

Hopf algebra approach to Feynman diagram calculations

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Abstract

The Hopf algebra structure underlying Feynman diagrams which governs the process of renormalization in perturbative quantum field theory is reviewed. Recent progress is briefly summarized with an emphasis on further directions of research.

Keywords: perturbative renormalization, Feynman diagrams, Hopf algebra of renormalization, Rota–Baxter algebra, Spitzer’s identity, Atkinson’s theorem, Baker–Campbell–Hausdorff formula, Birkhoff decomposition, Hochschild cohomology, Dyson–Schwinger equation

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1 Introduction and Overview

Quantum field theory (QFT) by now has a long and outstandingly successful history in all theories of physics. Merging the two major revolutionary achievements of early 20th century physics, quantum mechanics and special relativity, the founding fathers of QFT were setting out for an unified description of elementary particles phenomena. Its ideas and techniques found far reaching applications in different and very distinct areas of theoretical physics, and pure and applied mathematics.

Several approaches to QFT have been developed so far. Wightman's early axiomatic [111] setting leading to constructive QFT, together with Haag's mathematically elegant and rigorous algebraic formulation of QFT in terms of von Neumann algebras [63], best describes the nowadays common believe of what should be the general physical principles underlying any QFT. Still, despite the enormous and mathematically rigorous progress which has been made using these formulations, both approaches have several problems to make fruitful contact with experimental results, whilst they give a crucial insight into the structure of free quantum fields.

The perturbative approach to quantum field theory is the most successful. Theoretical predictions of physical quantities made by using their expansion in terms of –renormalized– Feynman graphs match experimental results with a vertiginous high precision. Nevertheless, in most, if not all, of the interesting and relevant 4-dimensional quantum field theories, performing even simple perturbative calculations one cannot avoid facing ill-defined integrals. The removal of these divergences in a sound way is the process of renormalization, better known by the illustrative description of "sweeping them under the carpet". The basic idea of perturbative renormalization in QFT goes back to Kramers [26], and was successfully applied

for the first time in a 1947 seminal paper by Bethe [12], dealing with the concrete problem of the self energy contribution for the Lamb shift in perturbative quantum electrodynamics (QED). The latter can nowadays be regarded as one of the best tested physics theories. Its modern extension to the standard model of elementary particles represents one of the cornerstones of our present understanding of the physical world. Here again the perturbative treatment together with renormalization is the bread-and-butter of the practitioner in high energy physics.

Maintaining the physical principles of locality, unitarity, and Lorentz invariance, renormalization theory may be summed up by the statement that to all orders in perturbation theory the (ultraviolet) divergencies can be absorbed in a redefinition of the parameters defining the QFT. Here two distinct concepts enter, that of renormalizability, and the process of renormalization. The former distinguishes those theories with only a finite number of parameters, lending them considerably more predictive power. The process of renormalization instead works indifferently of the number of parameters.

Soon after Bethe's paper on perturbative QED, there have been several approaches to establish that quantum field theories are renormalizable in general. Dyson [41, 42] was the first to do so, using integral equations and skeleton expansions for Green's functions. His work was then continued by Salam and Weinberg. Unfortunately, this attempt failed in the first instance, due to a problem related to a particular 14th order QED graph, but could be cured later. The second approach, based on earlier work by Stückelberg and Green, was taken by Bogoliubov and Parasiuk [17, 18], using a recursive subtraction method, known as Bogoliubov's \bar{R} -map. Also their proof contained a loophole, but eventually found its final and satisfying form with the work of Hepp [64] and later Zimmermann [113]. This standard result is nowadays well-known under the name Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) renormalization prescription. Later, Epstein and Glaser [51] presented a rigorous proof of renormalizability situated in the realm of the axiomatic treatment of QFT. A fourth approach was taken by Blaer and Young [15], using the renormalization group equations, going back to a suggestion by Callan. At this point we refer the interested reader to consult the work by Caswell and Kennedy [28], Brown [26], Delamotte [40], Collins [32], and Wightman [112] for more references and details.

