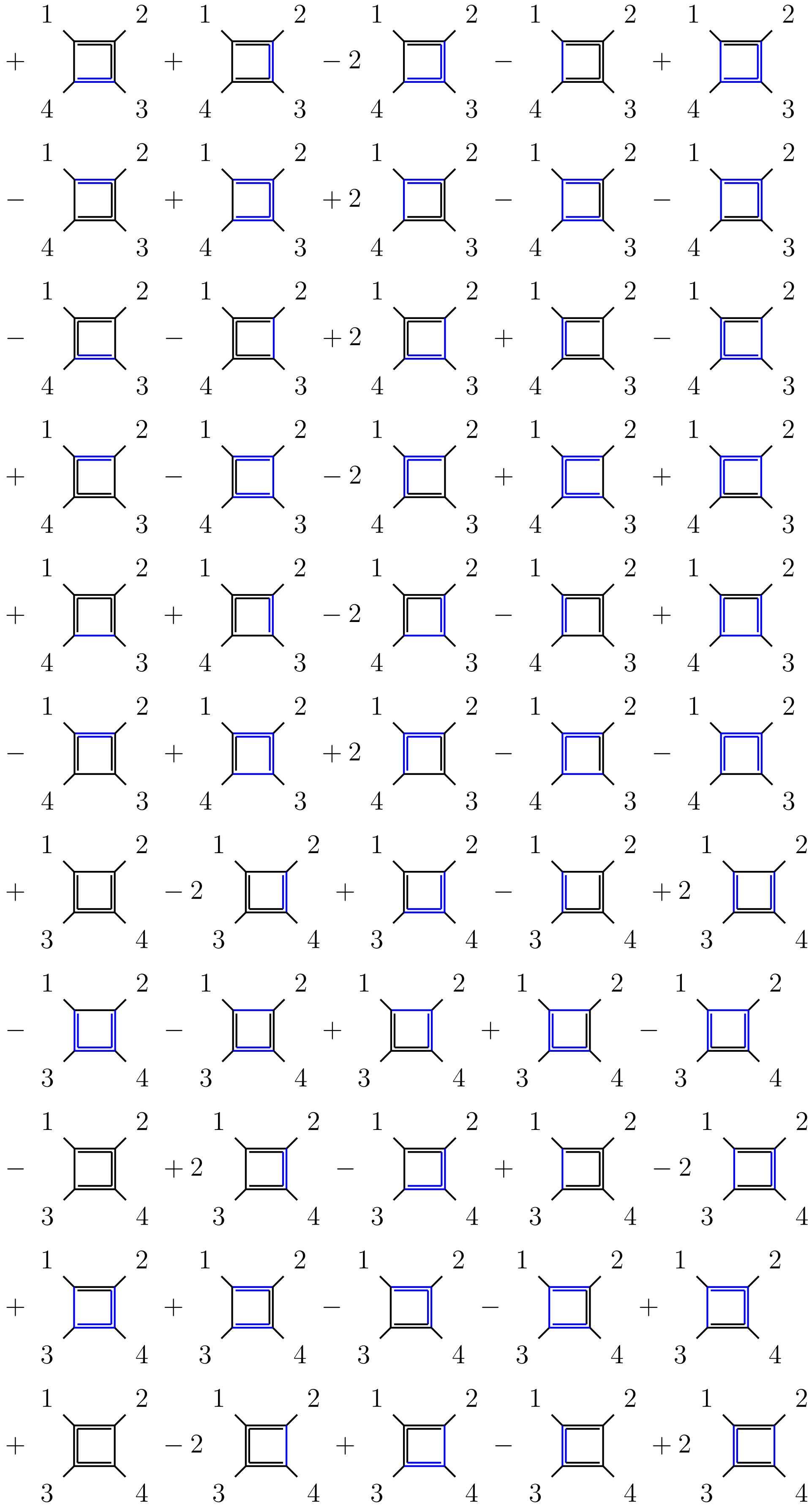
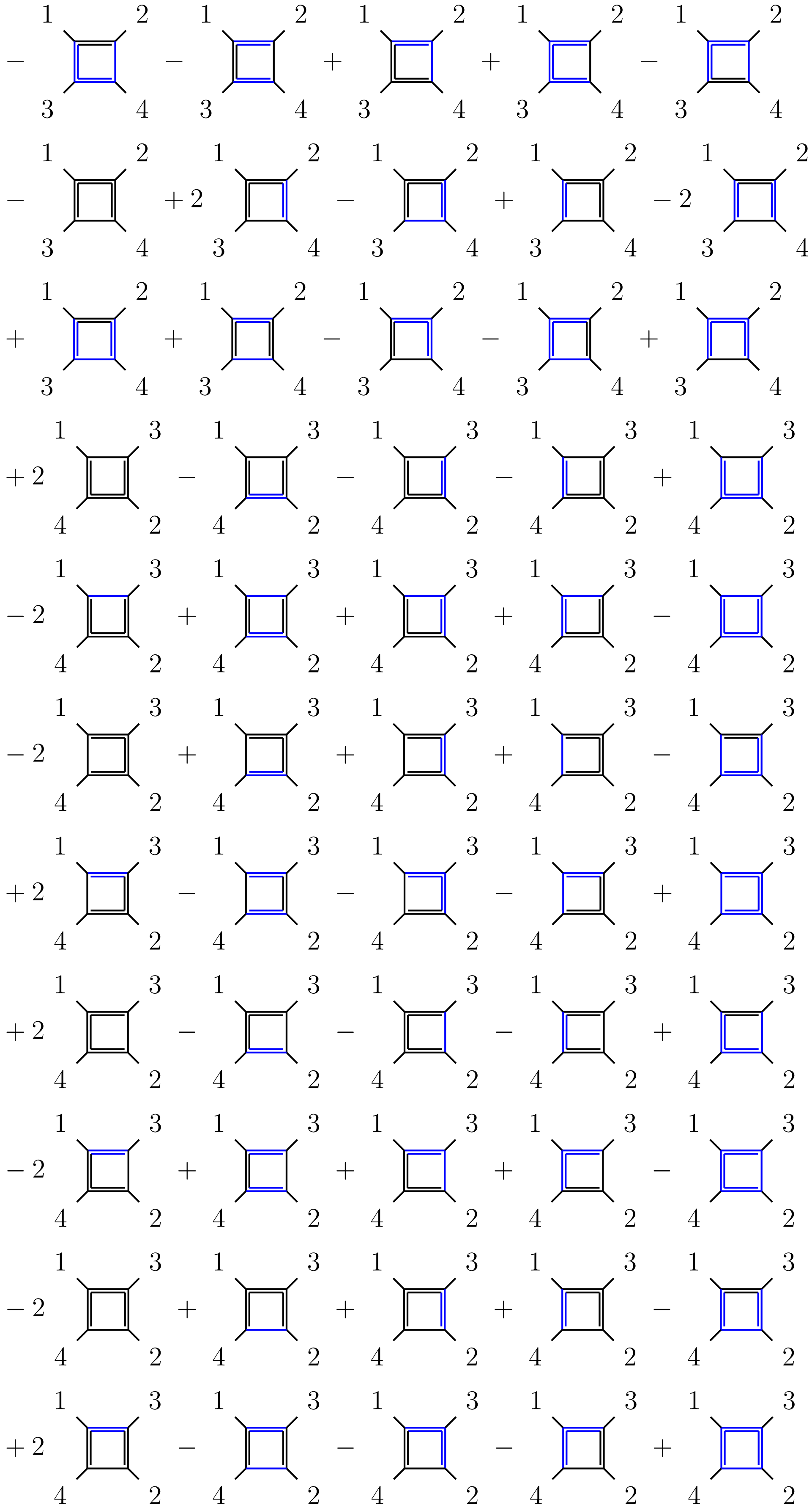




$g_9 =$ 





B.4 Generators of $H_2(X_{1,3,3};\mathbb{Q})$

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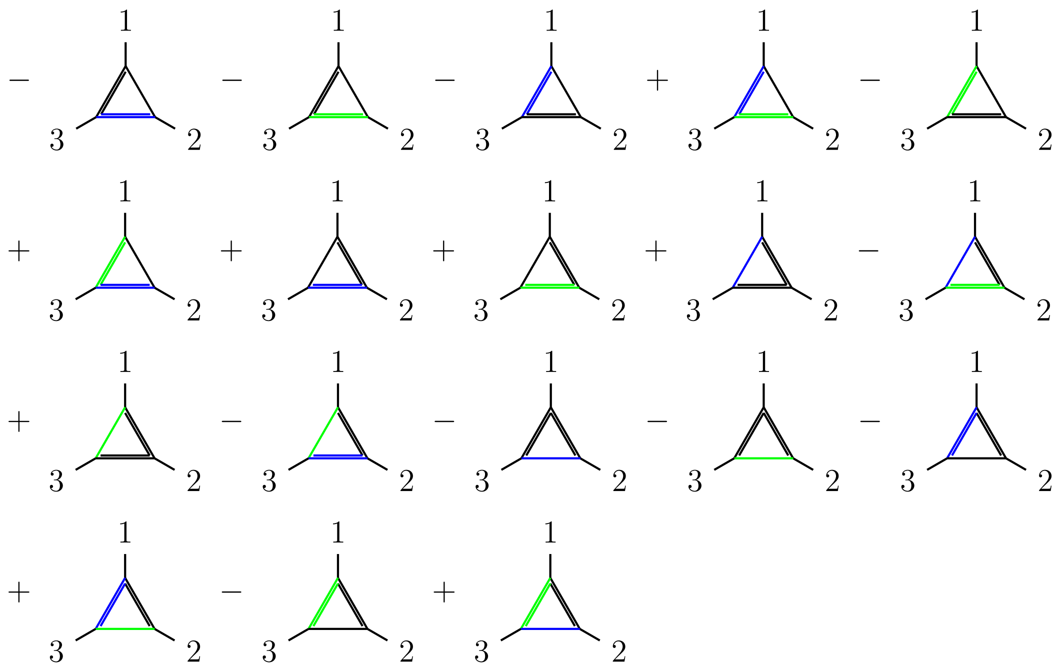

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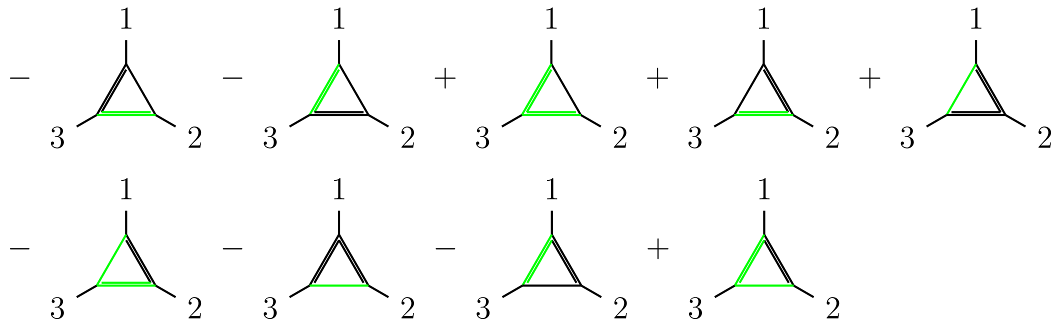

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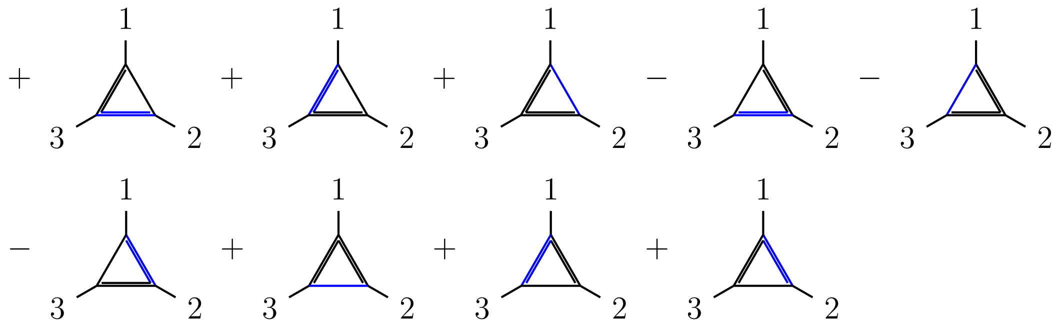

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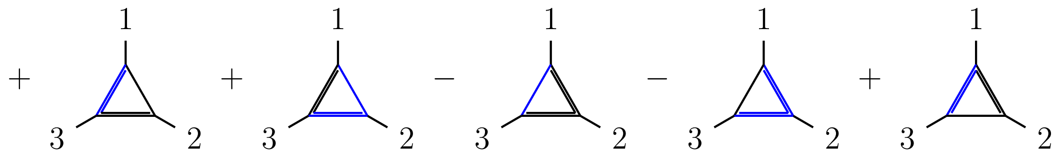
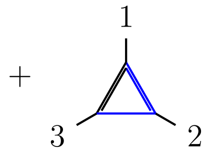
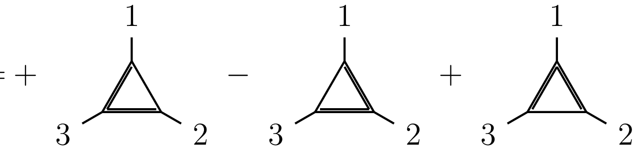
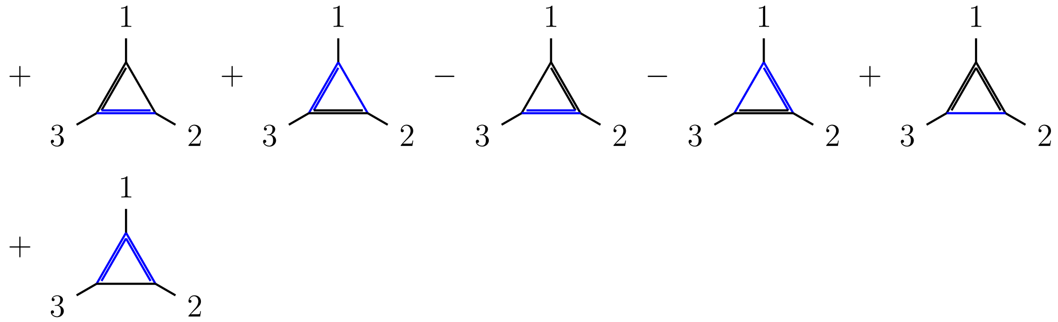
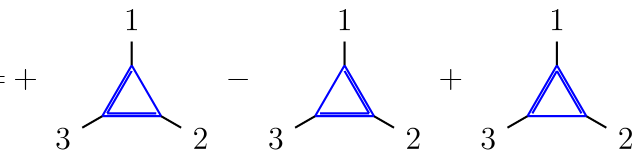
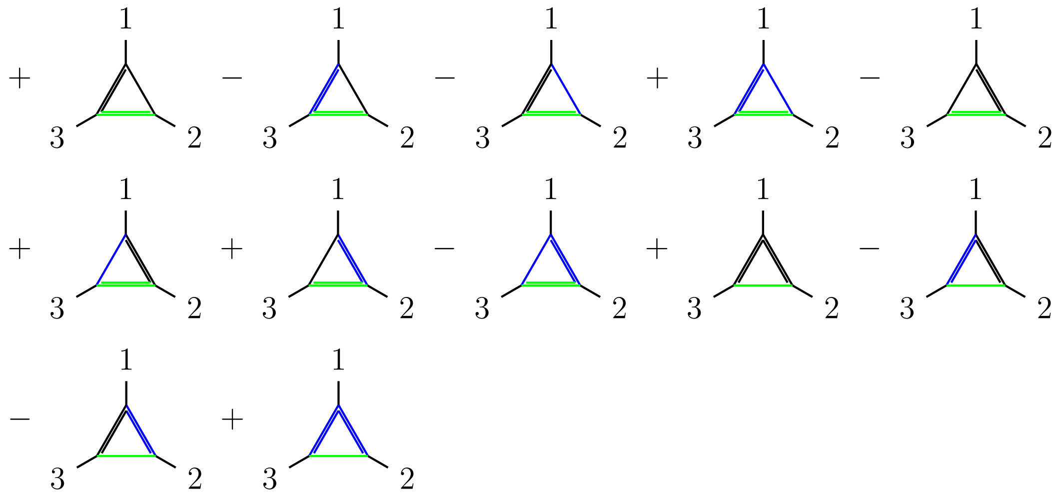


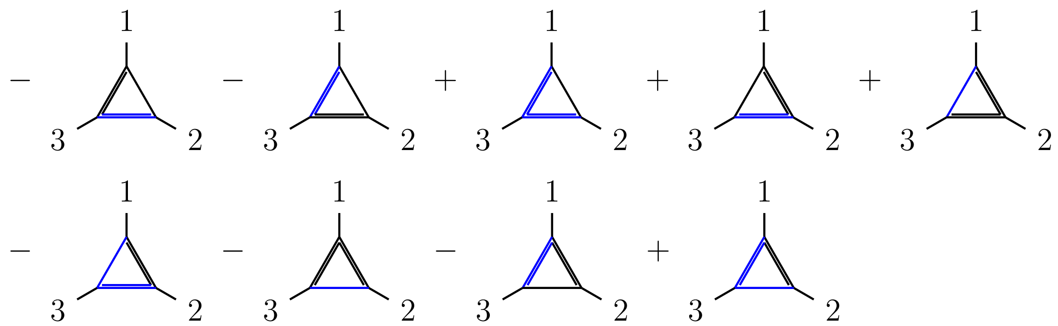


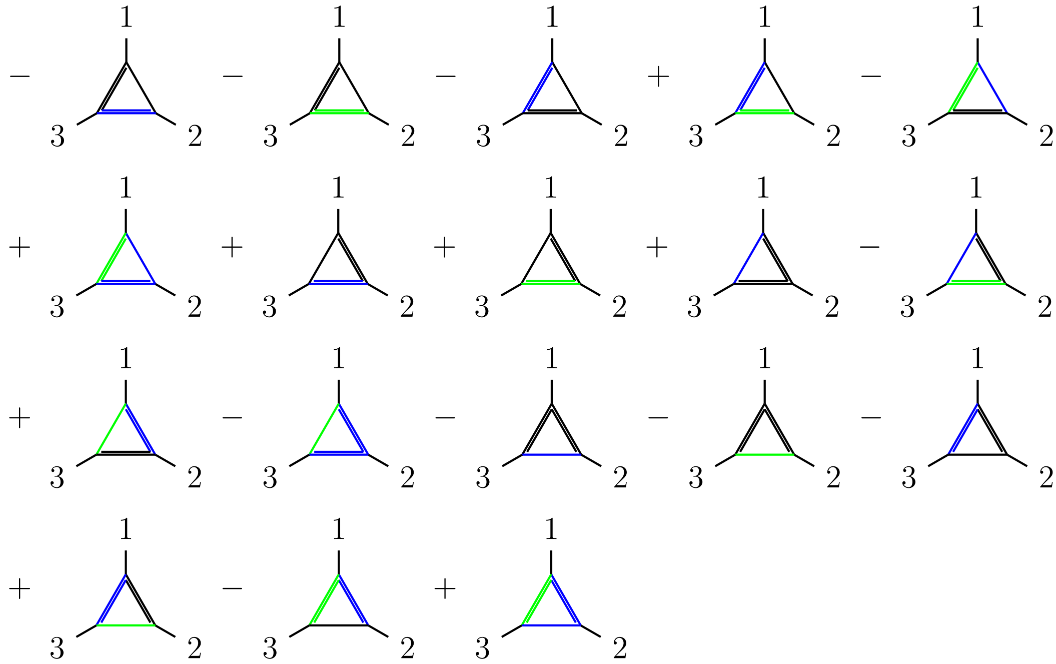
$g_5 =$ 

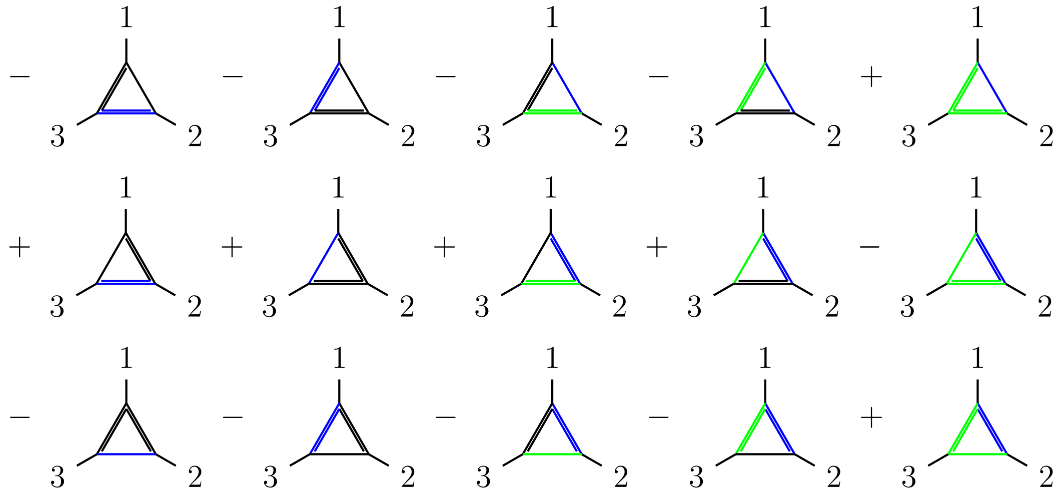
$g_6 =$ 

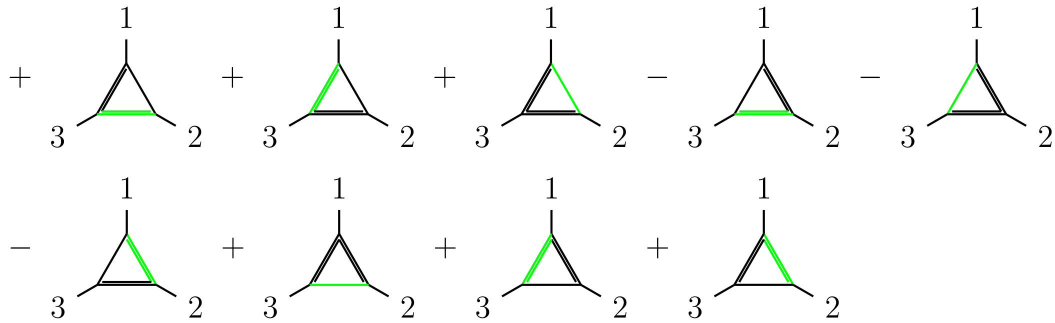
$g_7 =$ 

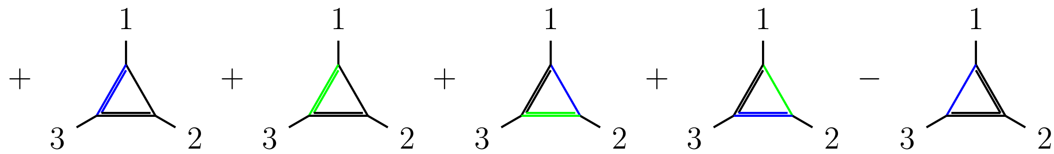
$g_8 =$ 

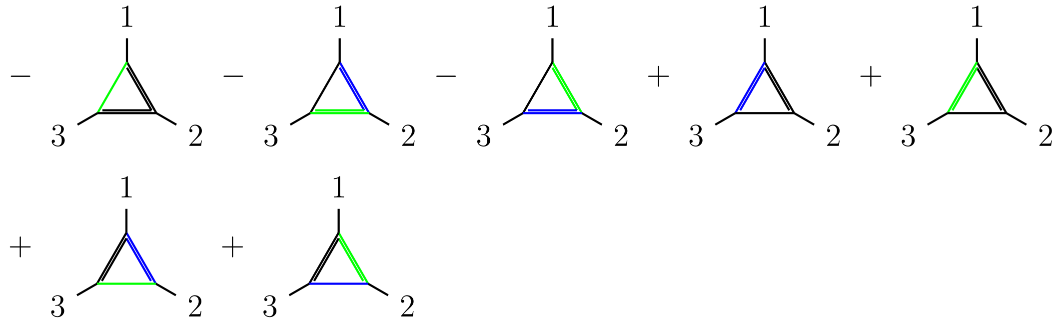
$g_9 =$ 

$g_{10} =$ 

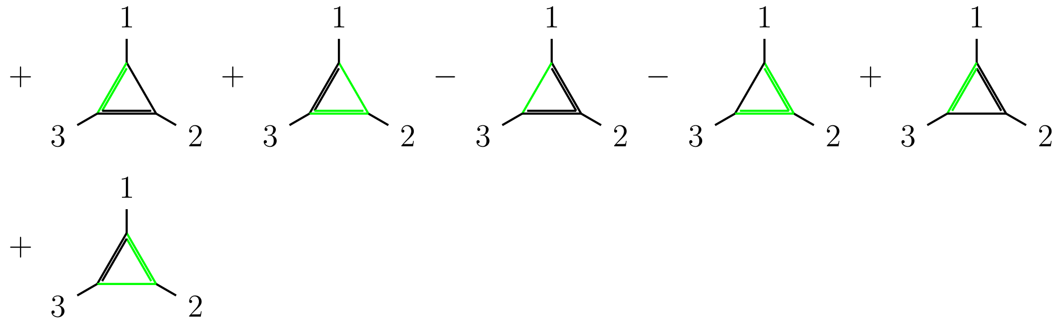
$g_{11} =$ 

$g_{12} =$ 

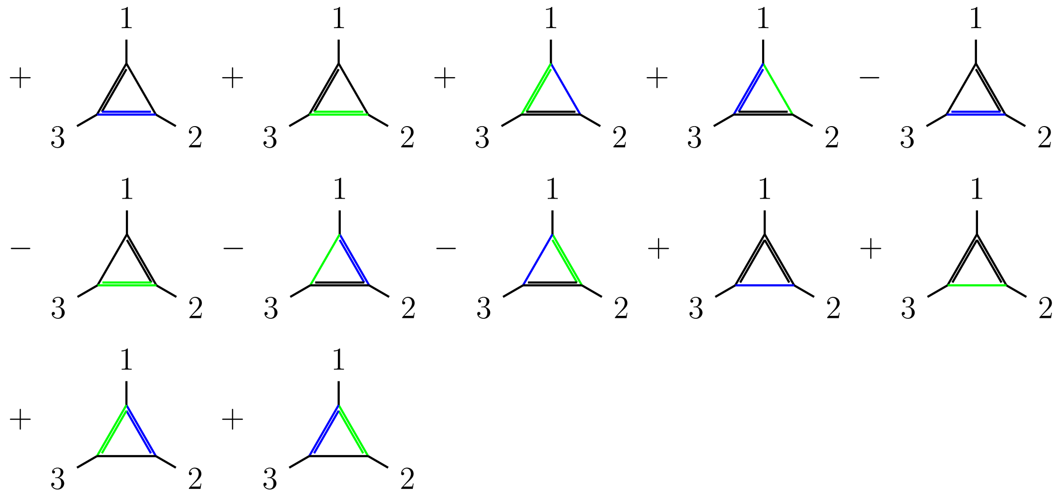
$g_{13} =$ 



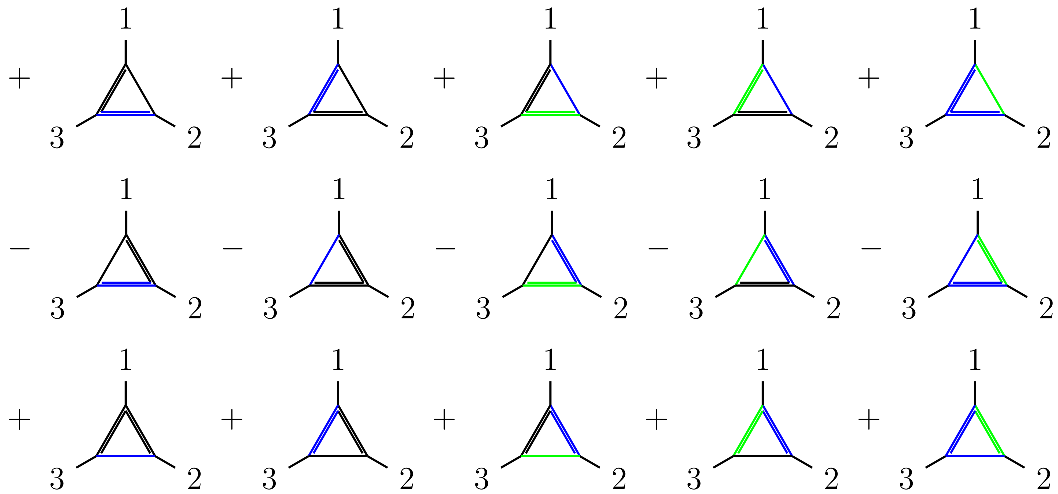
$g_{14} =$

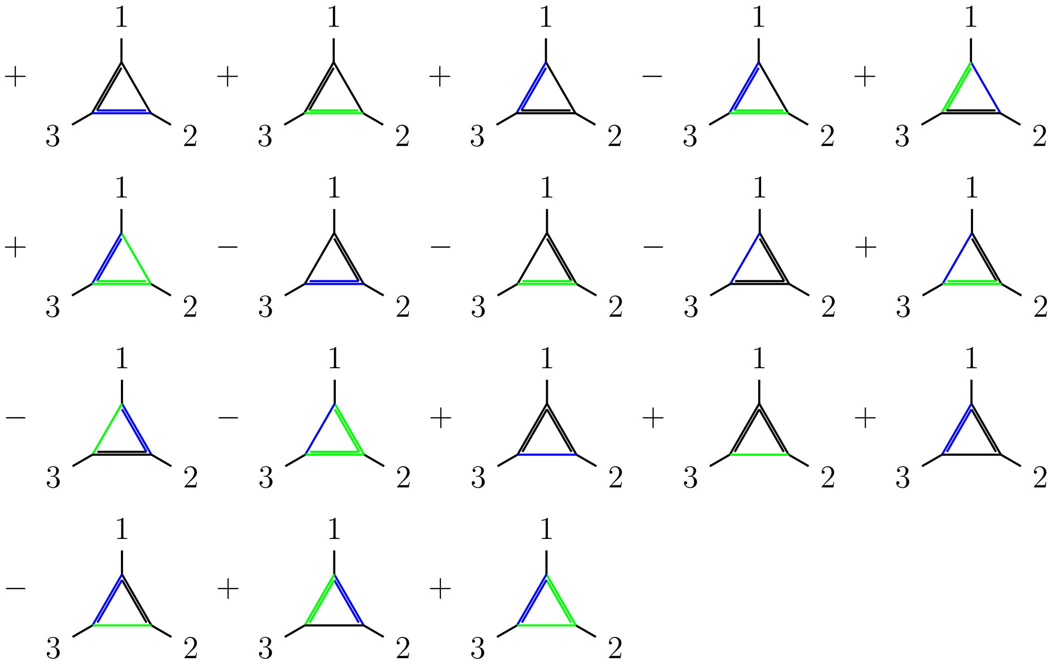


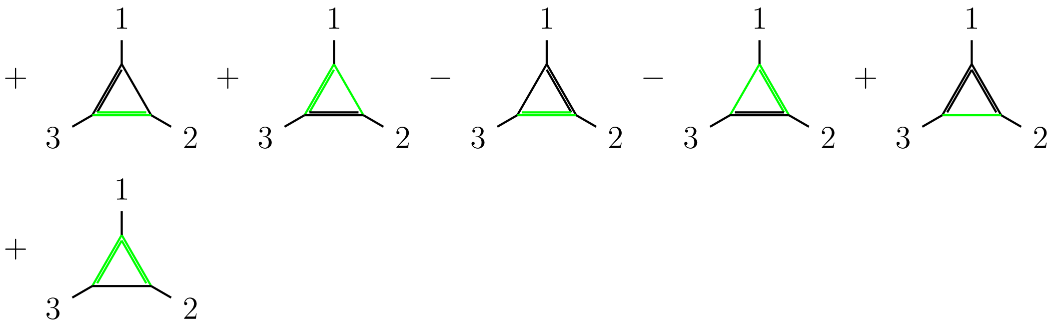
$g_{15} =$

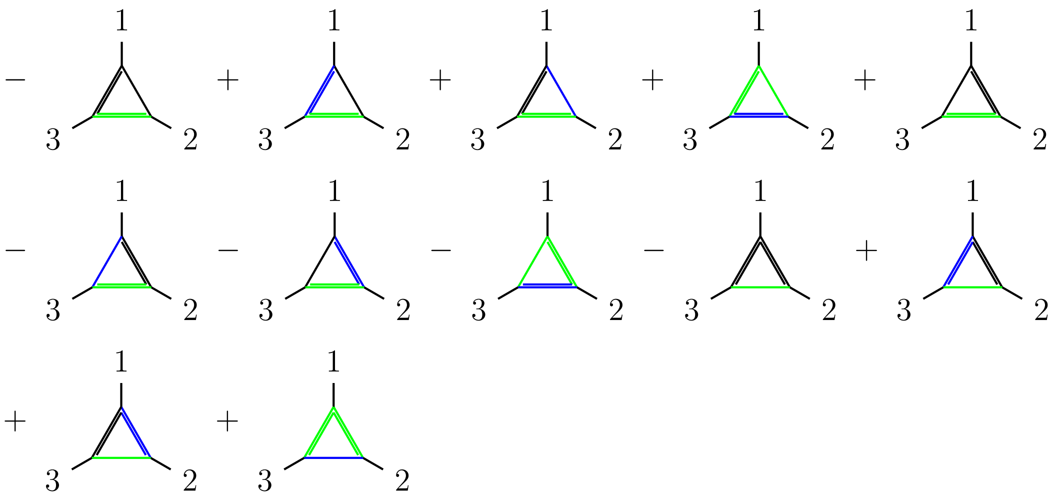


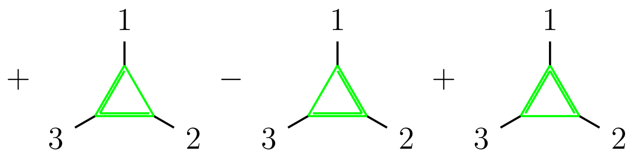
$g_{16} =$



$g_{17} =$ 

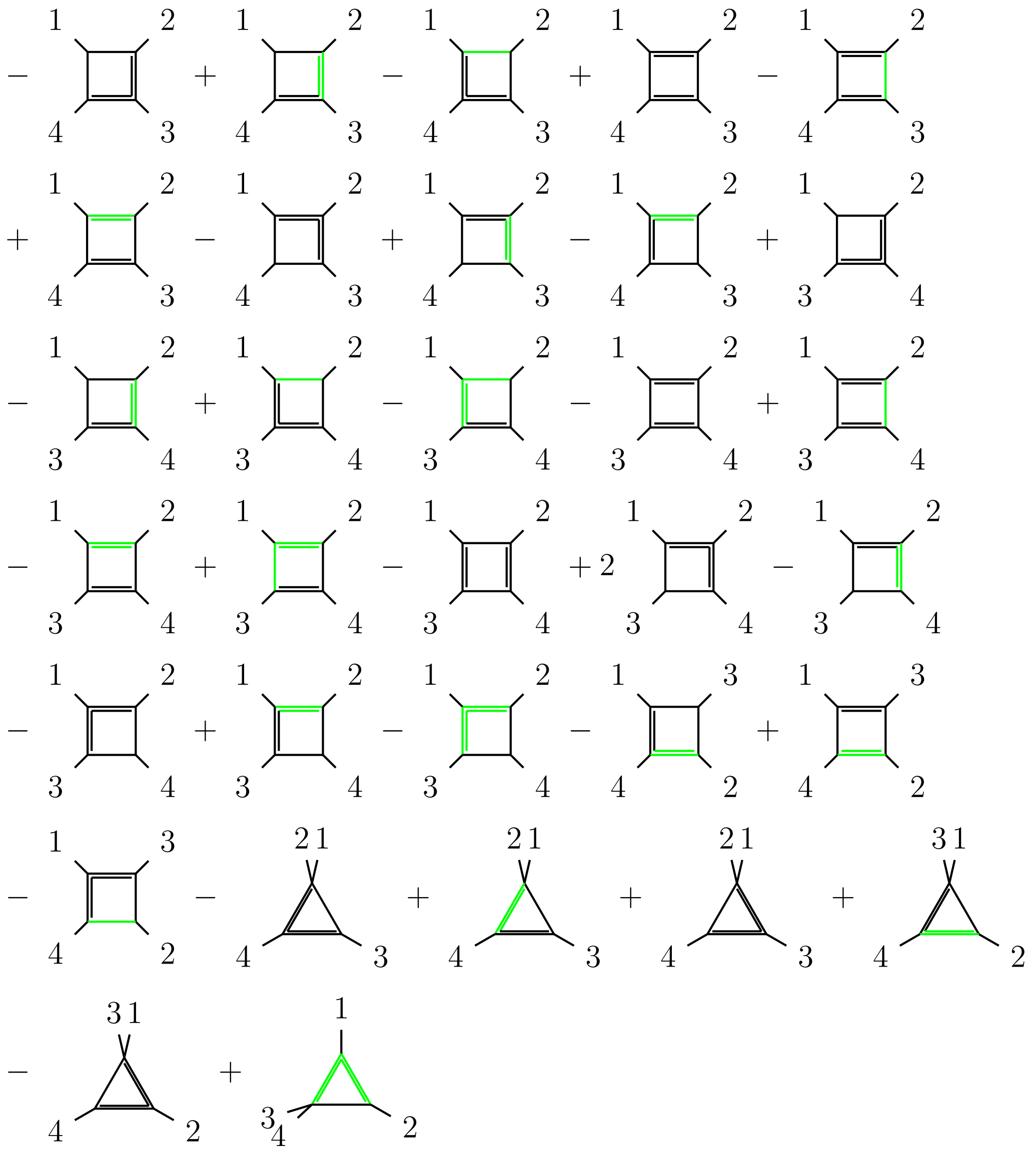
$g_{18} =$ 

$g_{19} =$ 

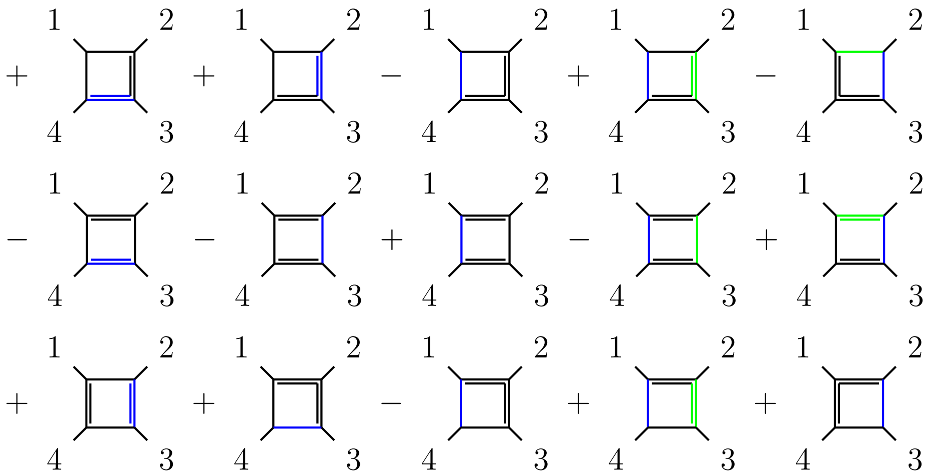
$g_{20} =$ 

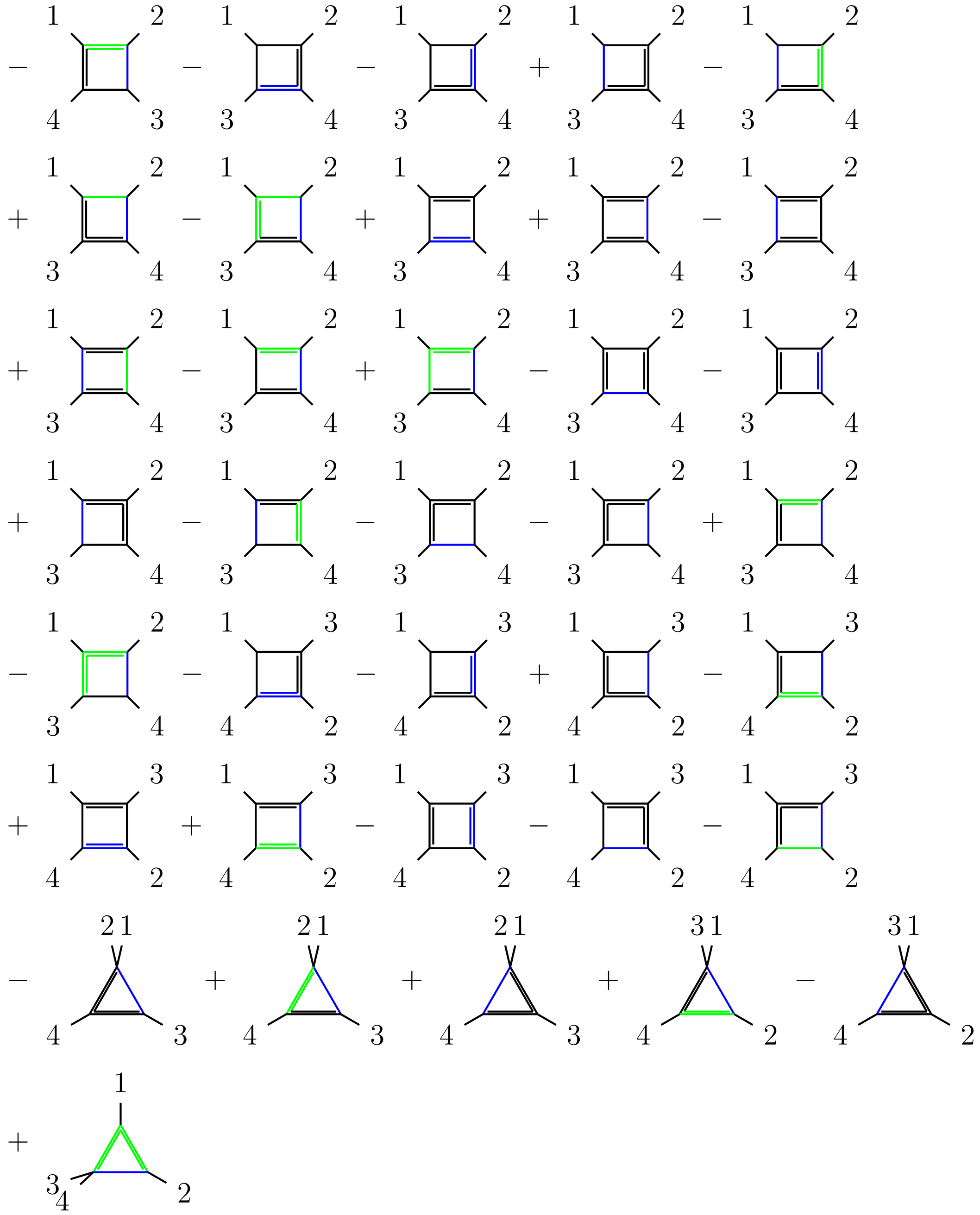
B.5 Generators of $H_2(X_{1,4,3};\mathbb{Q})$

$g_1 =$

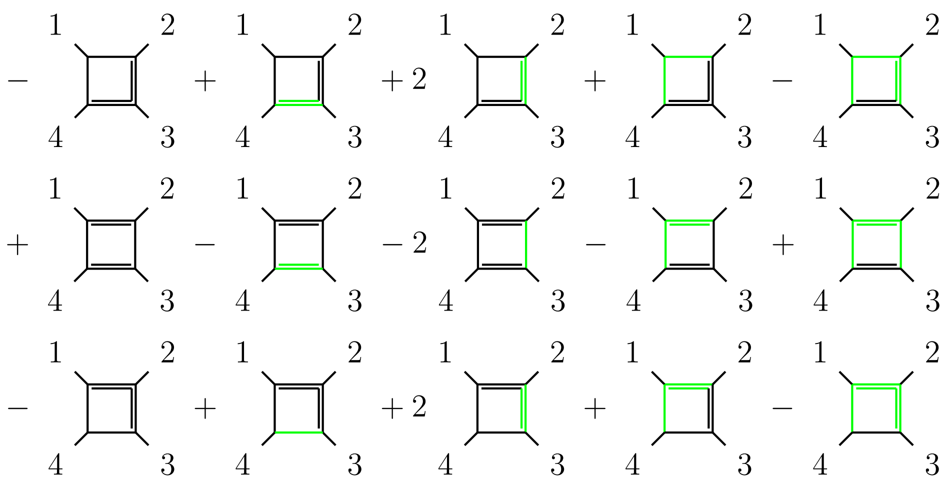


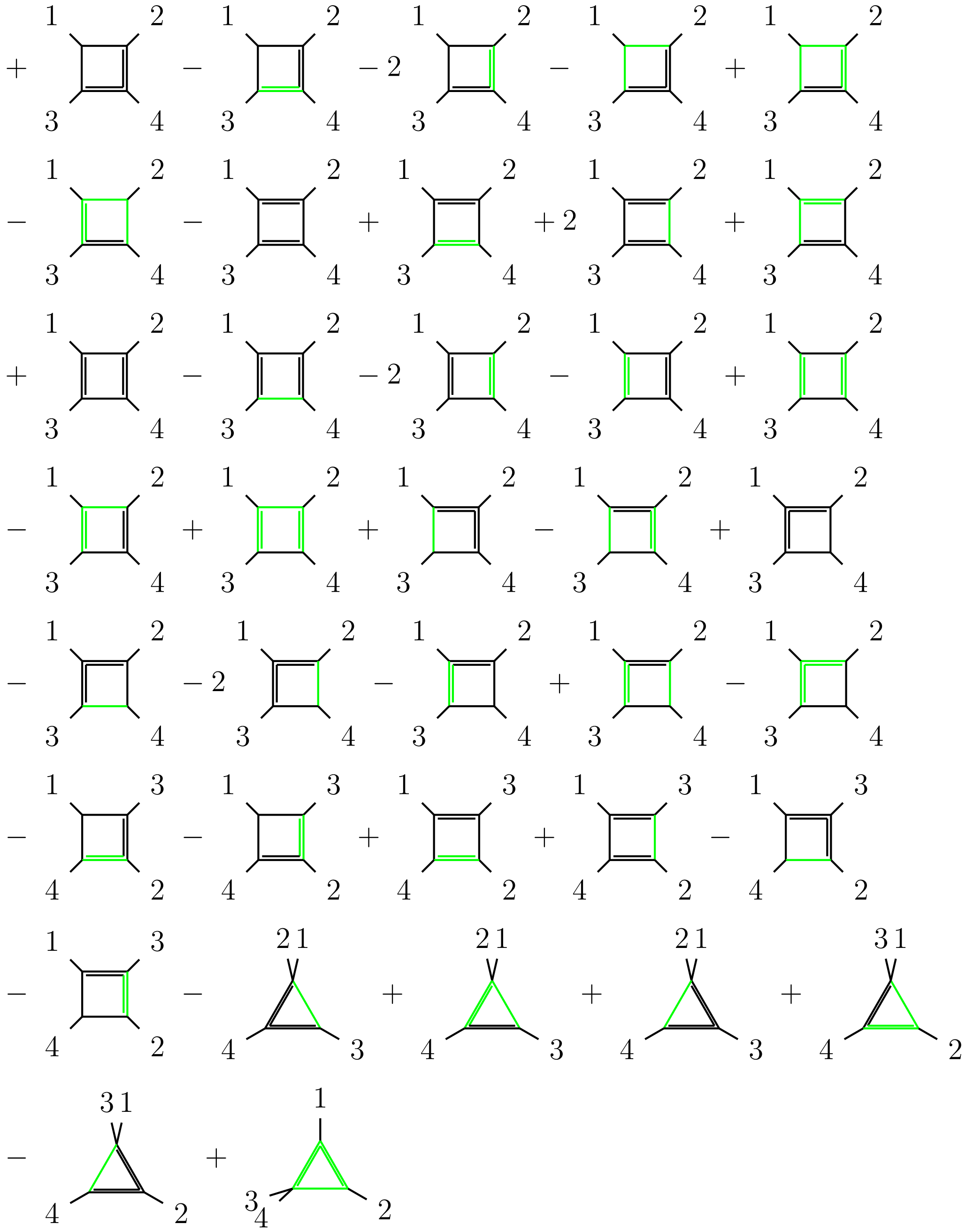
$g_2 =$



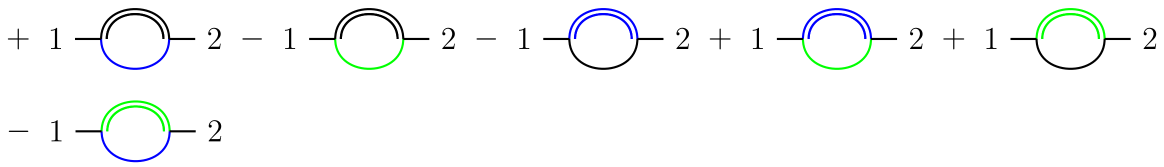


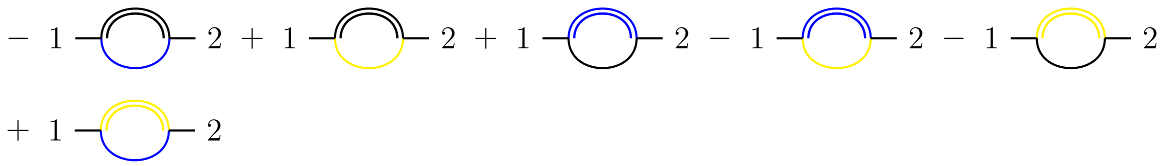
$g_3 =$

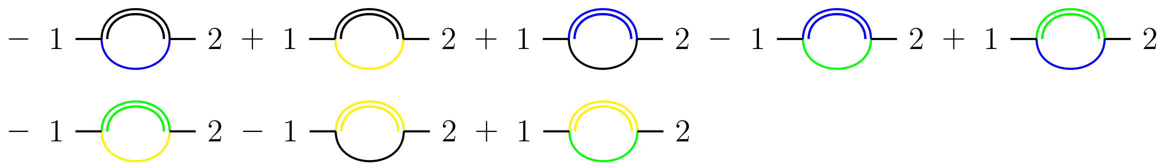




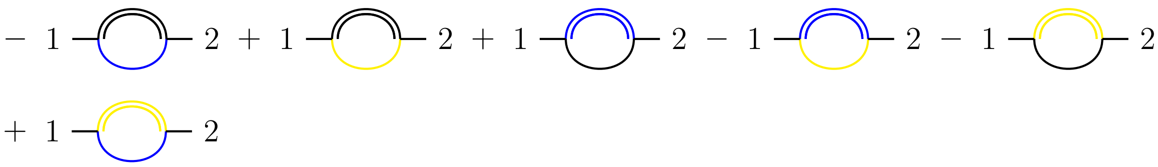
B.6 Generators of $H_1(X_{1,2,4};\mathbb{Q})$