Notwithstanding its somewhat notorious reputation, renormalization theory, together with the gauge principle, forms the backbone of the perturbative approach to physically relevant quantum field theories. These days, the modern point of view, represented by the concept of Wilson's renormalization group, elevates it even to a fundamental structure in the understanding of high energy physics.

Unfortunately, despite its accomplishments, renormalization theory was stigmatized, especially for its lack of a firm mathematical underpinning. Indeed, examining the current introductory and advanced literature on renormalization, as it is used in everyday applications in many branches of physics, one feels the need for a more conceptual picture unifying mathematical and computational aspects. A possible reason for this situation might have been the fact that its building blocks, the (one-particle irreducible) Feynman graphs in it-

self appeared to be unrelated to a sound mathematical structure that may underlie the renormalization prescription in perturbative QFT.

Almost five decades after Bethe's work, this changed to a great extent with the original paper by Kreimer [72] introducing the notion of Hopf algebra. The ensuing work by Kreimer [73, 77] and collaborators, especially those of Broadhurst and Kreimer [20, 21, 22, 23], and Connes and Kreimer [33, 34, 35] explored this new approach both in terms of its mathematical and physical content, as well as its computational aspects. The Hopf algebraic setting captures the combinatorial and algebraic aspects of the process for renormalization by organizing the Feynman graphs into a combinatorial Hopf algebra, \mathcal{H}_F , which is a connected graded commutative bialgebra, essentially characterized by its non-cocommutative coproduct structure map. The formulation of renormalization using Hopf algebras was completed in the work of Connes and Kreimer. It gives rise to an elegant and useful disentanglement of analytic and algebraic aspects of perturbative renormalization in general QFT, affirming the remark that "Few physicists object nowadays to the idea that diagrams contain more truth than the underlying formalism[...]" made by Veltman and 't Hooft in [108].

In this review we will focus on an elementary introduction to the Hopf algebra structure on Feynman graphs combined with the description of a completely algebraic formulation of renormalization in terms of a factorization problem valid for any renormalization scheme, and based on a theorem by Atkinson and Spitzer's identity [45, 46].

Let us continue with some more details. The restricted dual of the Hopf algebra of Feynman graphs, denoted by \mathcal{H}_F^* , contains the group $\mathcal{G} := \text{char}(\mathcal{H}_F, \mathbb{C})$ of characters, that is, algebra homomorphisms from \mathcal{H}_F to the underlying base field \mathbb{C} . Feynman rules are understood as such linear and multiplicative maps, associating to each Feynman graph, seen as a collection of vertices and edges, its corresponding Feynman integral. This group of characters possesses a corresponding Lie algebra of derivations, or infinitesimal characters, $\mathcal{L} := \partial \text{char}(\mathcal{H}_F, \mathbb{C})$, which comes from a fundamental pre-Lie algebra structure on Feynman graphs.

The ill-defined Feynman integrals are plagued with ultraviolet divergences in general, and demand for a regularization prescription, where we replace the base field \mathbb{C} by a (commutative and unital) algebra A of Feynman amplitudes. Alternatively, we might consider Taylor expansions on the level of the integrands. Whichever way, this leads us to consider the space of A -valued, or regularized, linear maps $\text{Hom}(\mathcal{H}_F, A)$, which contains $\mathcal{G}_A := \text{char}(\mathcal{H}_F, A)$, the group of regularized characters, respectively its associated Lie algebra $\mathcal{L}_A := \partial \text{char}(\mathcal{H}_F, A)$. As a principal example serves dimensional regularization, where $A := \mathbb{C}[\varepsilon^{-1}, \varepsilon]$, the field of Laurent series. In this context perturbative renormalization finds a compact formulation as a factorization problem in the group \mathcal{G}_A , to wit, the algebraic Birkhoff decomposition of Feynman rules [33, 34]. The initial proof of the Connes–Kreimer factorization of regularized Feynman rules uses the property that Laurent series actually form a commutative Rota–Baxter algebra [9, 99] with the pole part projection, $R := R_{ms}$, as linear Rota–Baxter operator (minimal subtraction scheme map) fulfilling the Rota–Baxter relation (of weight 1)

$$R(x)R(y) + R(xy) = R(R(x)y + xR(y)), \quad \forall x, y \in A.$$

The linearity of R permits to define a unital, but now non-commutative complete filtered Rota–Baxter algebra structure on the space $\text{Hom}(\mathcal{H}_F, A)$, with convolution as associative product. One of the fundamental results in the realm of commutative Rota–Baxter algebras is Spitzer’s classical identity [45, 46, 98, 106], and using its generalization to non-commutative Rota–Baxter algebras, together with Atkinson’s factorization theorem [7] for Rota–Baxter algebras, one can show that the multiplicative factorization of Connes–Kreimer follows from an additive decomposition through the exponential map [45, 46]. Hereby we realize Bogoliubov’s \bar{R} -map as a special case of Spitzer’s identity.

We hope that this brief review will guide the reader to crucial aspects of the recent developments related to the Hopf algebraic description of renormalization theory. The long list of references is meant to indicate the rich spectrum of research directions triggered by this approach. The modest mathematical style, i.e. we do not strive for a rigorous theorem-proof presentation might help the interested and novice reader to get a glimpse of the new aspects which opened with the Hopf algebra point of view on perturbative renormalization. Some of the remarks made during the expository writing indicate points to be further developed. But, we should underline that this article is neither meant to be an introduction to (perturbative) quantum field theory nor to renormalization theory in general. Rather, we would like to focus on the by now well-understood and established combinatorial-algebraic picture that makes renormalization theory in perturbative QFT such a challenging and venerable subject from both, mathematical and physical perspectives. After reading this article and going back to the vast existing physics literature on renormalization theory in all its facets, the reader may get an idea of the interesting open questions related to its Hopf algebraic description.

In the following we will comment on assorted references with respect to their research directions in this field, in the hope to facilitate access to this developing subject. Due to the review character of this work and limited space none of these topics could be treated in full detail. We start by mentioning two recent papers [55, 89] devoted in great detail to the general Hopf algebra structure in renormalization theory. A more mathematical, but shorter summary was given in [19].

The initial discovery of the Hopf algebra structure grew out of a study of the number-theoretic properties of graphs with many subdivergences, see [80], which provides an overview of some of the results up to the year 1999, including the link between knot theory, Feynman graphs and number theory [24].

The original work of Kreimer, and Connes and Kreimer [33, 34, 35, 36, 72, 73] explores and settles the Hopf algebraic formulation of renormalization for general perturbative QFT, and links it to non-commutative geometry. It thereby establishes the Birkhoff decomposition for Feynman rules giving rise to an unexpected correspondence with the Riemann–Hilbert problem. In [31, 53, 54, 57, 69, 71, 75, 97] further details were given. Malyshev’s work [87] shows the general character of Connes’ and Kreimer’s combinatorial Hopf algebra, applying it to Riemann surfaces in the context of ribbon Feynman graphs.

The link to Connes’ non-commutative geometry becomes evident in terms of a Hopf algebra of non-planar rooted trees [33], solving a universal problem in Hochschild cohomology,

and forming the role model for the Hopf algebraic structure of renormalization. This work renewed considerably the interest in Hopf algebraic aspects of combinatorics such as rooted tree Hopf algebras. In this context one must point out the work of G.-C. Rota [102] and his school, especially Rota's and A. Joni's seminal work [67] from the late 1970ies, forming the starting point for the theory of incidence Hopf algebras, further developed in [50, 104], see also [55]. Holtkamp [66]³ showed that the non-commutative version of Connes–Kreimer's Hopf algebra of rooted trees is isomorphic to Loday and Ronco's [86]. Aguiar and collaborators explored in more detail rooted tree Hopf algebras [3, 4, 5]. Hoffman in [65] improved a result of Panaite [95], showing the isomorphism between the dual of Connes–Kreimer's Hopf algebra and Grossmann–Larson's [59] rooted tree Hopf algebra (see also [56]). Brouder [25] explored the relation to Butcher's seminal work on Runge–Kutta integration methods [27]. Turaev in [109, 110] extended some combinatorial aspects of the Connes–Kreimer results, especially with respect to the notion of pre-Lie coalgebras, and thereby also gave a neat description of Connes–Kreimer's rooted tree Hopf algebra.