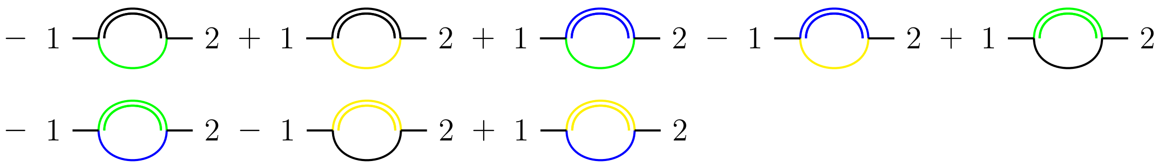
$g_1 =$


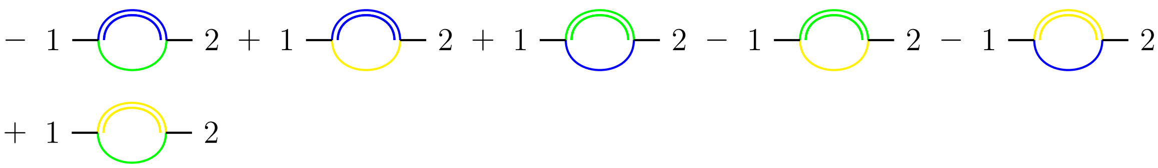
$g_2 =$


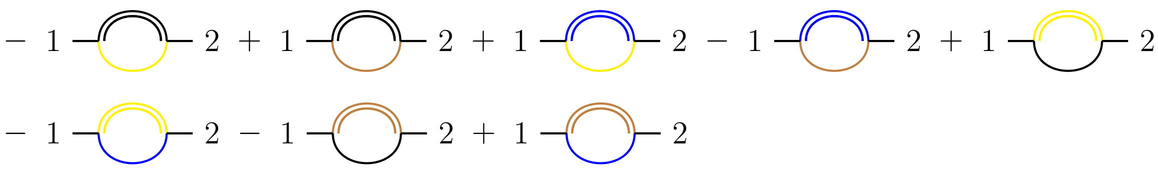
$g_3 =$


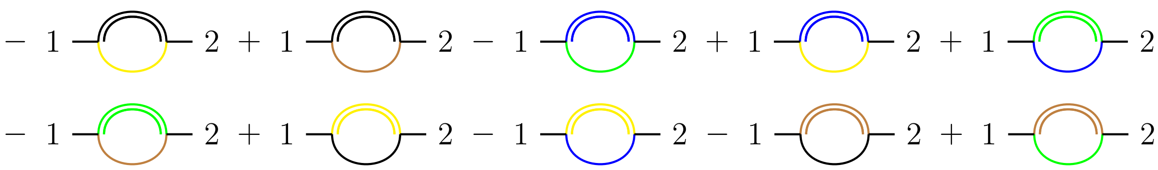
B.7 Generators of $H_1(X_{1,2,5};\mathbb{Q})$