Chapoton and Livernet [30] described free pre-Lie algebras in terms of rooted tree operads. Mencattini and Kreimer [91, 92] further analyzed the insertion and elimination Lie and pre-Lie algebraic structures of Feynman graphs [36] in terms of infinite matrix representations.

In [45, 46] the meaning of the Rota–Baxter relation in the context of Connes–Kreimer's Birkhoff decomposition is investigated in detail, pointing out some parallels to the theory of classical integrable systems [8, 105]. It thereby provides the algebraic underpinning for the factorization in terms of complete filtered Rota–Baxter algebras, Spitzer's identity and Atkinson's multiplicative decomposition theorem for Rota–Baxter algebras. This work was further extended in [48, 49] describing the combinatorics of renormalization in terms of unipotent triangular matrix representations, and their factorization capturing the process of renormalization analogously to the Birkhoff decomposition of Connes–Kreimer. An interesting application of renormalization techniques and Rota–Baxter algebras as described here to the iteration of symbols of pseudodifferential operators can be found in [90].

The work of Broadhurst and Kreimer [20, 21, 22, 23] develops many computational and physics aspects. They show how to use the coproduct structure of the Hopf algebra to efficiently compute the forest formula and use the Hochschild cohomology of the algebra to resum the perturbative series. The latter two of the aforementioned references hence form the starting point for the latest work of Kreimer [78, 84], and Bergbauer and Kreimer [11], putting emphasis on Hochschild cohomology of Hopf algebras as a source of locality, the Dyson–Schwinger equations, and even the Slavnov–Taylor identities for the couplings in generic gauge theories.

Finally, the authors in [58, 88, 103] started to analyze some aspects of renormalization group calculations in the Hopf algebra context.

The work [13, 14] (see also [94]) is of more computational character, indicating the efficiency of the use of Hopf algebras in perturbative renormalization.

³See also L. Foissy's PhD-thesis.

Several people [10, 96] investigated the link between the Hopf algebra of renormalization to the most rigorous approach to renormalization in perturbative QFT, provided by the Epstein–Glaser prescription.

Recently, progress has been made in the mathematical context of number theory, and motivic structures of Feynman integrals. The notion of equisingular connections was used to explore Tannakian categories and Galois symmetries in the spirit of differential Galois theory in [37, 38, 39]. Underlying the notion of an equisingular connection is the locality of counterterms, which itself results from Hochschild cohomology. The resulting Dyson–Schwinger equation allows for gradings similar to the weight- and Hodge filtrations for the polylogarithm [74, 79]. More concretely, the motivic nature of primitive graphs has been established very recently by Bloch, Esnault and Kreimer [16].

Let us briefly outline the organization of the paper. Section 2 introduces briefly the basic Lie and Hopf algebra structures in perturbative renormalization, including the pre-Lie composition of Feynman graphs and Bogoliubov’s \bar{R} -operation. The next section uses perturbative QED as a simple example to manifest the aforementioned notions. In Section 4 we formulate the process of renormalization in perturbative QFT as a factorization problem in purely algebraic terms. Emphasis is put on the freedom in choosing a particular regularization prescription, captured via the notion of commutative unital Rota–Baxter algebra. Section 5 outlines the use of the Birkhoff decomposition introduced in the former section on the level of diffeomorphisms of coupling constants, in the realm of dimensional regularization together with the minimal subtraction scheme as a particular useful renormalization prescription. The review ends with a brief section on the role of Hochschild cohomology in perturbative renormalization. In an appendix we collect some general facts about Rota–Baxter algebras as they form the main ingredient for Section 4.

2 From the Lie and Hopf algebras of graphs to Bogoliubov’s formula

In this section we describe the elementary Lie and Hopf algebra structures underlying perturbation theory. The reader looking for a mathematical rigorous and detailed presentation of Hopf algebras and related aspects is referred to the standard texts such as [1, 52, 68, 107] (see also [55, 89]).

Let \mathbb{K} be a field of characteristic zero. All \mathbb{K} -algebras, denoted by a triple (A, m, η) , where A is a \mathbb{K} -vector space with a product $m : A \otimes A \rightarrow A$ and a unit map $\eta : \mathbb{K} \rightarrow A$, are supposed to be associative. Similarly for coalgebras, denoted by the triple (C, Δ, \bar{e}) , where the coproduct map $\Delta : C \rightarrow C \otimes C$ fulfills coassociativity, i.e. $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$, and $\bar{e} : C \rightarrow \mathbb{K}$ is the counit map. The identity will be denoted by \mathbb{I} . All algebra homomorphisms are supposed to be unital. A bialgebra, denoted by a quintuple $(H = \bigoplus_{i=0}^{\infty} H_{(i)}, m, \eta, \Delta, \bar{e})$ consists of an algebra and coalgebra structure in a compatible way. Here, $\Delta : H \rightarrow H \otimes H$ is the coproduct, $m : H \otimes H \rightarrow H$ the product. These maps together with the counit

$\bar{e} : H \rightarrow \mathbb{K}$ fulfil the standard bialgebra axioms. See the above general references for details. It is called connected graded if $H_{(i)}H_{(j)} \subset H_{(i+j)}$ and $\Delta(H_{(i)}) \subset \bigoplus_{j+k=i} H_{(j)} \otimes H_{(k)}$, and if $\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$ and $H_{(0)} = \mathbb{K}\mathbb{I}$. The counit simply is $\bar{e}(\mathbb{I}) = 1 \in \mathbb{K}$ and $\bar{e} = 0$ on $\bigoplus_{i=1}^{\infty} H_{(i)}$.

We call $\ker \bar{e}$ the augmentation ideal of H and denote by P the projection $H \rightarrow \ker \bar{e}$ onto the augmentation ideal, $P = \text{id} - \eta \bar{e}$. Furthermore, we use Sweedler's notation $\Delta(h) = \sum h' \otimes h''$ for the coproduct. Let us define

$$\text{Aug}^{(k)} = \left(\underbrace{P \otimes \cdots \otimes P}_{k \text{ times}} \right) \Delta^{k-1}, \quad H \rightarrow \{\ker \bar{e}\}^{\otimes k}, \quad (1)$$

as a map into the k -fold tensor product of the augmentation ideal. Here, Δ^{k-1} is defined inductively by $\Delta^0 := \text{id}$, and $\Delta^n := (\Delta^{n-1} \otimes \text{id}) \circ \Delta$ for $n > 0$. We let

$$H^{(k)} = \ker \text{Aug}^{(k+1)} / \ker \text{Aug}^{(k)}, \quad (2)$$

$\forall k \geq 1$. All bialgebras considered here are bigraded in the sense that

$$H = \bigoplus_{i=0}^{\infty} H_{(i)} = \bigoplus_{k=0}^{\infty} H^{(k)}, \quad (3)$$

where $H_{(k)} \subset \bigoplus_{j=1}^k H_{(j)}$ for all $k \geq 1$. $H_{(0)} \simeq H^{(0)} \simeq \mathbb{K}$.

While these algebraic notions may seem rather abstract, they indeed govern the structure of quantum field theory. To understand how, we have first to study the pre-Lie algebra structure of one-particle irreducible Feynman graphs.