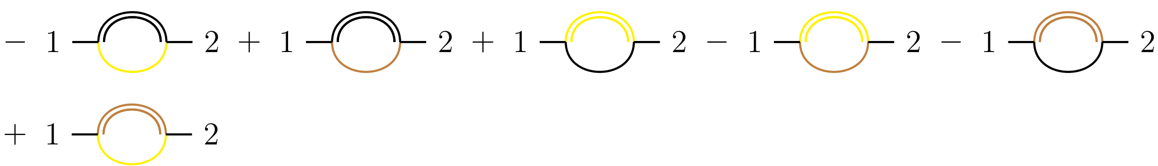
$g_1 =$ 

$g_2 =$ 

$g_3 =$ 

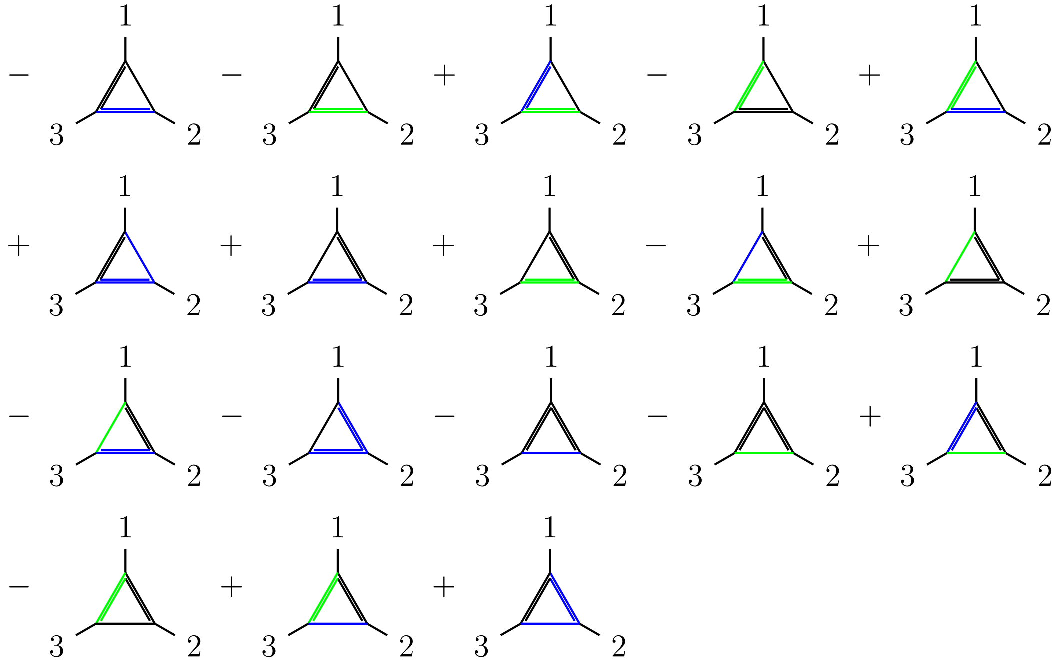
$g_4 =$ 

$g_5 =$ 

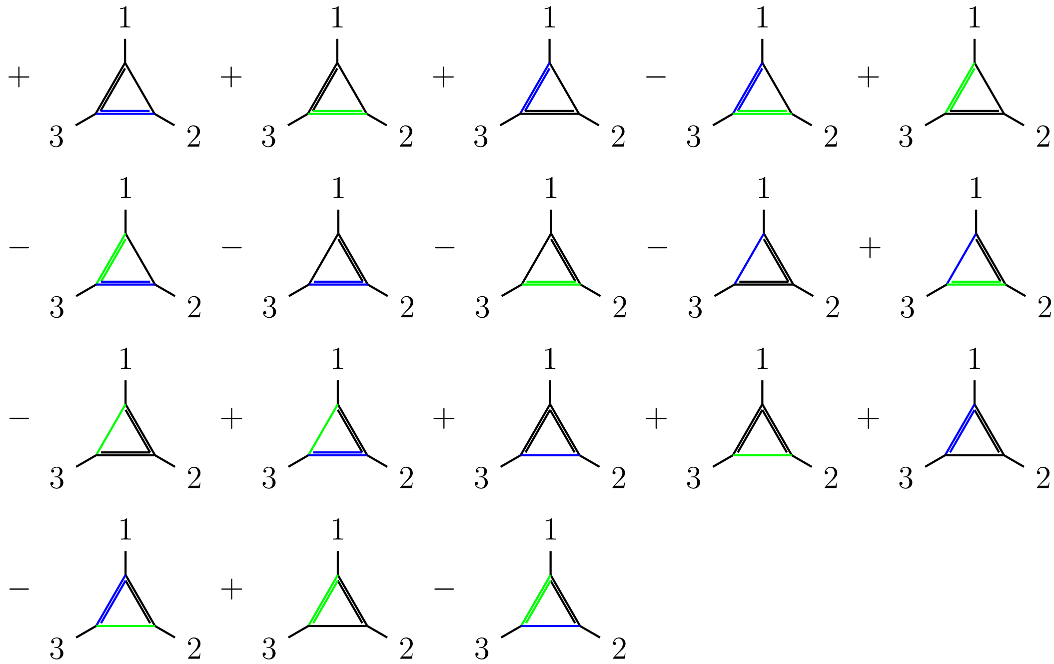
$g_6 =$ 

B.8 Generators of $H_1(X_{1,3,5};\mathbb{Q})$

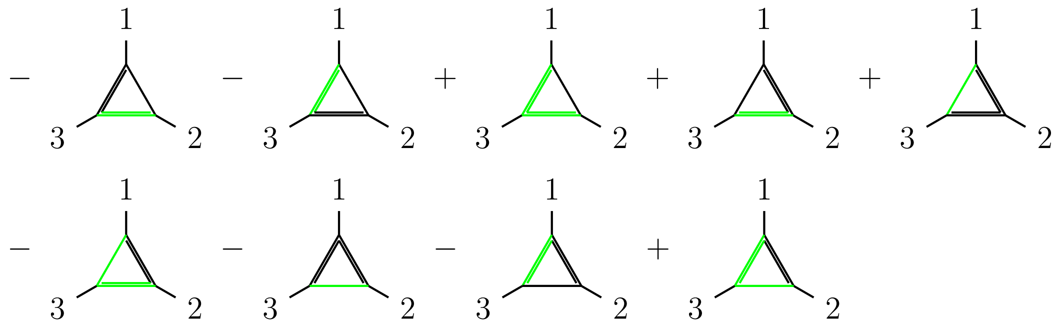
$g_1 =$

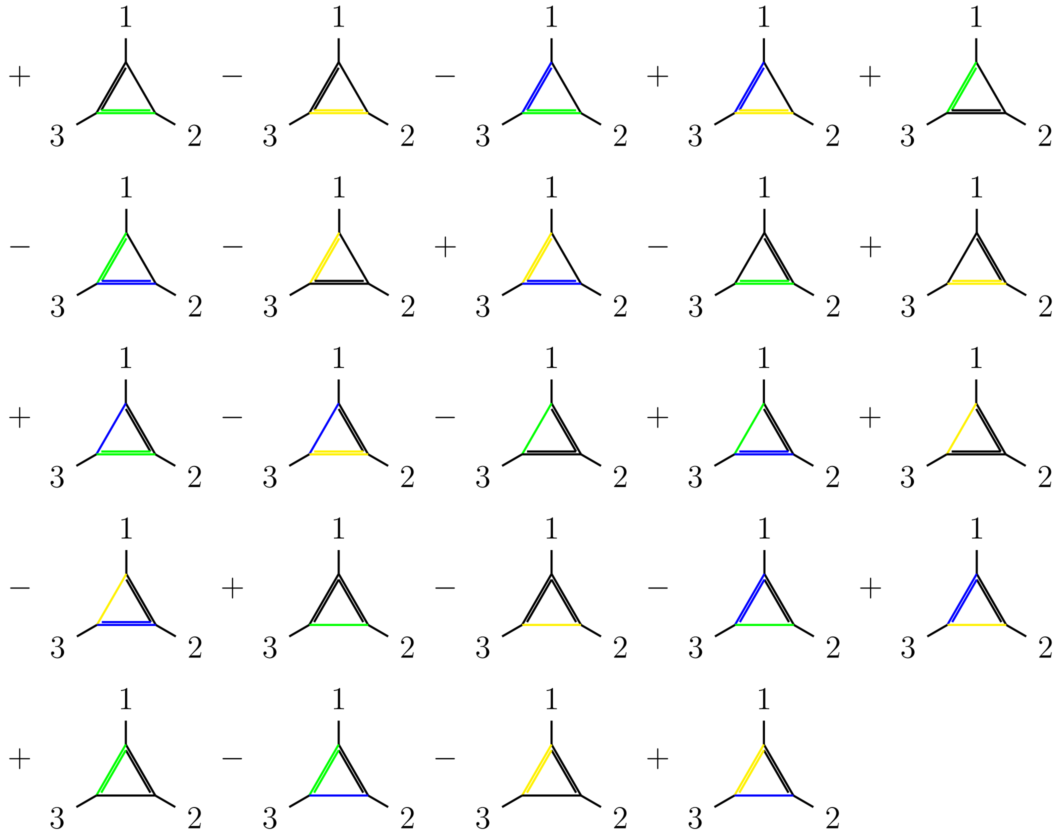


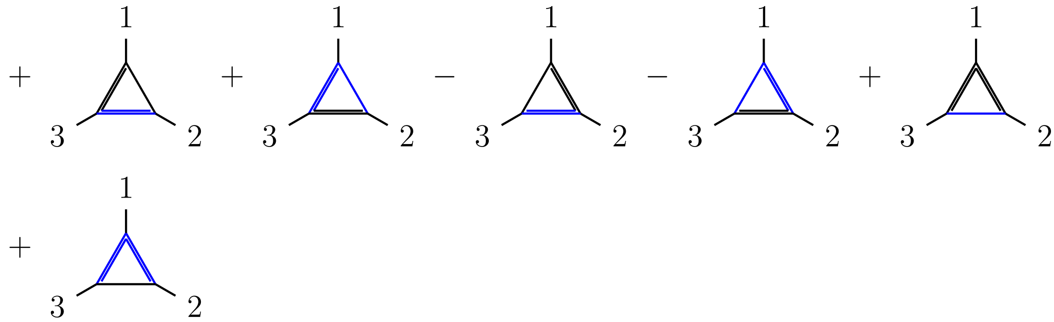
$g_2 =$

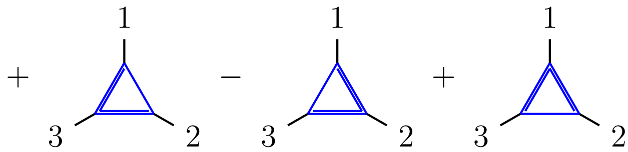


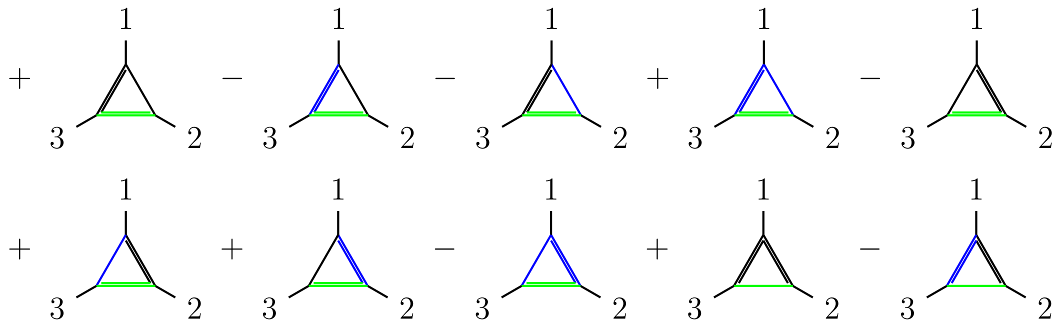
$g_3 =$

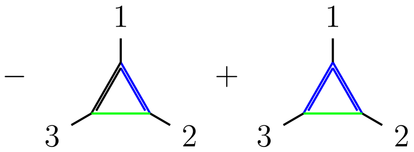


$g_4 =$ 

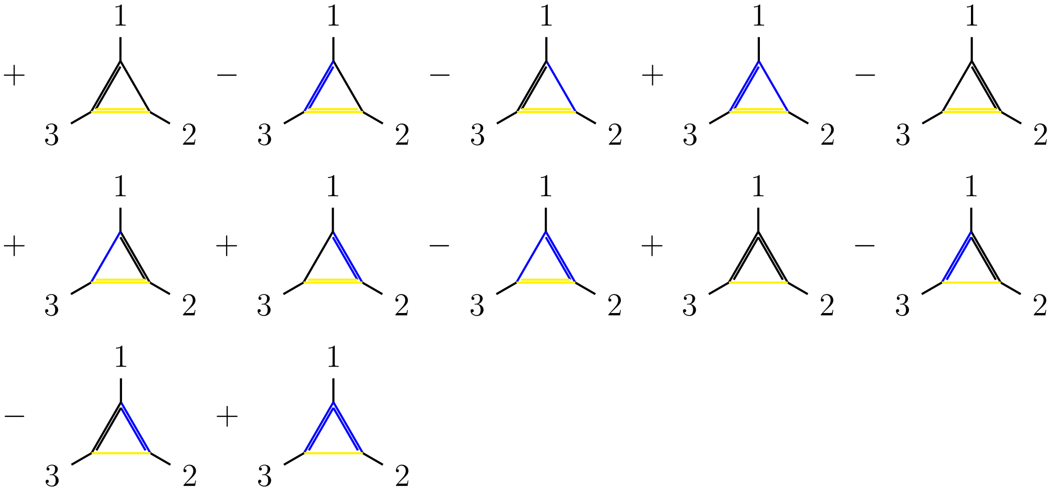
$g_5 =$ 

$g_6 =$ 

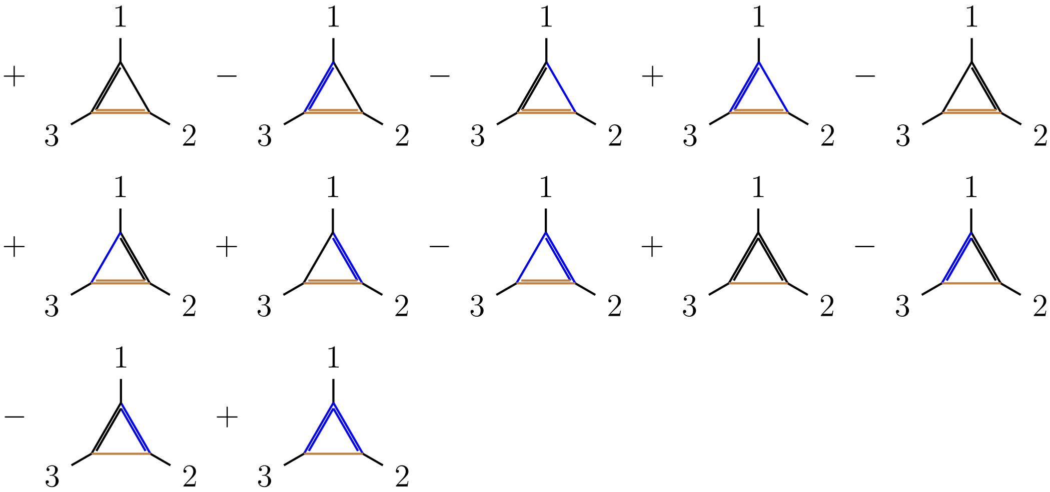
$g_7 =$ 



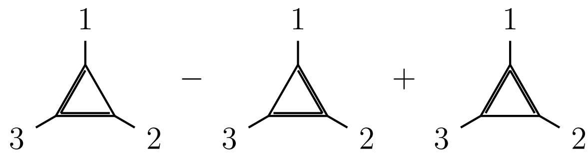
$g_8 =$



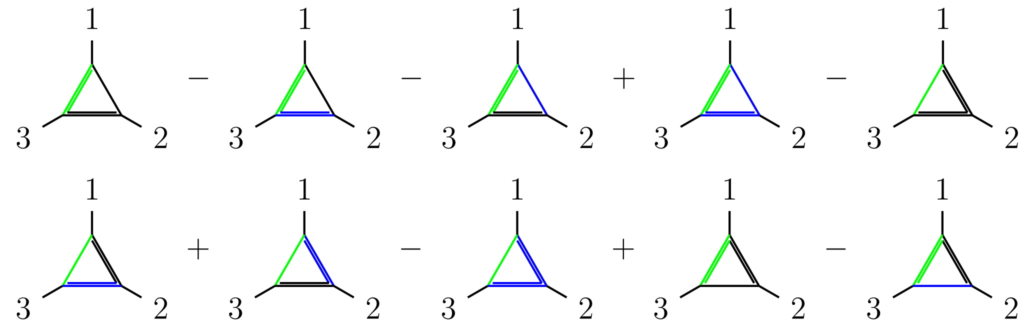
$g_9 =$

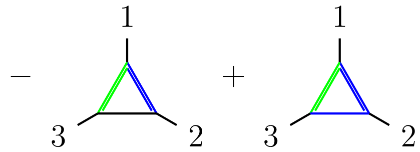


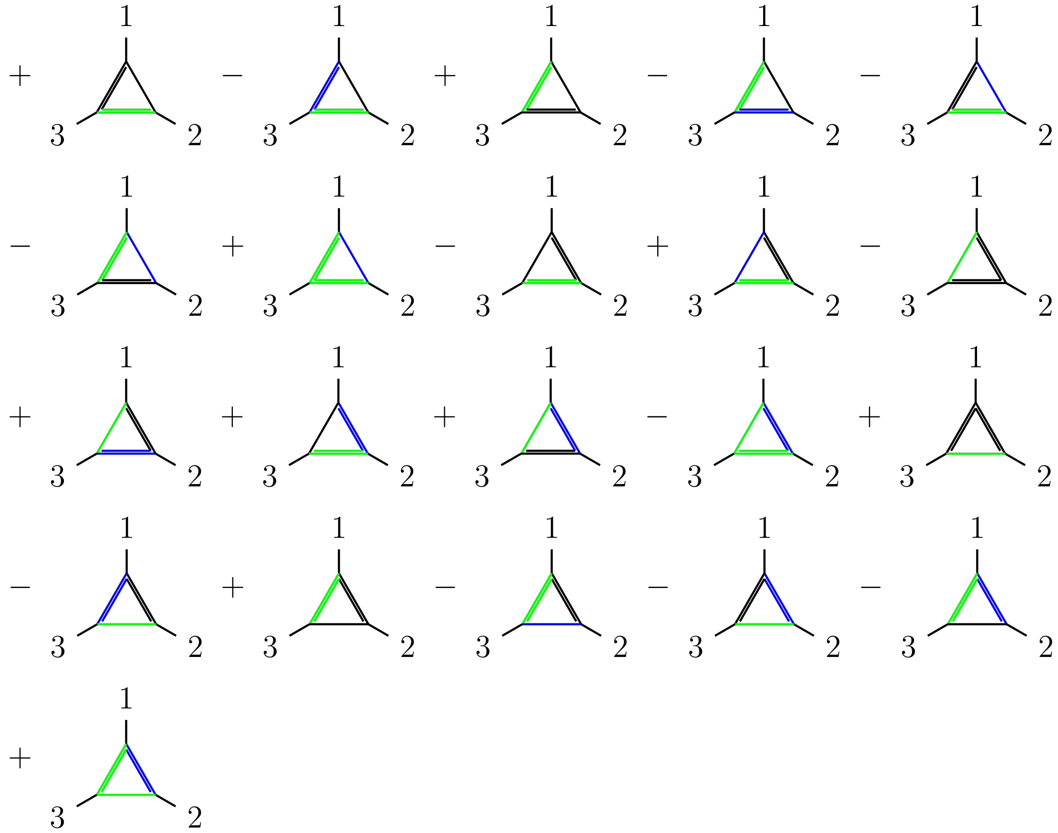
$g_{10} = +$

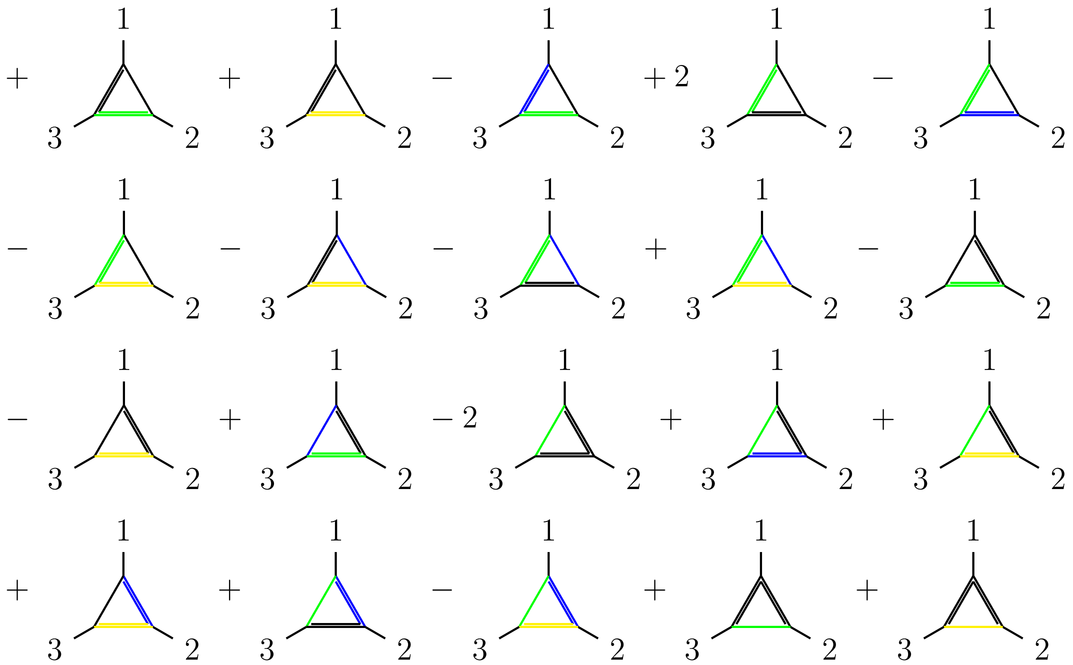


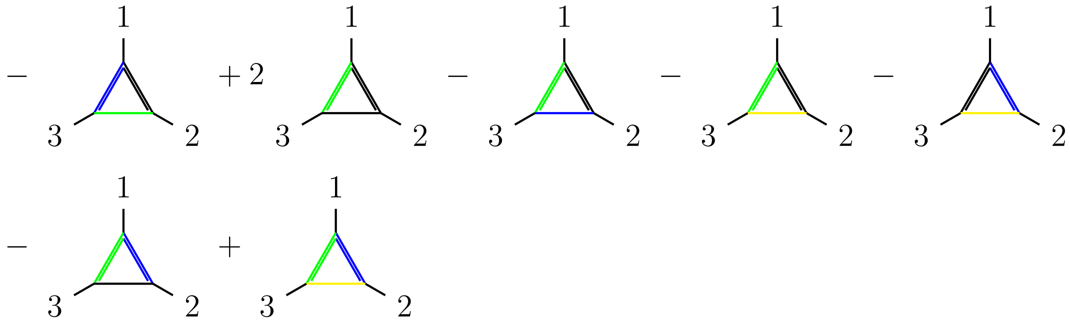
$g_{11} = +$



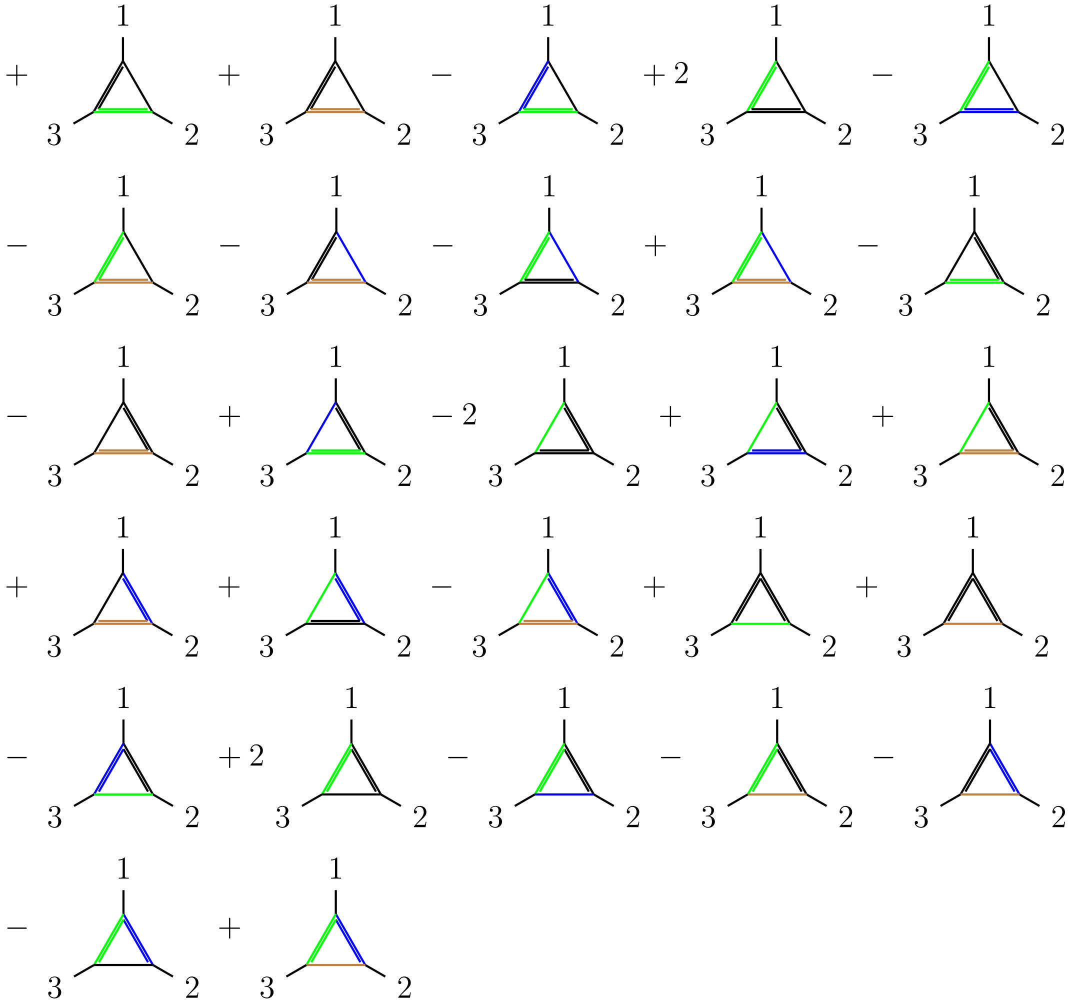


$g_{12} =$ 

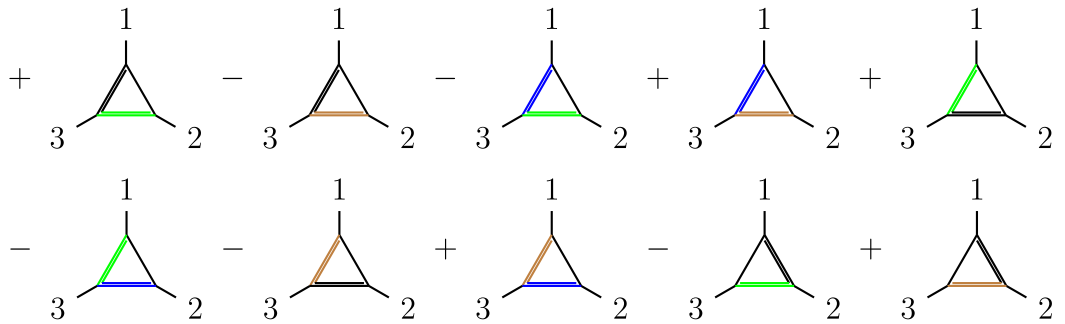
$g_{13} =$ 

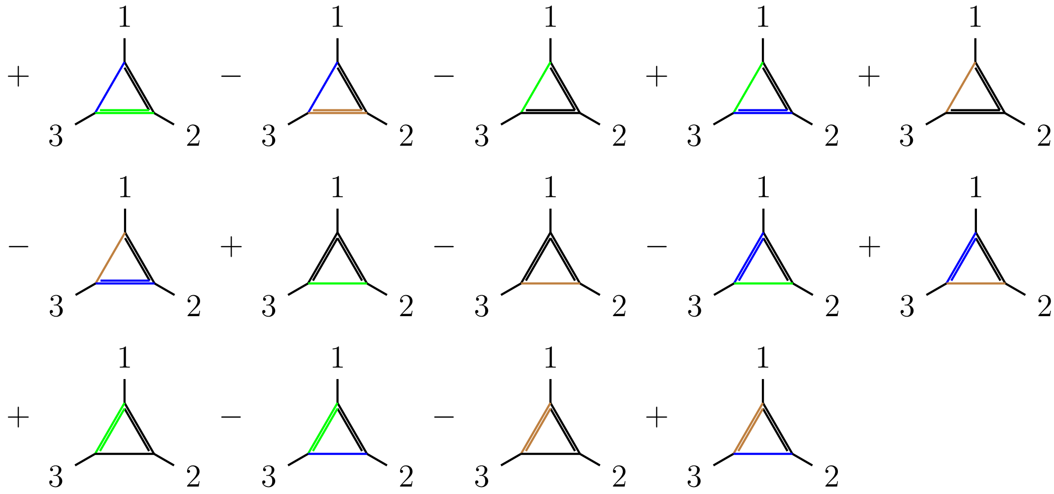


$g_{14} =$

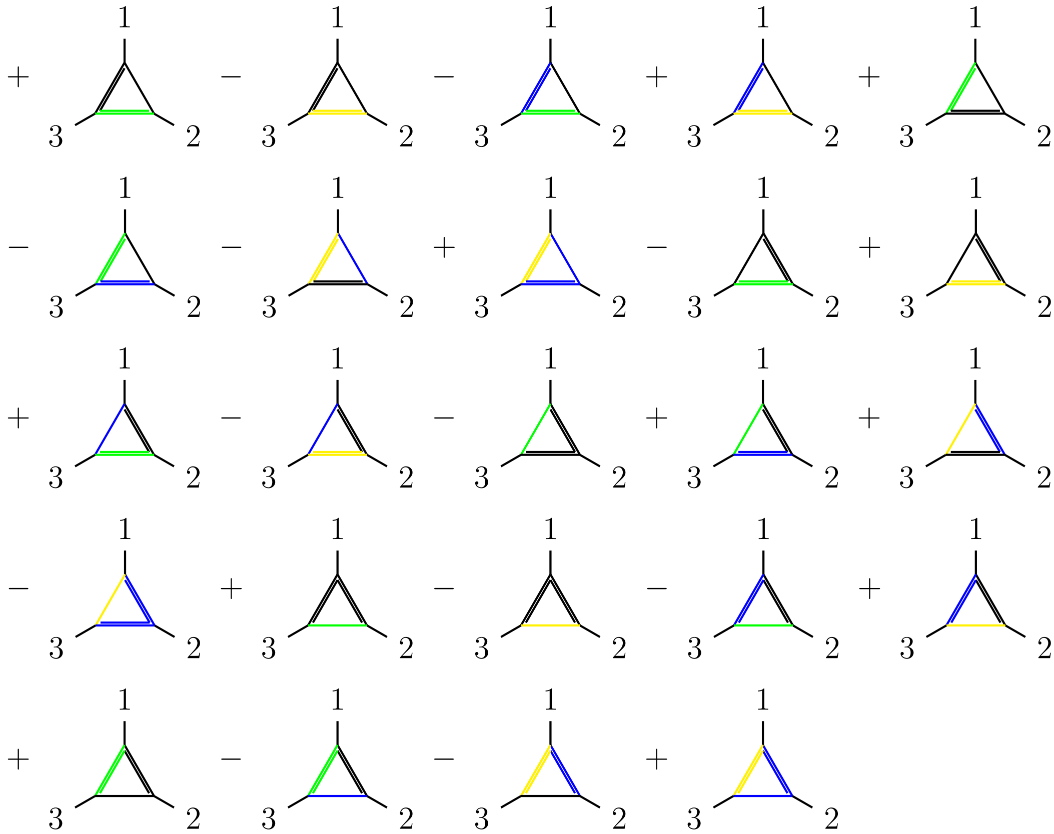


$g_{15} =$

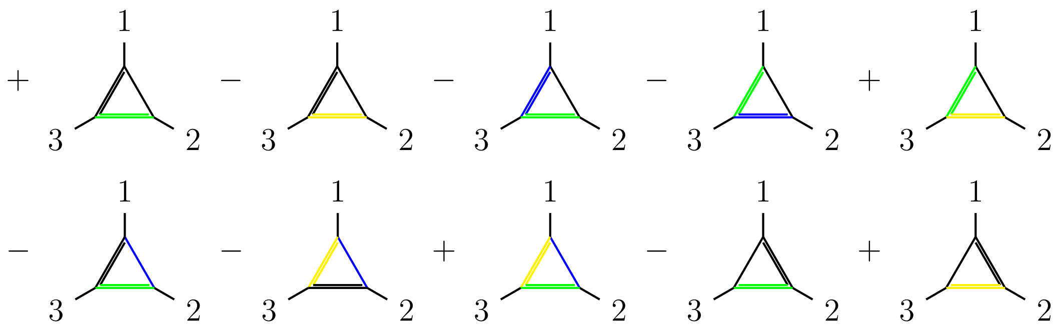


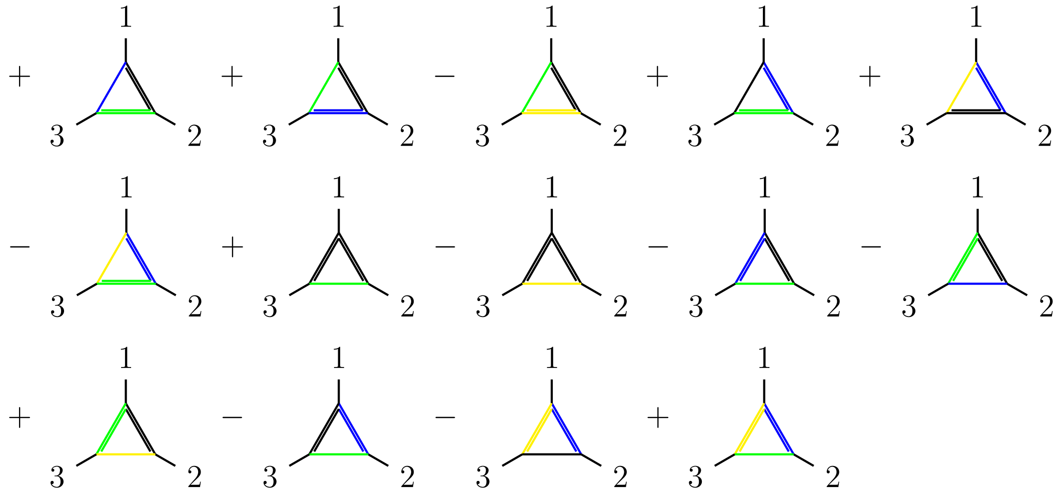


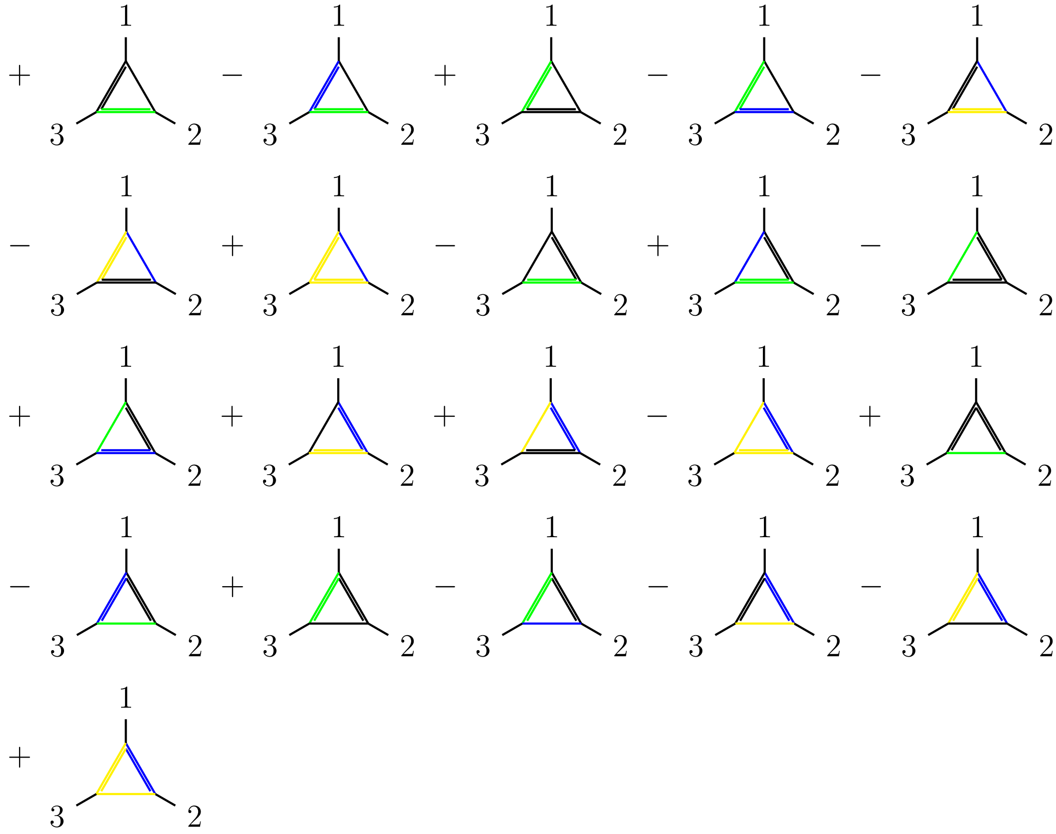
$g_{16} =$

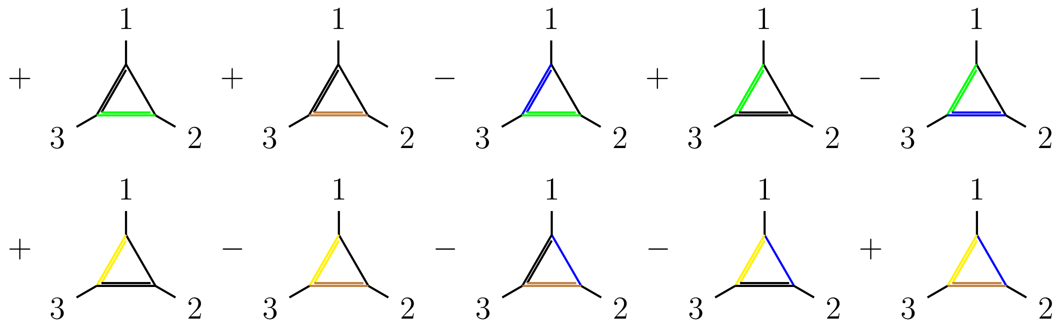


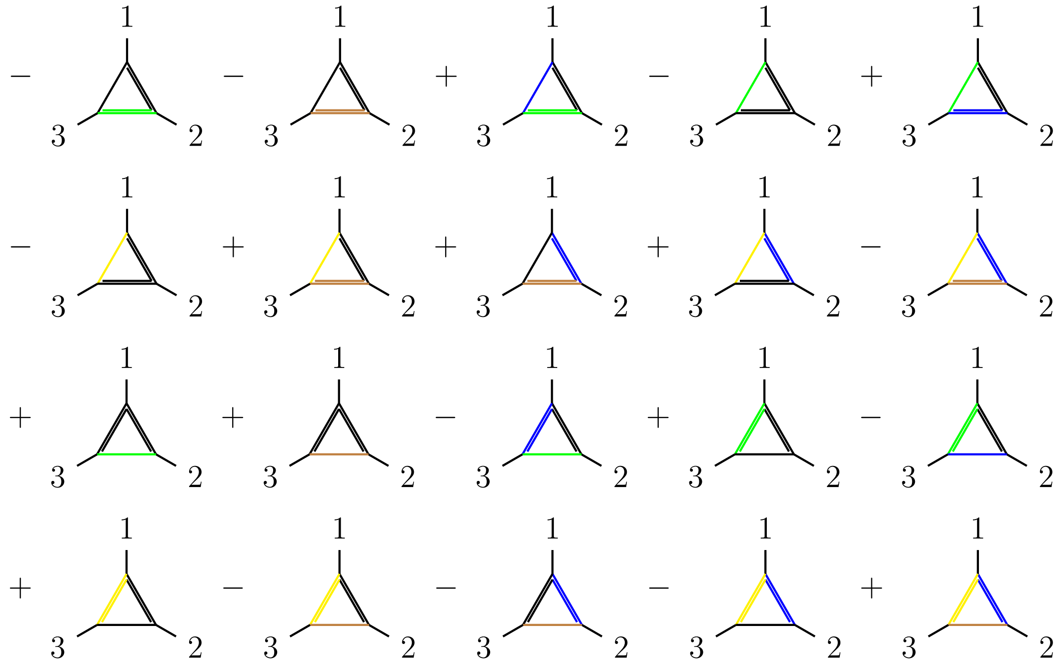
$g_{17} =$



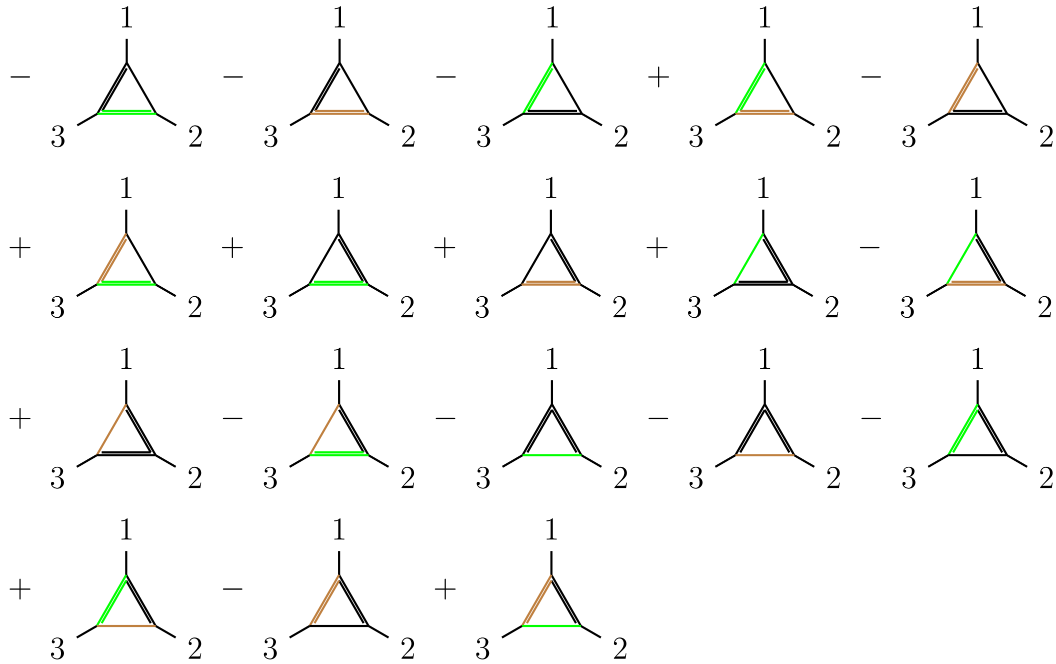


$g_{18} =$ 

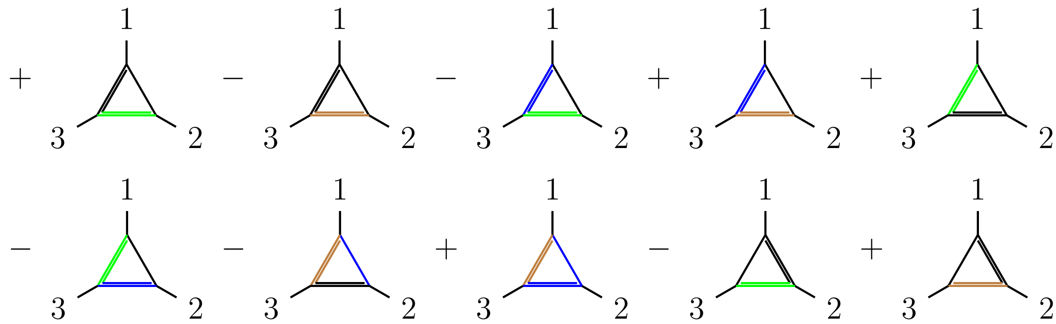
$g_{19} =$ 

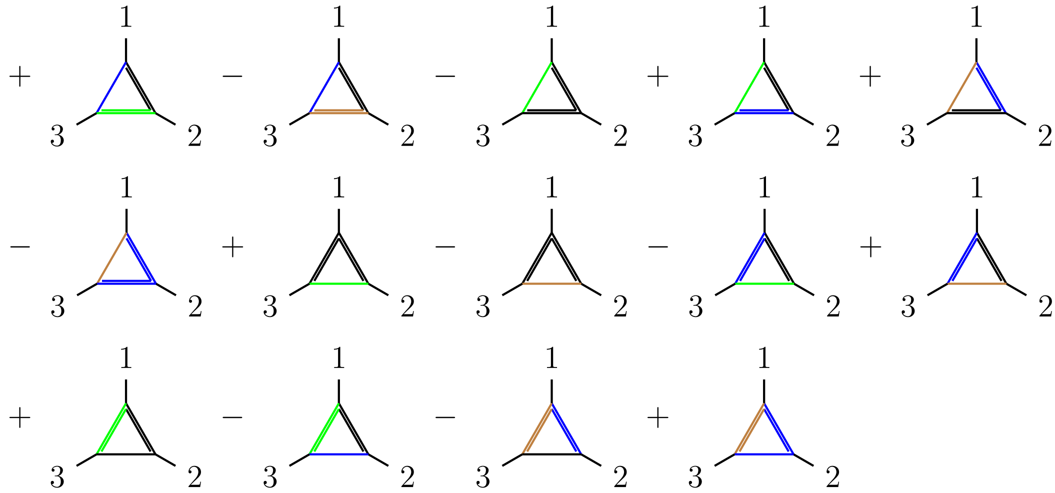


$g_{20} =$

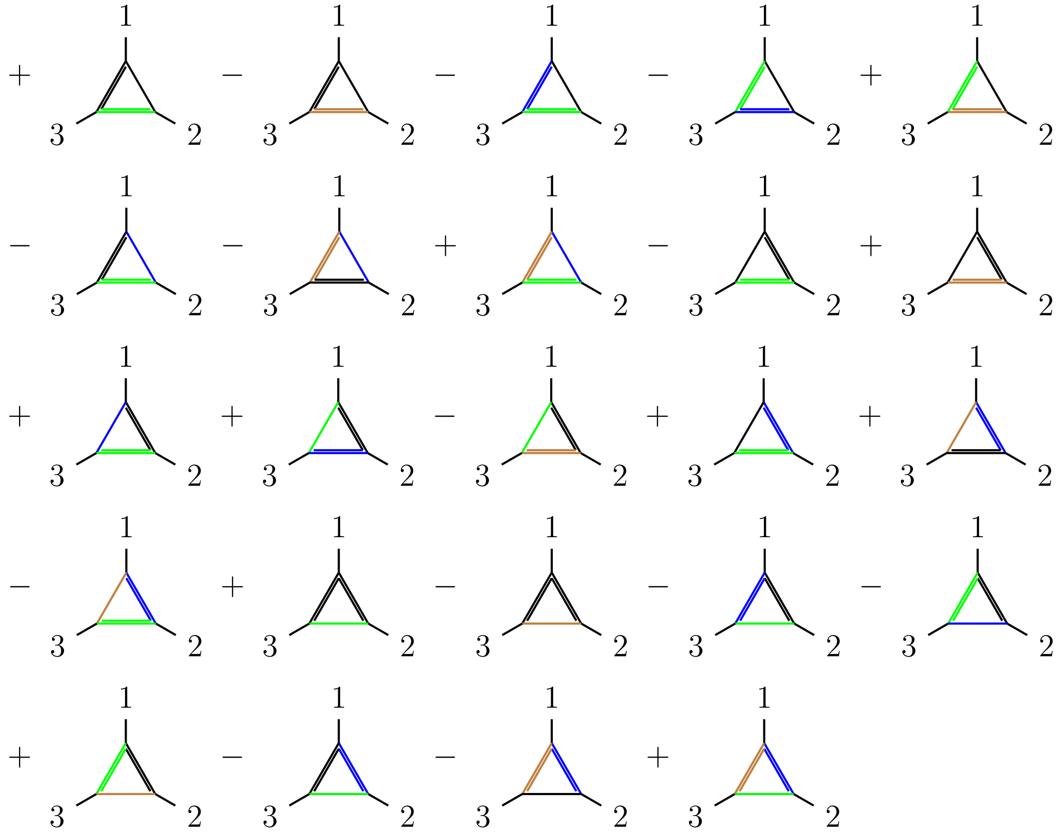


$g_{21} =$

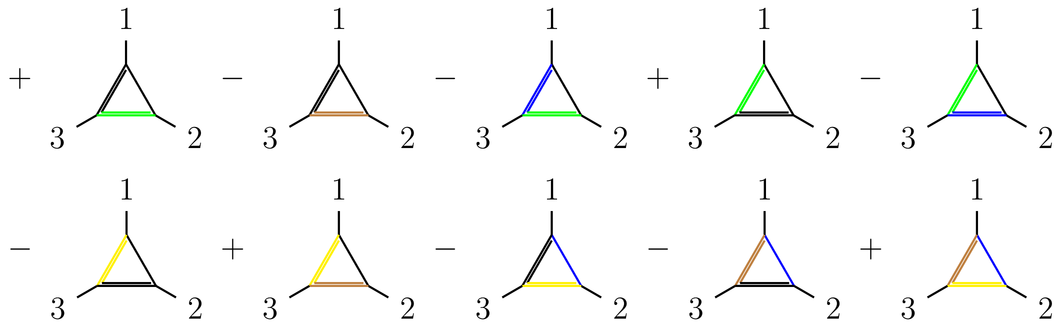


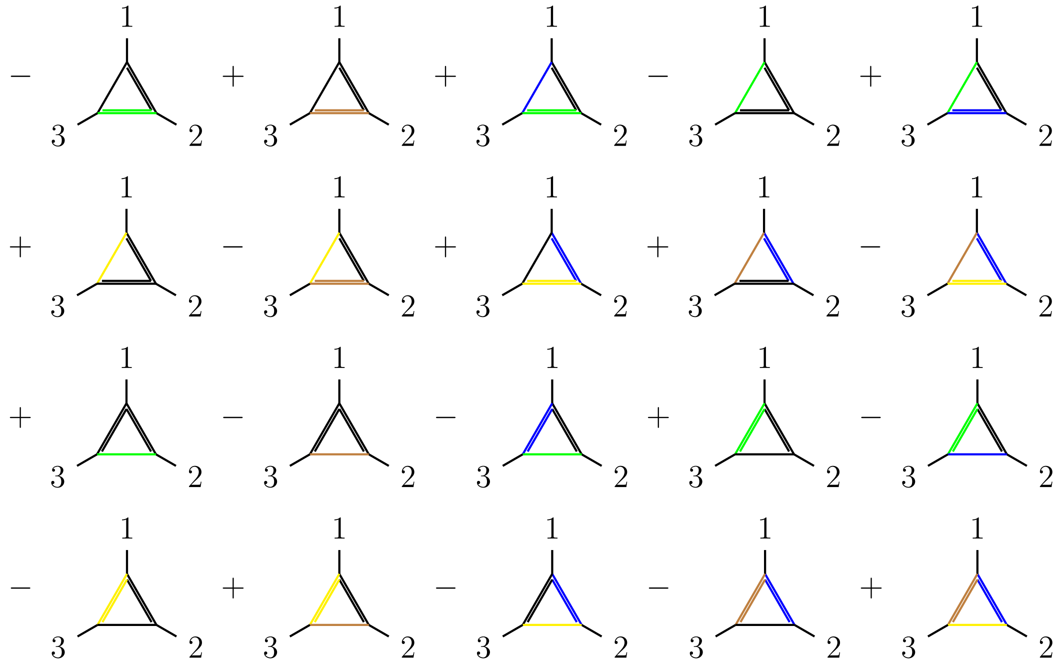


$g_{22} =$

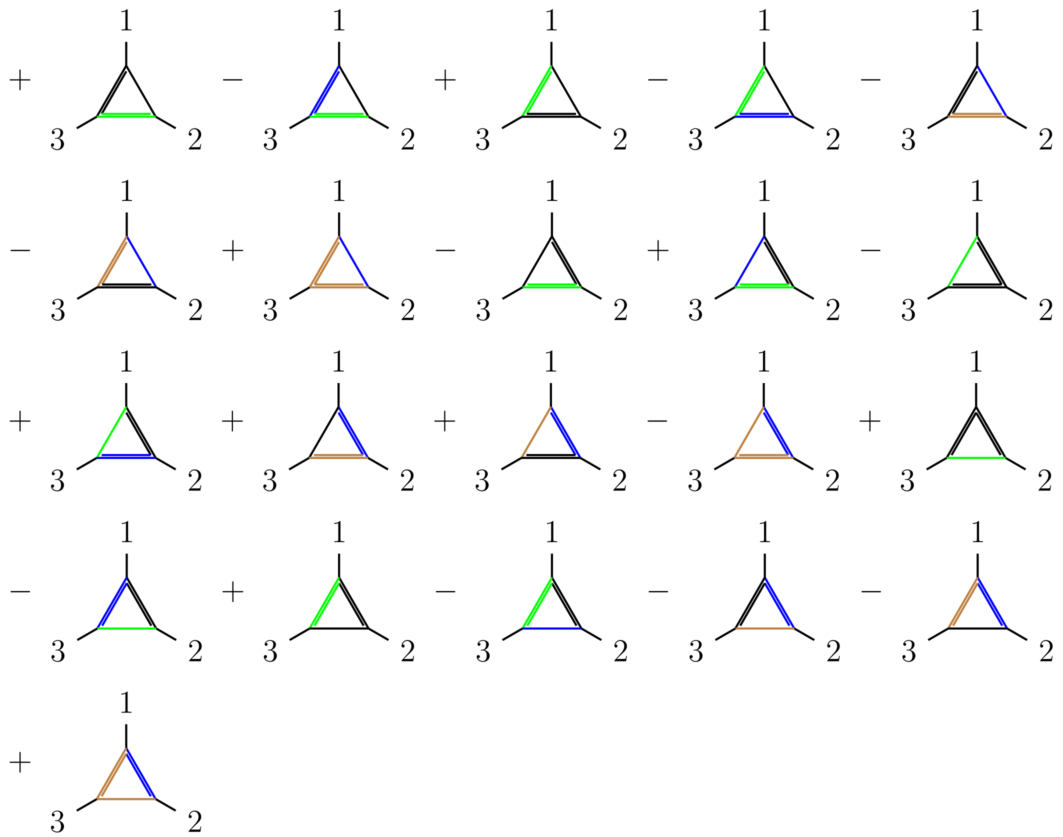


$g_{23} =$

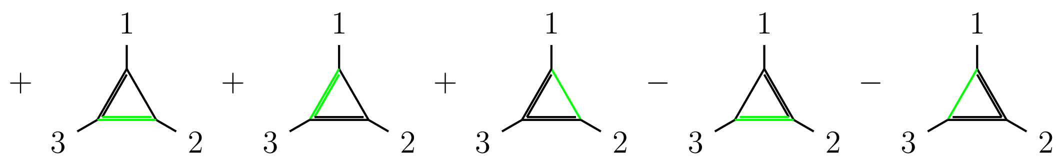


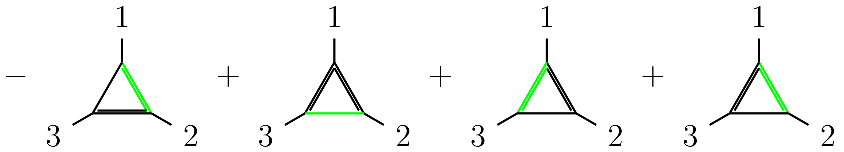


$g_{24} =$

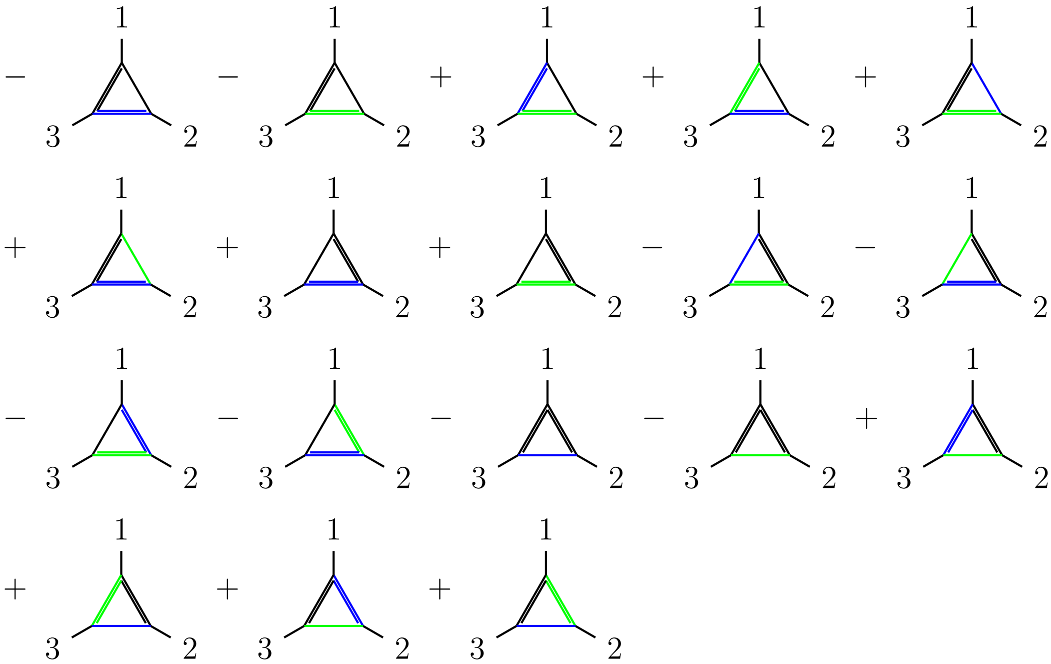


$g_{25} =$

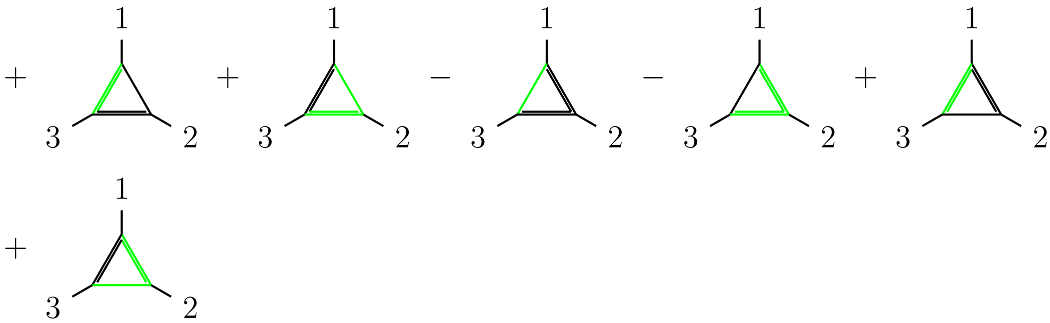




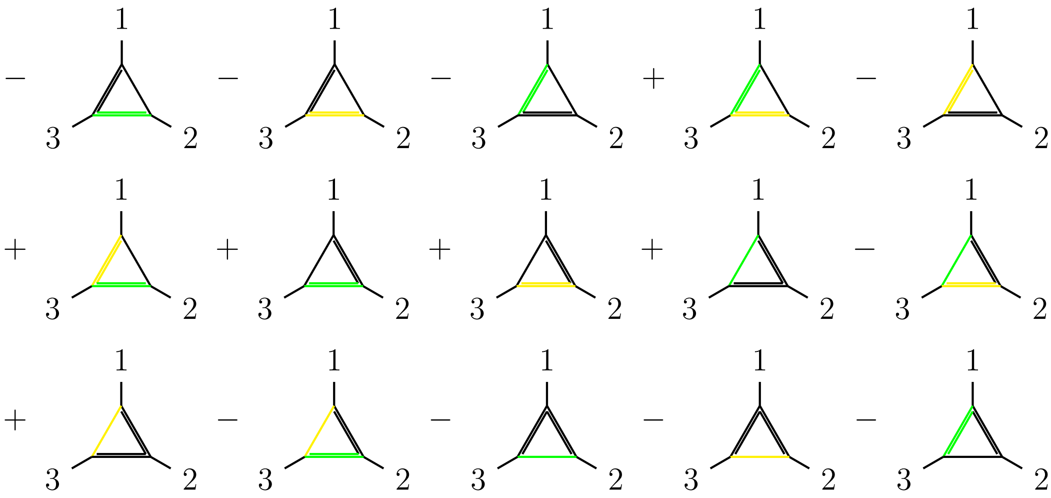
$g_{26} =$

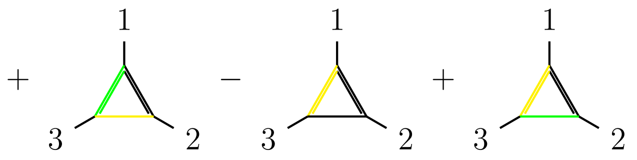


$g_{27} =$

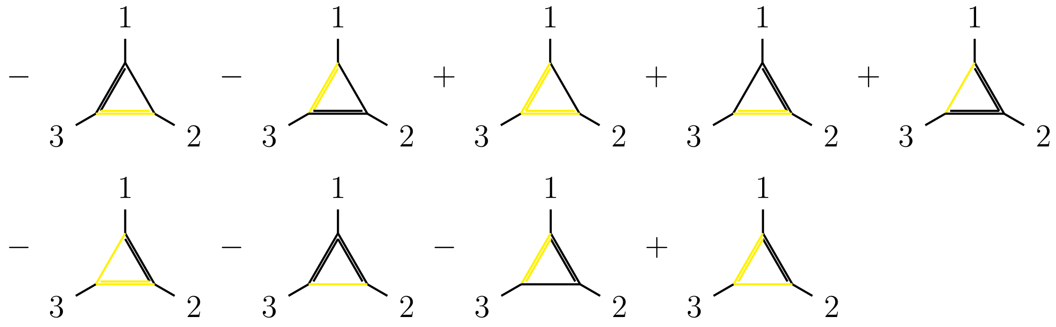


$g_{28} =$

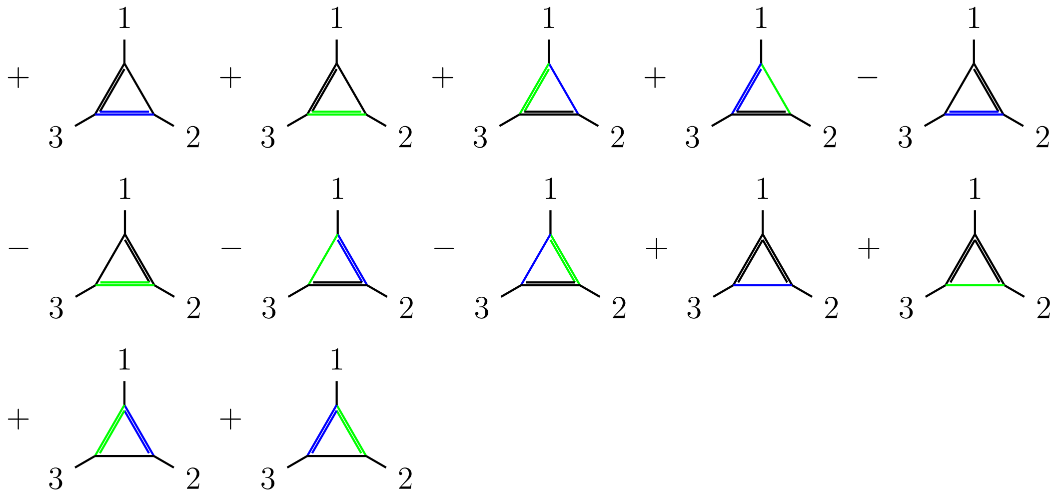




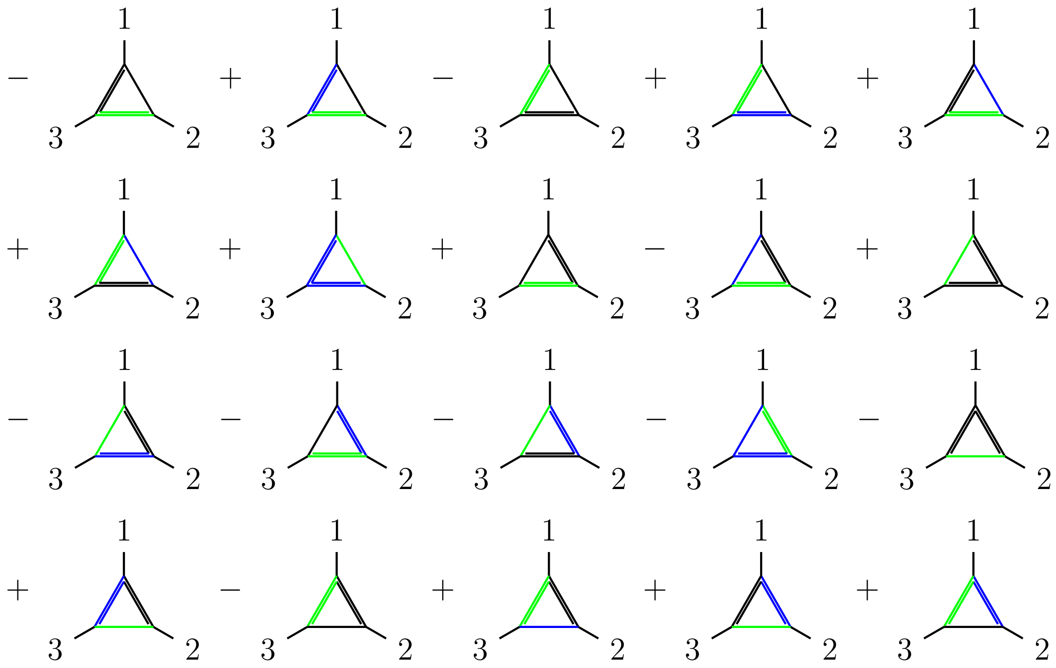
$g_{29} =$

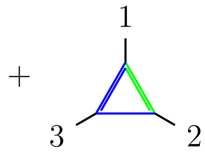


$g_{30} =$



$g_{31} =$





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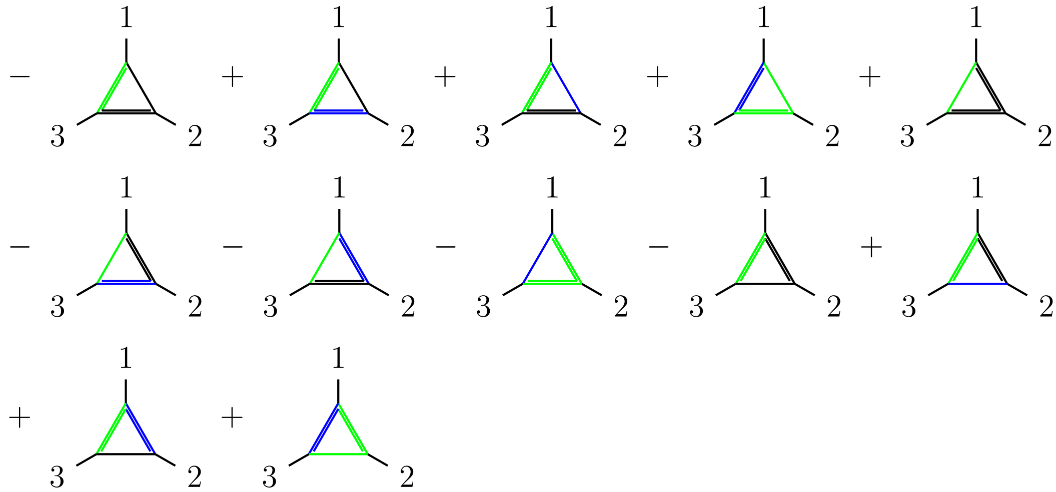

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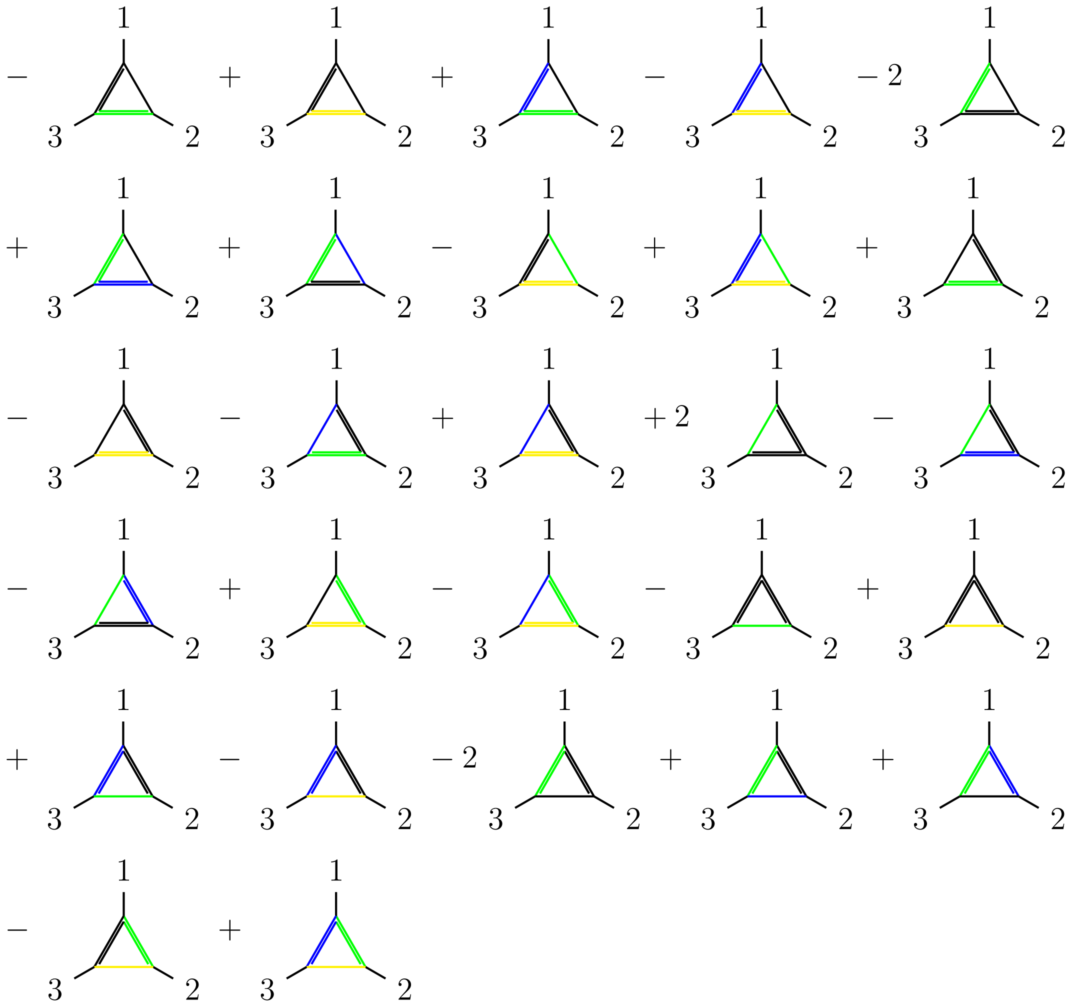

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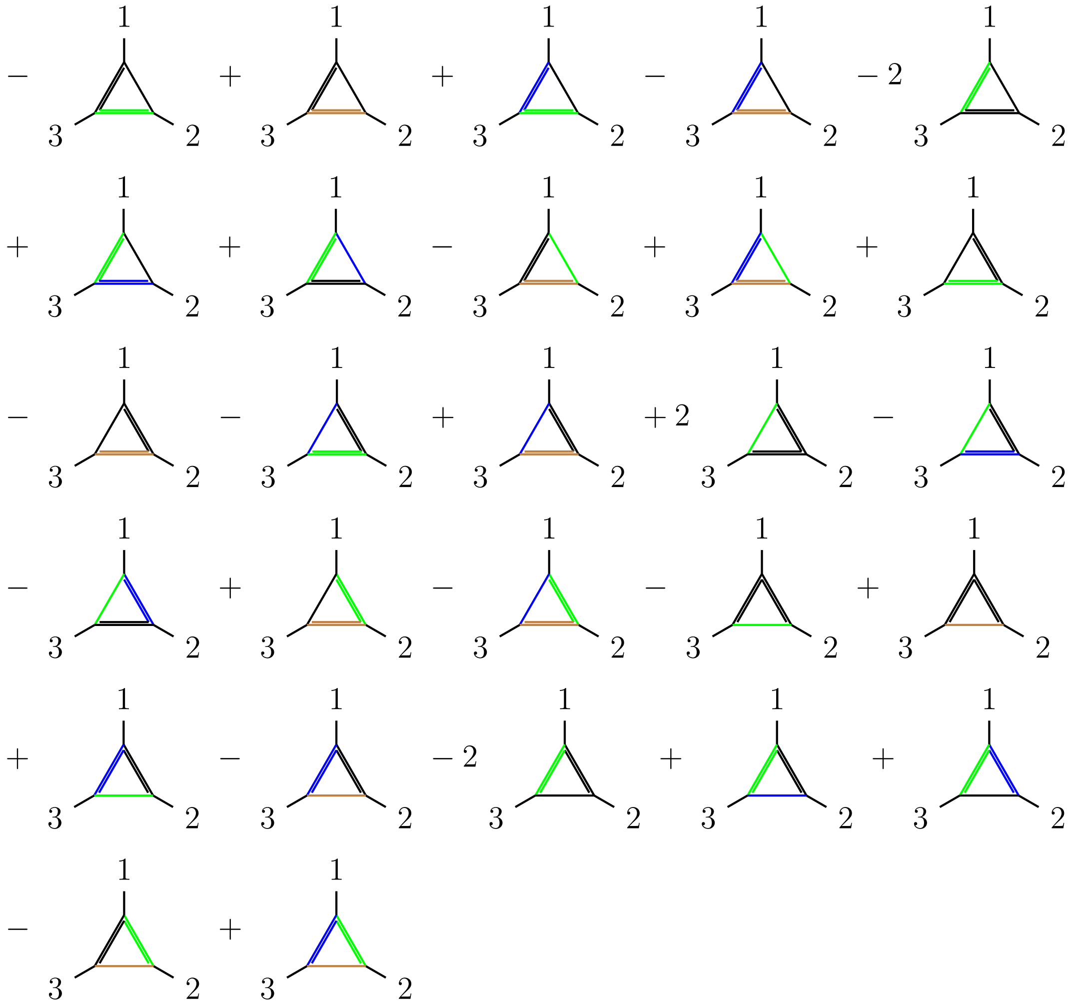

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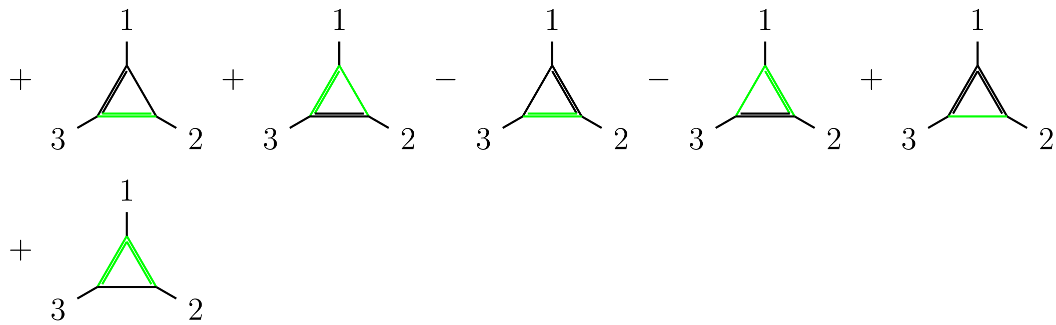

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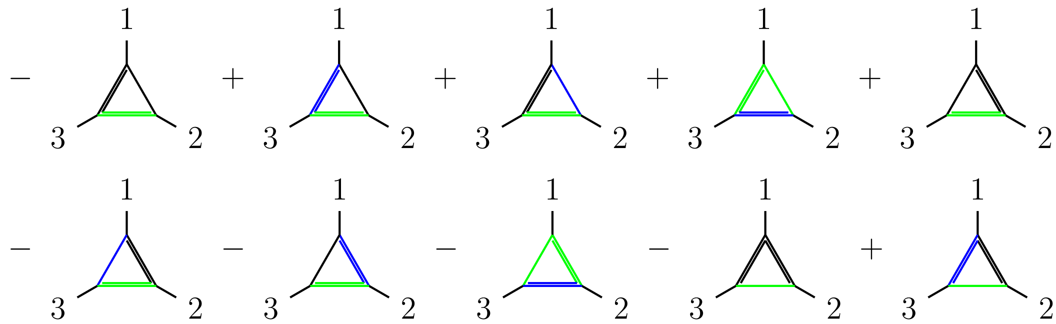
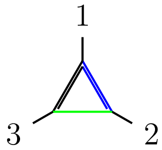
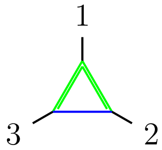
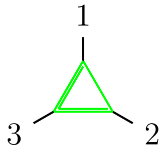
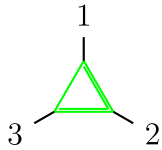
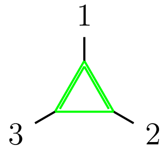
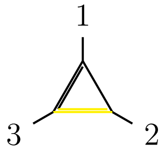
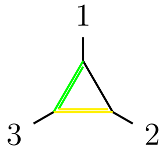
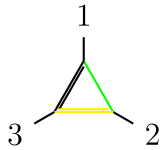
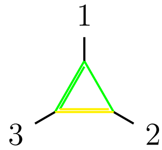
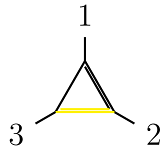
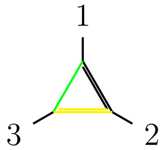
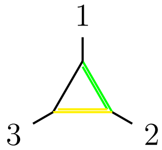
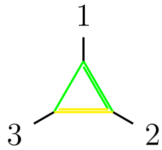
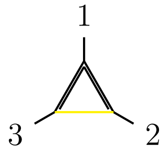
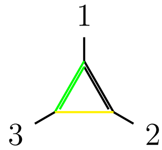
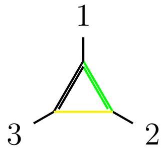
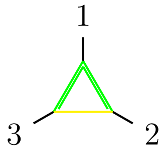


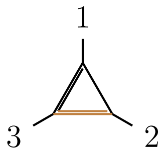
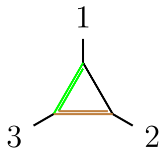
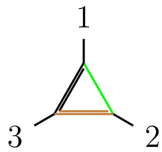
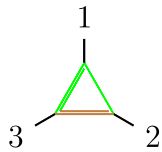
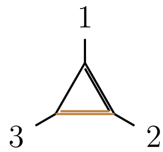
$+$  $+$ 

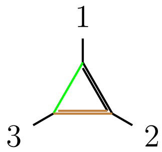
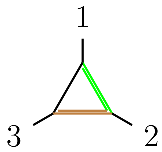
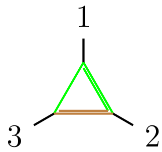
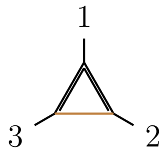
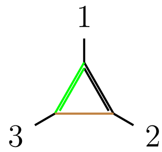
$g_{37} = +$  $-$  $+$ 

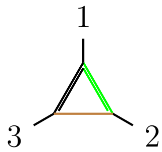
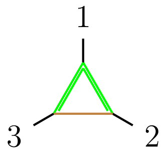
$g_{38} = +$  $-$  $-$  $+$  $-$ 

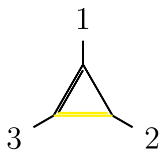
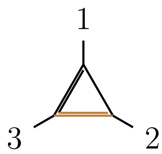
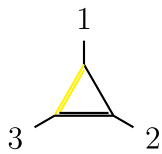
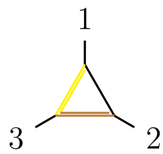
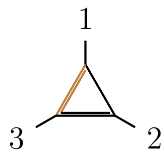
$+$  $+$  $-$  $+$  $-$ 

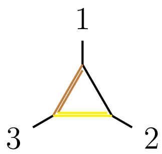
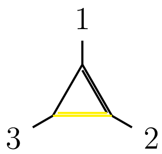
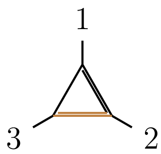
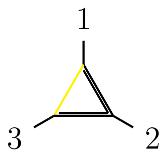
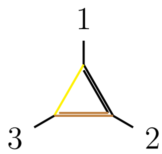
$-$  $+$ 

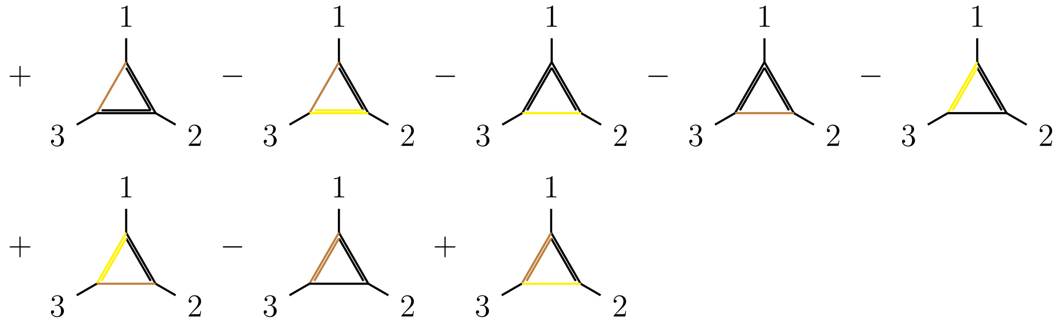
$g_{39} = +$  $-$  $-$  $+$  $-$ 

$+$  $+$  $-$  $+$  $-$ 

$-$  $+$ 

$g_{40} = -$  $-$  $-$  $+$  $-$ 

$+$  $+$  $+$  $+$  $-$ 



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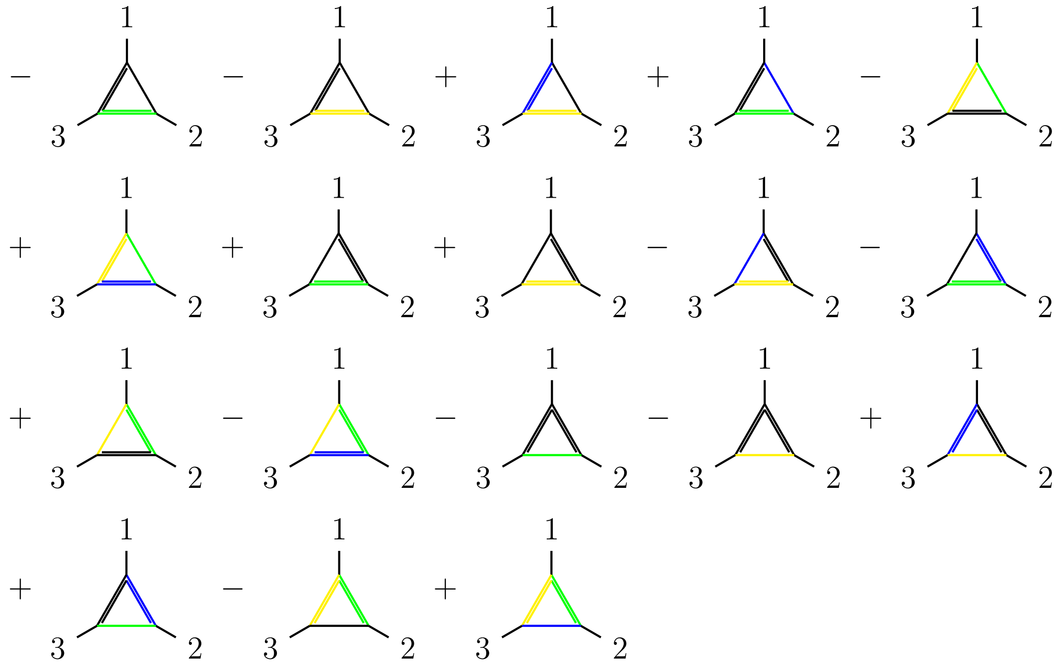

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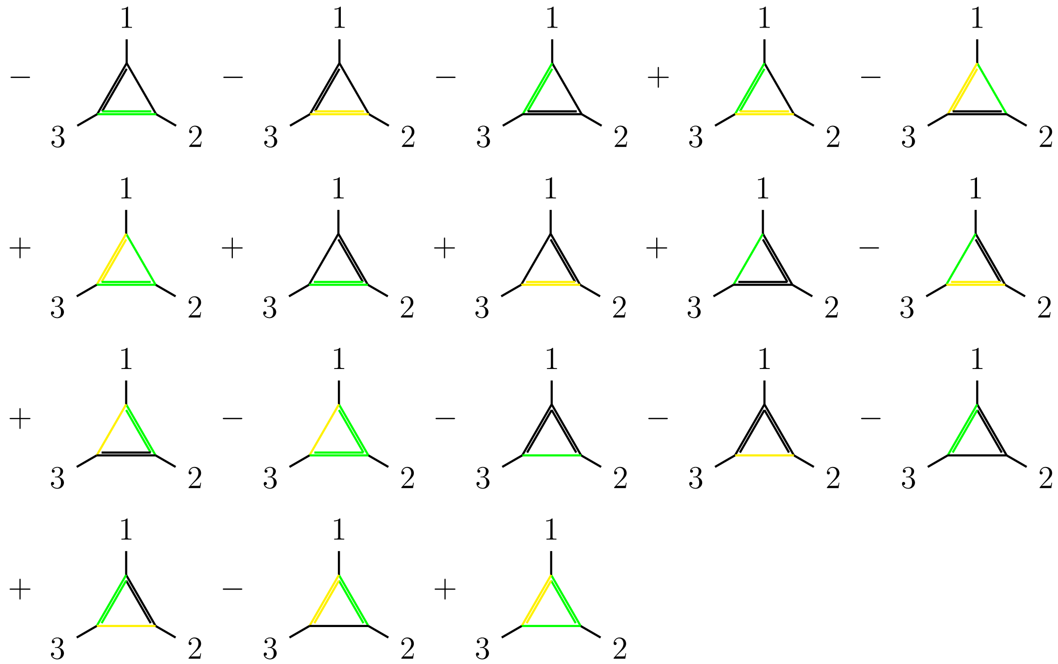

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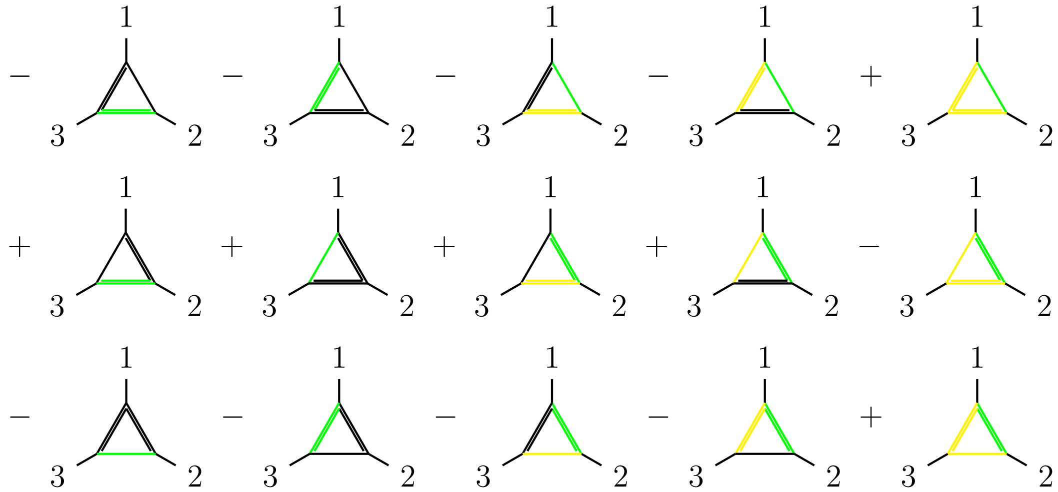

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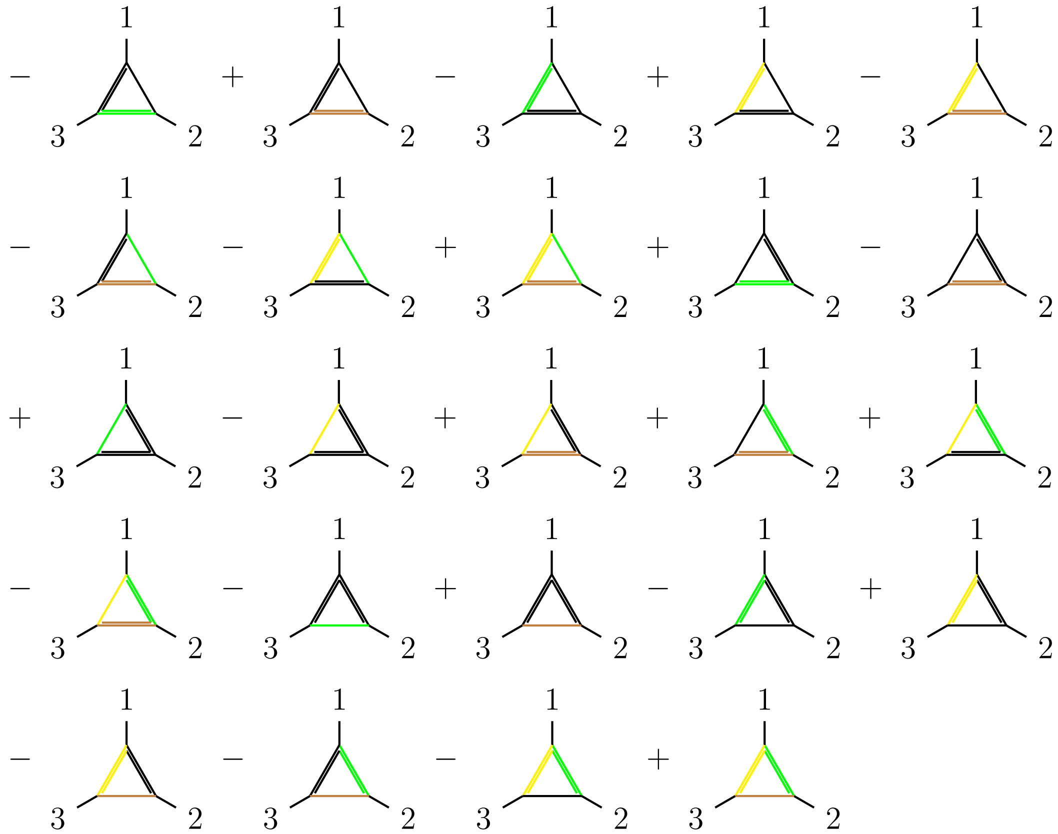

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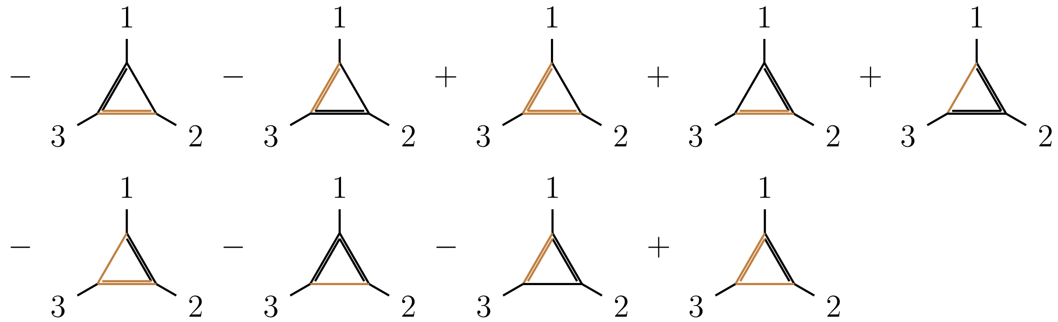

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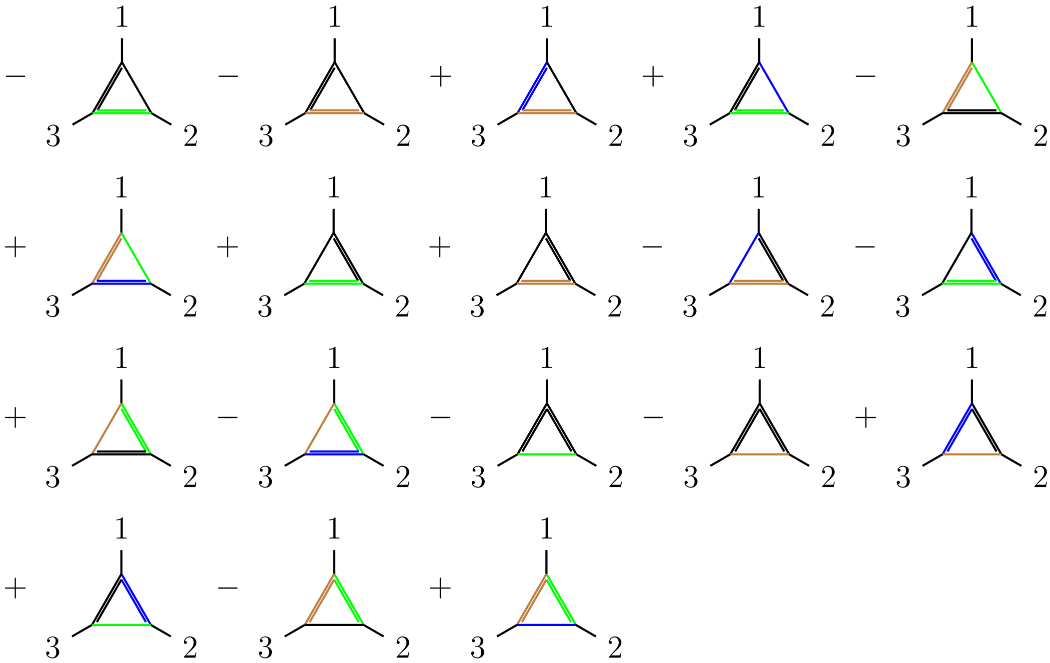

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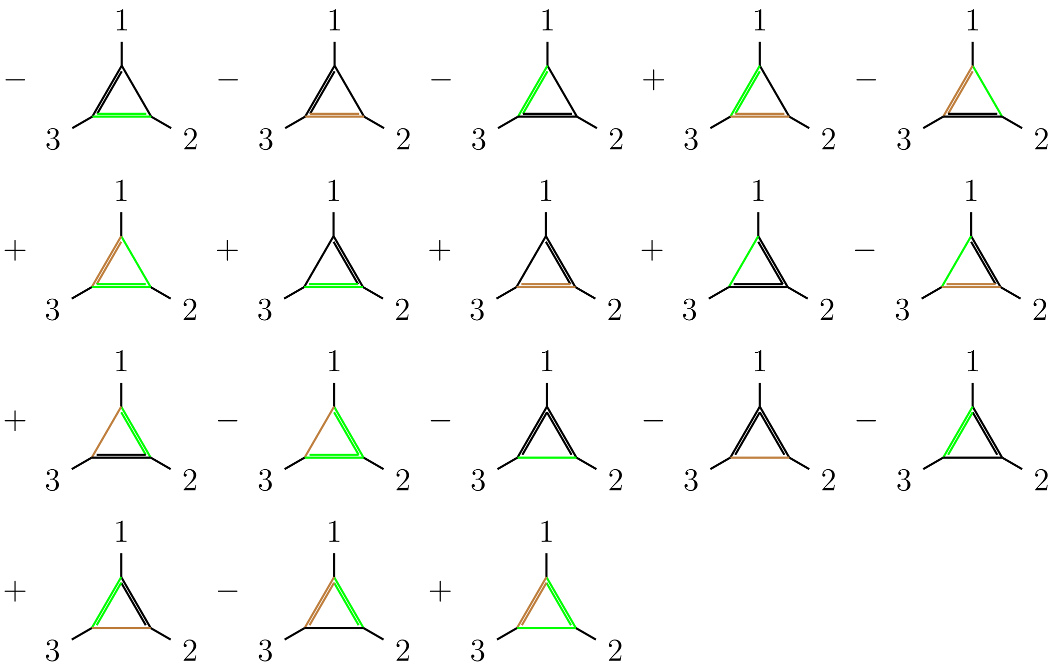

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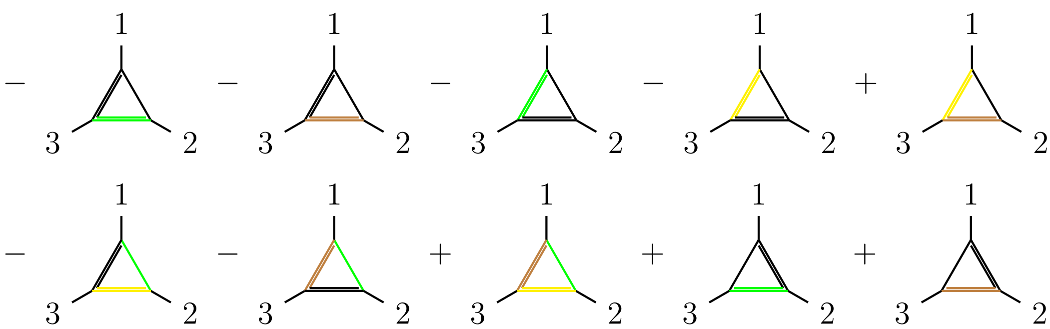

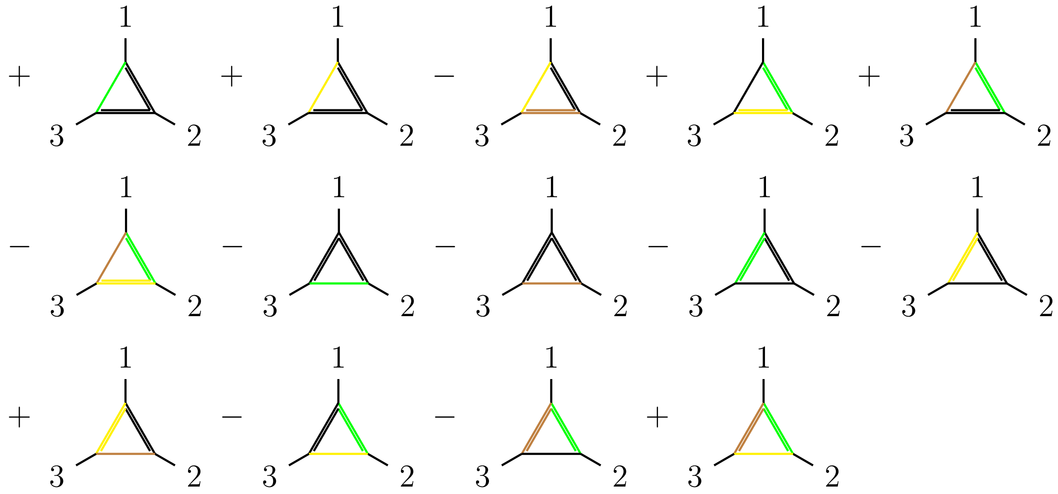

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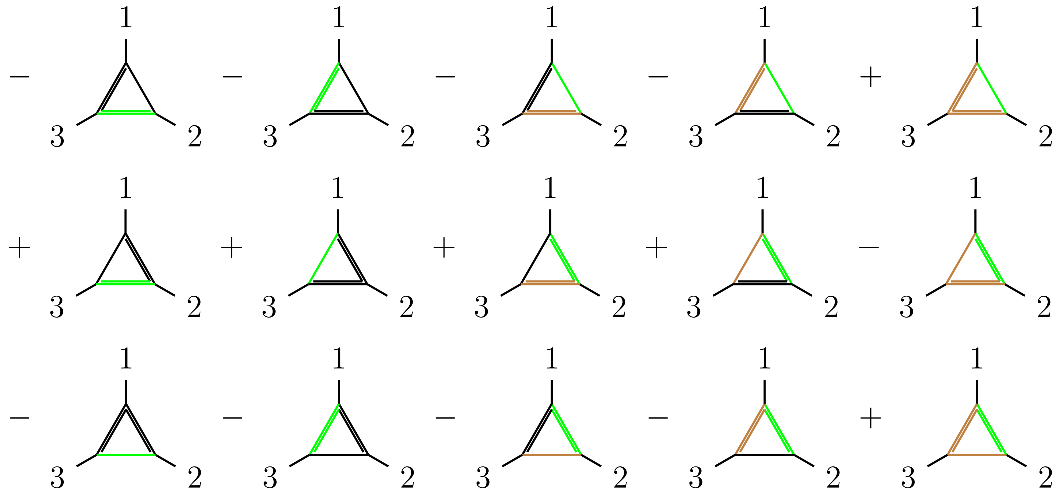

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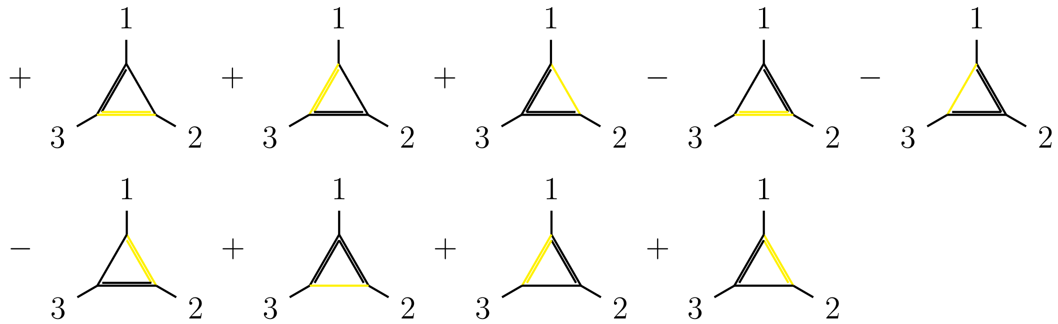

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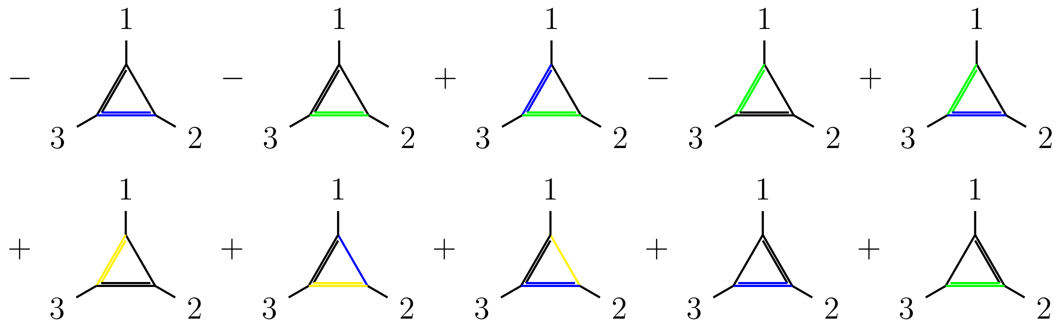

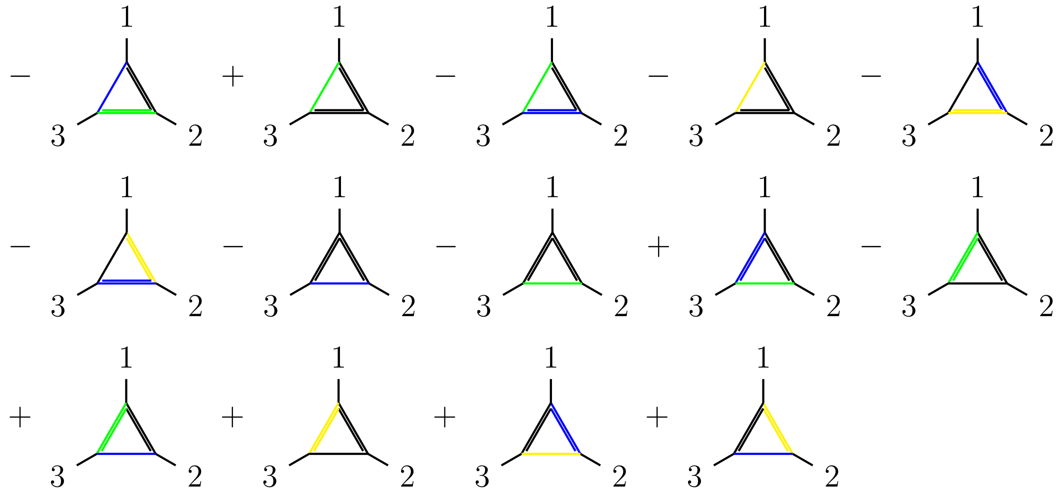

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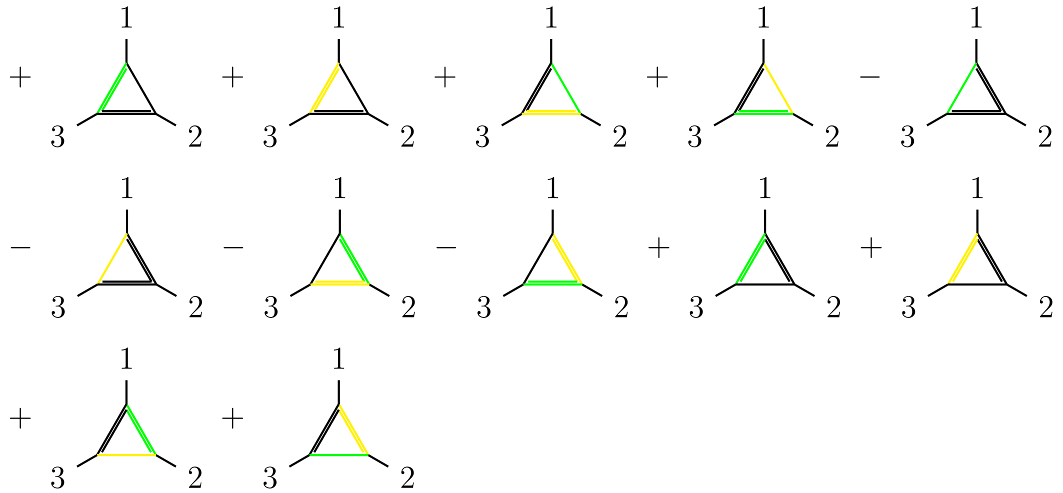

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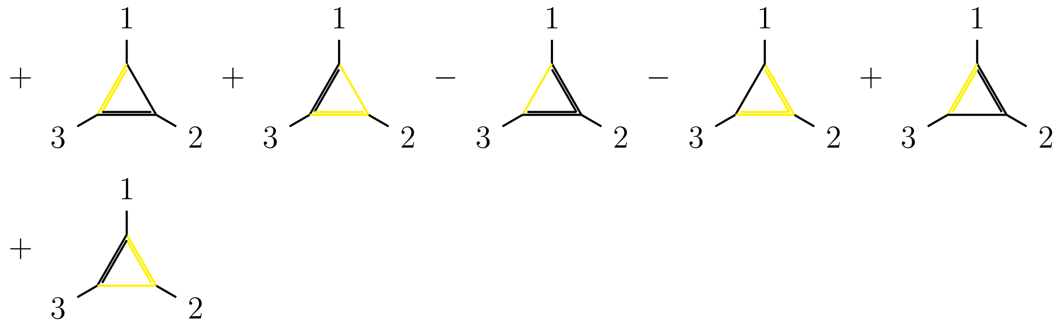

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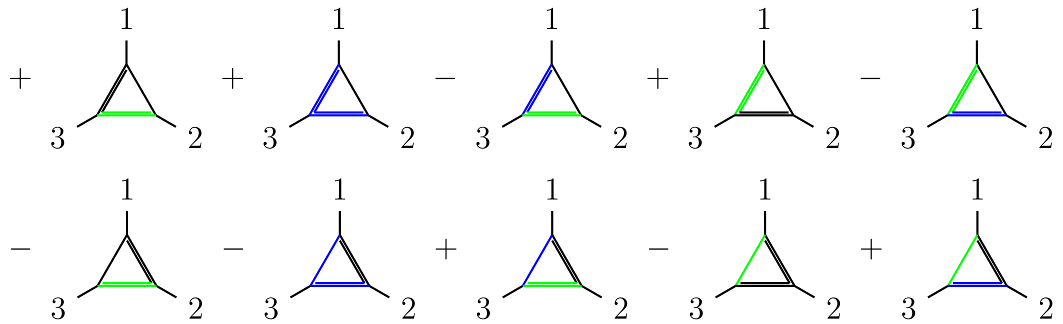

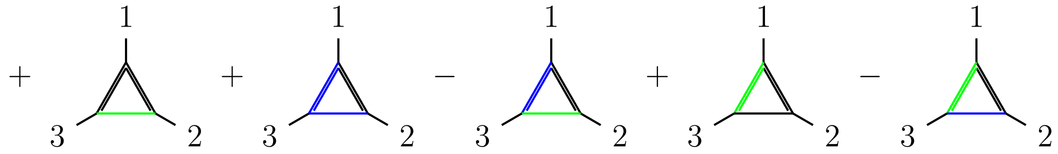

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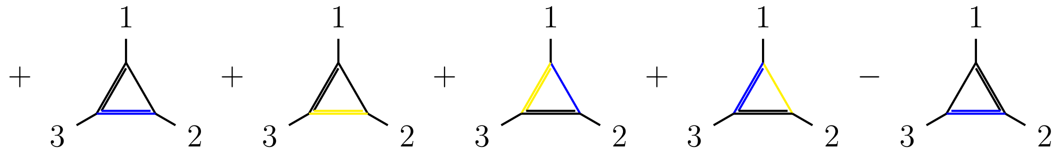

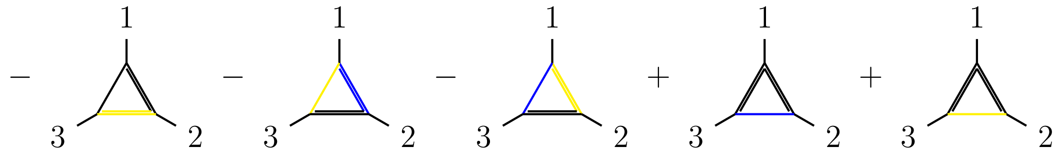

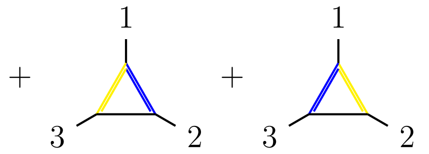

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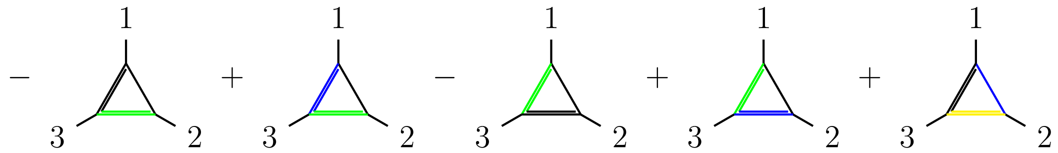

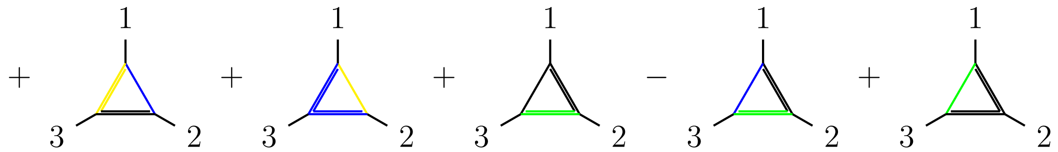

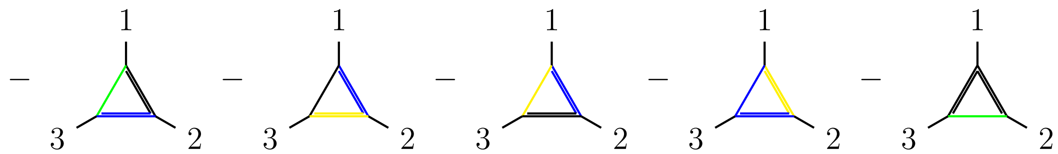

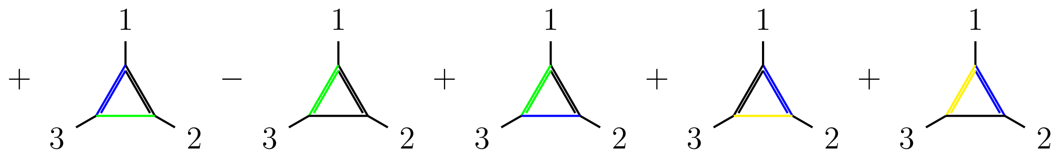

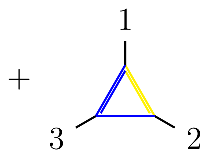

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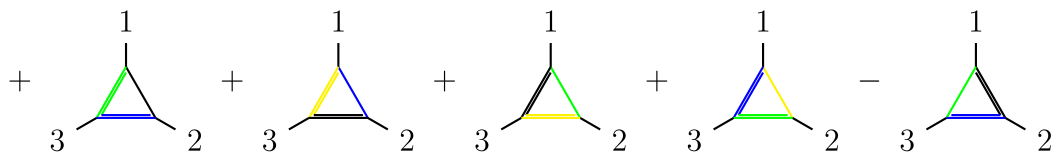

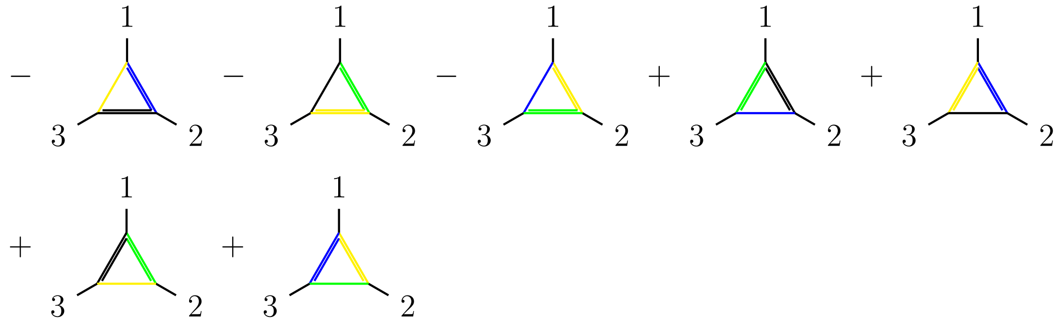

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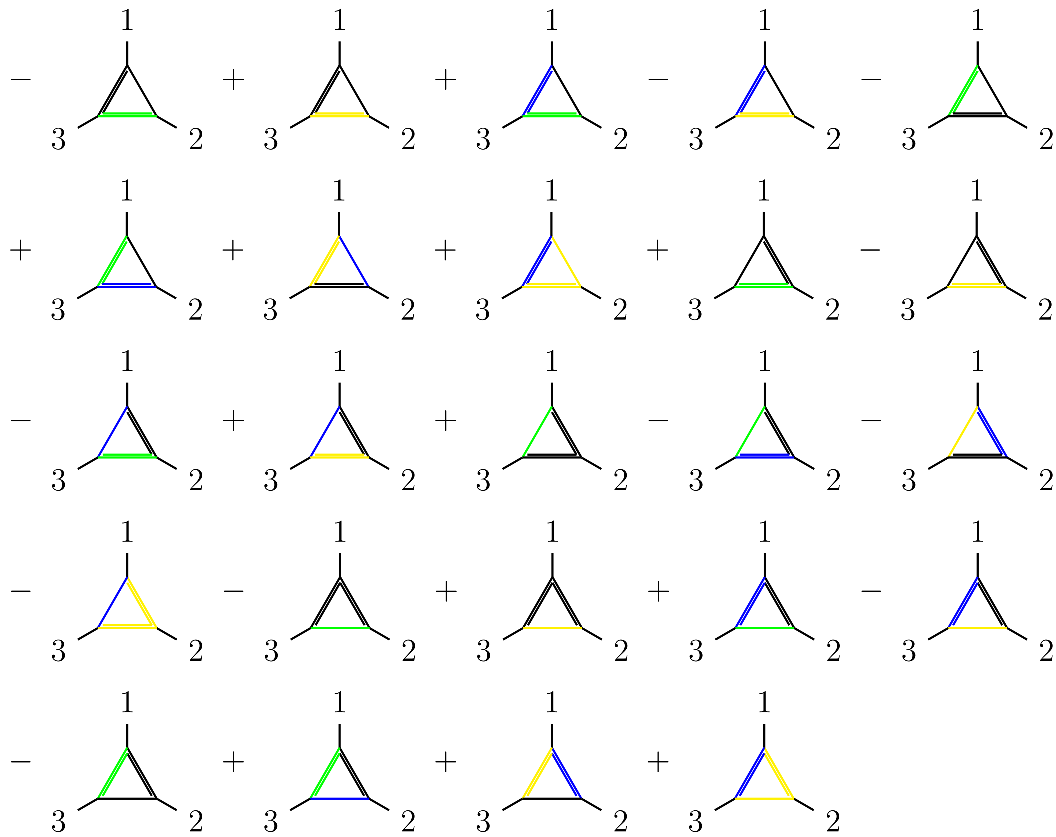

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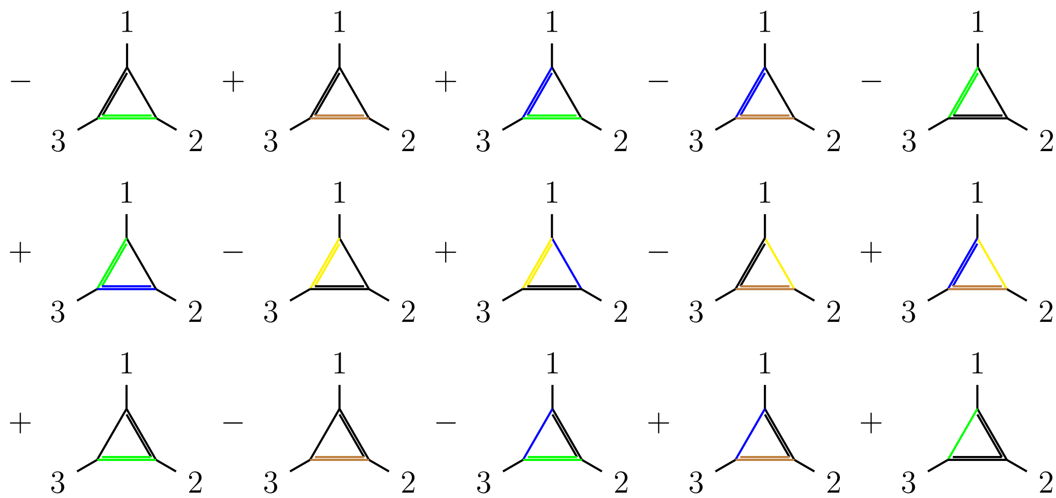

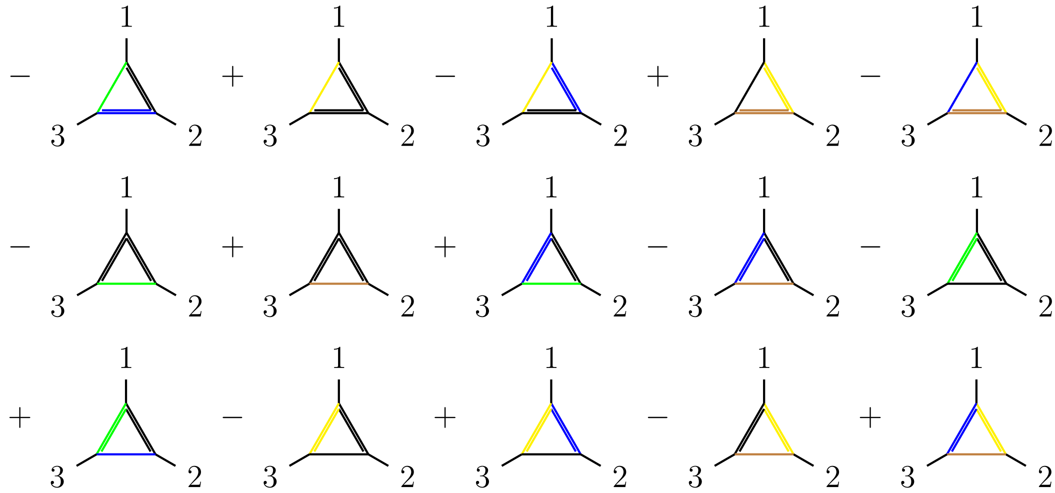

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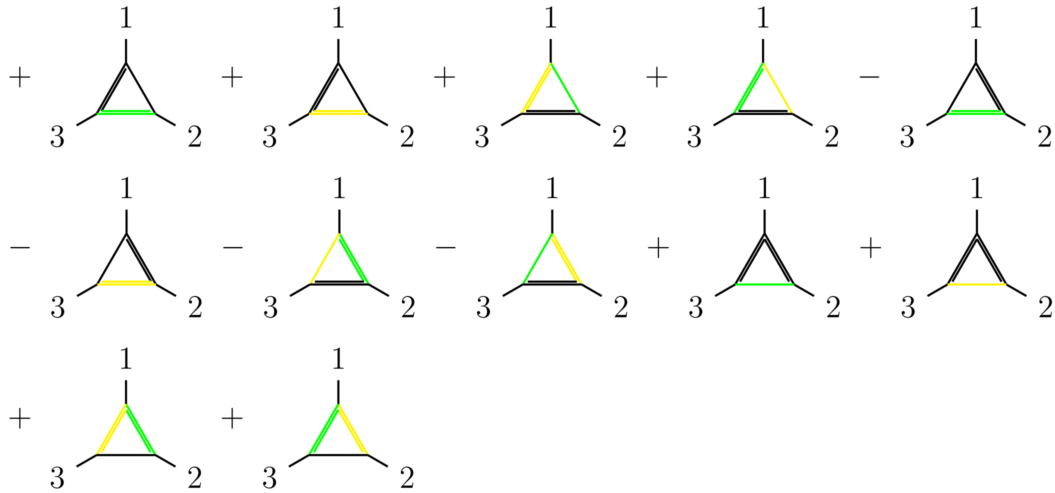

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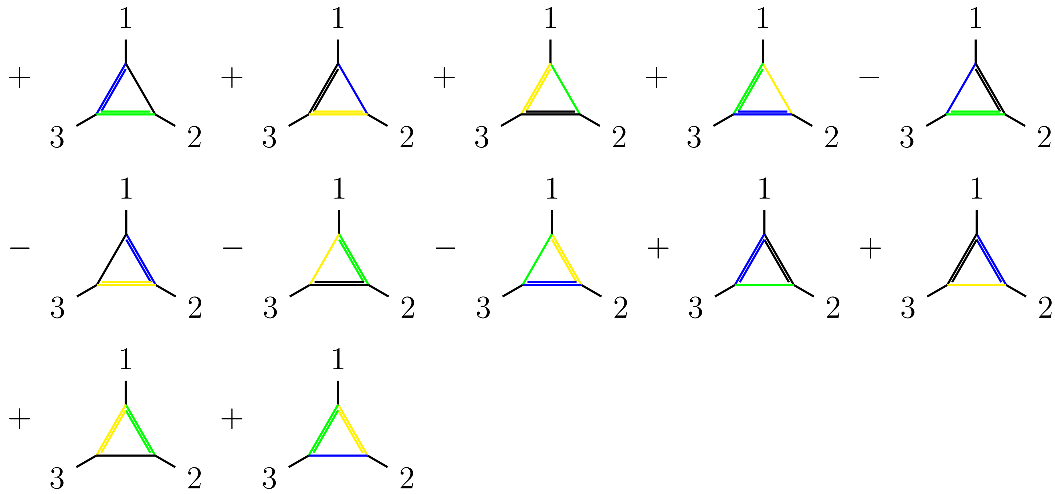

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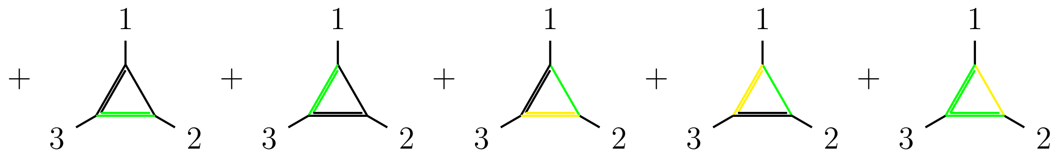

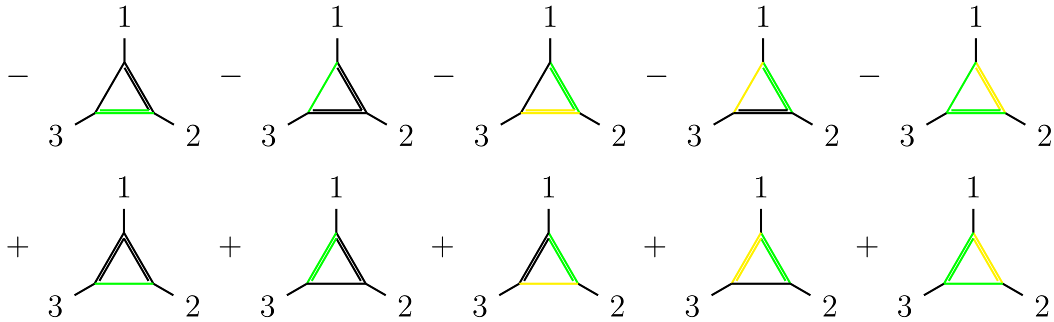

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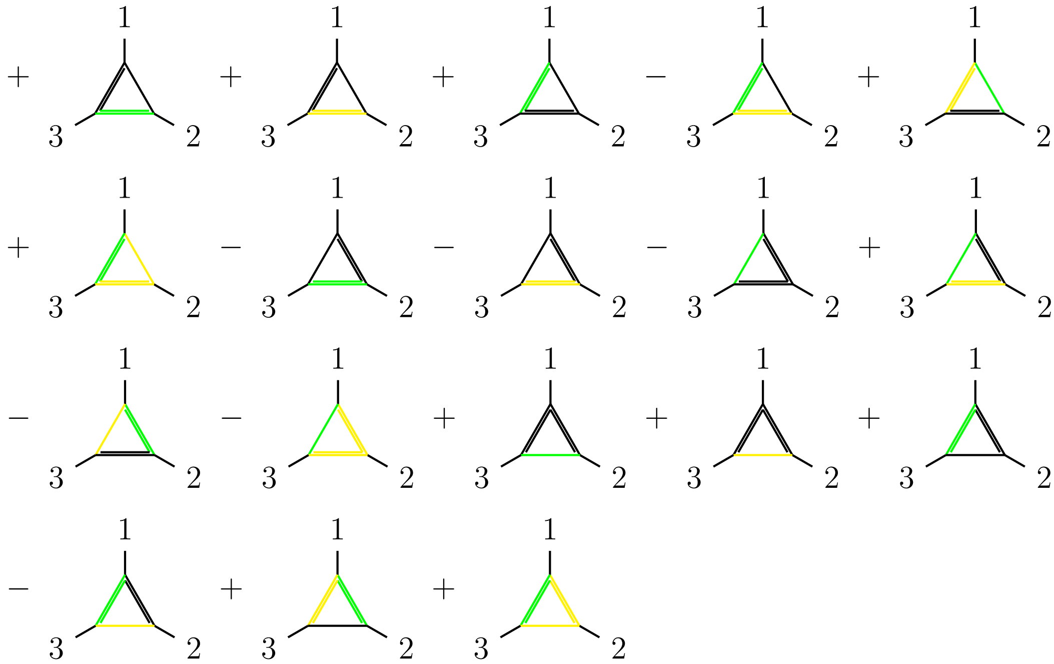

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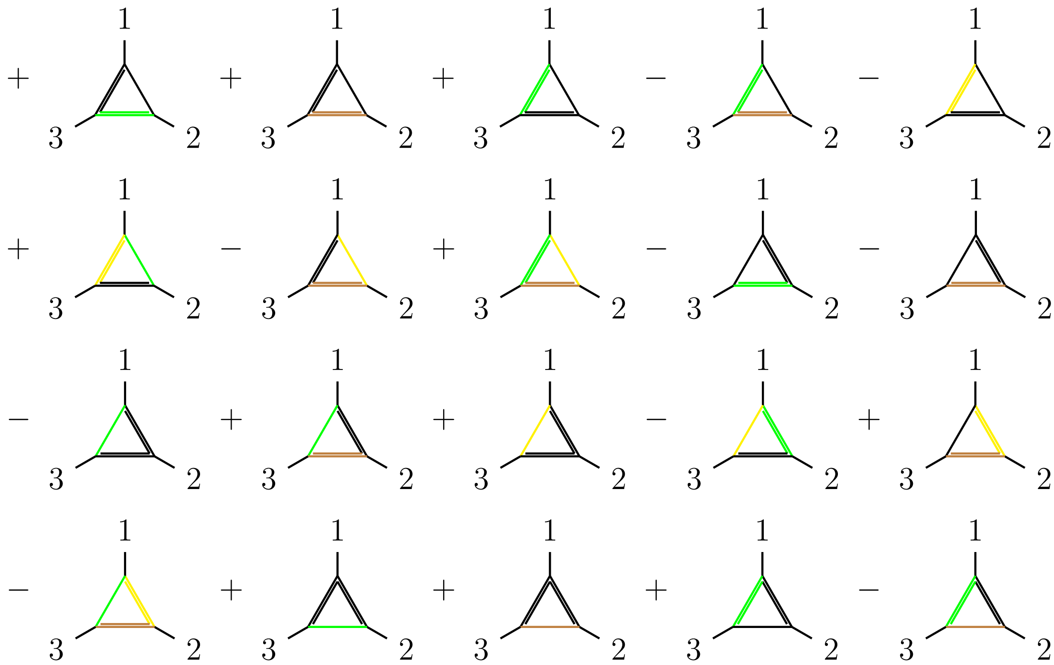

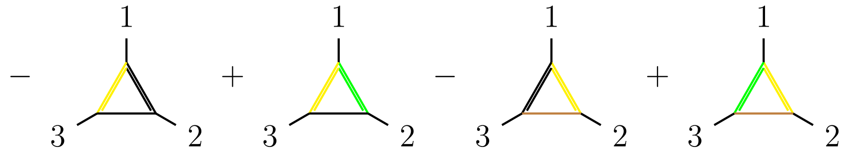

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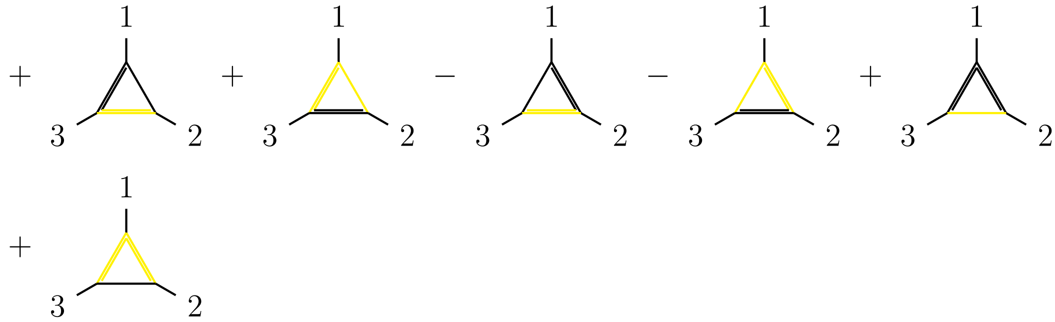

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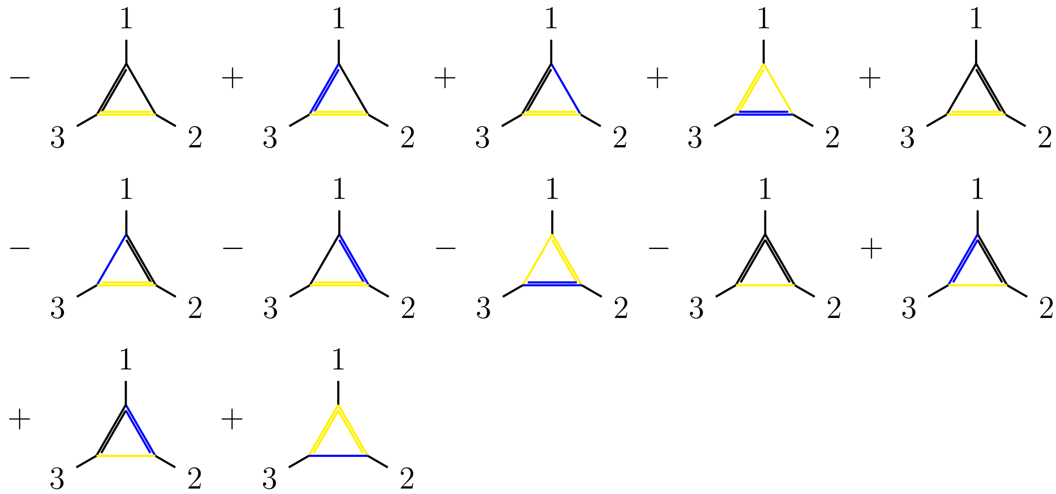

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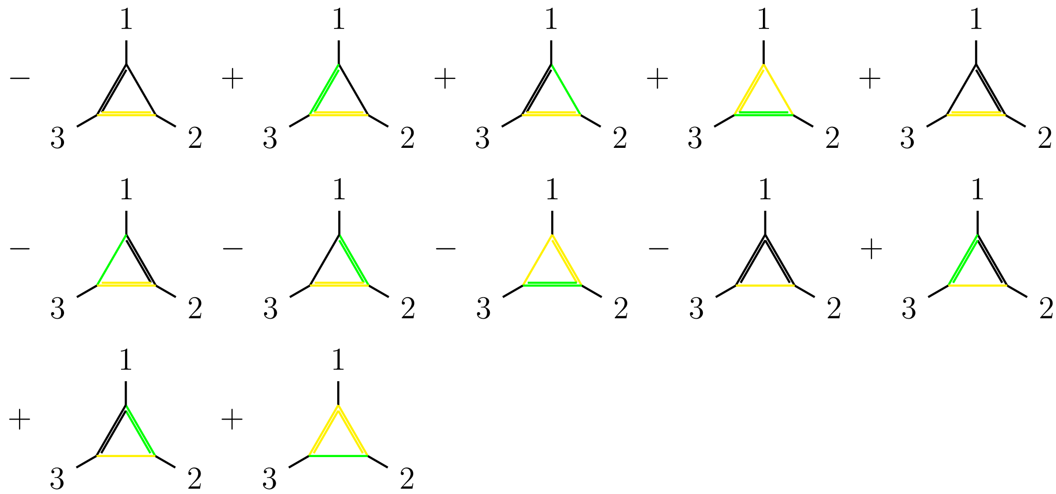

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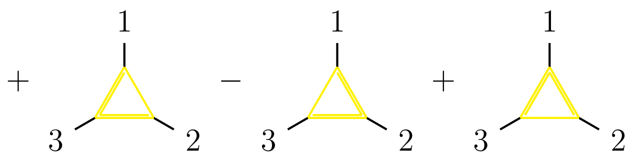

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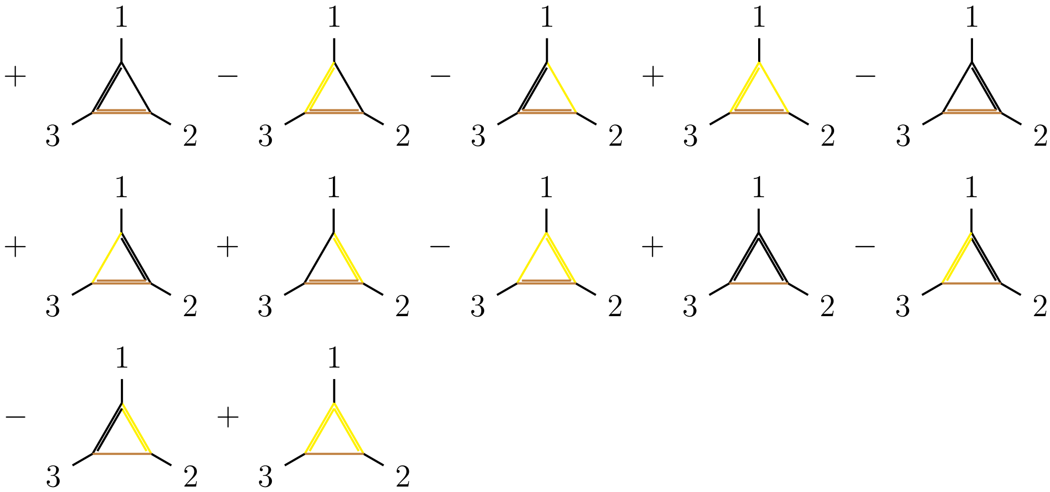

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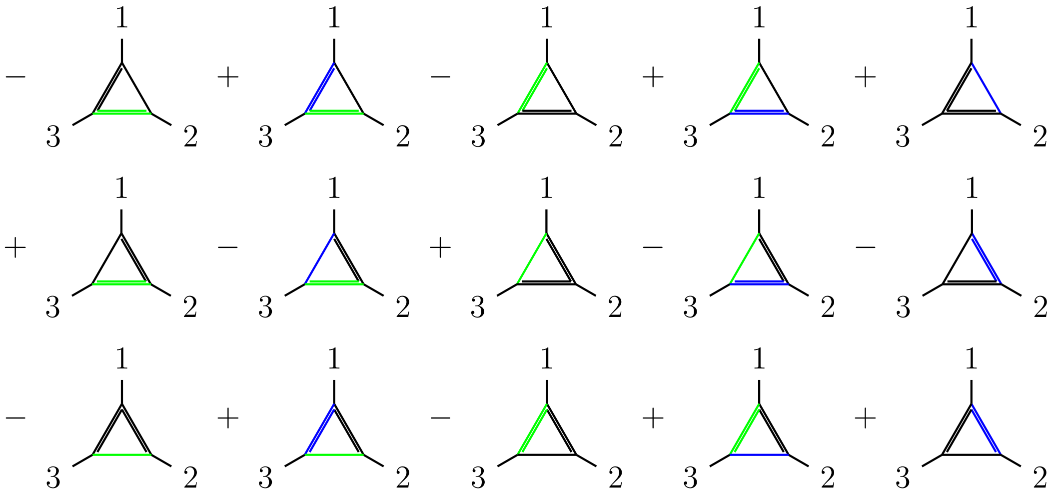

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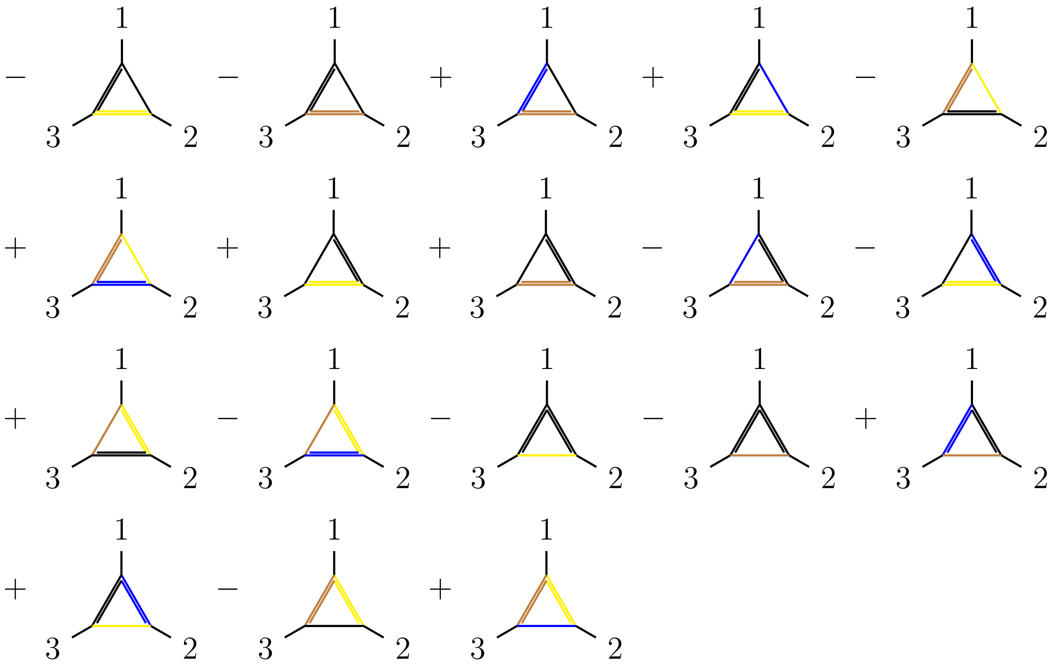
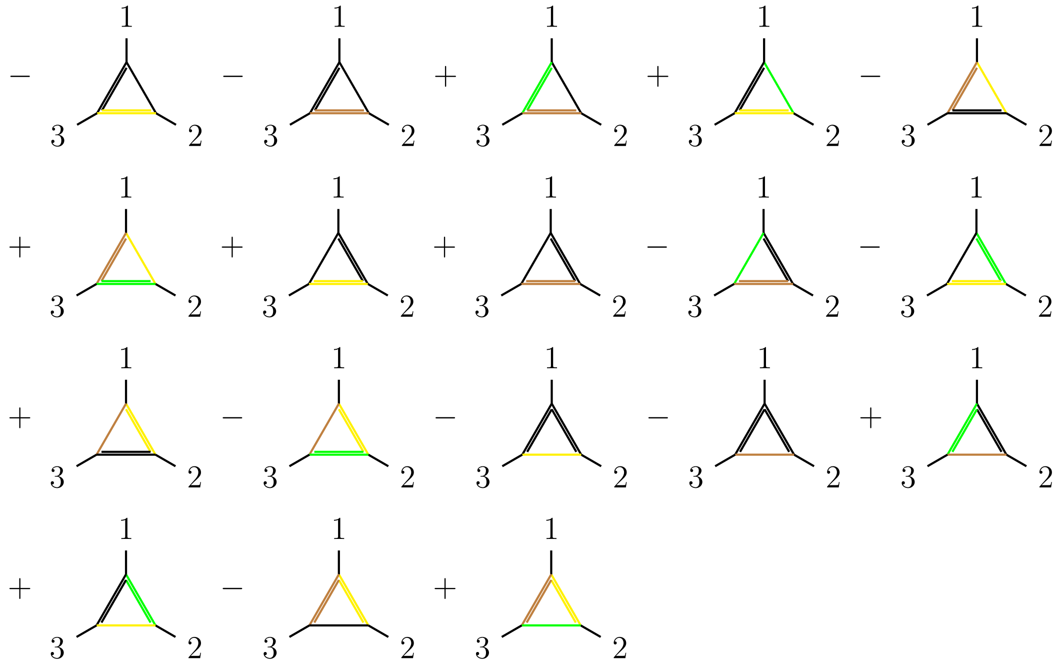
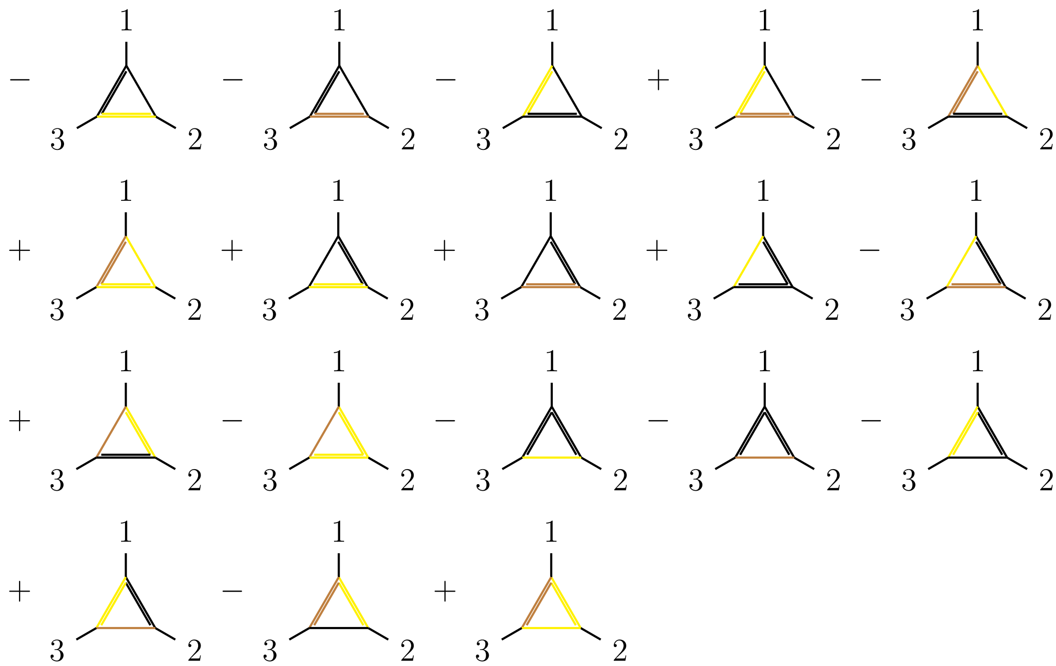
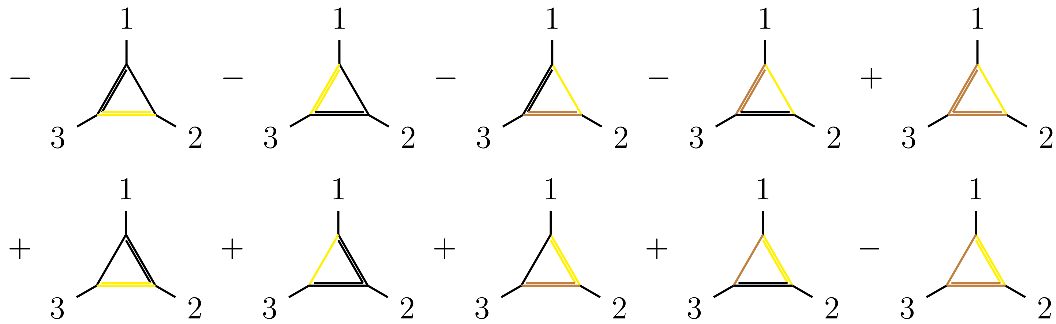
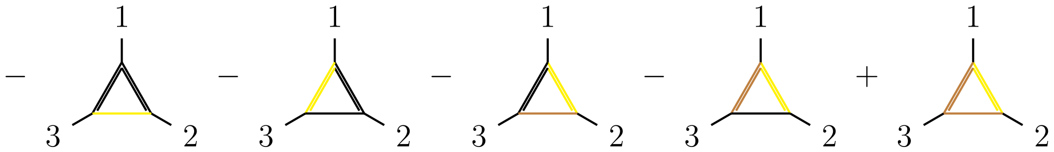
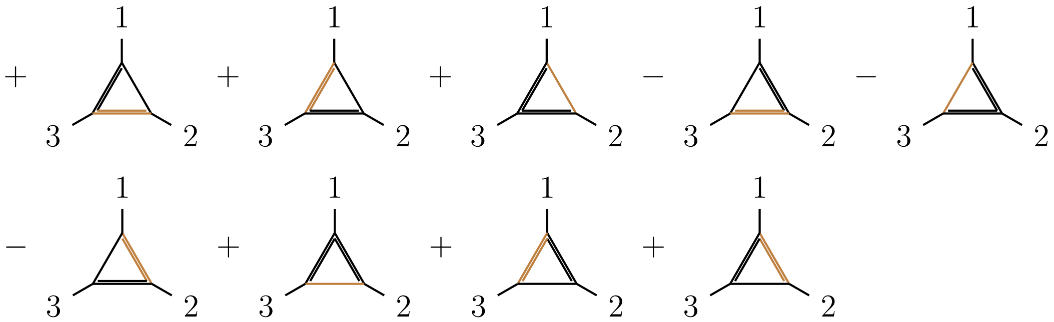


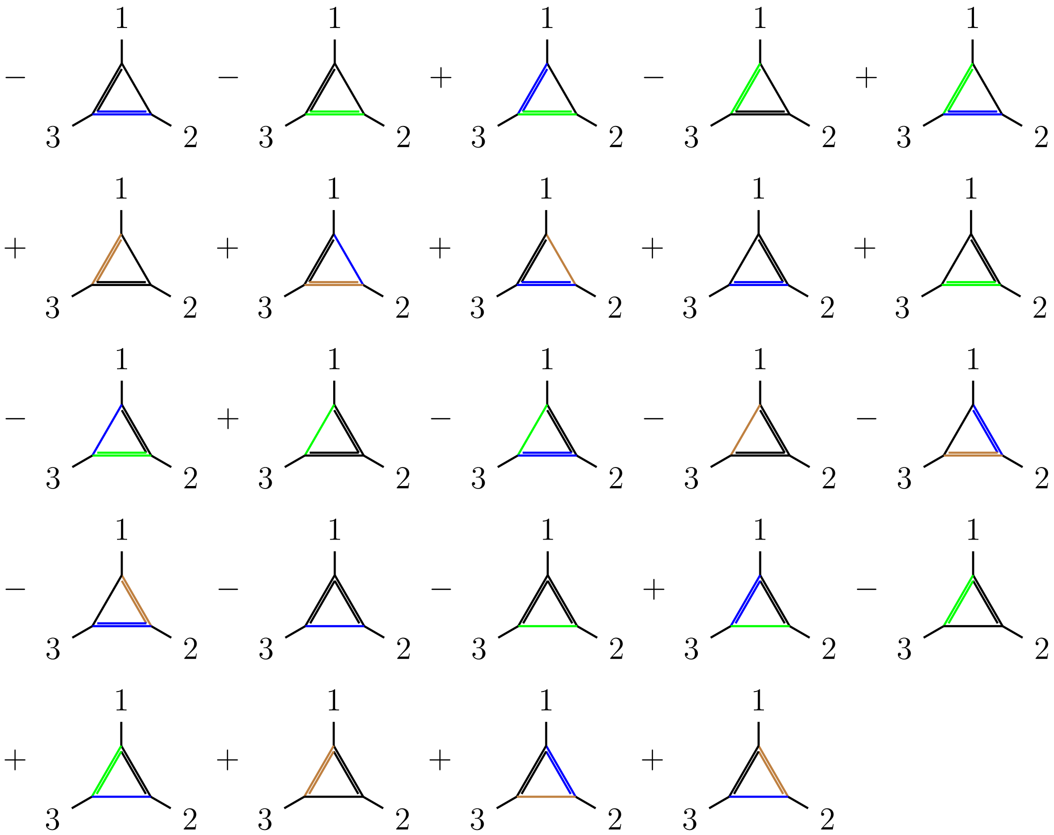
$g_{72} =$ 

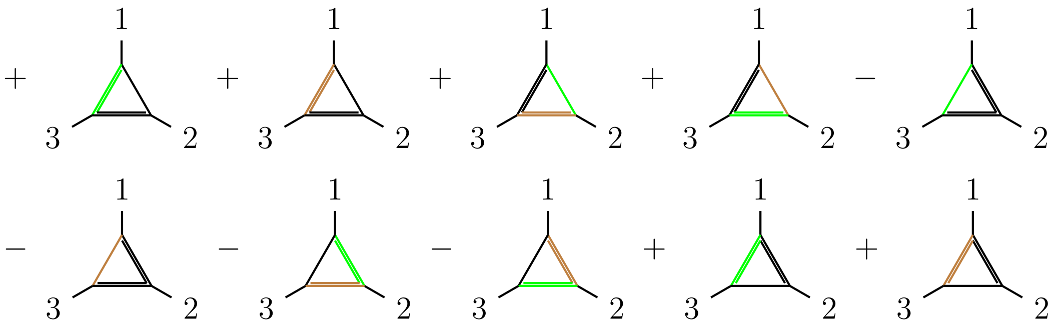
$g_{73} =$ 

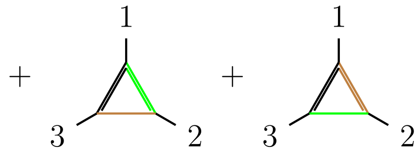
$g_{74} =$ 



$g_{75} =$ 

$g_{76} =$ 

$g_{77} =$ 



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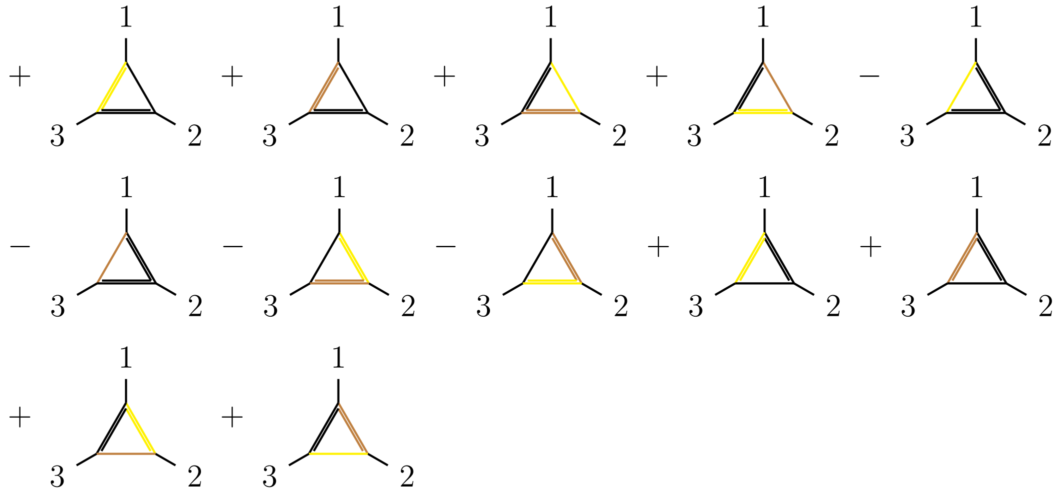