2.1 The Pre-Lie Structure of Feynman Graphs

For each quantum field theory, we have an underlying free theory which provides propagators and hence Feynman rules for edges in Feynman graphs⁴. The request for local interactions and a renormalizable theory then gives us Feynman rules for interactions. With those graphs come the sets of one-particle irreducible (1PI) graphs contributing to a chosen amplitude. The amplitudes are distinguished by the external fields, asymptotically free fields represented by external edges in the graphs. We call this an external leg structure, denoted by \underline{r} . For a renormalizable theory, there is a finite number of such external leg structures, one for each monomial in the Lagrangian.

For each such Feynman graph we hence have vertices as well as internal and external edges. External edges are edges which have an open end not connected to a vertex. They indicate the particles participating in the scattering amplitude under consideration and each such edge carries the quantum numbers of the corresponding free field. The internal edges and vertices form a graph in their own right. For an internal edge, both ends of the edge are connected to a vertex. For a graph Γ we denote by $\Gamma^{[0]}$ its set of vertices and by

⁴We assume that the reader has already seen a Feynman graph, otherwise he might take a brief look on Eq. (9).

$\Gamma^{[1]} := \Gamma_{\text{int}}^{[1]} \cup \Gamma_{\text{ext}}^{[1]}$ its set of internal and external edges. Furthermore, $\omega_{\underline{r}}$ is the number of space-time derivatives appearing in the corresponding monomial in the Lagrangian.

We are considering 1PI Feynman graphs. By definition a graph Γ is 1PI if and only if all graphs, obtained by removal of any one of its internal edges, are still connected. Such 1PI graphs are naturally graded by their number of independent loops, the rank of their first homology group $H_{[1]}(\Gamma, \mathbb{Z})$. We write $|\Gamma|$ for this degree of a graph Γ . Note that $|\mathbf{res}(\Gamma)| = 0$, where we let $\mathbf{res}(\Gamma)$, called the residue of Γ , be the graph obtained when all edges in $\Gamma_{\text{int}}^{[1]}$ shrink to a point. The graph we obtain in this manner consists of a single vertex, to which the edges $\Gamma_{\text{ext}}^{[1]}$ are attached. In case the initial graph was a self-energy graph, we regard its residue as a single edge. We denote the set of all external leg structures \underline{r} by \mathfrak{R} . For a renormalizable QFT it consists of the edges and vertices corresponding to the monomials in the Lagrangian.

Having specified free quantum fields and local interaction terms between them, one immediately obtains the set of 1PI graphs, and can consider for a given external leg structure \underline{r} the set $M_{\underline{r}}$ of graphs with that external leg structure. The Green's function for the corresponding amplitude is then obtained as the evaluation under the Feynman rules of the formal sum

$$\Gamma^{\underline{r}} := 1 + \sum_{\mathbf{res}(\Gamma)=\underline{r}} \alpha^{|\Gamma|} \frac{\Gamma}{\text{sym}(\Gamma)}, \quad (4)$$

where we divide by the symmetry factor $\text{sym}(\Gamma)$ and α is a small parameter like (the square of) a coupling constant. These sums exhibit rich structure thanks to the algebraic structures of the single graphs [84] to be discussed now.

For a renormalizable theory, we can define a superficial degree of divergence

$$\omega = \sum_{\underline{r} \in \Gamma_{\text{int}}^{[1]} \cup \Gamma^{[0]}} \omega_{\underline{r}} - 4|H_{[1]}(\Gamma, \mathbb{Z})|, \quad (5)$$

for each such external leg structure: $\omega(\Gamma) = \omega(\Gamma')$ if $\mathbf{res}(\Gamma) = \mathbf{res}(\Gamma')$, all graphs with the same external leg structure have the same superficial degree of divergence. Only for a finite number of distinct external leg structures $\underline{r} \in \mathfrak{R}$ will this degree indeed signify a divergence. Our first observation is that there is a natural pre-Lie algebra structure on 1PI graphs.