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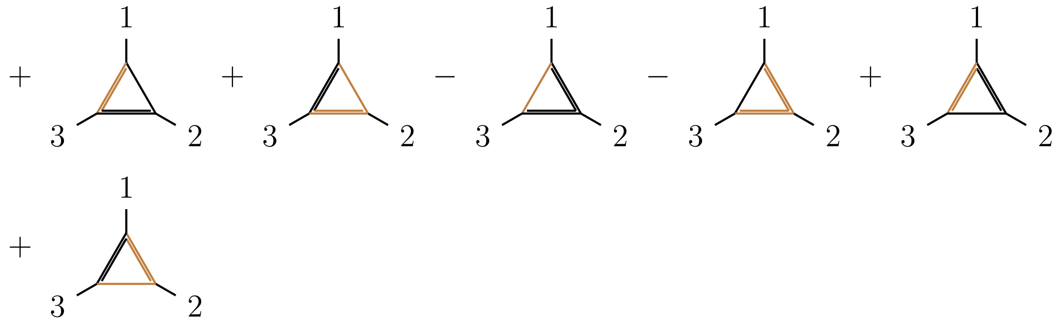

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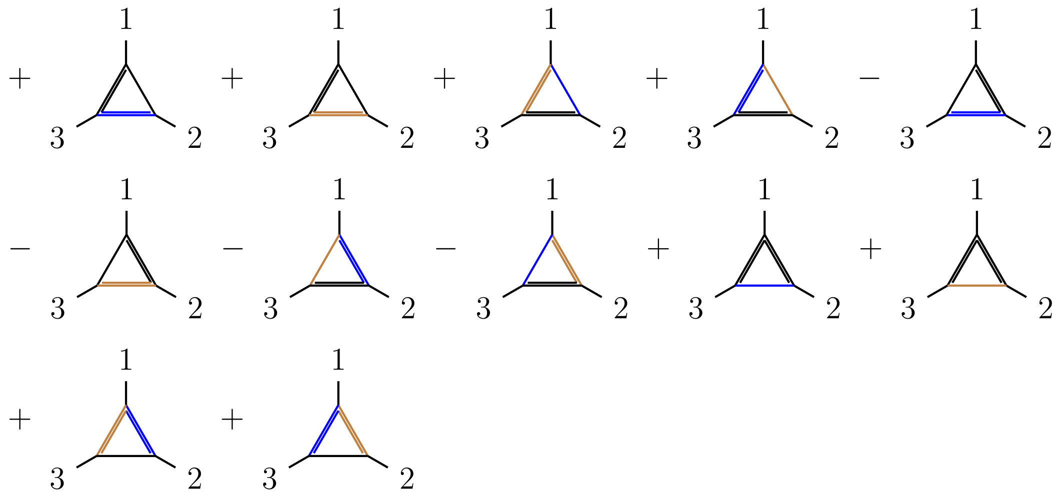

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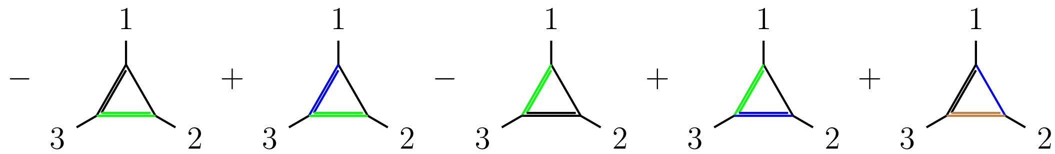

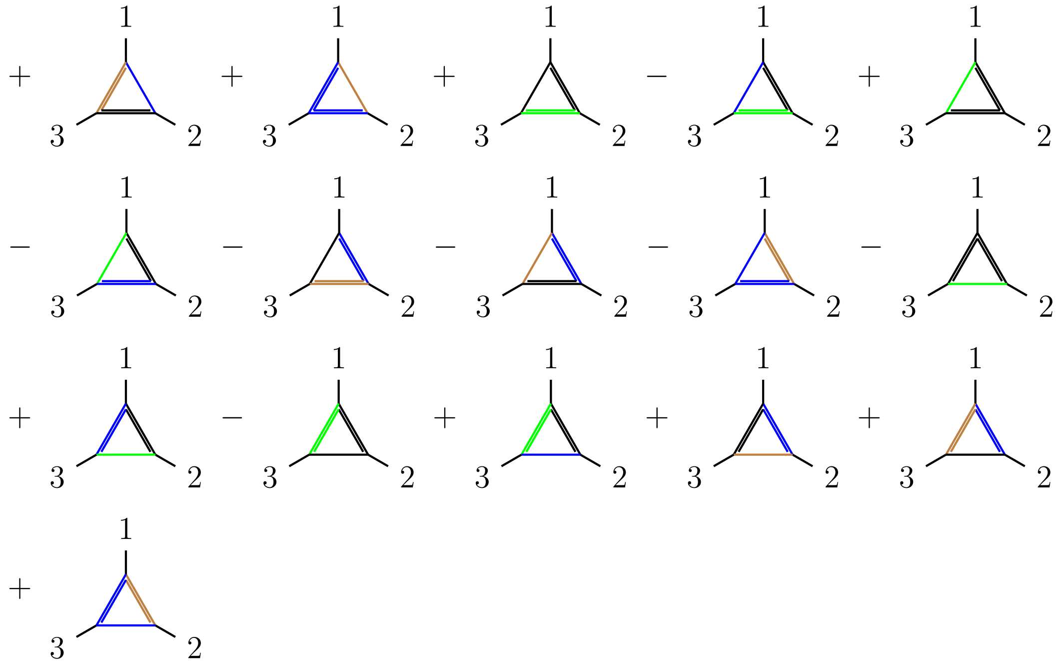

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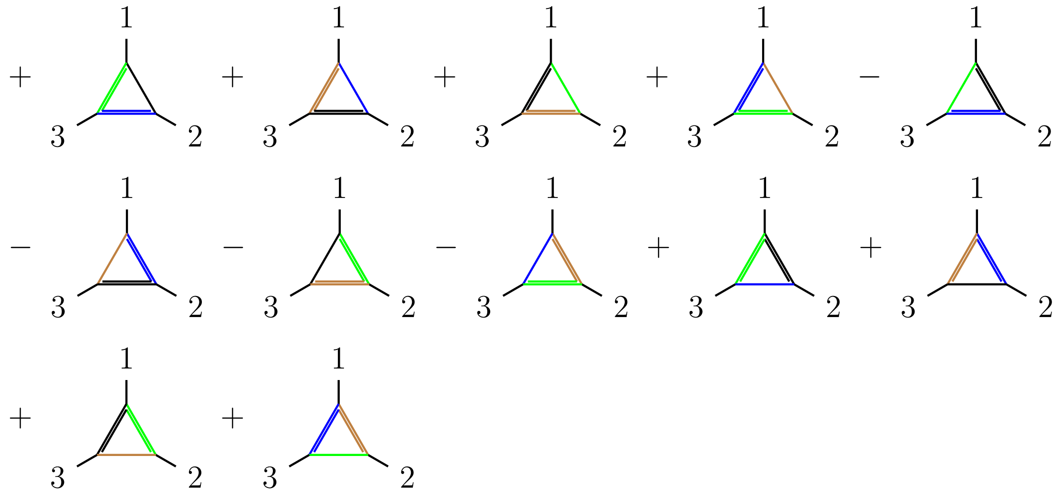

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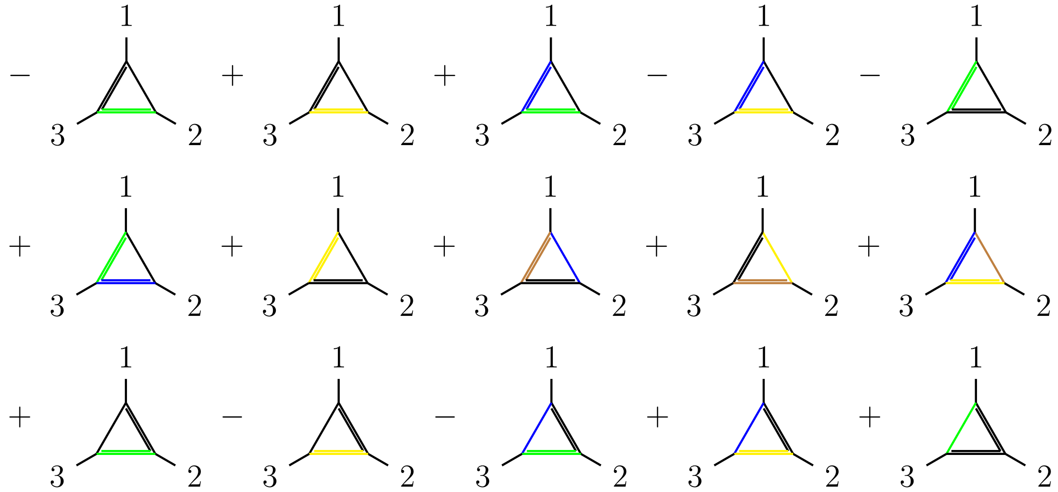

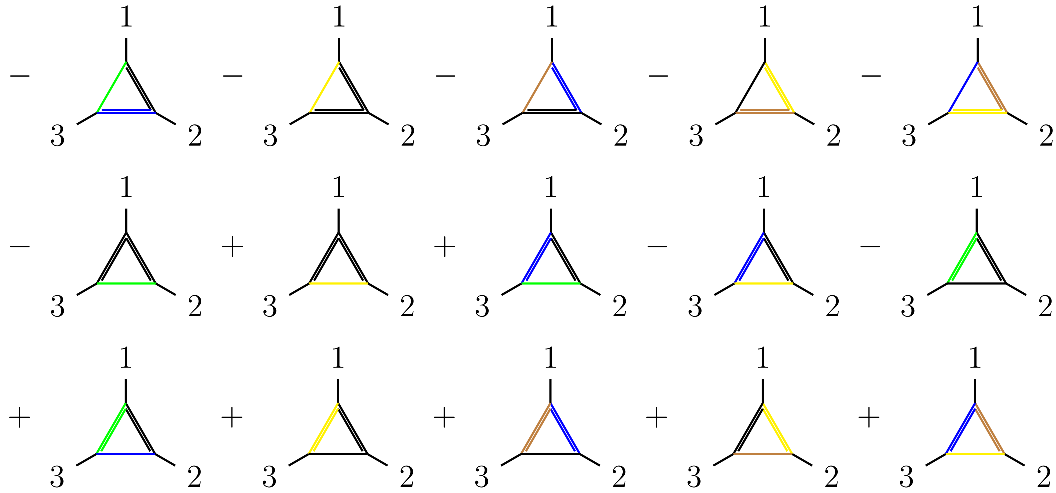

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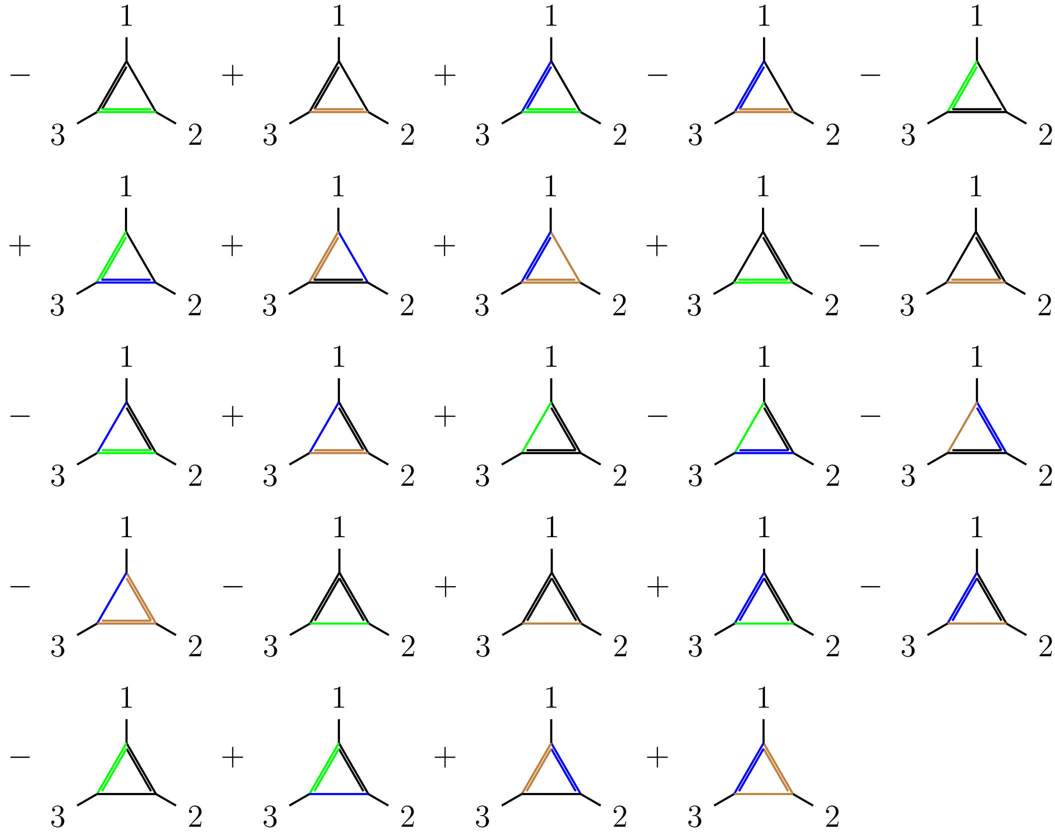

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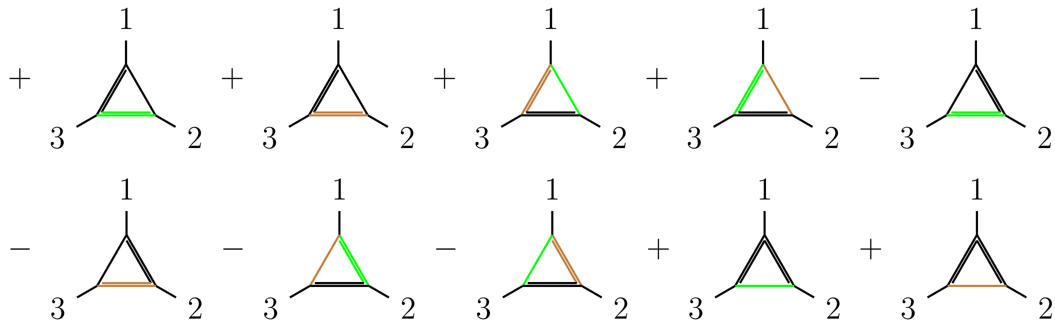
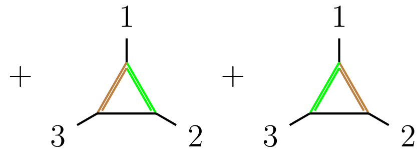
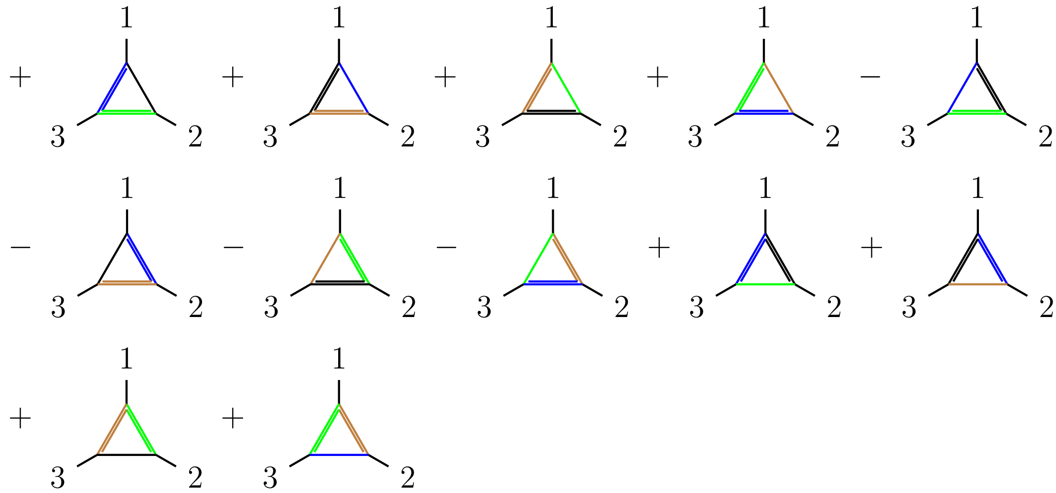
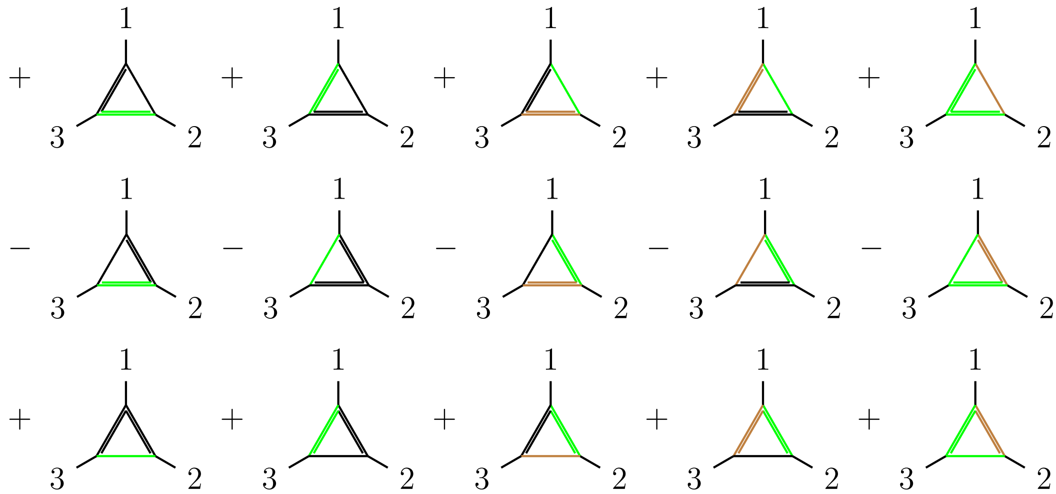
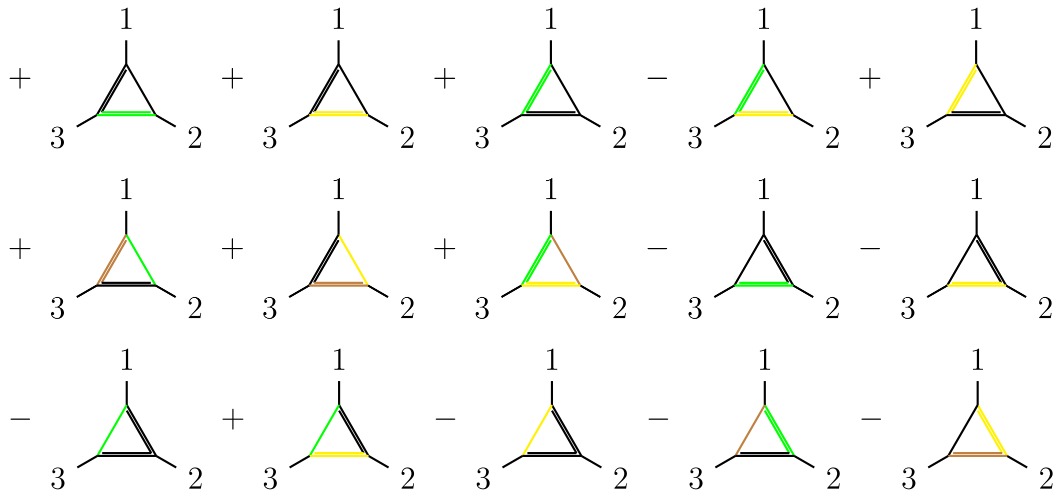
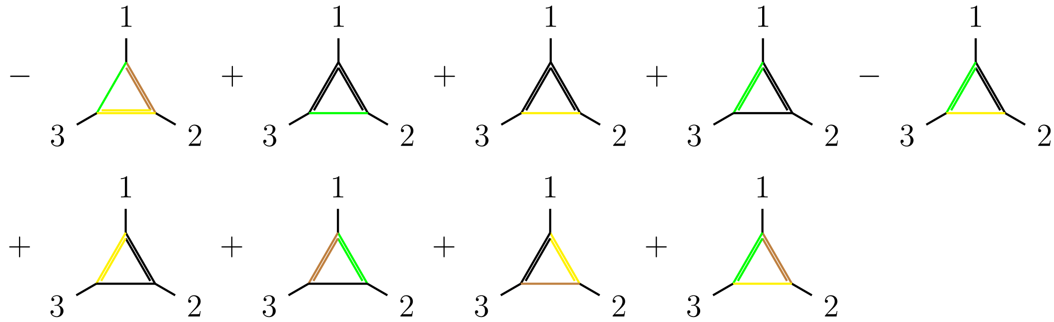




$g_{86} =$ 

$g_{87} =$ 

$g_{88} =$ 



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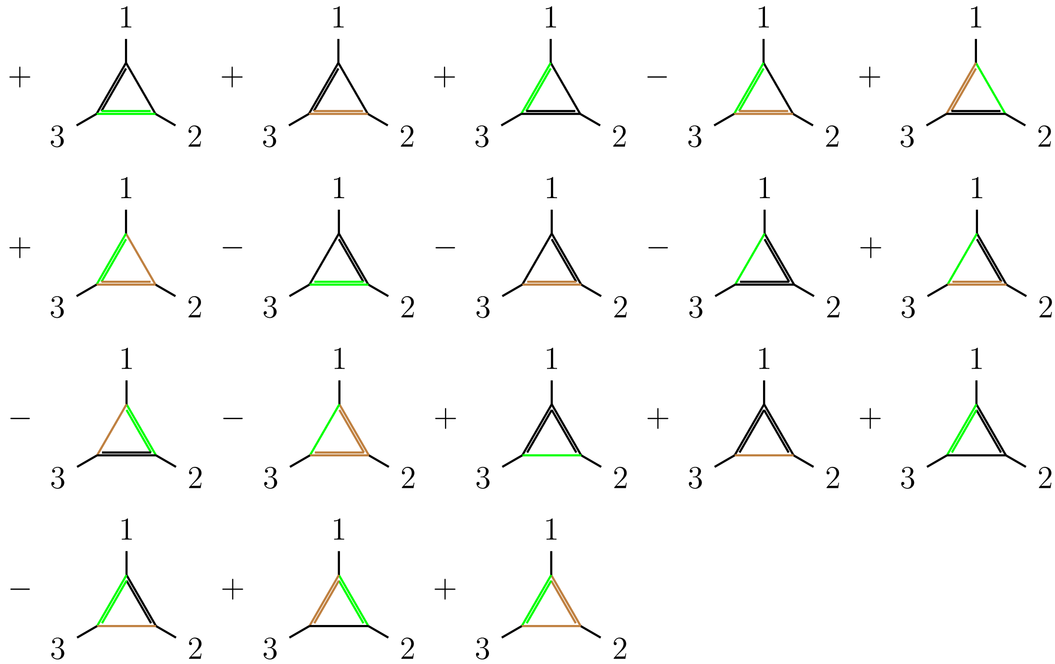

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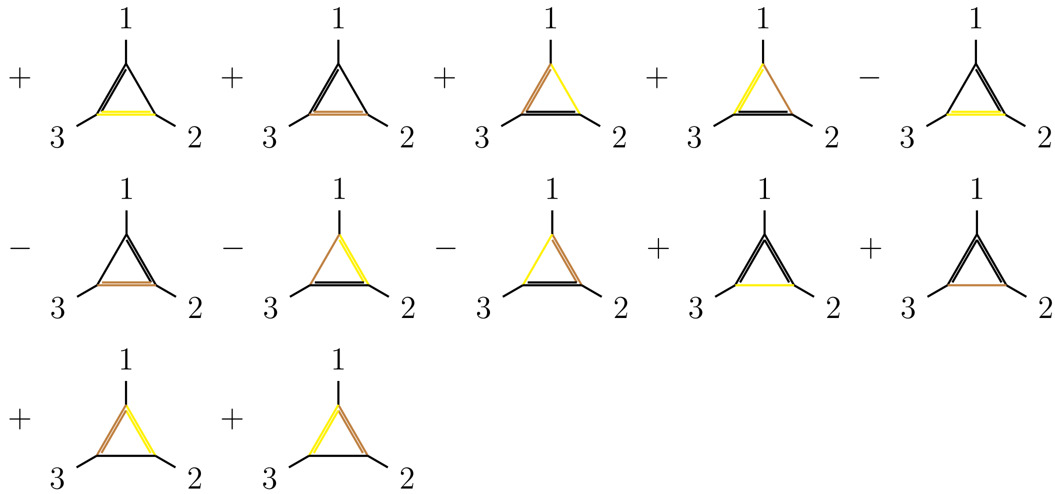

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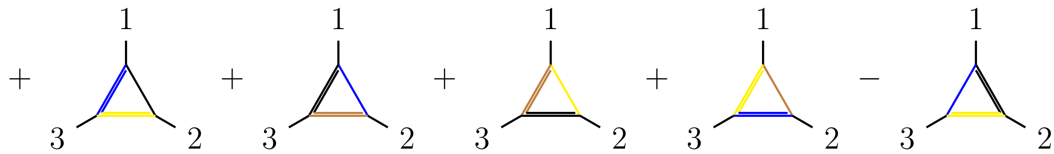

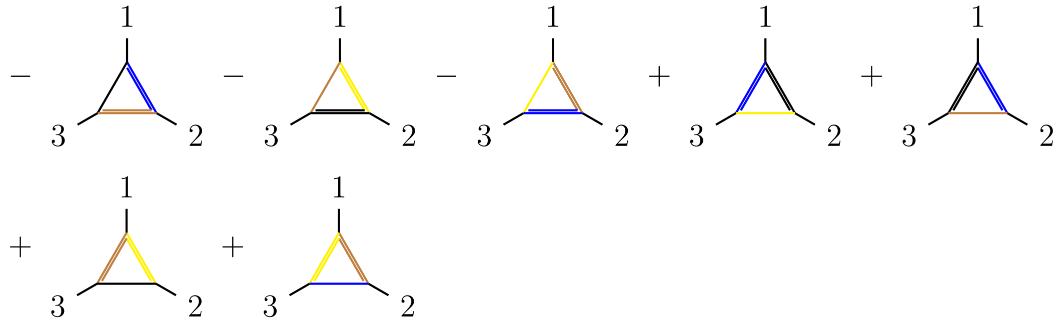

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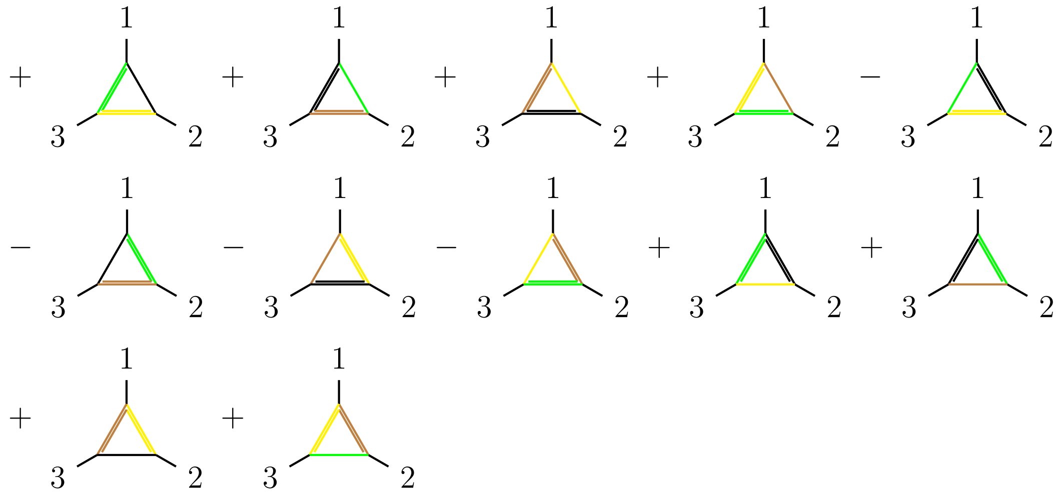

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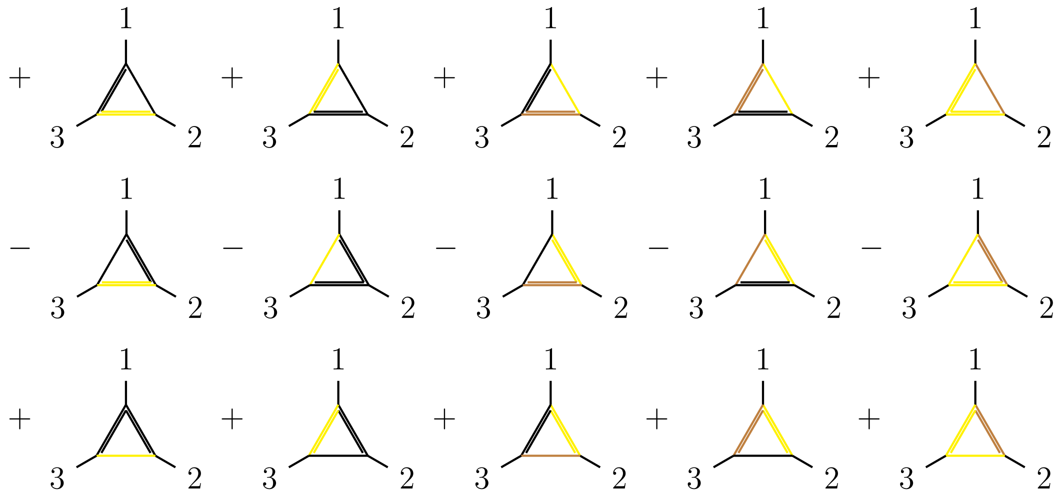

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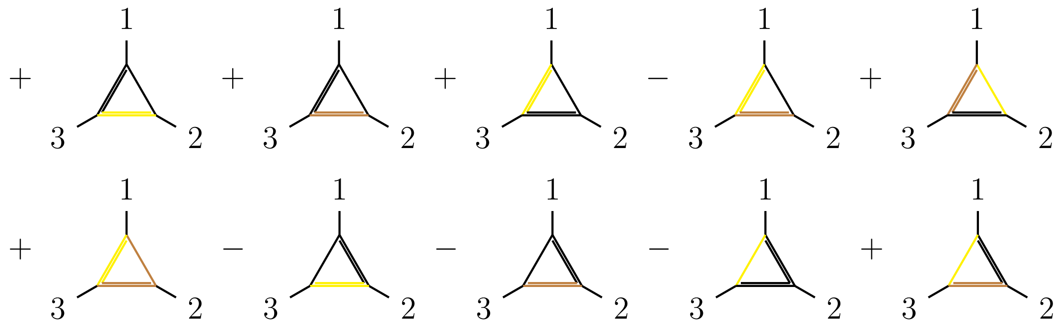
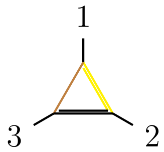
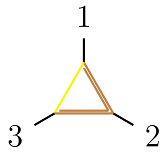
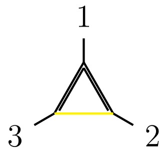
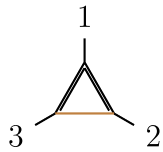
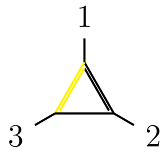
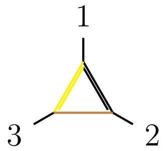
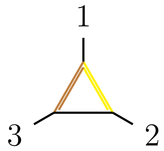
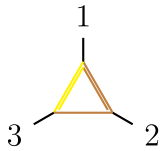
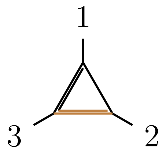
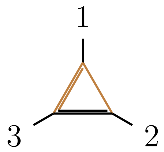
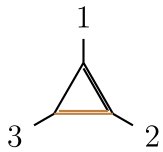
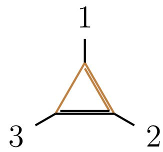
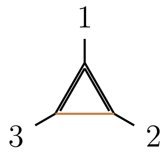
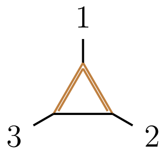
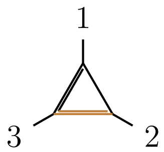
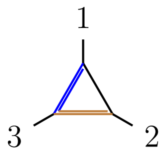
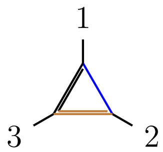
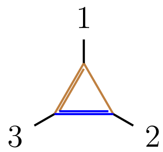
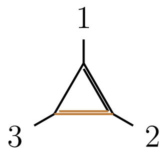


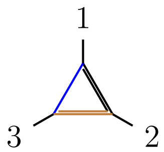
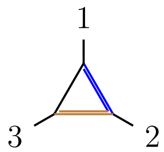
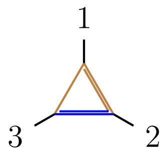
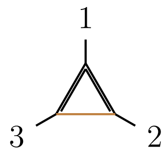
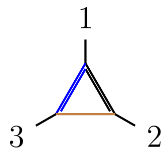
$-$  $-$  $+$  $+$  $+$ 

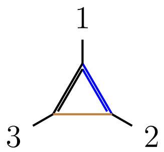
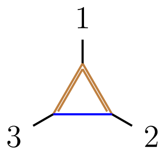
$-$  $+$  $+$ 

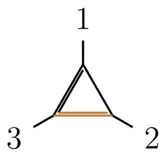
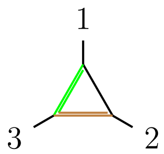
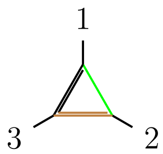
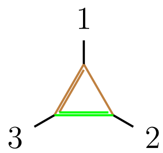
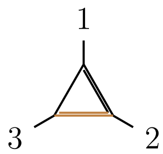
$g_{95} = +$  $+$  $-$  $-$  $+$ 

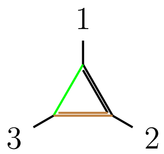
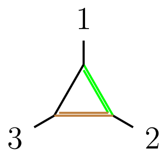
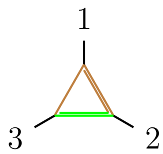
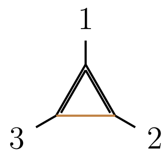
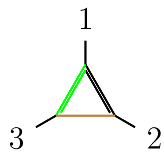
$+$ 

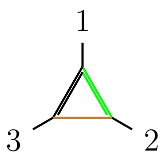
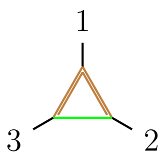
$g_{96} = -$  $+$  $+$  $+$  $+$ 

$-$  $-$  $-$  $-$  $+$ 

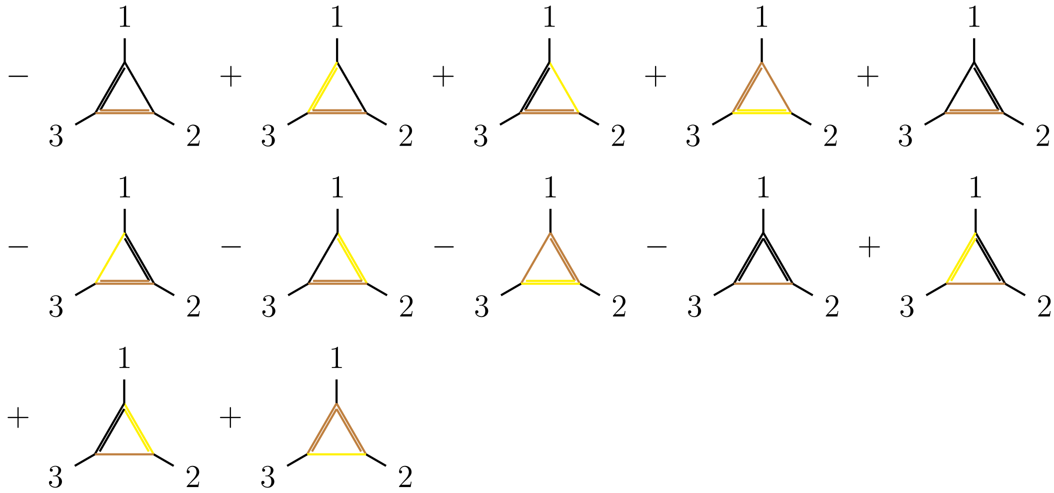
$+$  $+$ 

$g_{97} = -$  $+$  $+$  $+$  $+$ 

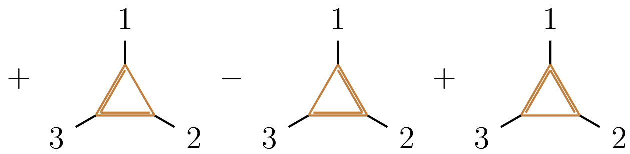
$-$  $-$  $-$  $-$  $+$ 

$+$  $+$ 

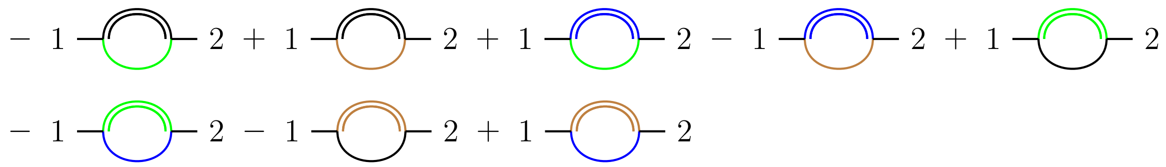
$g_{98} =$

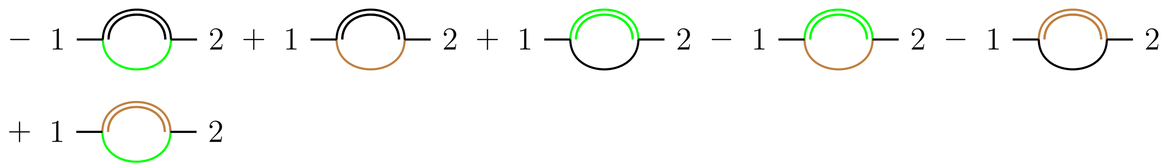


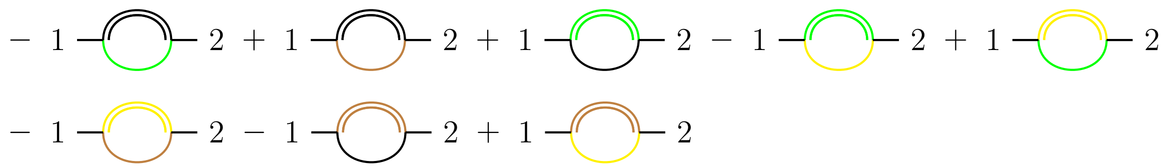
$g_{99} =$

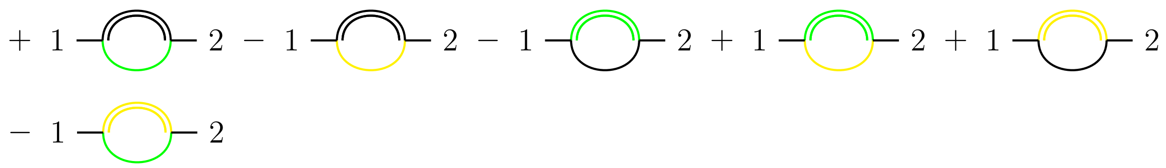


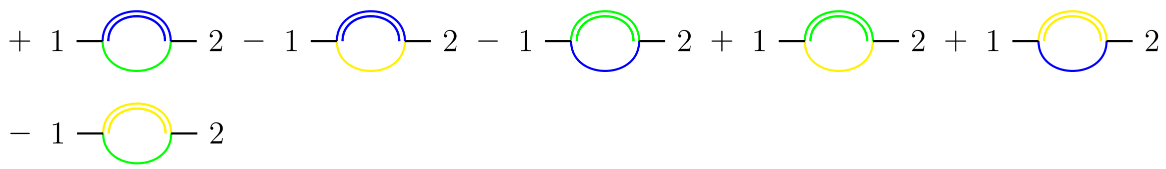
B.9 Generators of $H_1(X_{1,2,6};\mathbb{Q})$

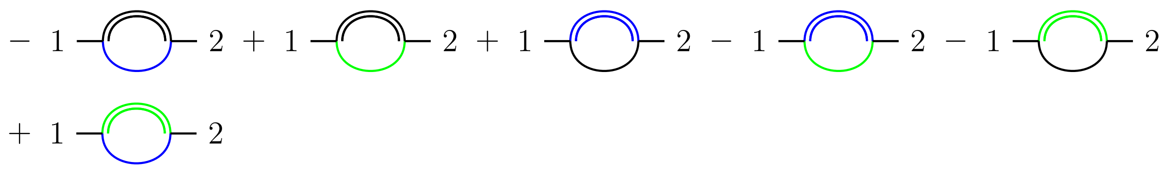
$g_1 =$


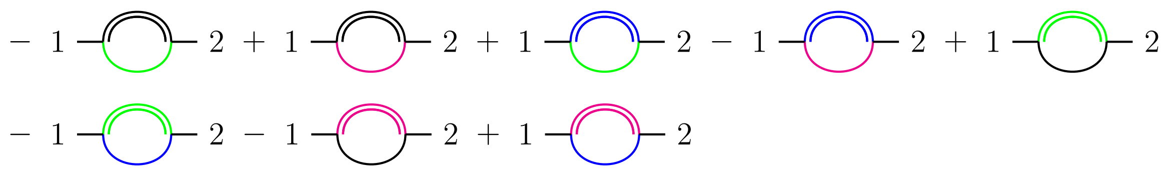
$g_2 =$


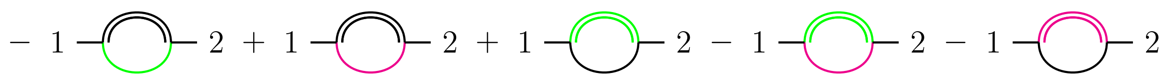
$g_3 =$


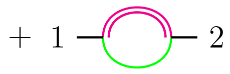
$g_4 =$


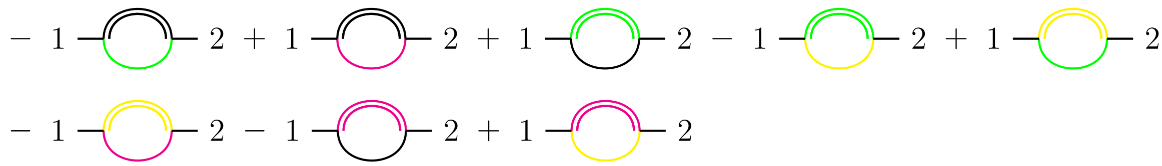
$g_5 =$


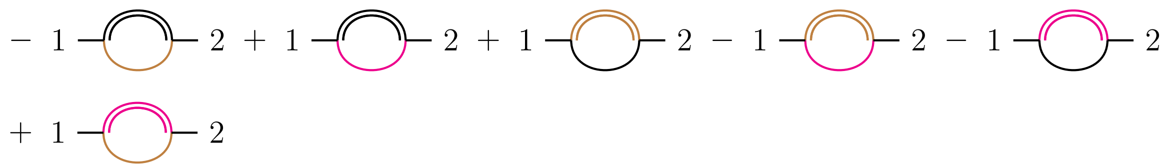
$g_6 =$


$g_7 =$


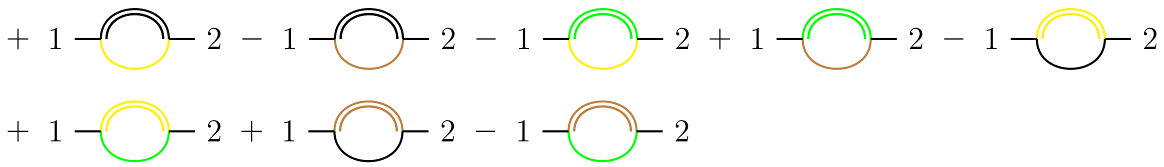
$g_8 =$


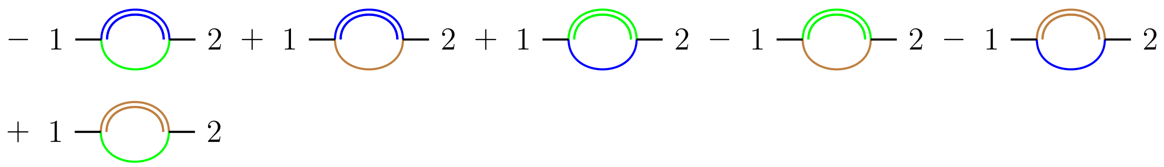


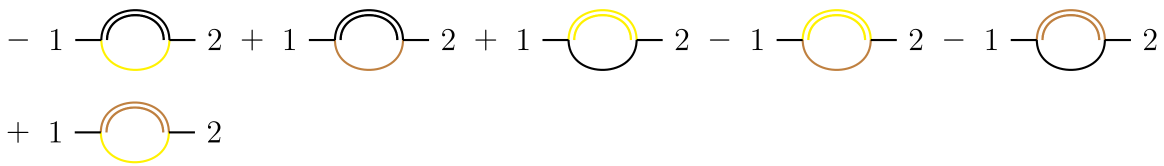
$g_9 =$ 

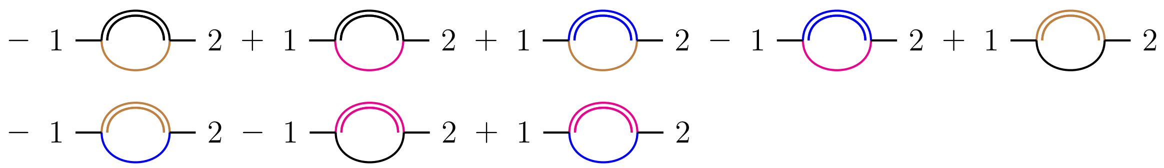
$g_{10} =$ 

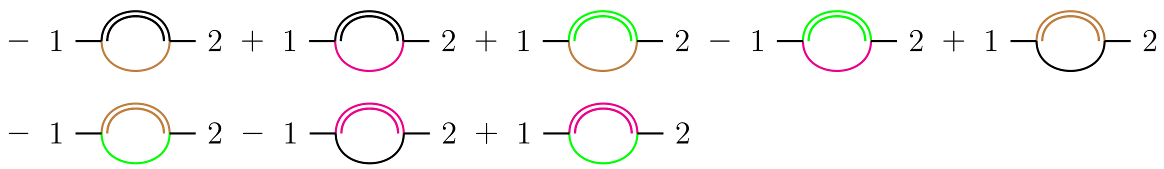
B.10 Generators of $H_1(X_{1,2,7};\mathbb{Q})$

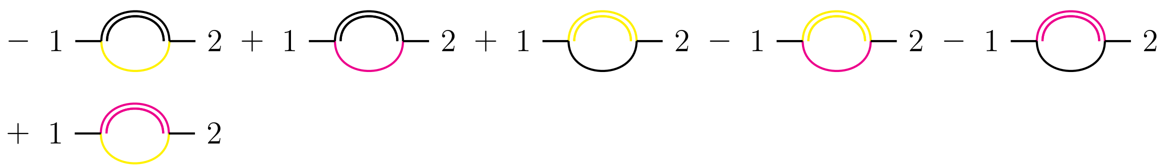
$g_1 =$ 

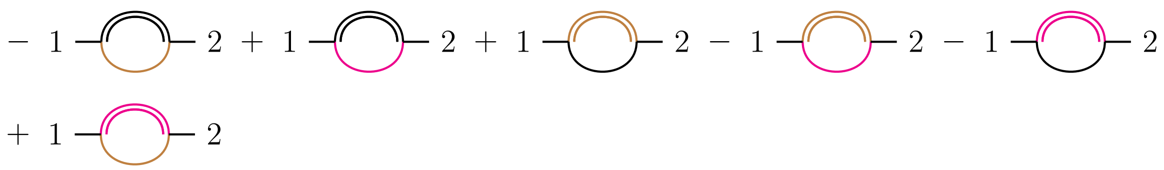
$g_2 =$ 

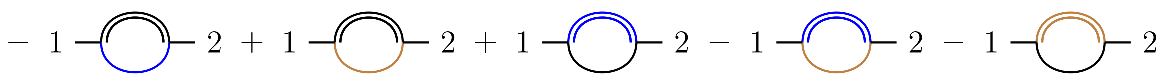
$g_3 =$ 

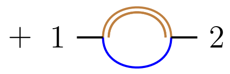
$g_4 =$ 

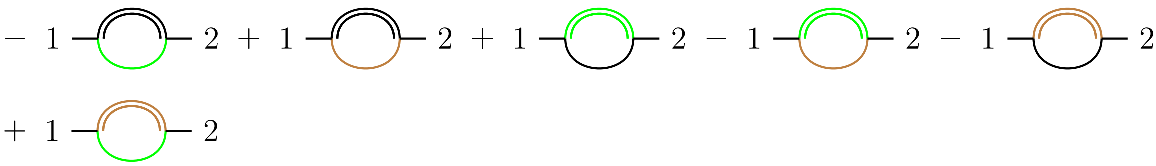
$g_5 =$ 

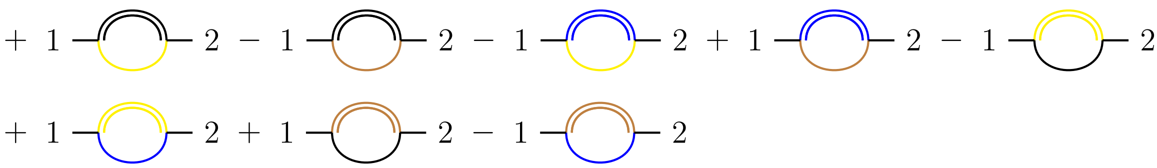
$g_6 =$ 

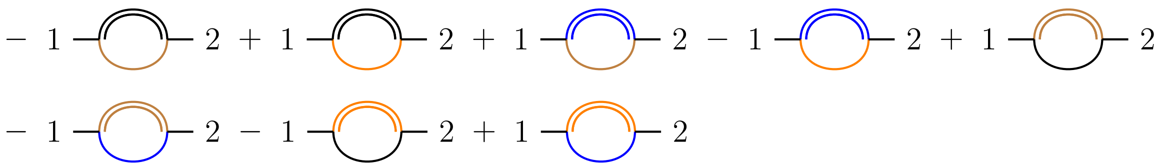
$g_7 =$ 

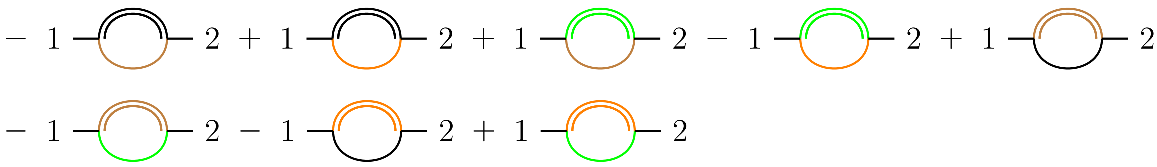
$g_8 =$ 

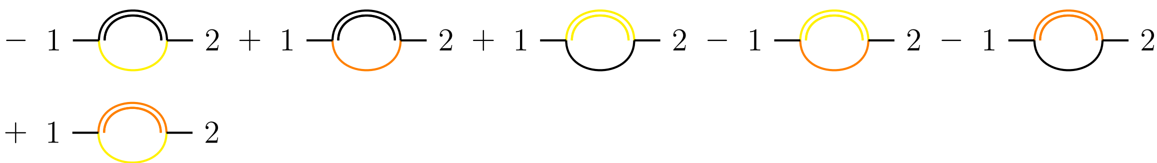


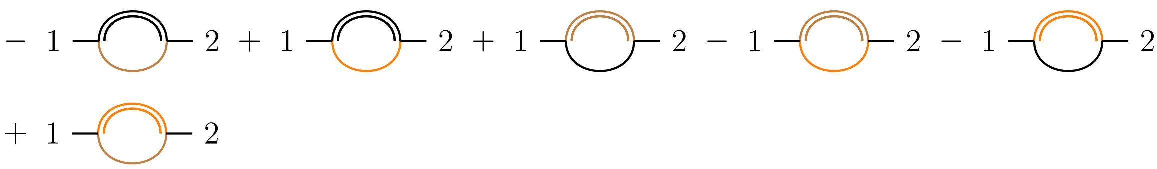
$g_9 =$ 

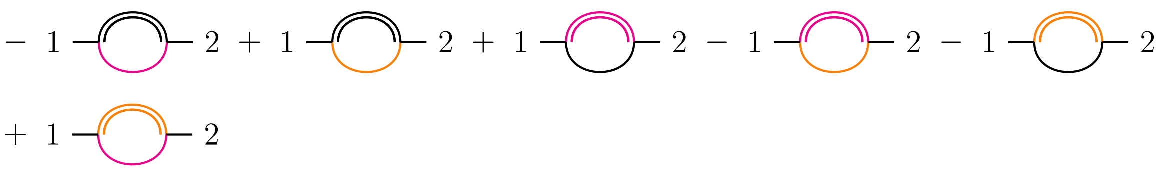
$g_{10} =$ 

$g_{11} =$ 

$g_{12} =$ 

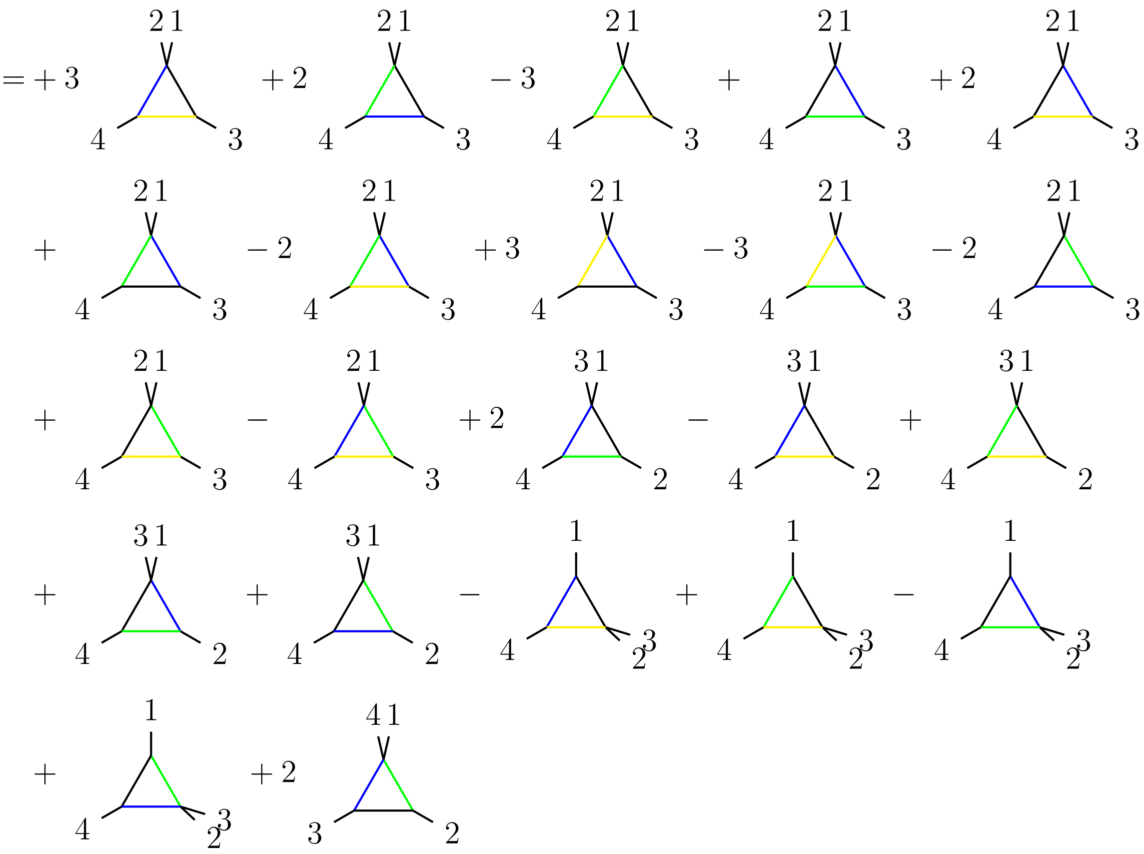
$g_{13} =$ 

$g_{14} =$ 

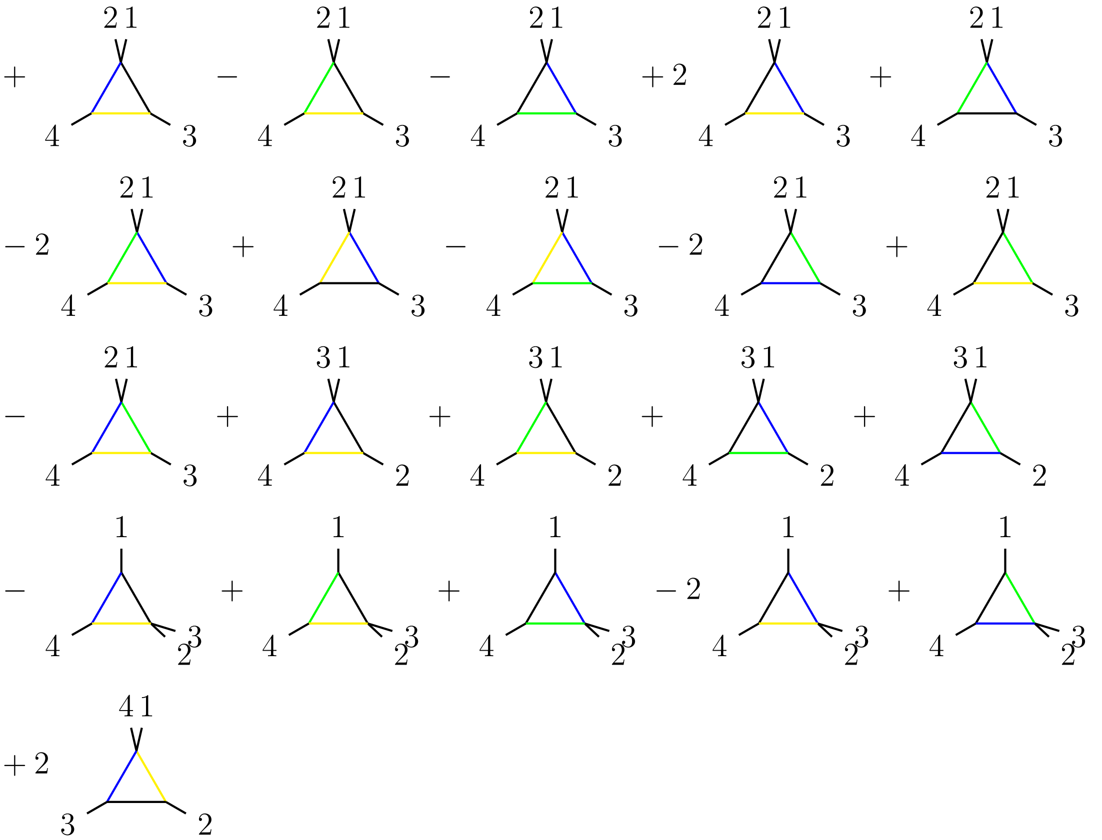
$g_{15} =$ 

B.11 Generators of $H_2(\tilde{X}_{1,4};\mathbb{Q})$

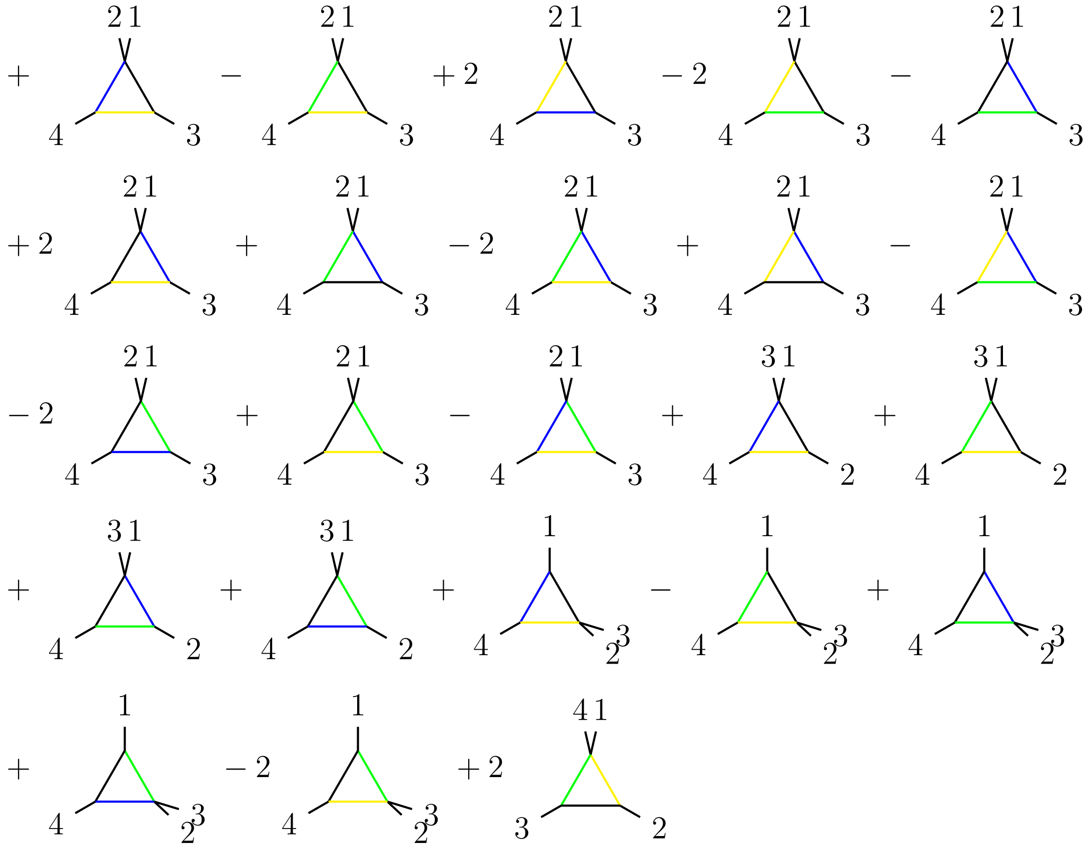
$g_1 =$



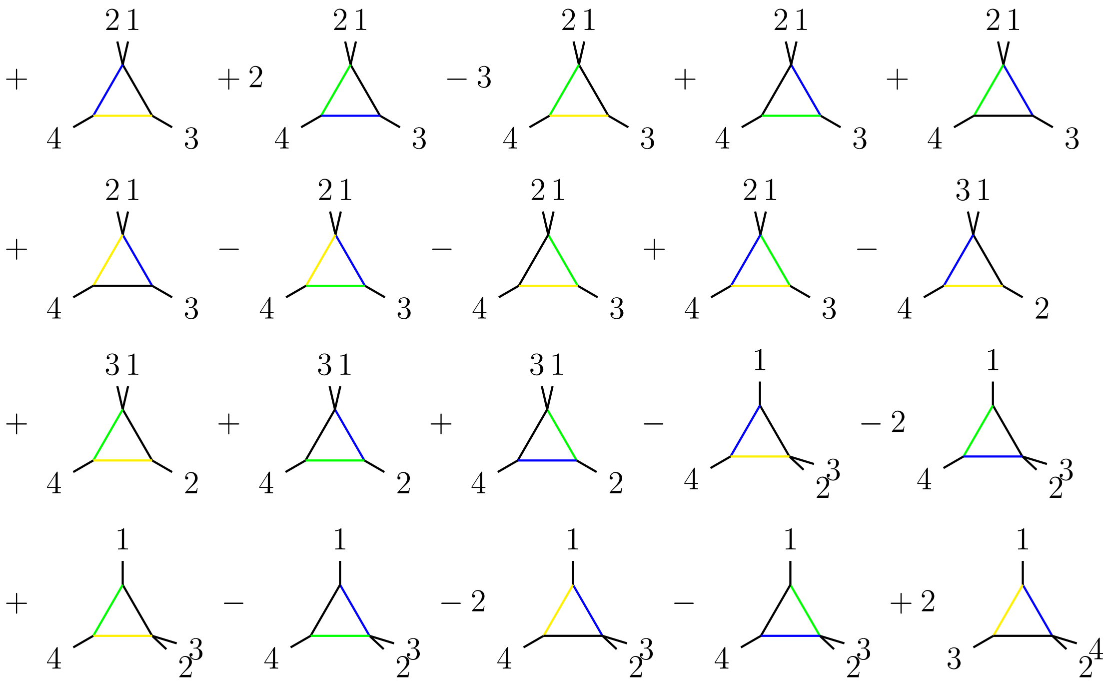
$g_2 =$



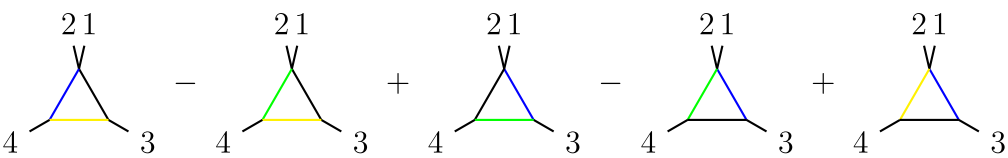
$g_3 =$

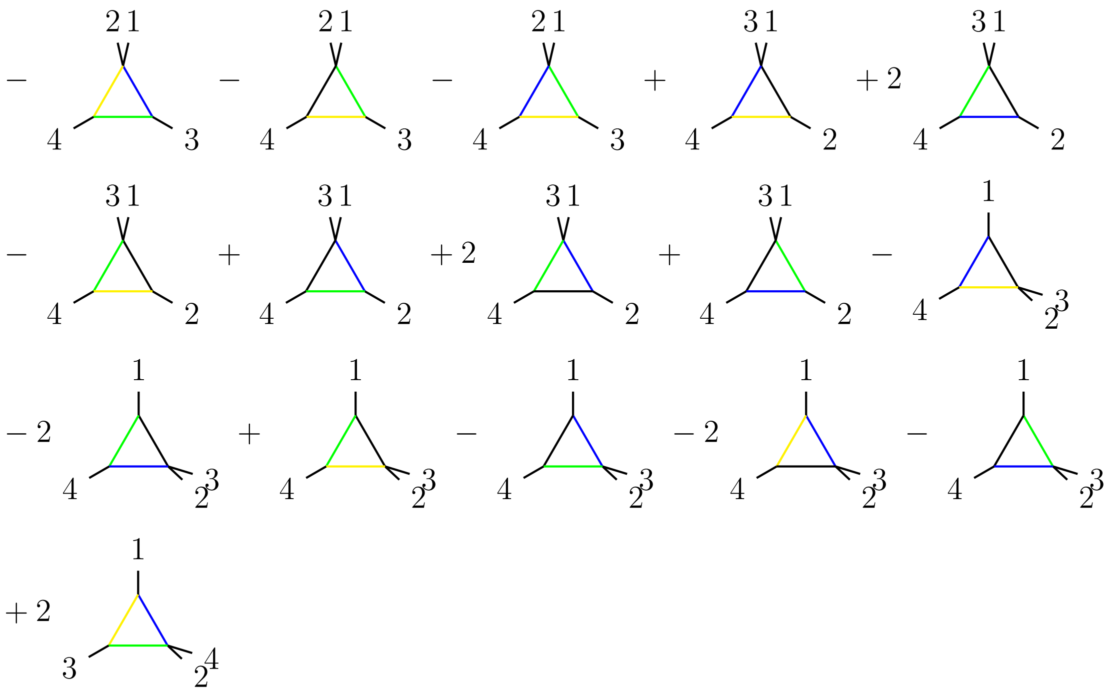


$g_4 =$

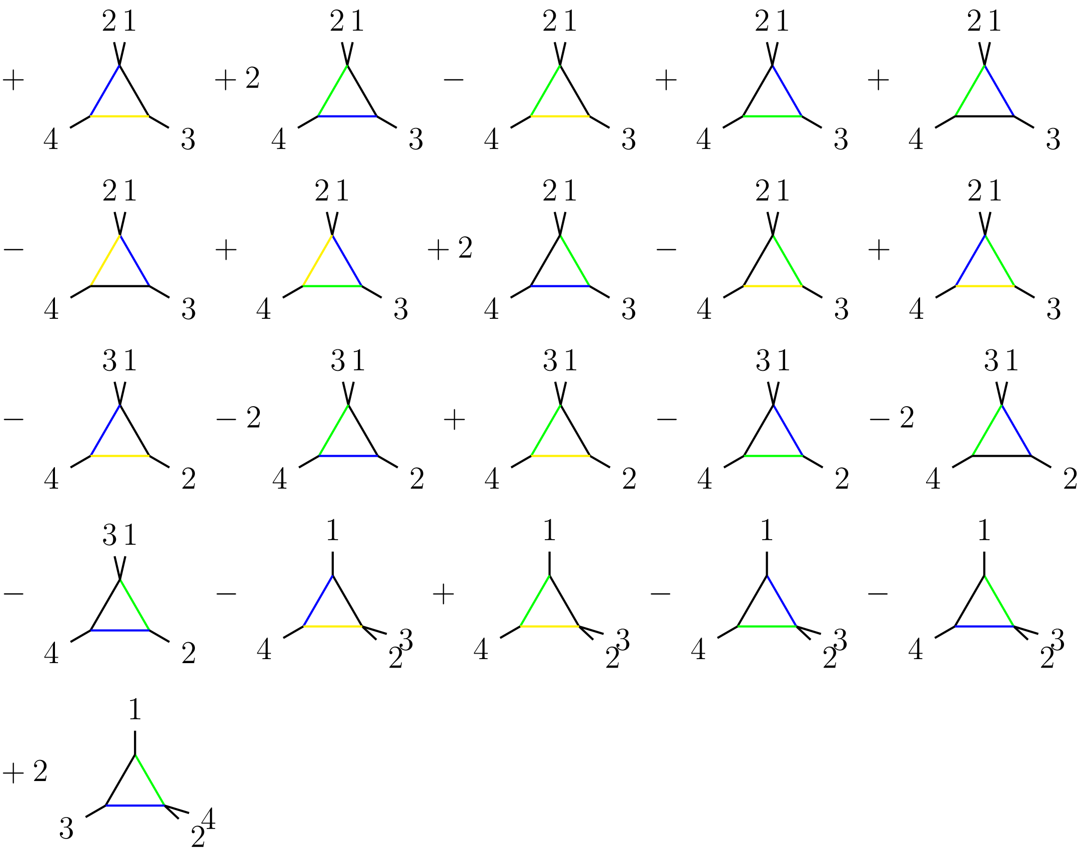


$g_5 =$

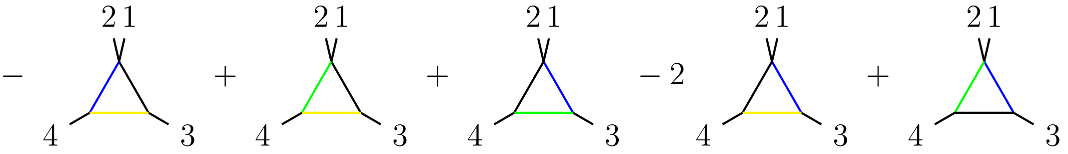


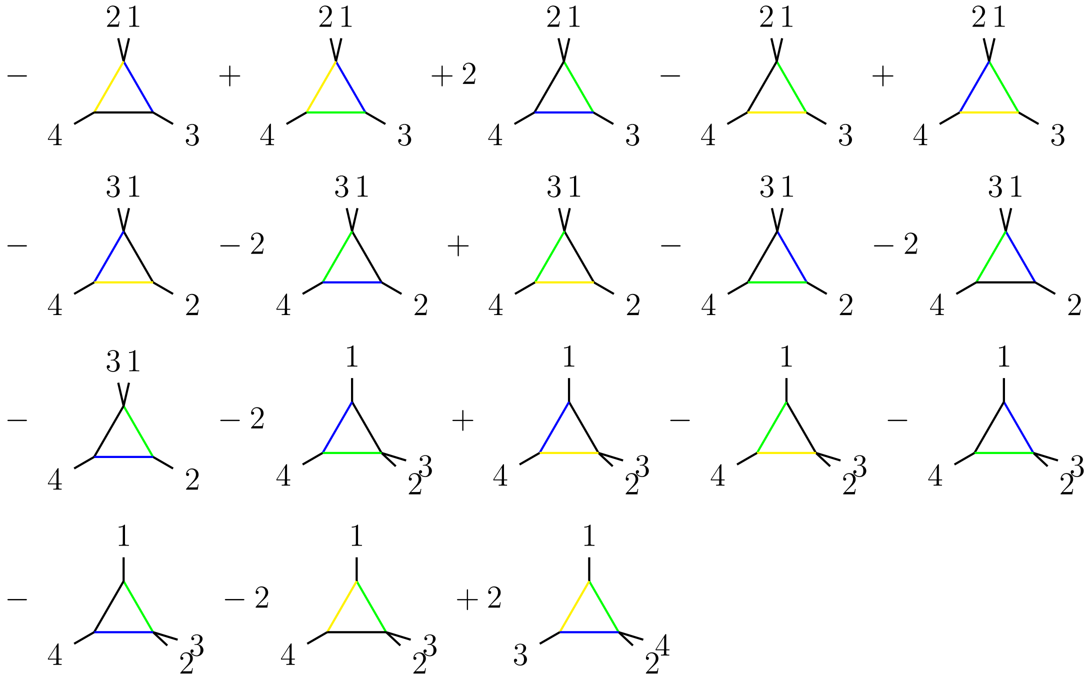


$g_6 =$

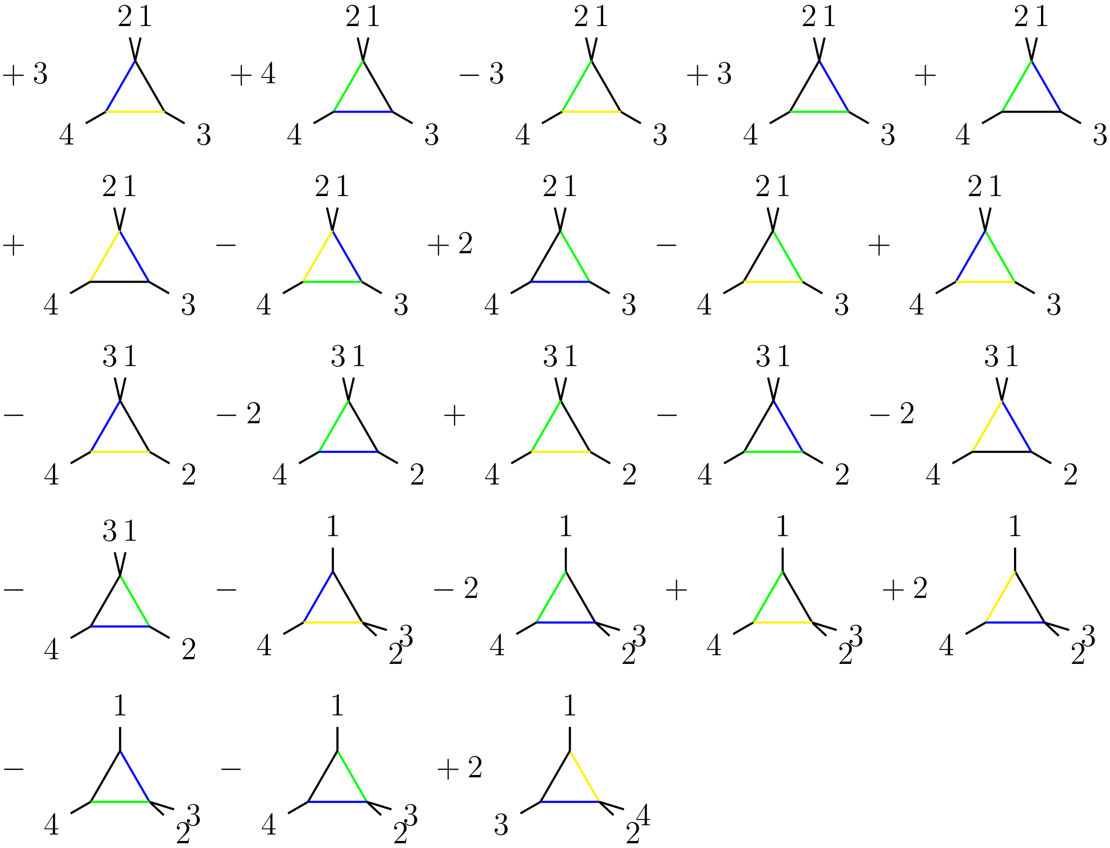


$g_7 =$

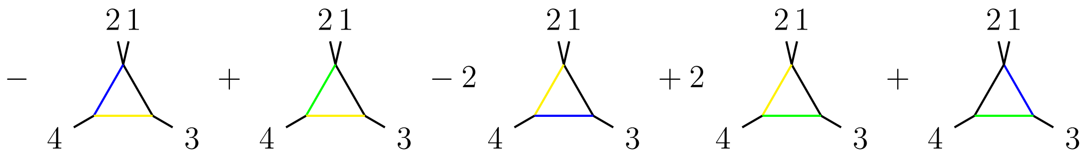


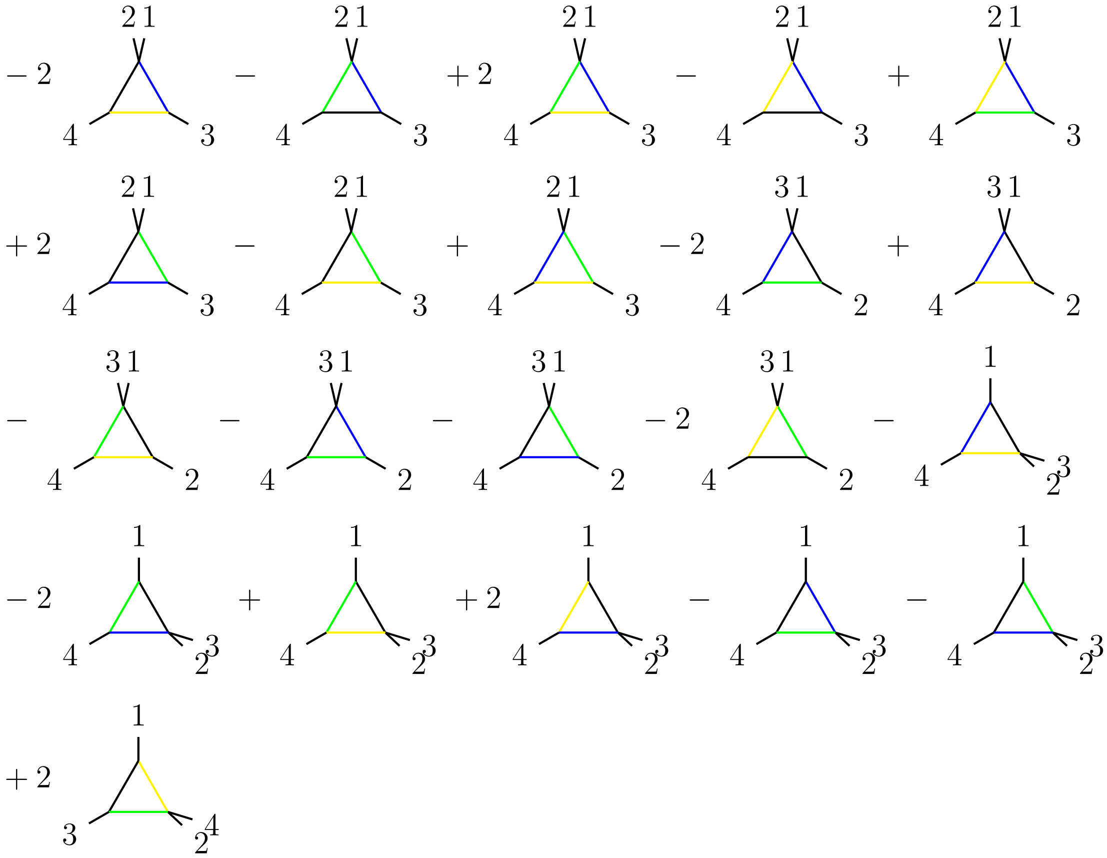


$g_8 =$

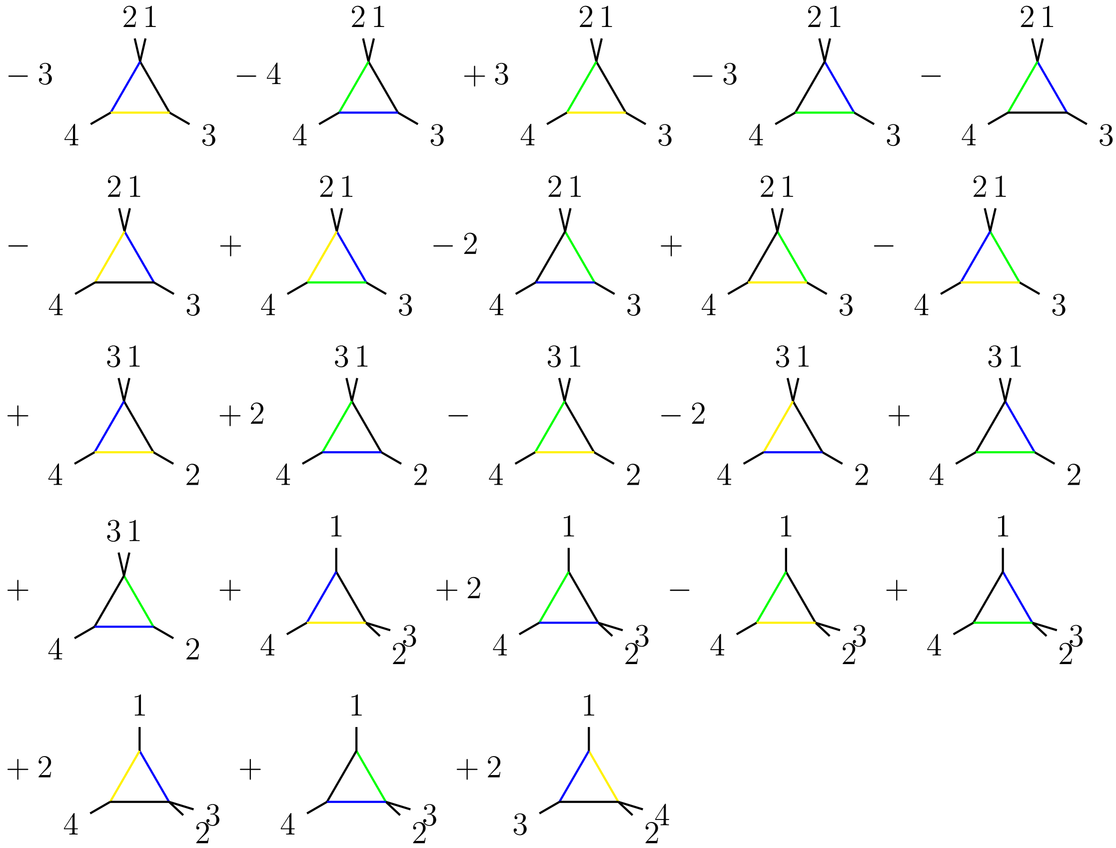


$g_9 =$

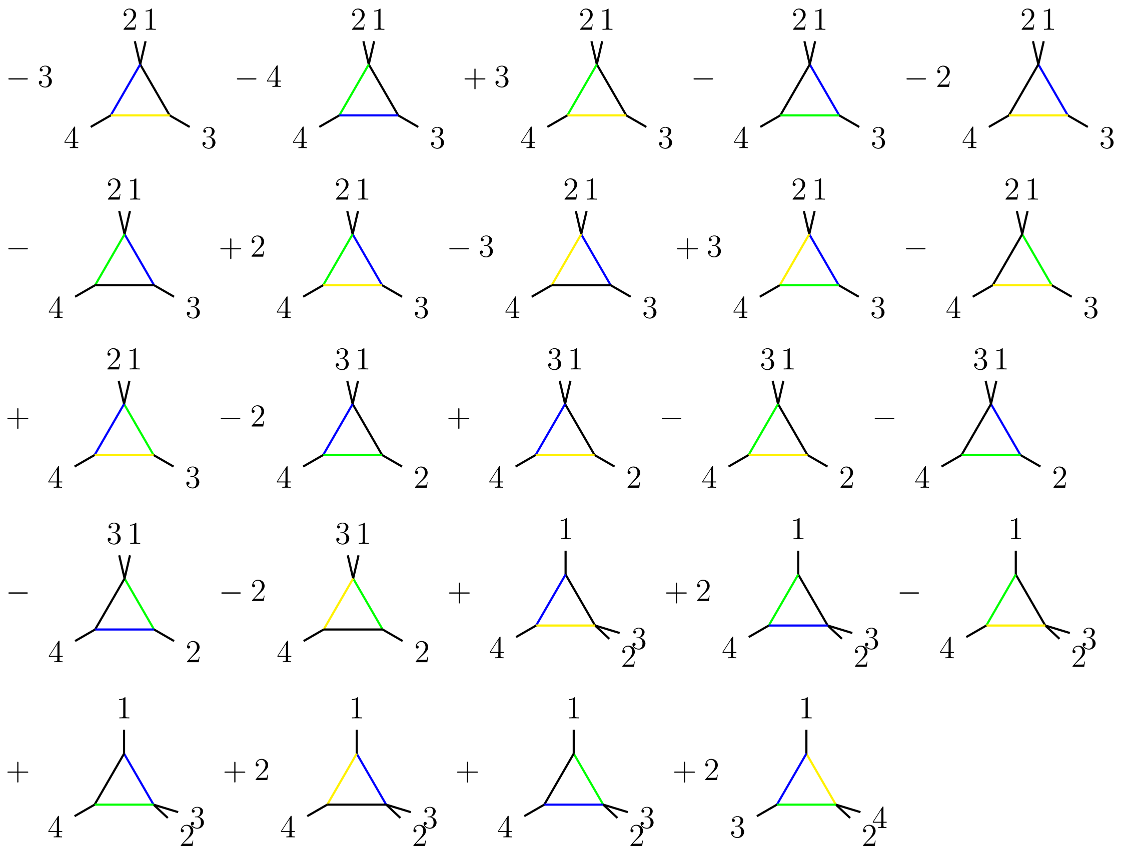




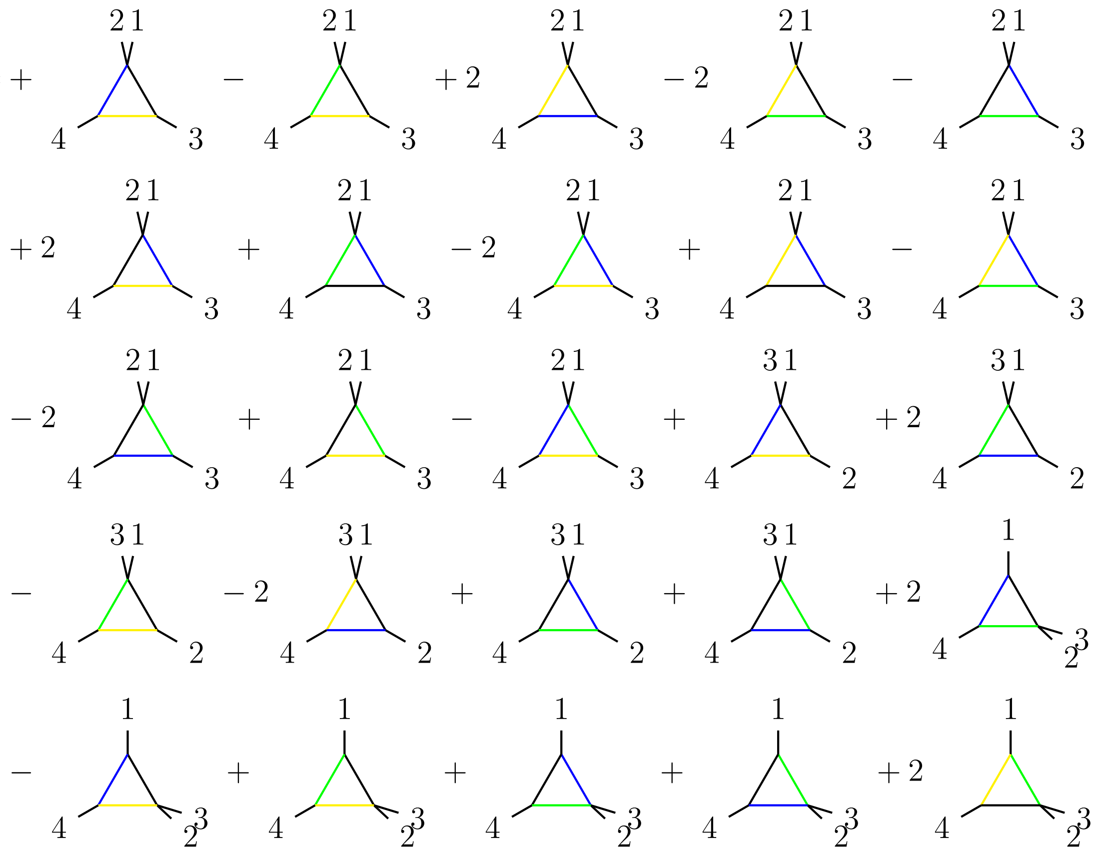
$g_{10} =$

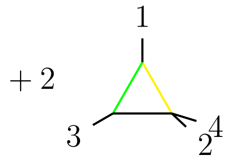


$g_{11} =$

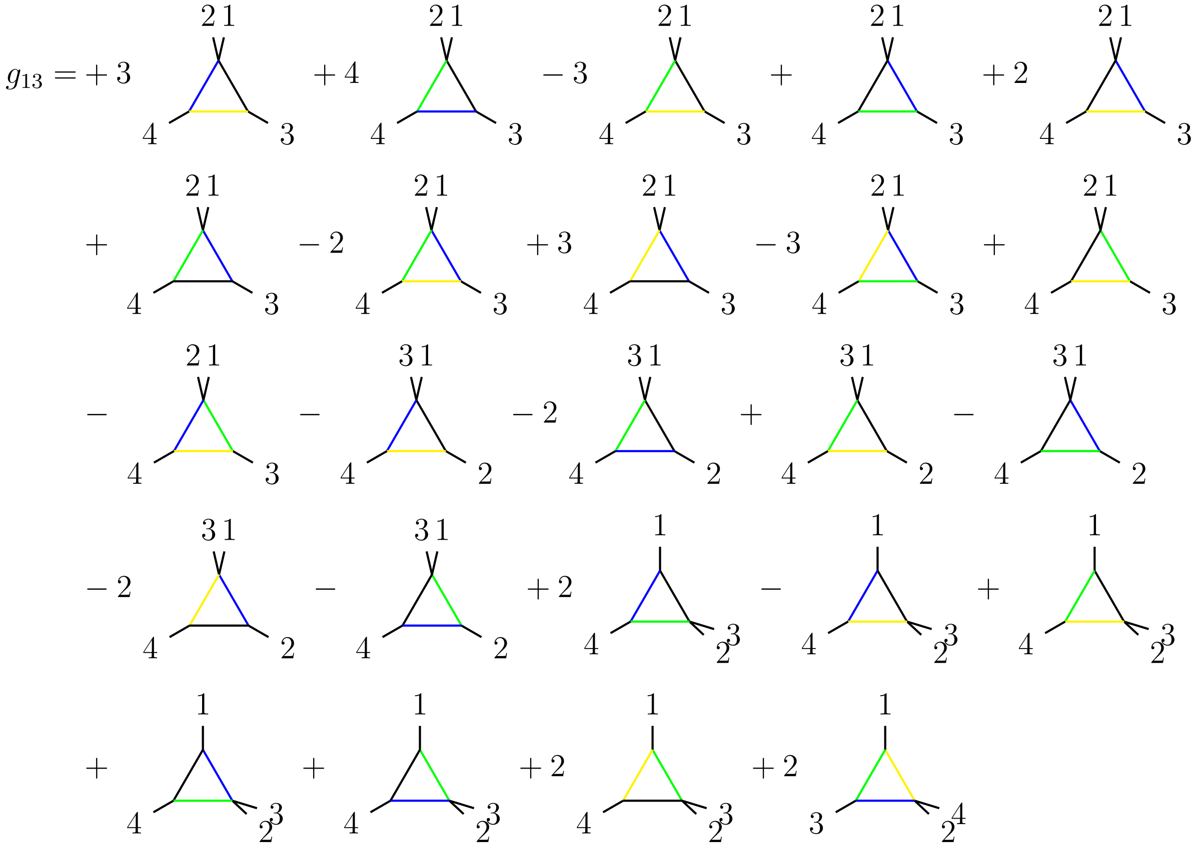


$g_{12} =$

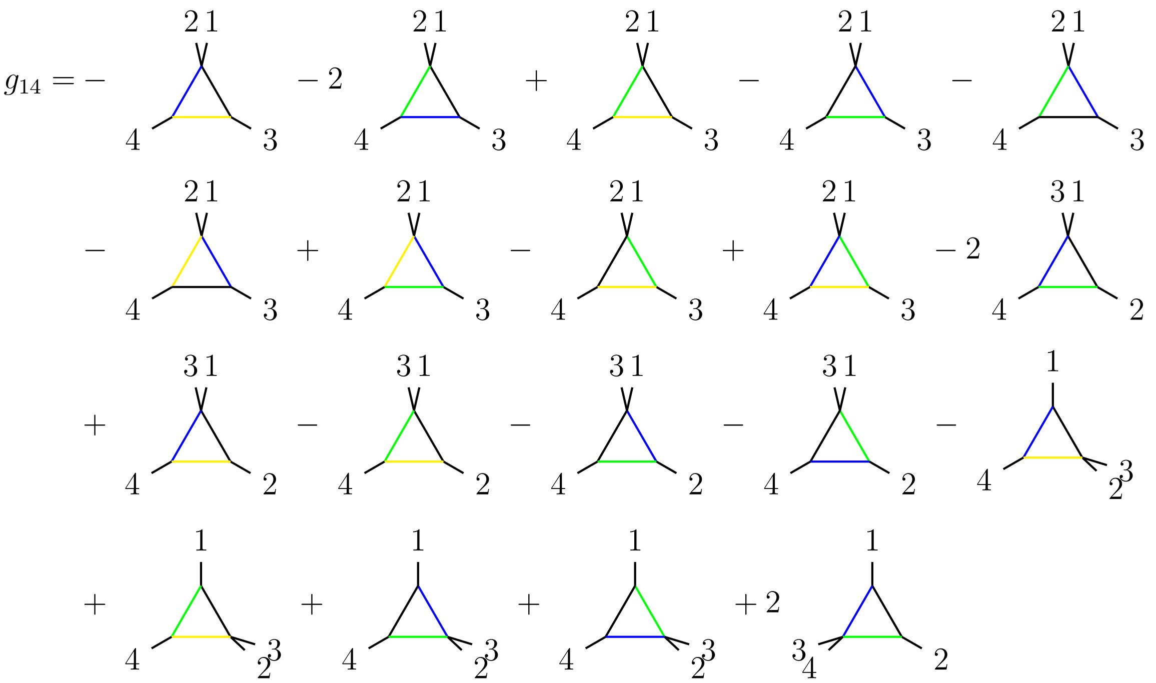




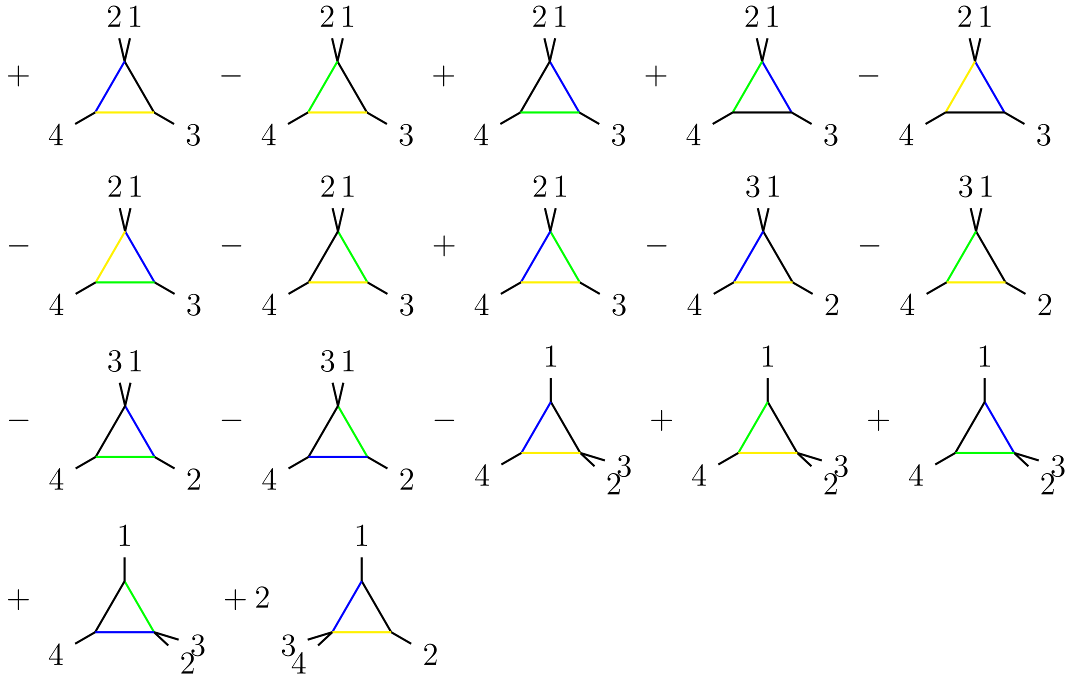
$g_{13} =$



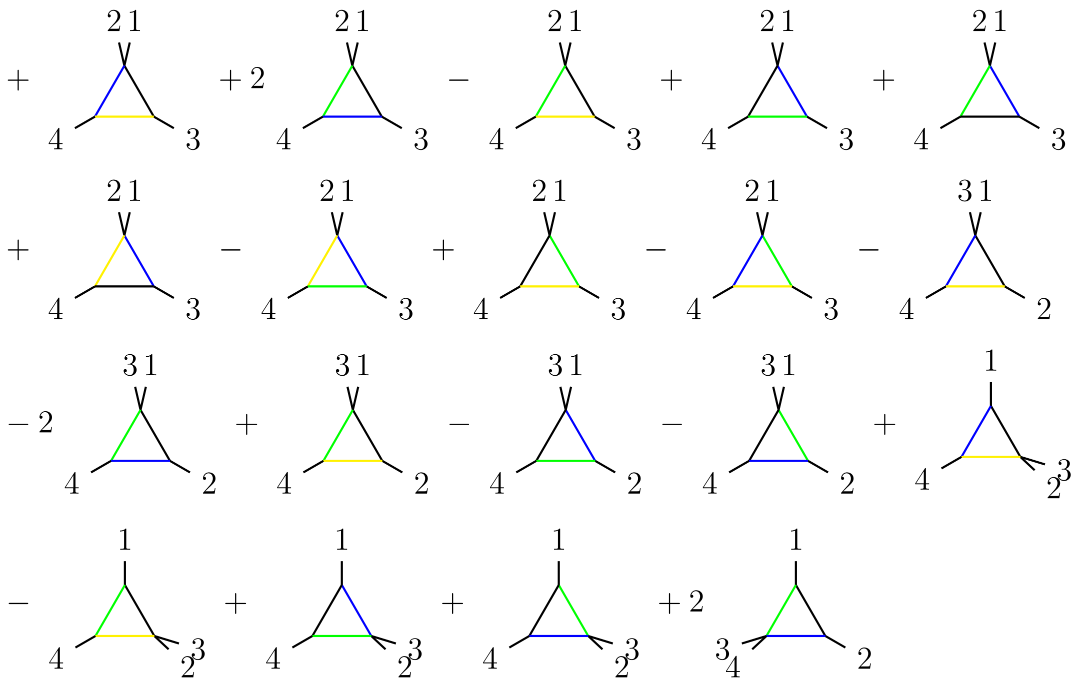
$g_{14} =$



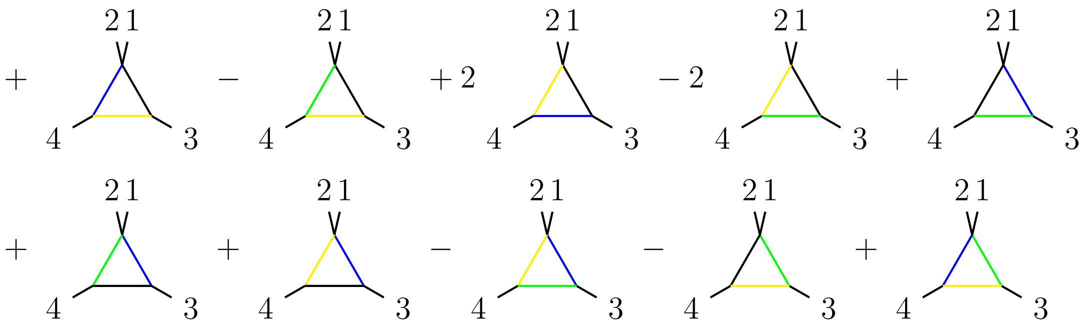
$g_{15} =$

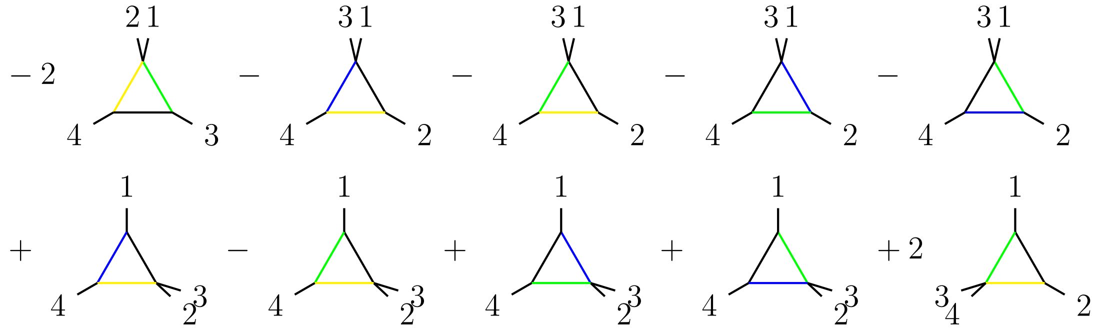


$g_{16} =$

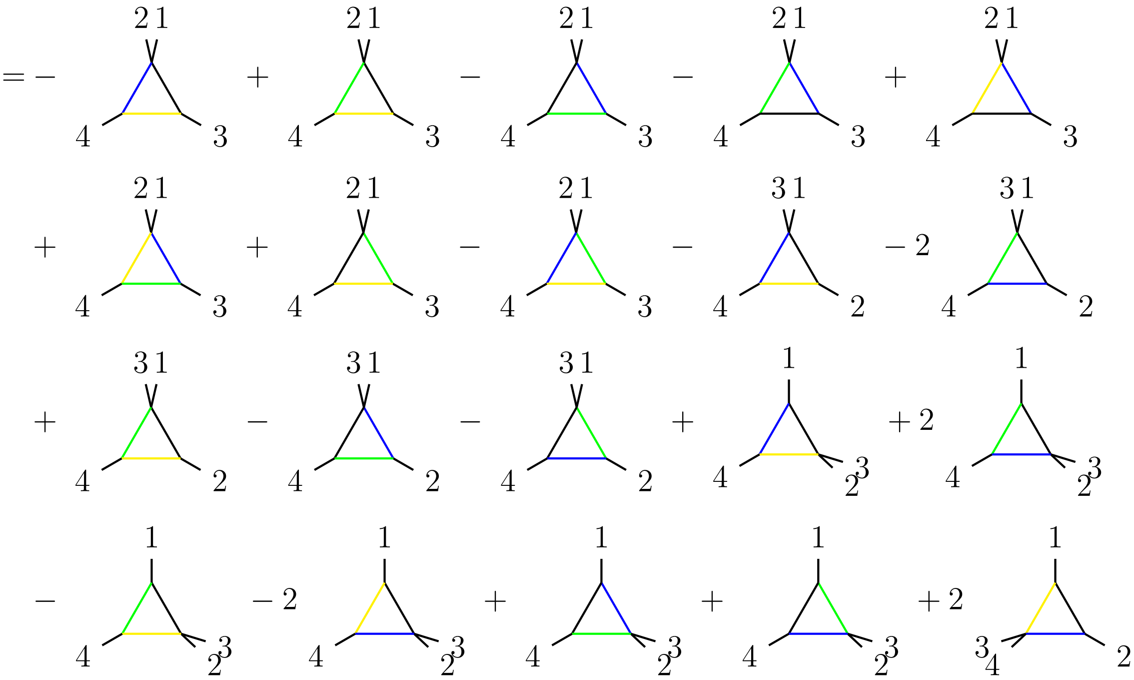


$g_{17} =$

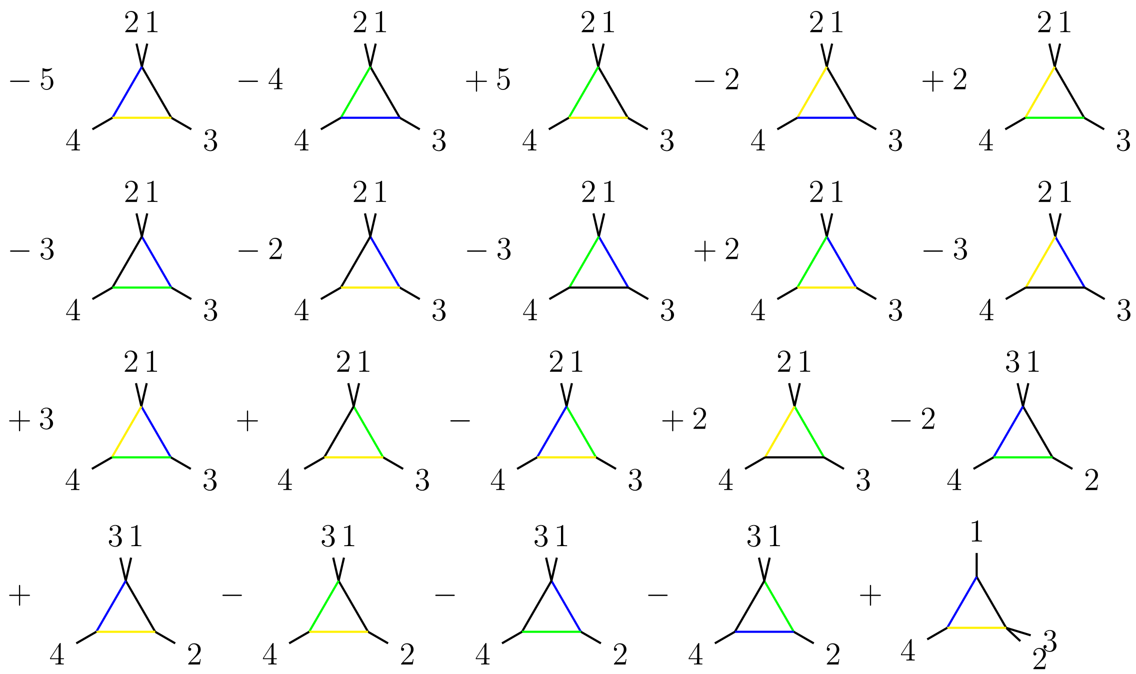


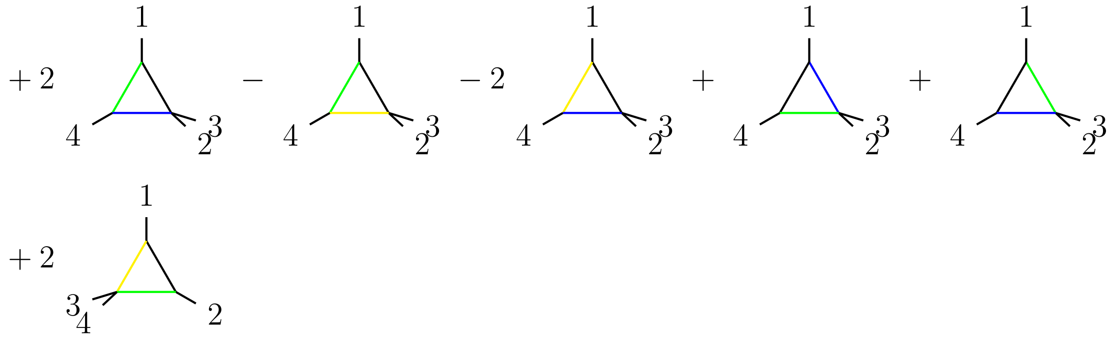


$g_{18} =$

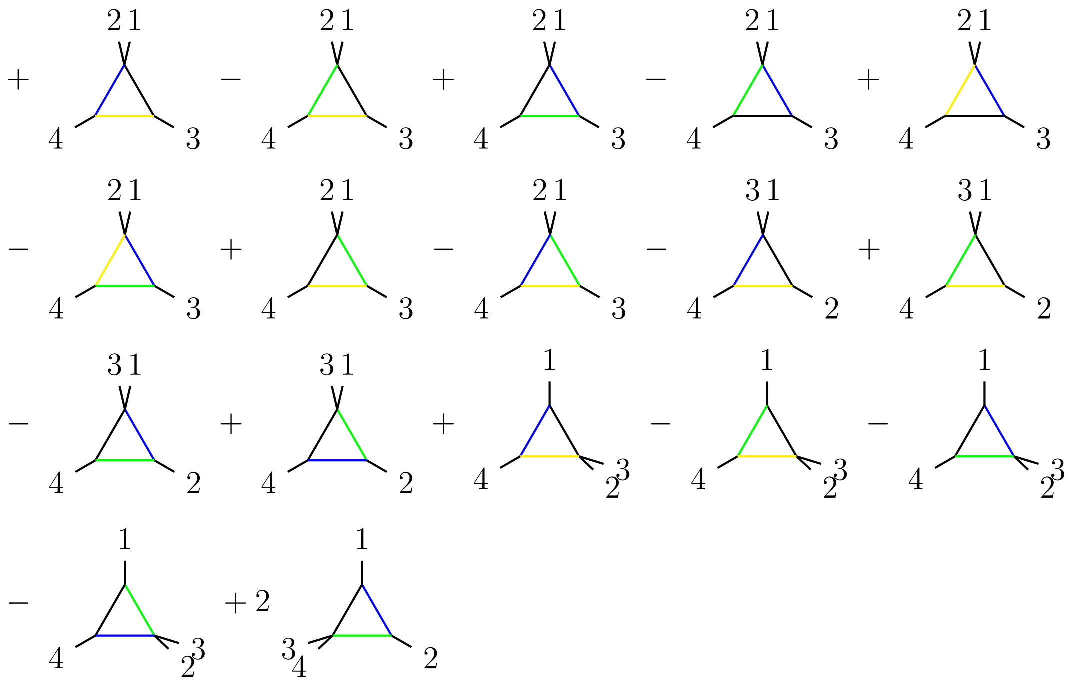


$g_{19} =$

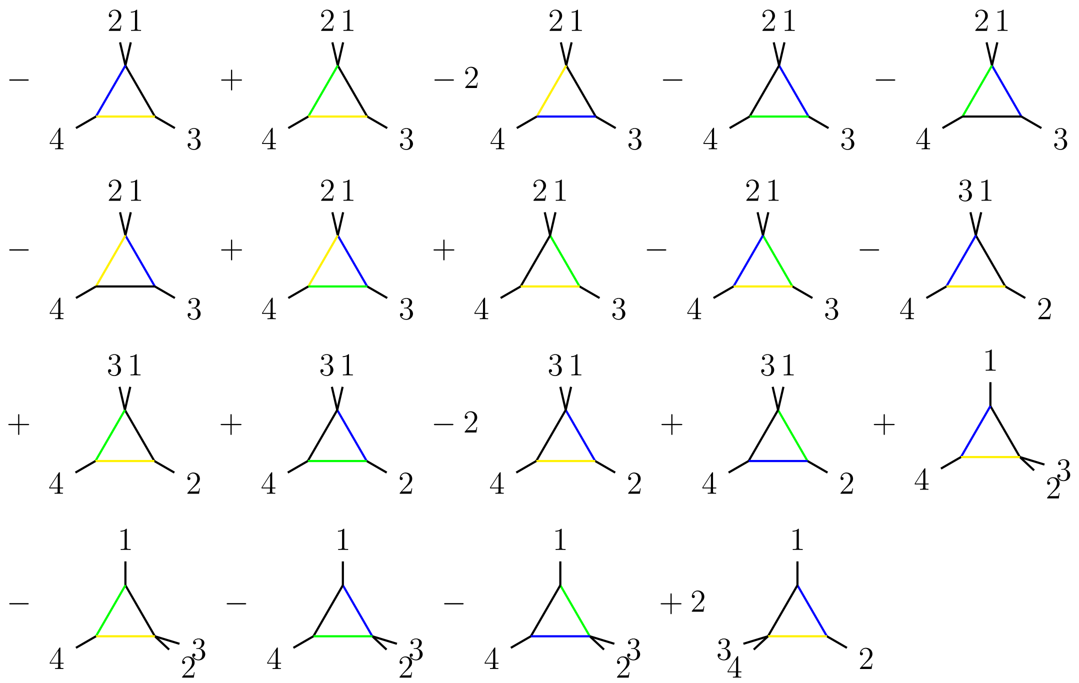




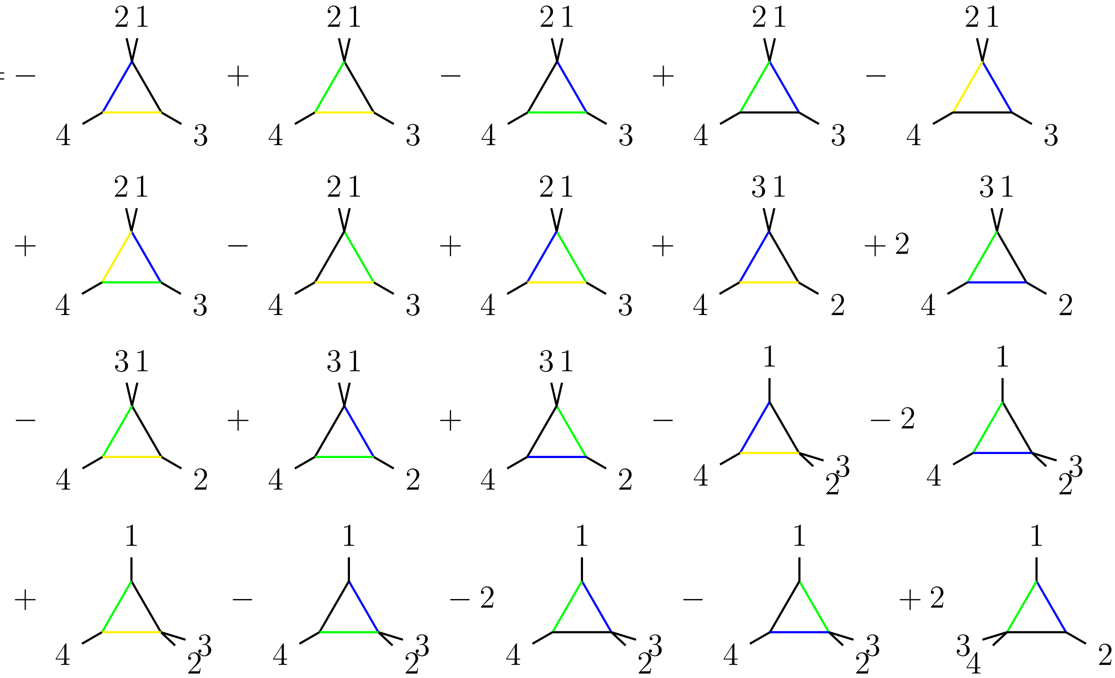
$g_{20} =$



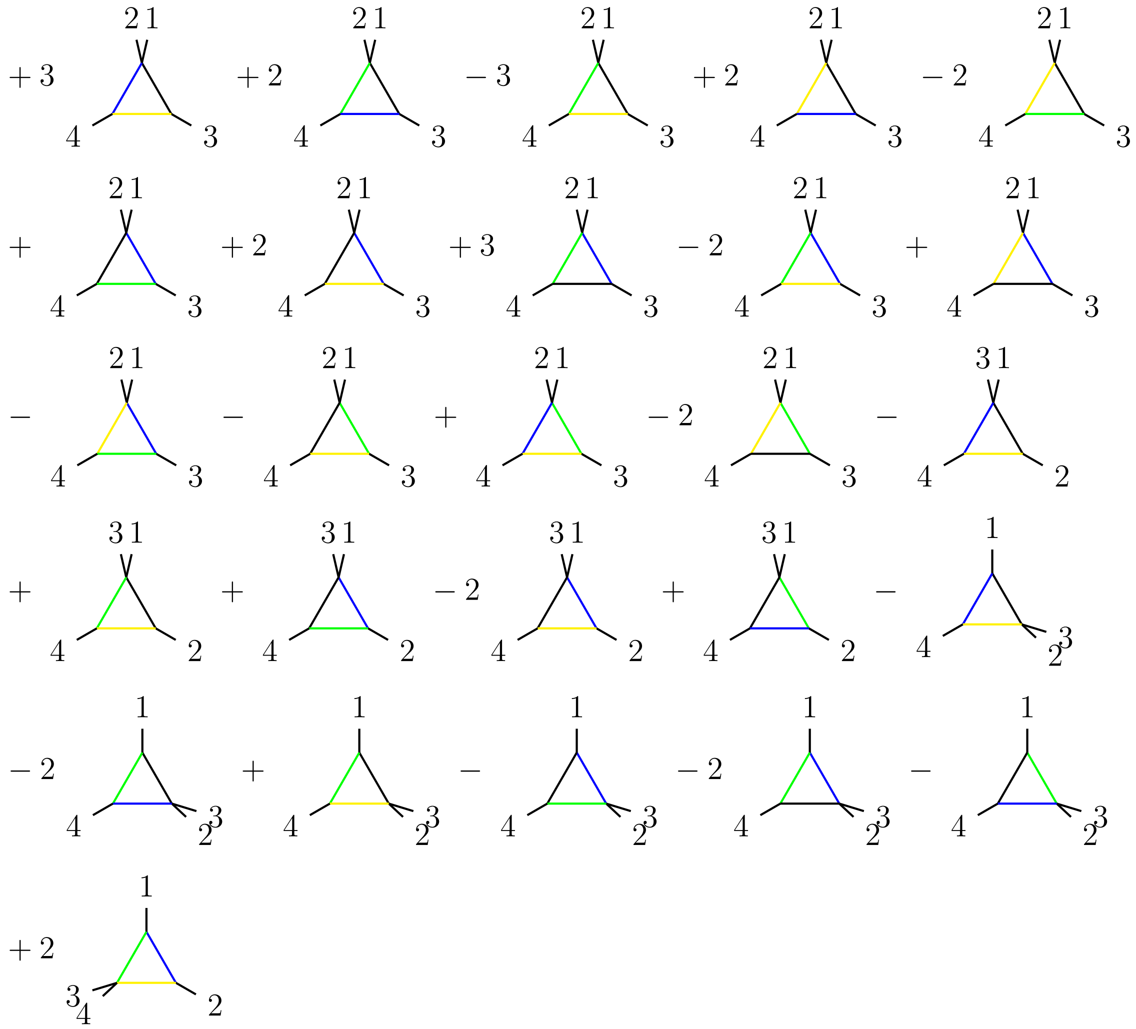
$g_{21} =$



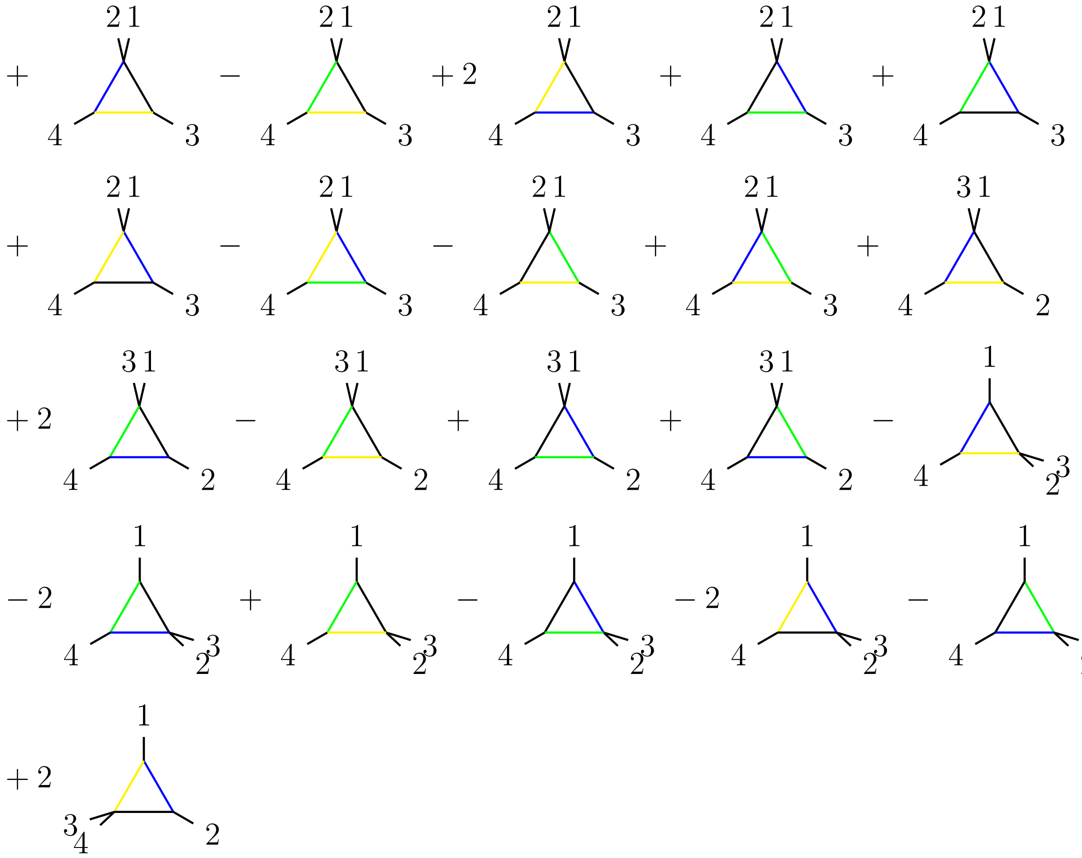
$g_{22} =$



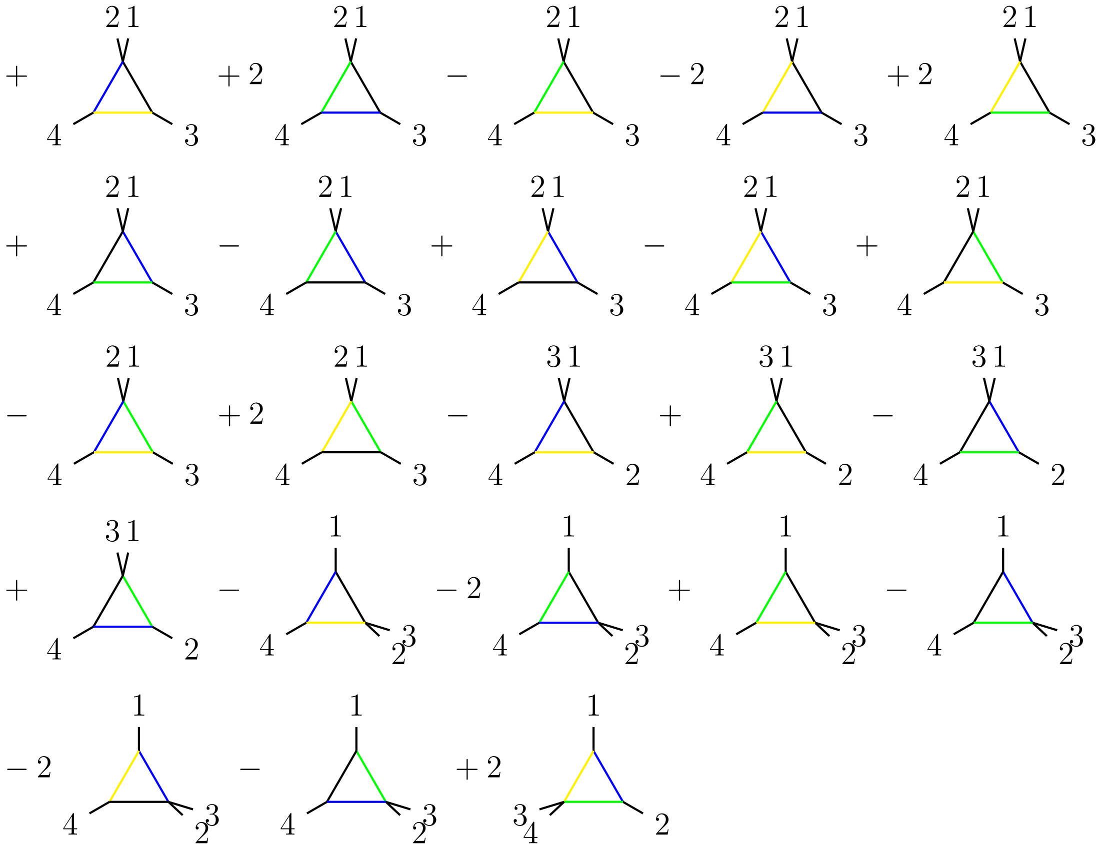
$g_{23} =$



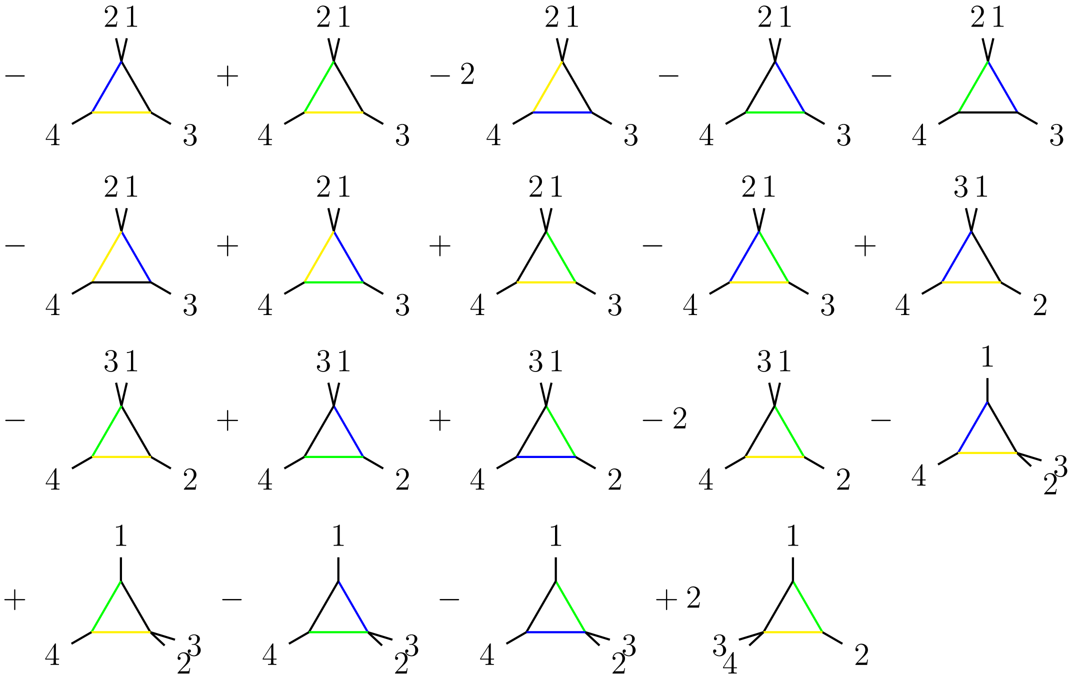
$g_{24} = +$



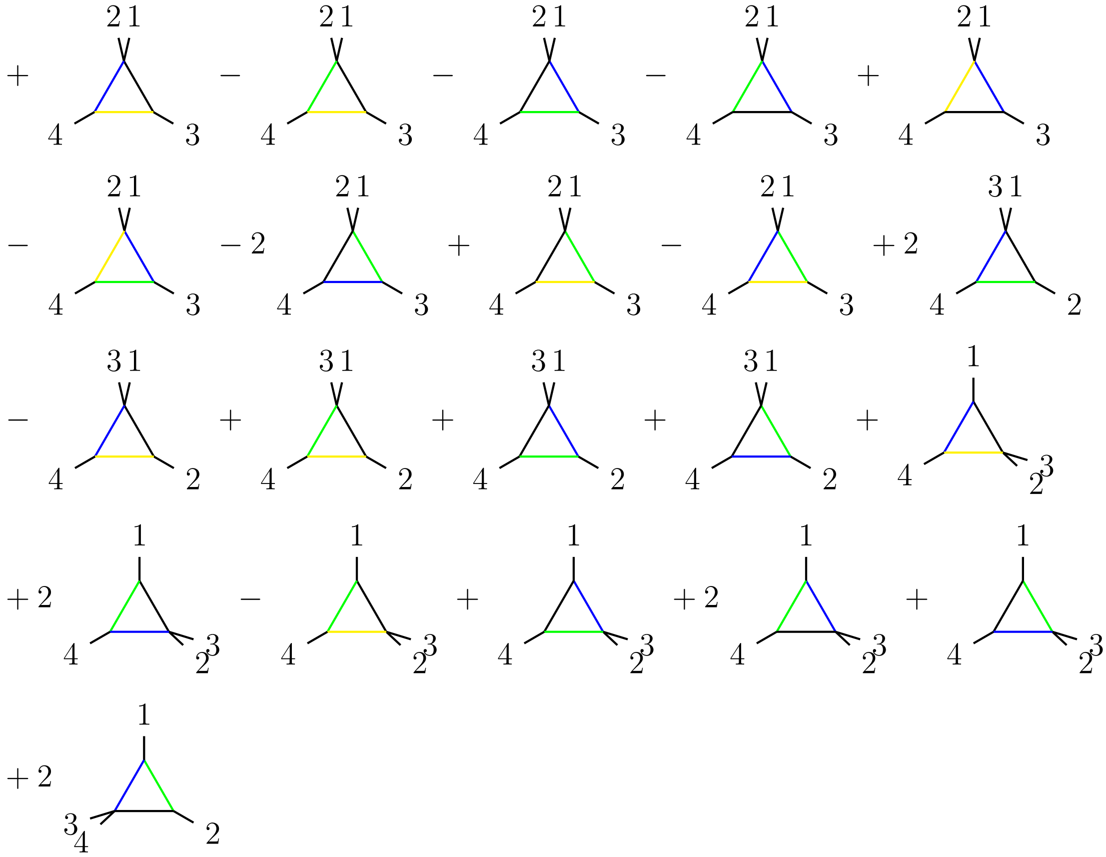
$g_{25} = +$



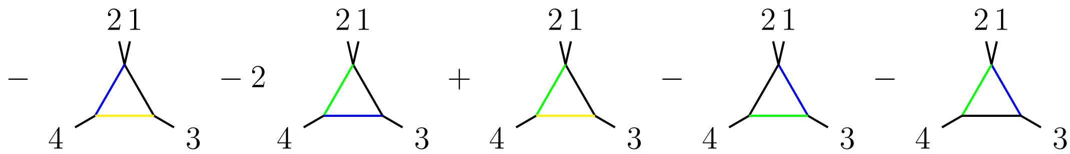
$g_{26} =$

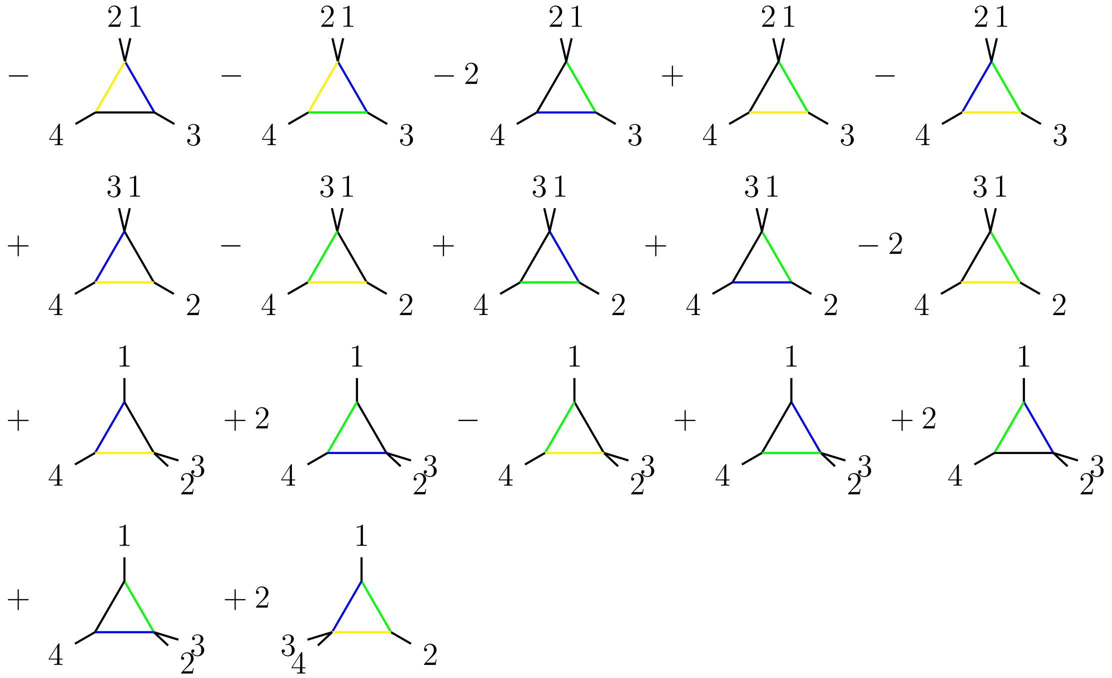


$g_{27} =$

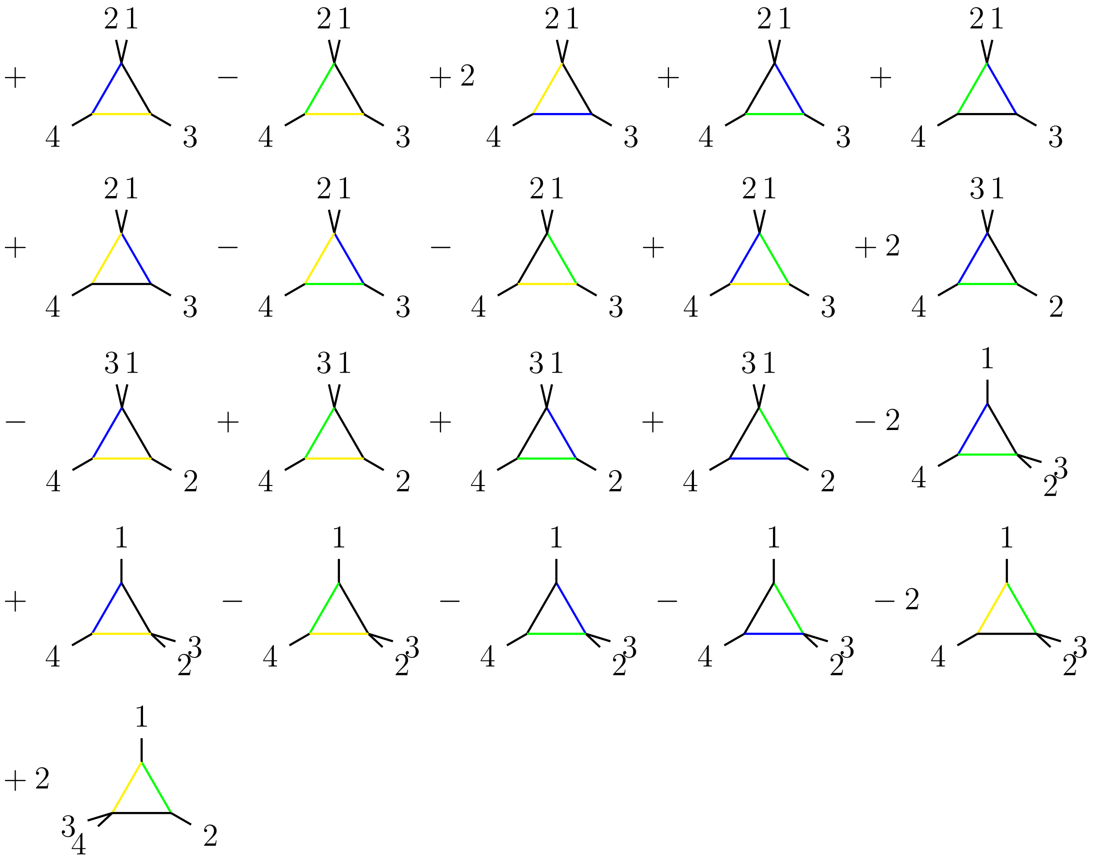


$g_{28} =$

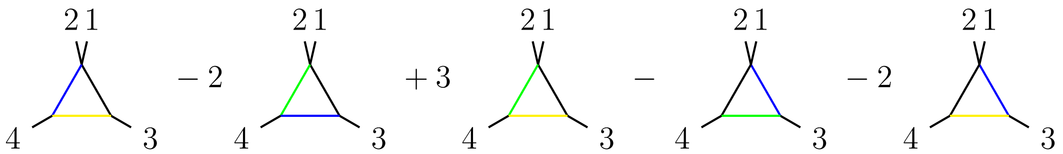


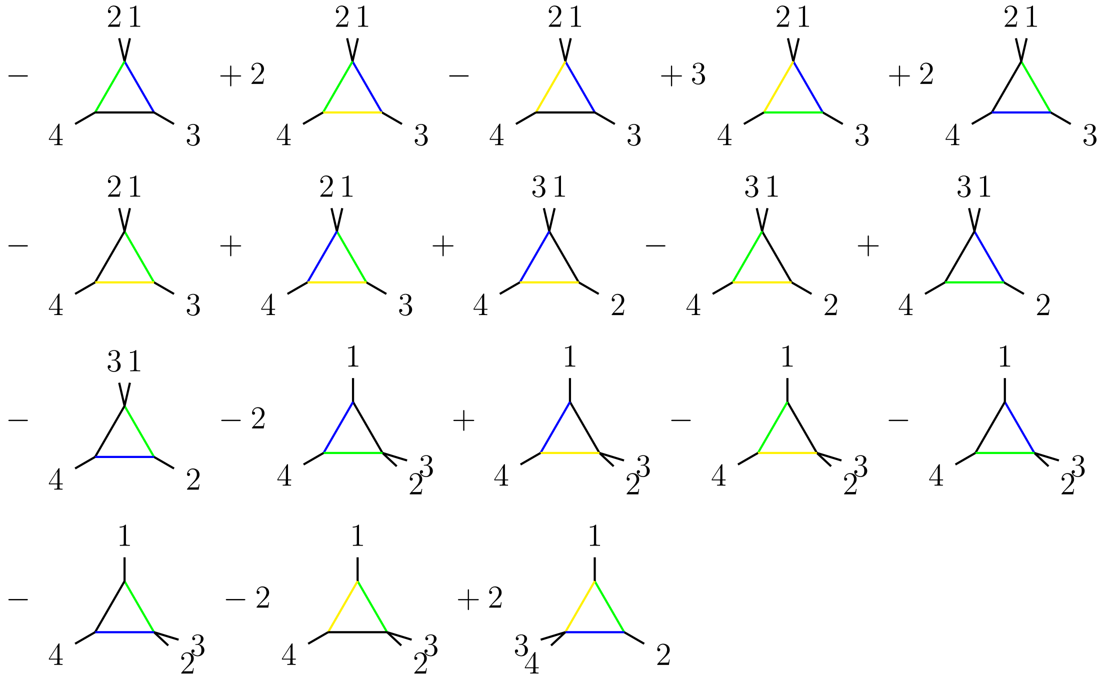


$g_{29} = +$

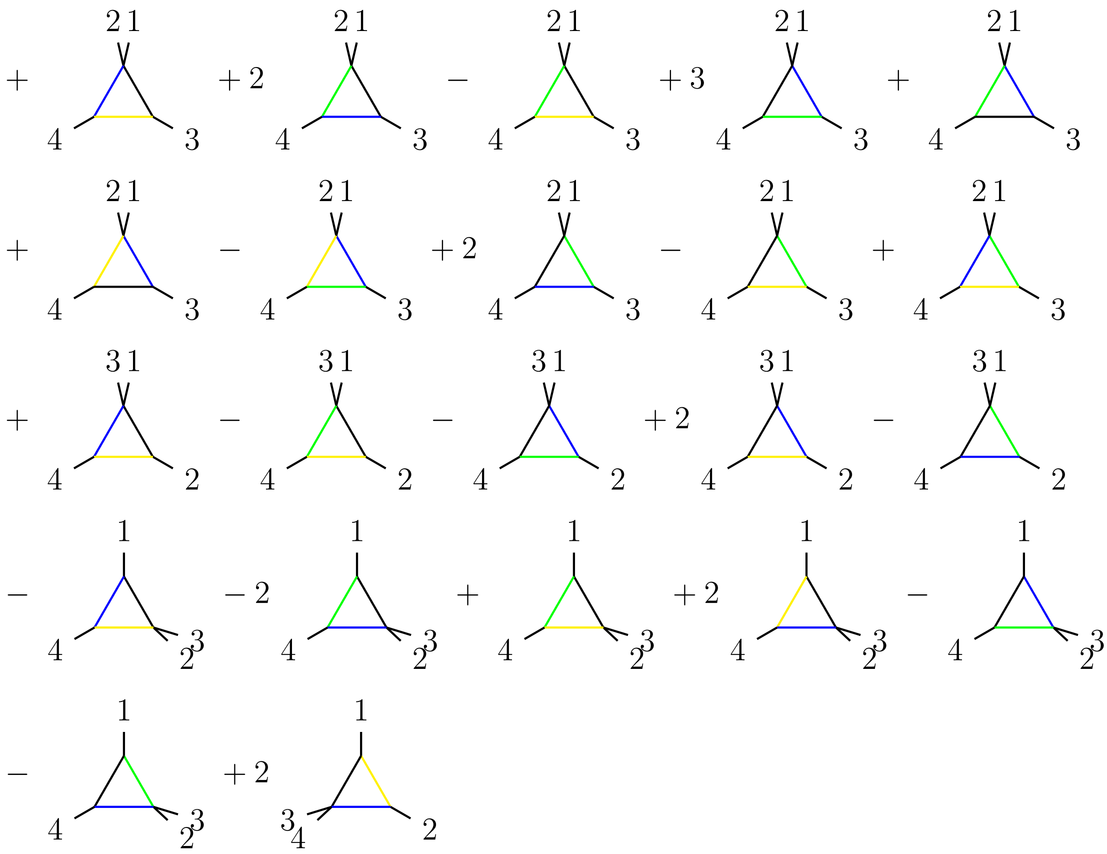


$g_{30} = -3$

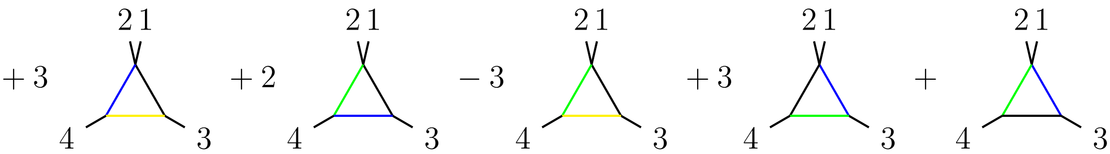


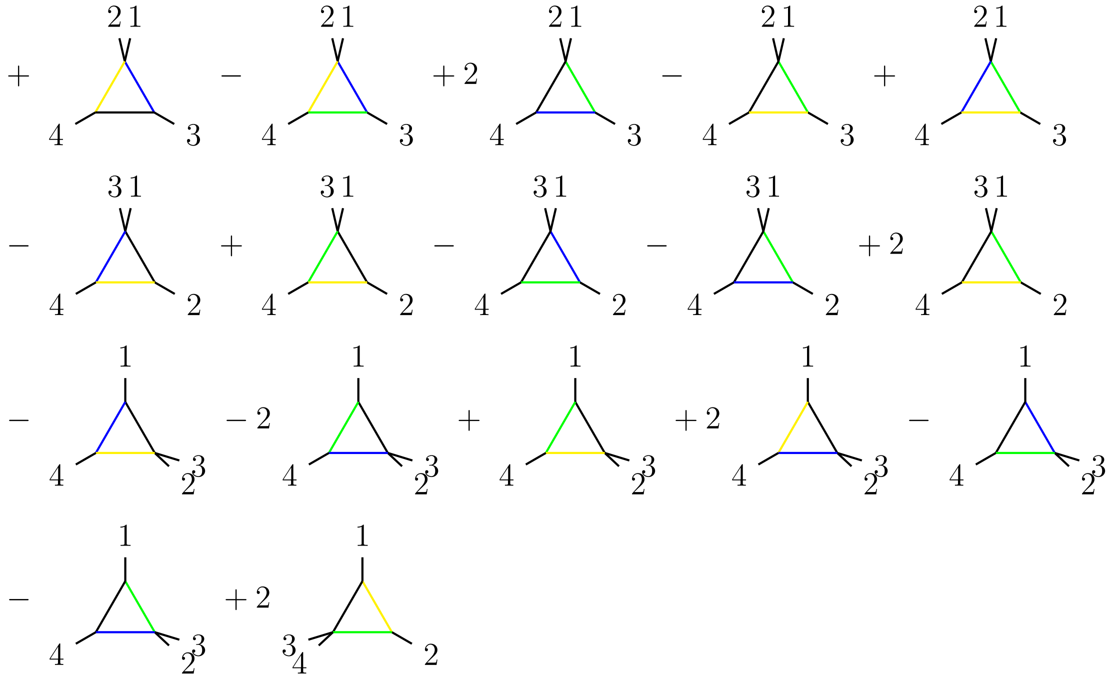


$g_{31} =$

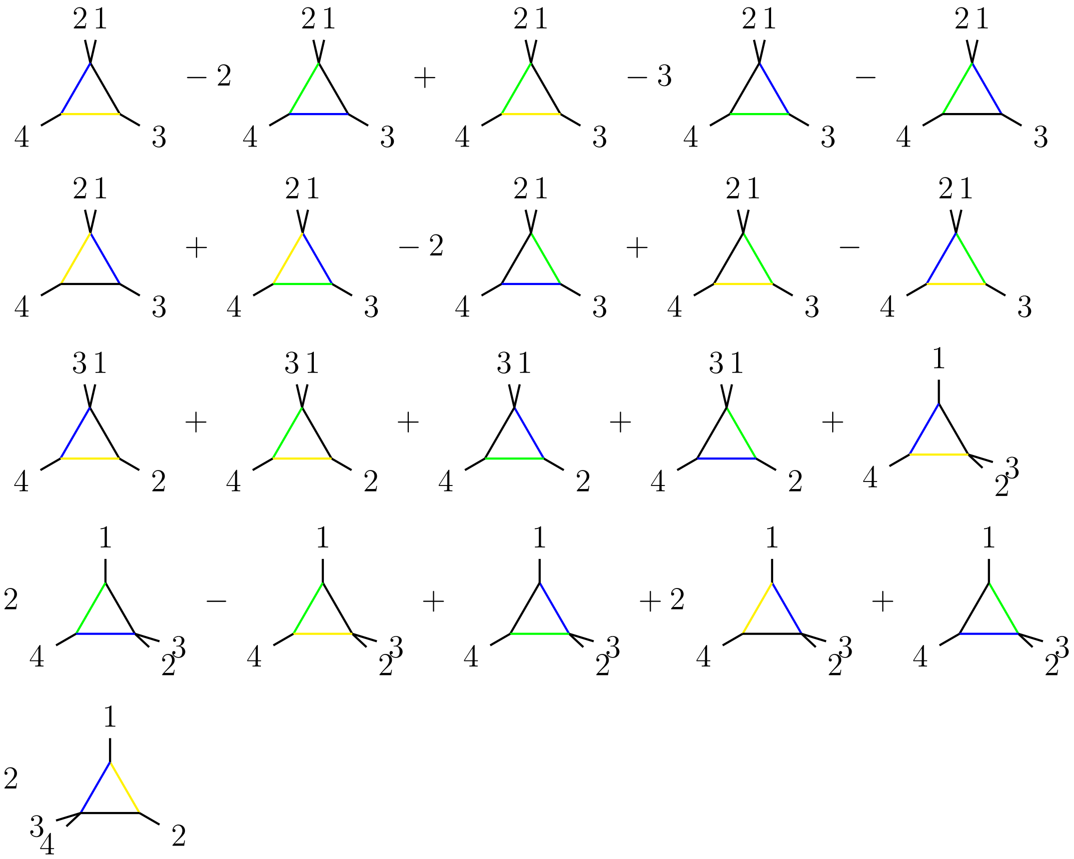


$g_{32} =$

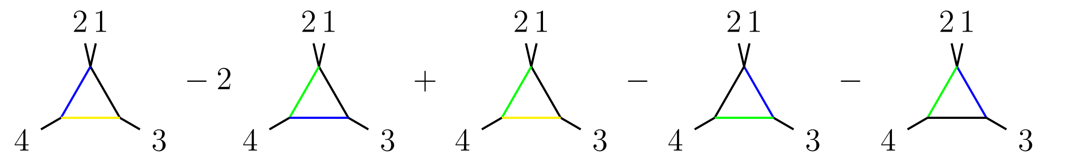


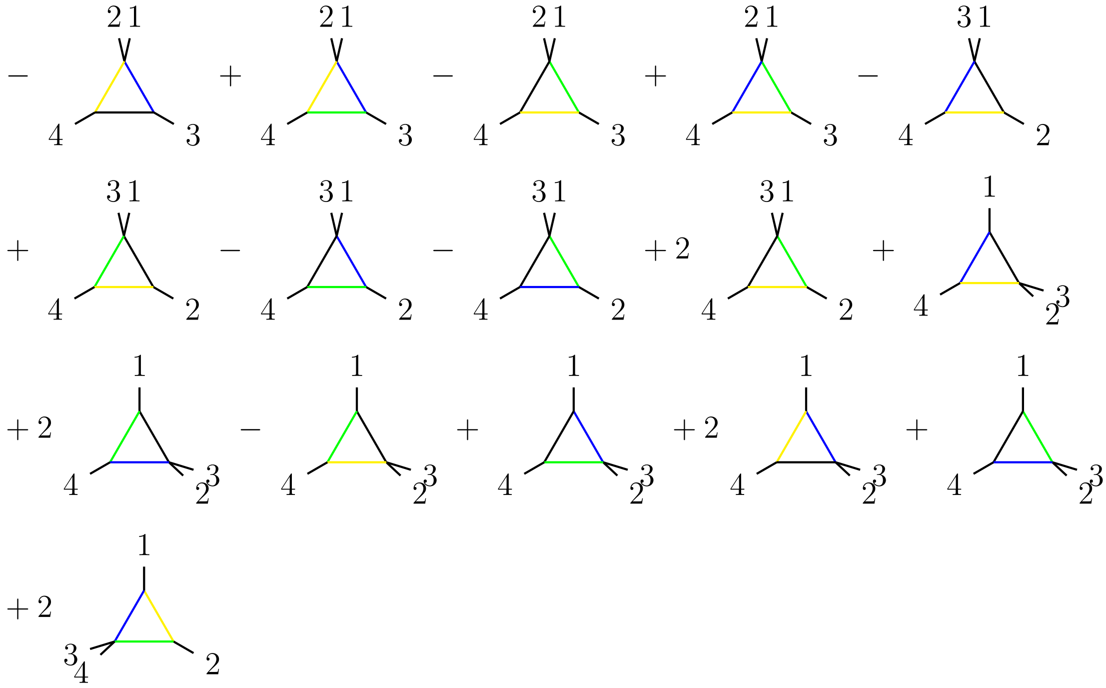


$g_{33} = -$

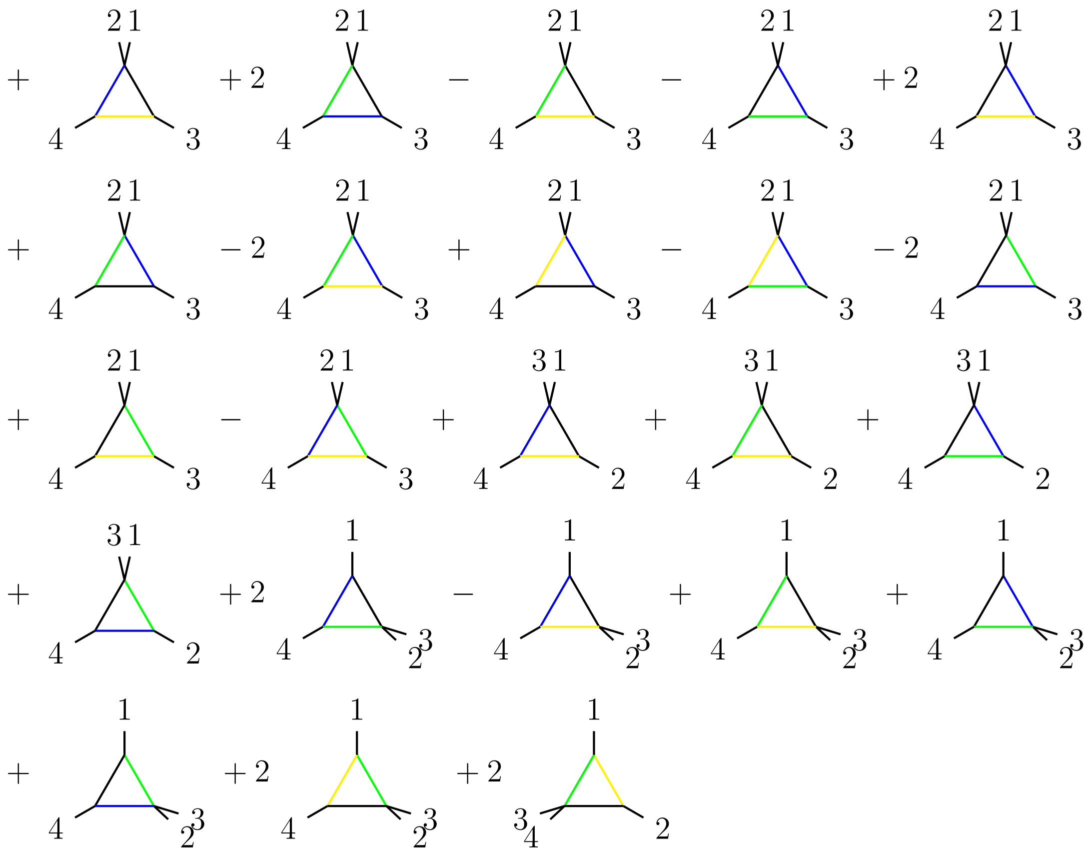


$g_{34} = -$

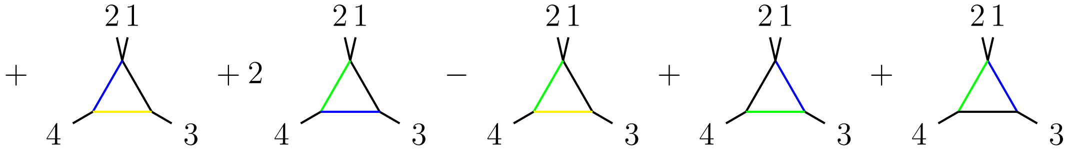


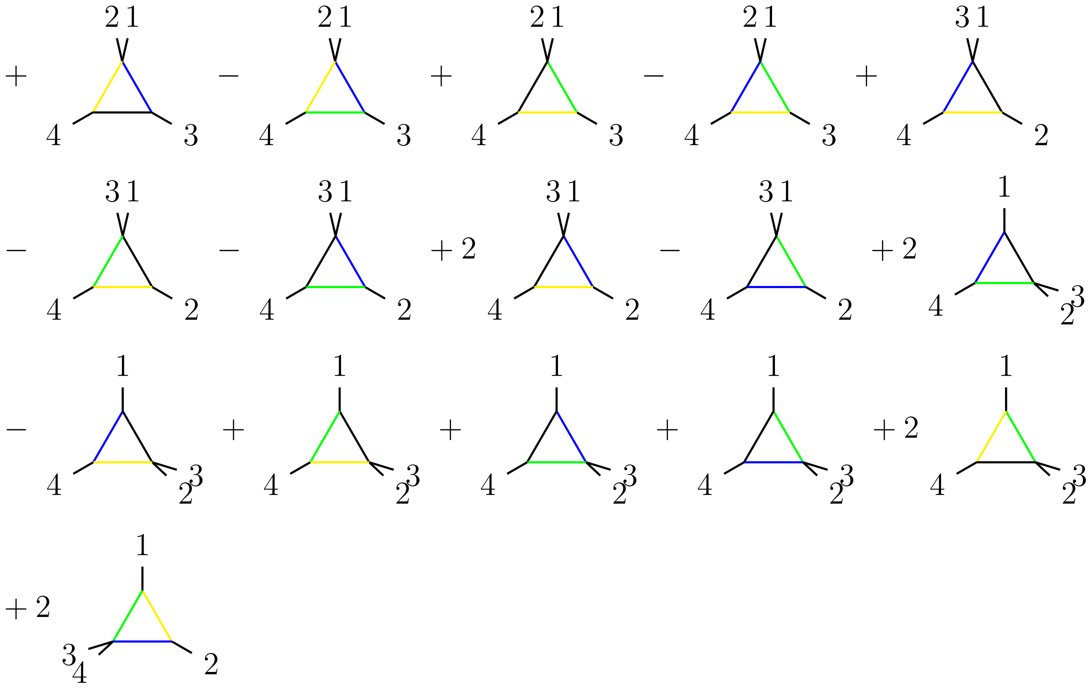


$g_{35} =$



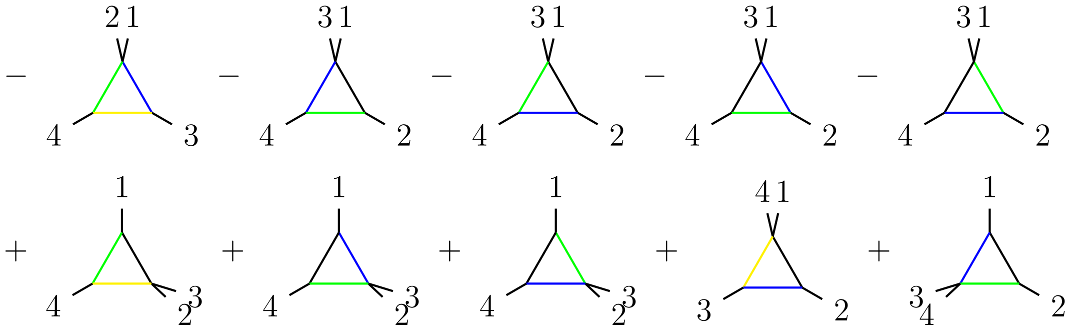
$g_{36} =$



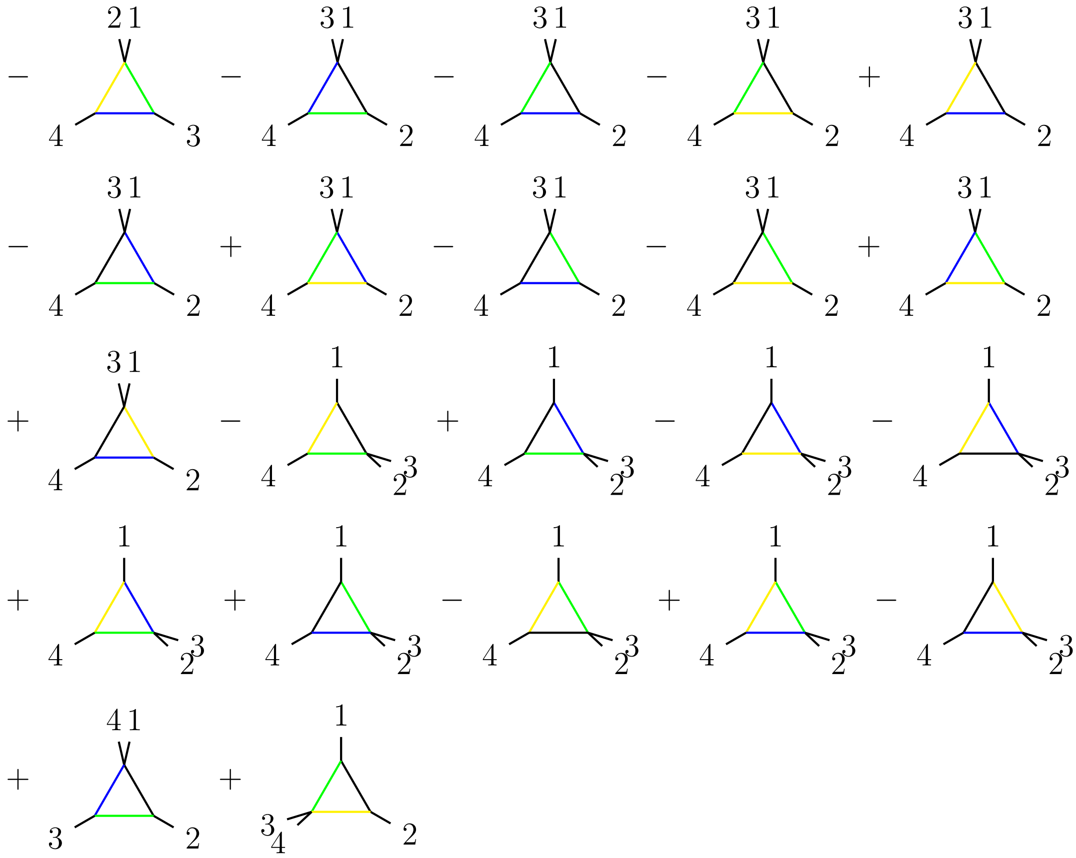


B.12 Generators of $H_2(\bar{X}_{1,4};\mathbb{Q})$

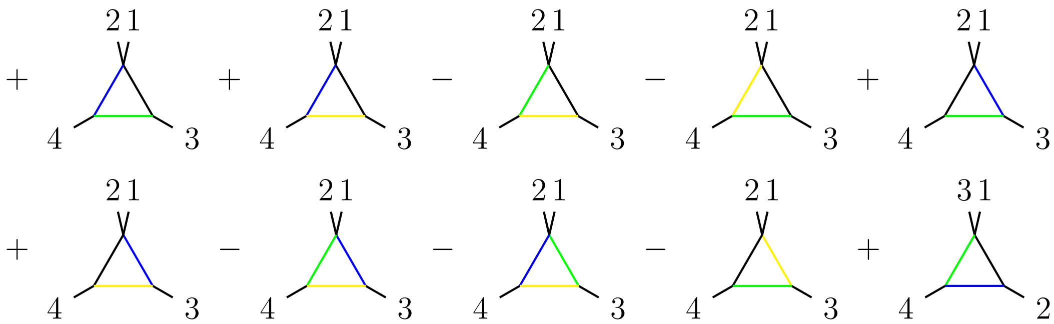
$g_1 =$

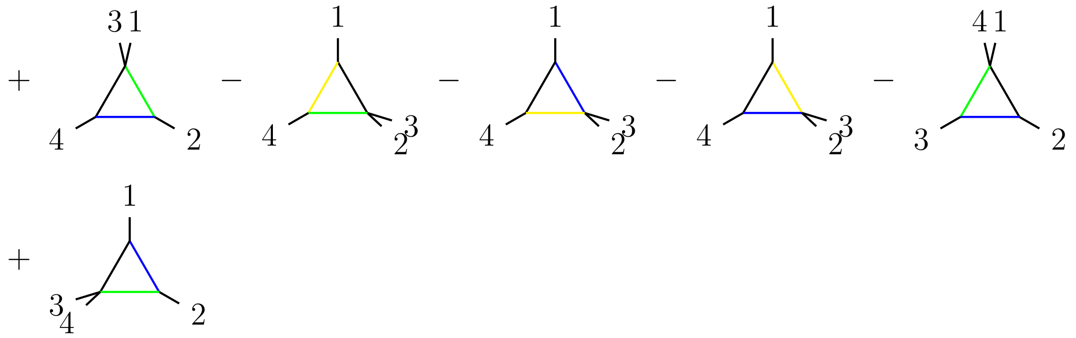


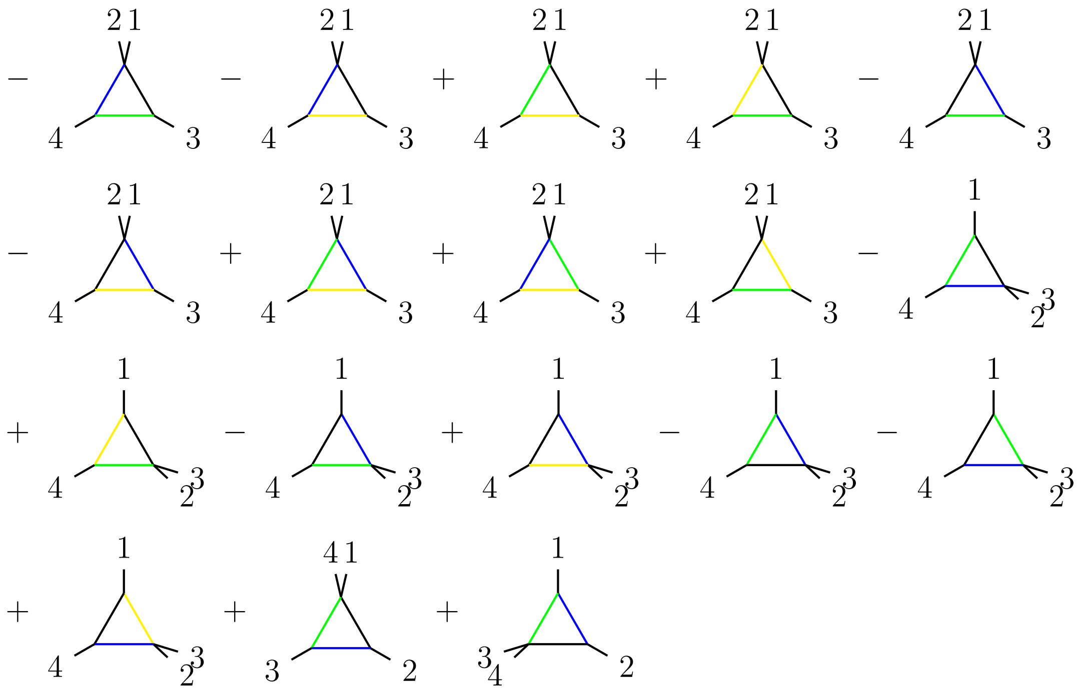
$g_2 =$

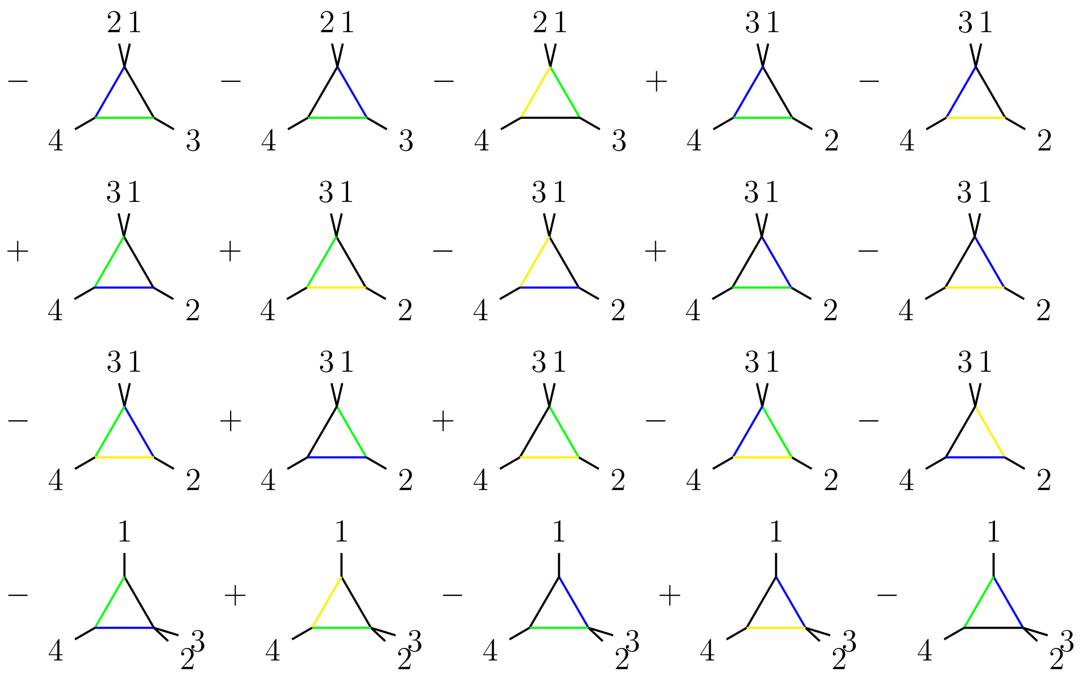


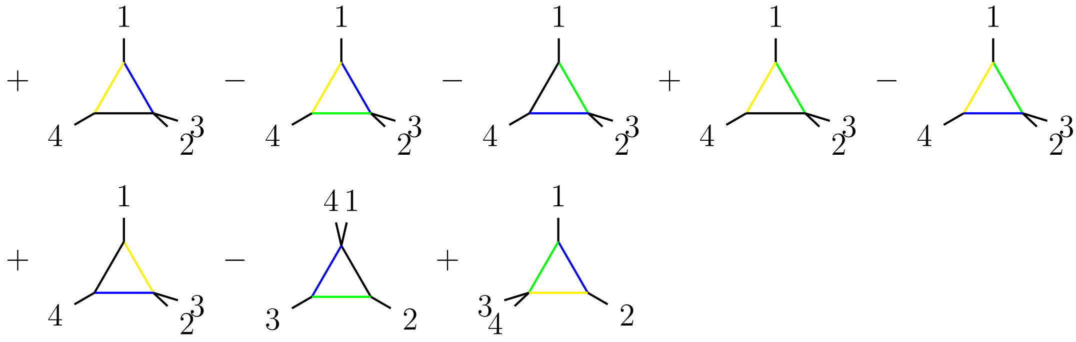
$g_3 =$

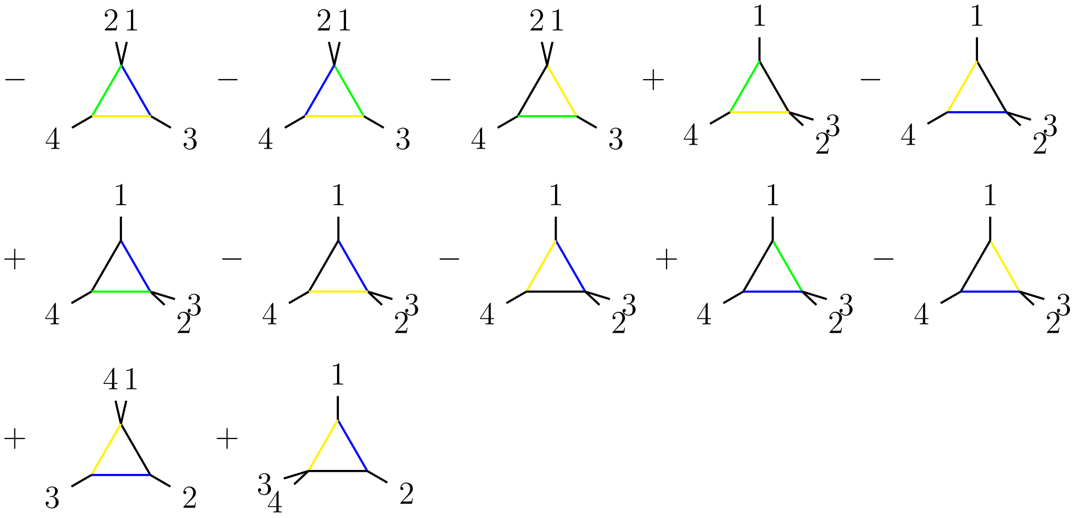


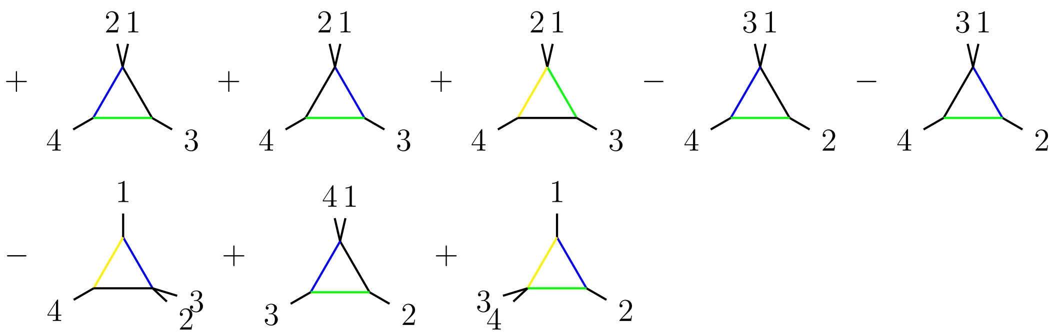


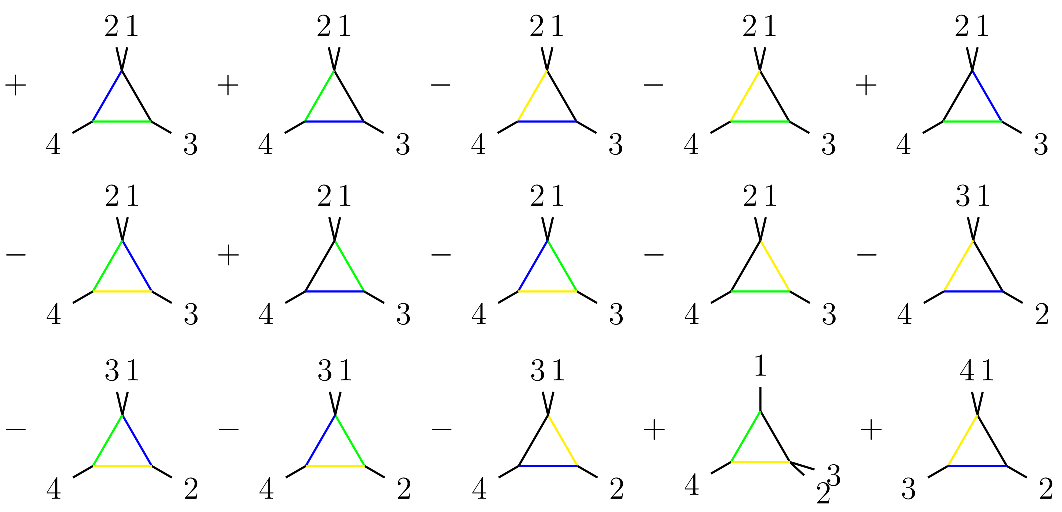
$g_4 =$ 

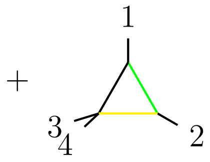
$g_5 =$ 



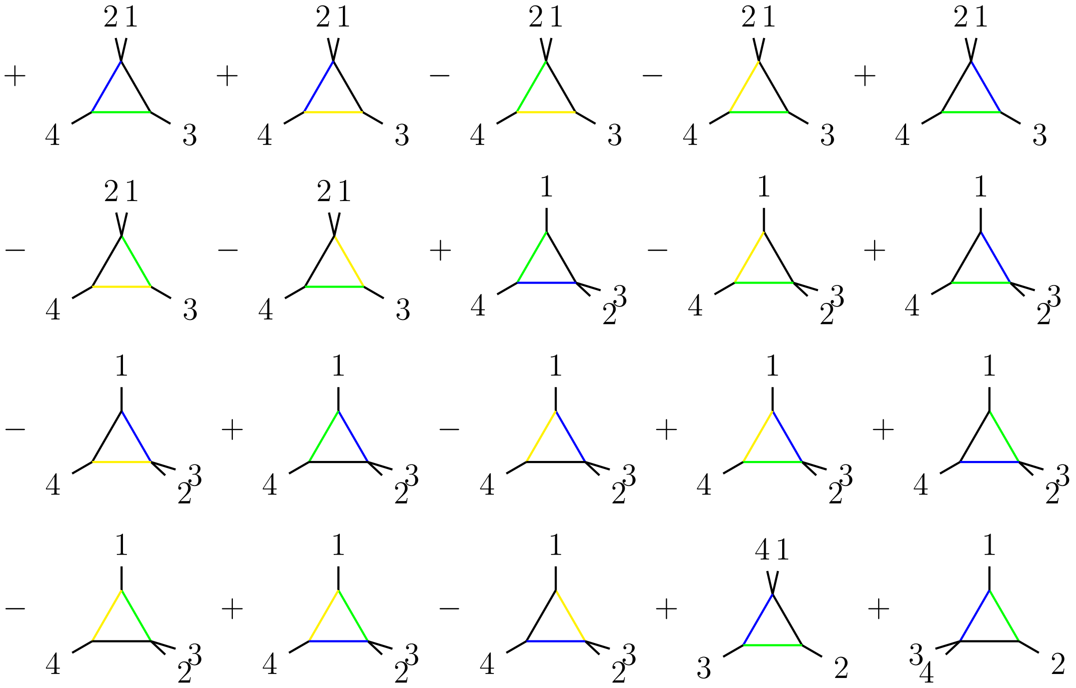
$g_6 =$ 

$g_7 =$ 

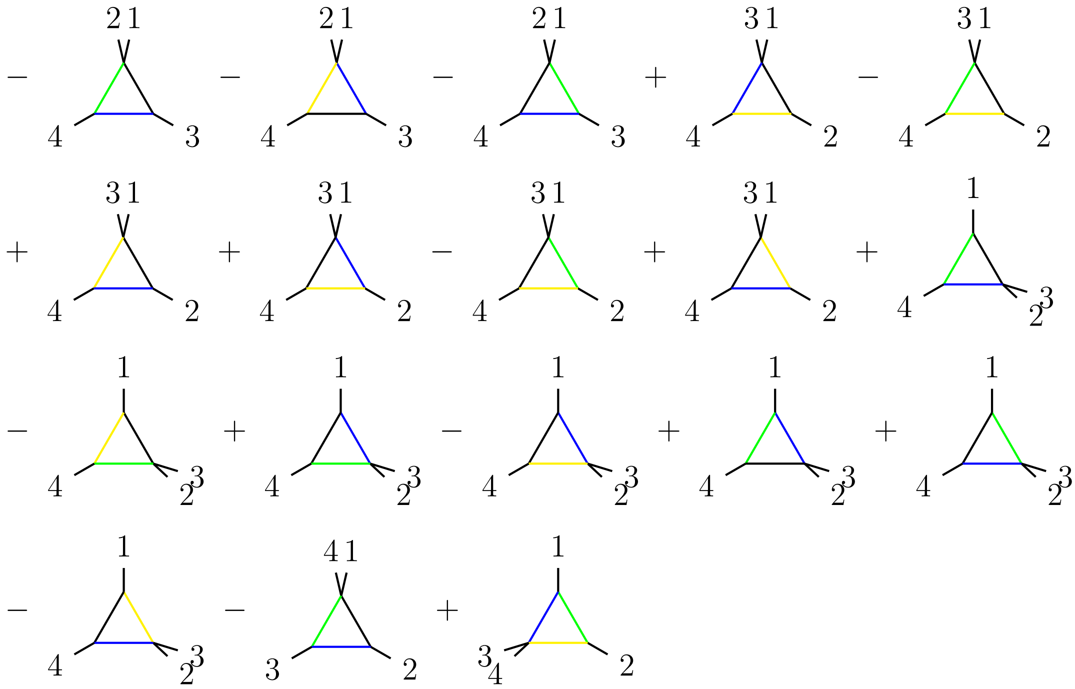
$g_8 =$ 



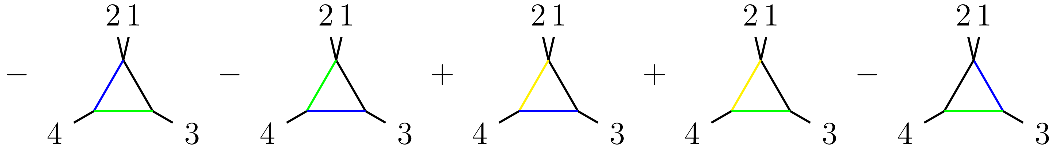
$g_9 =$

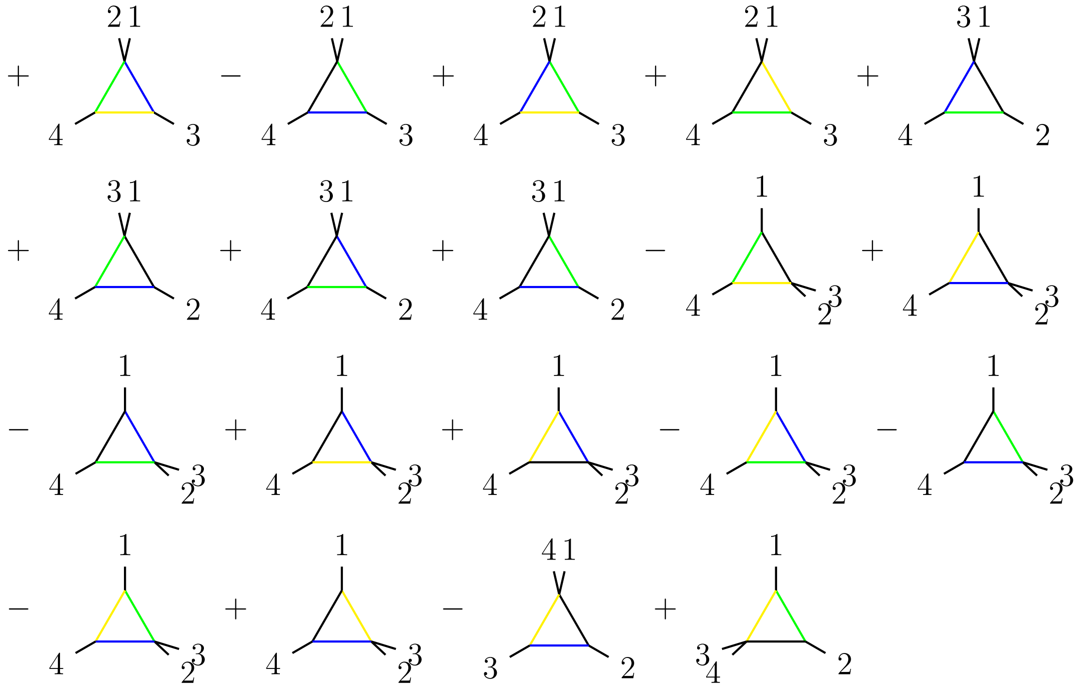


$g_{10} =$

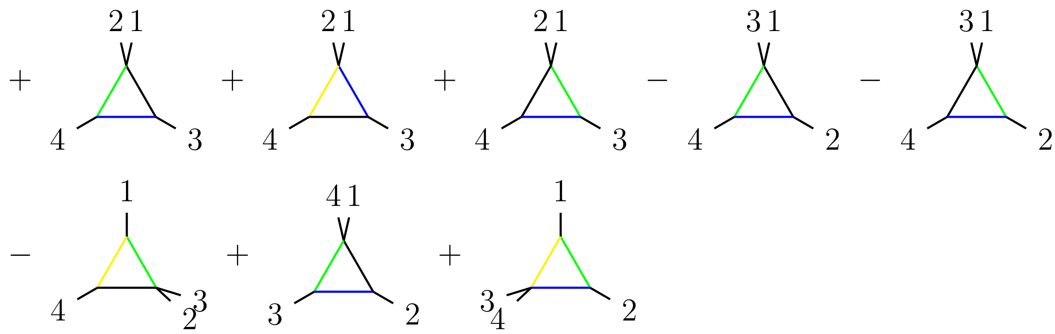


$g_{11} =$

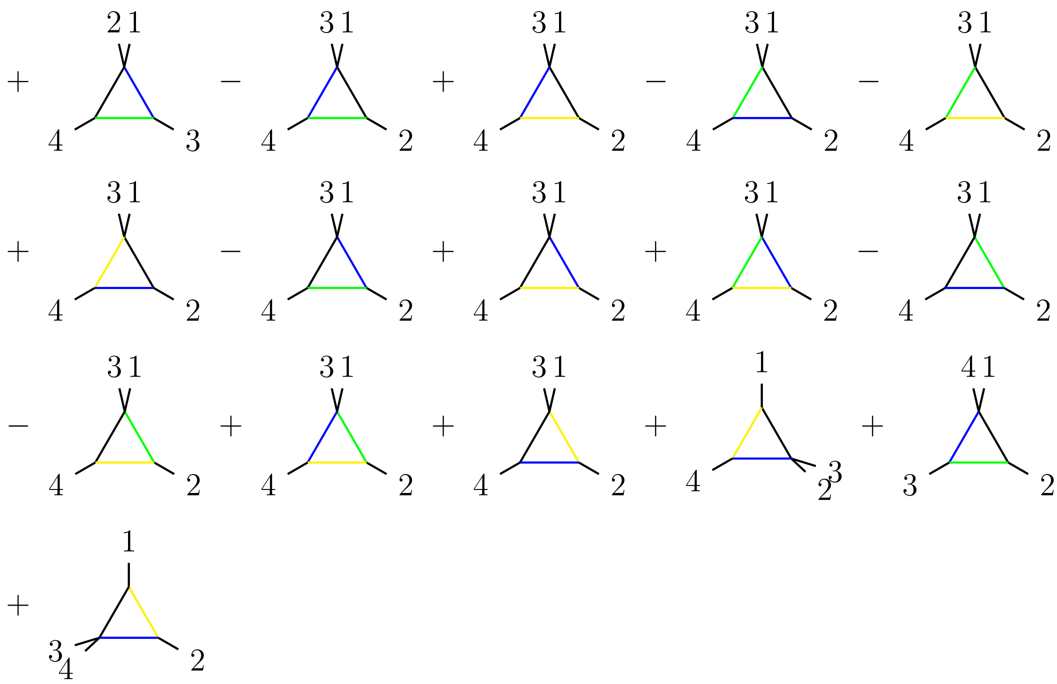




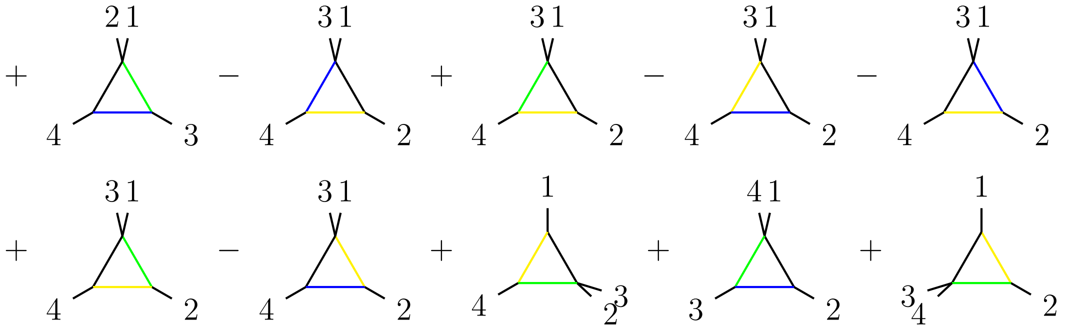
$g_{12} =$



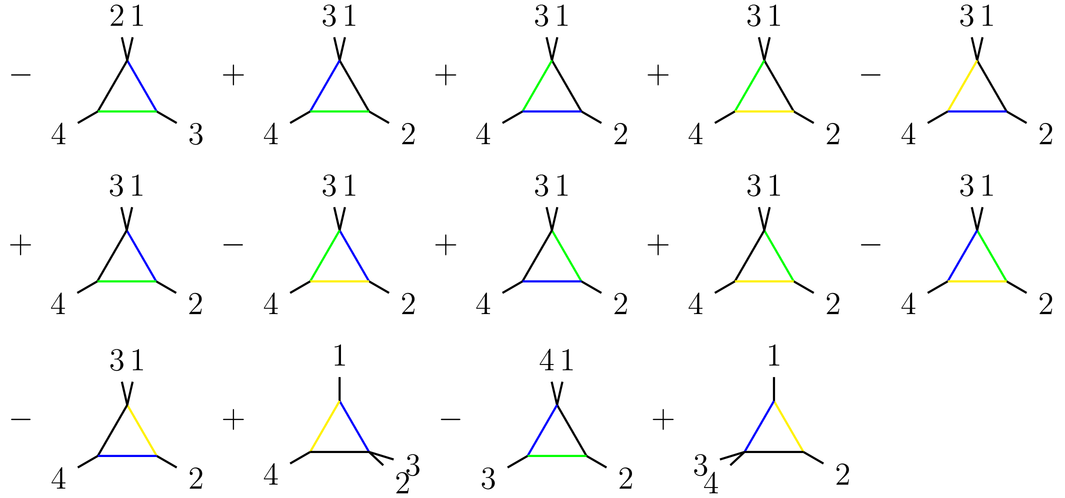
$g_{13} =$



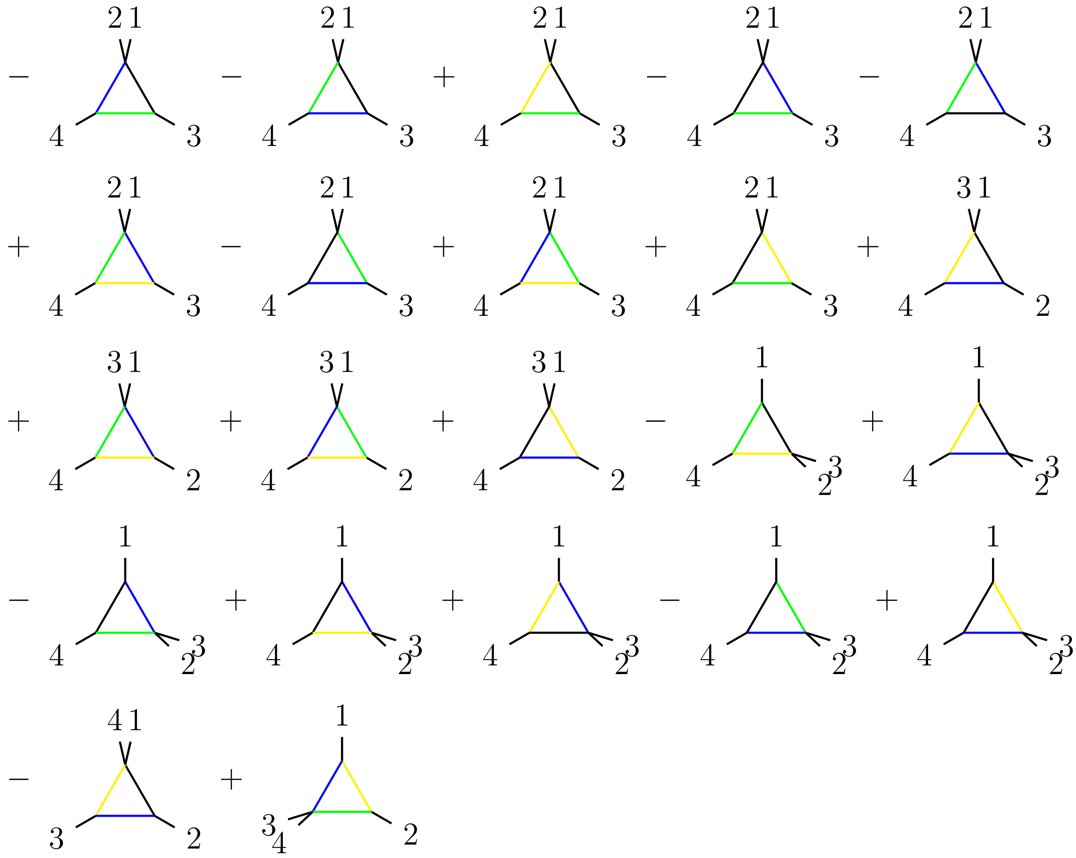
$g_{14} =$

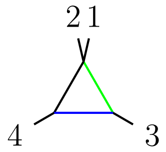
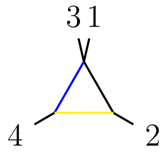
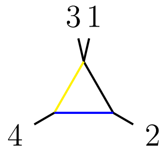
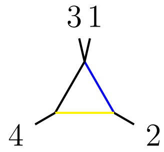
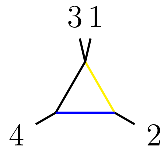


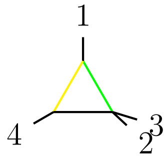
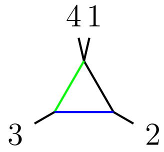
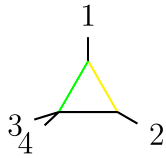
$g_{15} =$

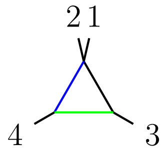
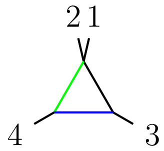
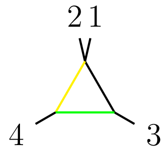
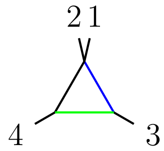
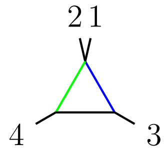


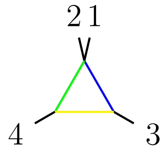
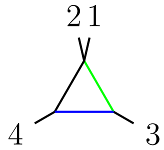
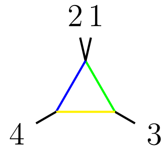
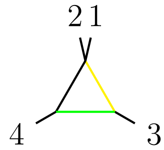
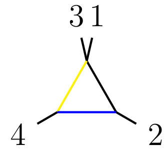
$g_{16} =$

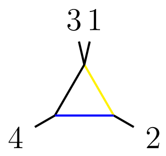
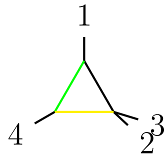
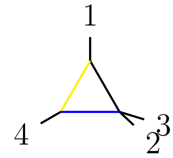
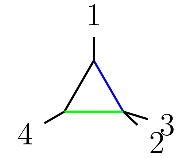
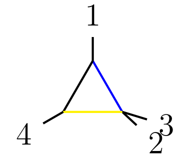


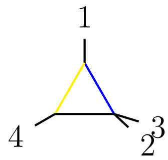
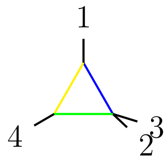
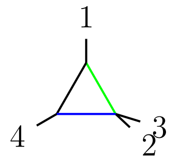
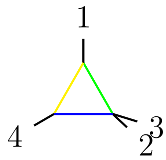
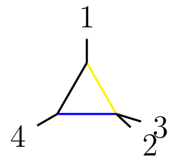
$g_{17} = -$  $+$  $+$  $+$  $+$ 

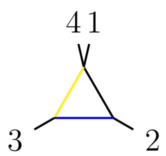
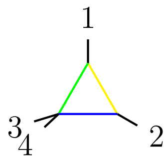
$+$  $-$  $+$ 

$g_{18} = +$  $+$  $-$  $+$  $+$ 

$-$  $+$  $-$  $-$  $-$ 

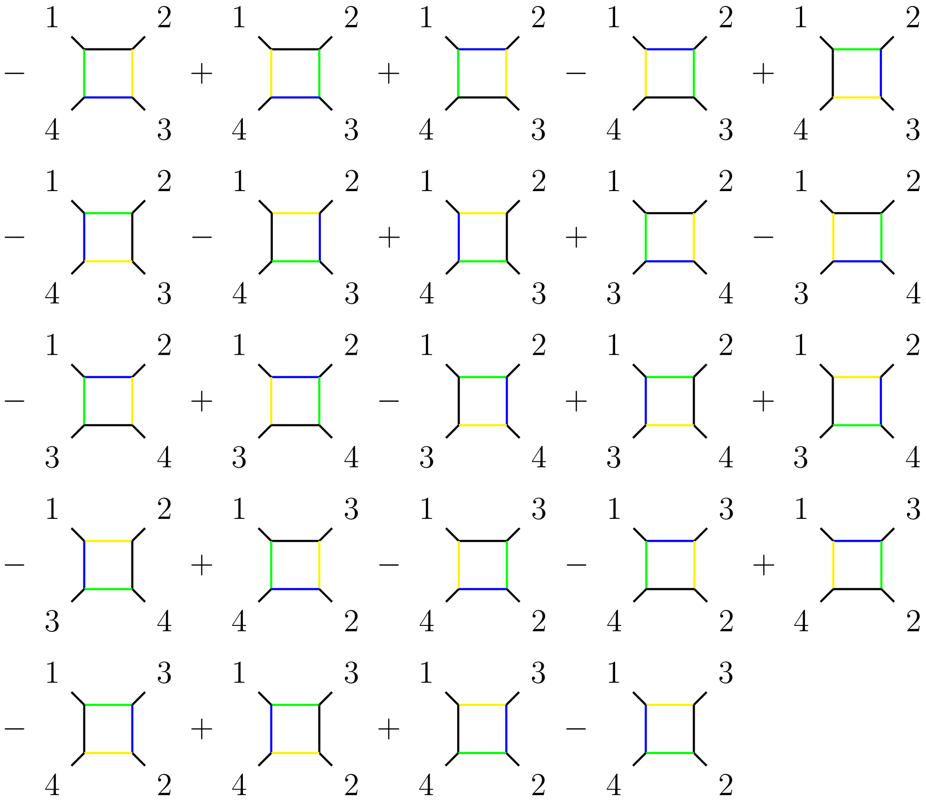
$-$  $+$  $-$  $+$  $-$ 

$-$  $+$  $+$  $+$  $-$ 

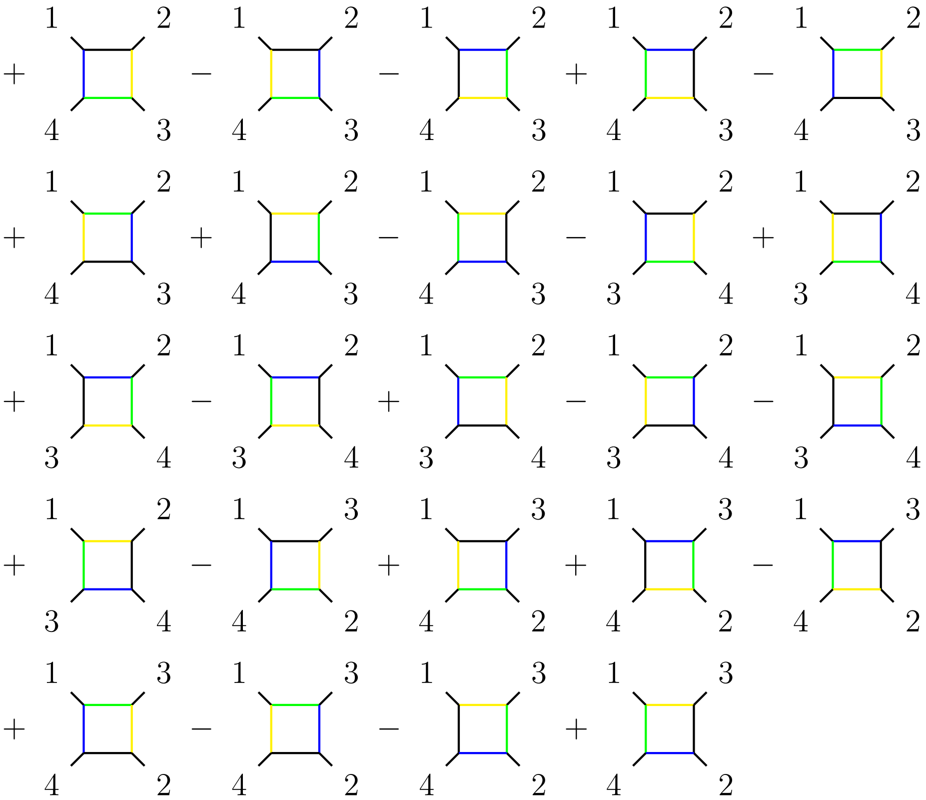
$+$  $+$ 

B.13 Generators of $H_3(\tilde{X}_{1,4};\mathbb{Q})$ and $H_3(\bar{X}_{1,4};\mathbb{Q})$

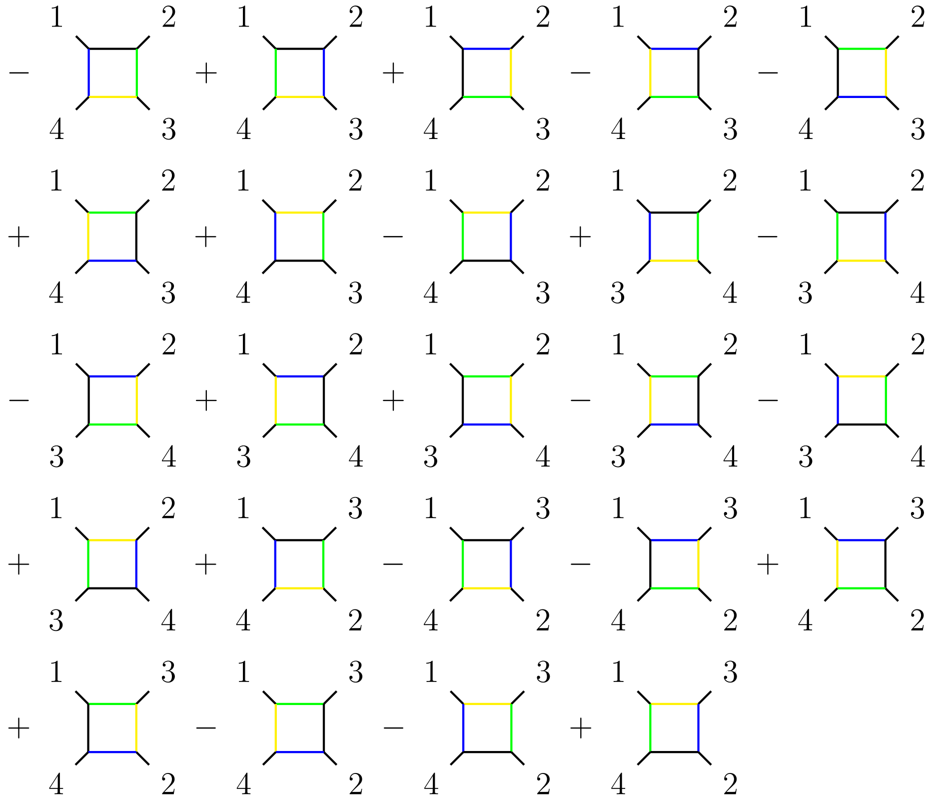
$g_1 =$



$g_2 =$



$g_3 =$



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Declaration of Authorship

I hereby confirm that I have authored this Master's thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

Berlin, June 20, 2018

Maximilian Mühlbauer