To see this, we define a bilinear operation on graphs

$$\Gamma_1 * \Gamma_2 = \sum_{\Gamma} n(\Gamma_1, \Gamma_2; \Gamma) \Gamma, \quad (6)$$

where the sum is over all 1PI graphs Γ . Here, $n(\Gamma_1, \Gamma_2; \Gamma)$ is a section coefficient which counts the number of ways a subgraph Γ_2 in Γ can be reduced to a point such that Γ_1 is obtained. The above sum is evidently finite as long as Γ_1 and Γ_2 are finite graphs, and the graphs which contribute necessarily fulfill $|\Gamma| = |\Gamma_1| + |\Gamma_2|$ and $\mathbf{res}(\Gamma) = \mathbf{res}(\Gamma_1)$.

One then has:



Theorem 2.1. *The operation $*$ is pre-Lie:*

$$[\Gamma_1 * \Gamma_2] * \Gamma_3 - \Gamma_1 * [\Gamma_2 * \Gamma_3] = [\Gamma_1 * \Gamma_3] * \Gamma_2 - \Gamma_1 * [\Gamma_3 * \Gamma_2]. \quad (7)$$

This is evident when one rewrites the $*$ product in suitable gluing operations, using the dichotomy of inserting in nested or disjoint manner. See [44, 76, 81, 82] for more details.

Note that the equation claims that the lack of associativity in the bilinear operation $*$ is invariant under permutation of the elements indexed by 2, 3. This suffices to show that the anti-symmetrization of this map fulfils the Jacobi identity. Hence we get a Lie algebra \mathcal{L} by anti-symmetrizing this operation:

$$[\Gamma_1, \Gamma_2] = \Gamma_1 * \Gamma_2 - \Gamma_2 * \Gamma_1. \quad (8)$$

This Lie algebra is graded and of finite dimension in each degree. Let us look at a couple of examples for pre-Lie products. We take graphs from quantum electrodynamics (QED) as a rather self-evident example. For the graphs  and  with residues $\text{res}(\text{circle with wavy lines}) = \text{wavy line}$ respectively $\text{res}(\text{circle with dashed line}) = \text{dashed line}$, we find

$$\text{circle with wavy lines} * \text{circle with dashed line} = 2 \text{ circle with wavy and dashed lines} . \quad (9)$$

Together with \mathcal{L} one is led to consider the dual of its universal enveloping algebra $\mathcal{U}(\mathcal{L})$ using the theorem of Milnor and Moore [93]. For this we use the above grading by the loop number.

This universal enveloping algebra $\mathcal{U}(\mathcal{L})$ is build from the tensor algebra

$$\mathbf{T} = \bigoplus_k T^k, \quad T^k = \underbrace{\mathcal{L} \otimes \dots \otimes \mathcal{L}}_{k \text{ times}}, \quad (10)$$

by dividing out the ideal generated by the relations

$$a \otimes b - b \otimes a = [a, b] \in \mathcal{L}. \quad (11)$$

Note that in $\mathcal{U}(\mathcal{L})$ we have a natural concatenation product m_* . Even more, $\mathcal{U}(\mathcal{L})$ carries a natural Hopf algebra structure with this product. For that, the Lie algebra \mathcal{L} furnishes the primitive elements:

$$\Delta_*(a) = a \otimes 1 + 1 \otimes a, \quad \forall a \in \mathcal{L}. \quad (12)$$

It is by construction a connected finitely graded Hopf algebra which is cocommutative but not commutative.

We can then consider its graded dual which will be a Hopf algebra $\mathcal{H}_F(m, \eta, \Delta, \bar{e})$ which is commutative but not cocommutative. One finds the coproduct Δ upon using a Kronecker pairing

$$\langle Z_\Gamma, \delta_{\Gamma'} \rangle = \begin{cases} 1, & \Gamma = \Gamma' \\ 0, & \text{else.} \end{cases} \quad (13)$$

From there, one determines all other structure maps with ease, demanding that

$$\langle Z_{[\Gamma_2, \Gamma_1]}, \delta_\Gamma \rangle = \langle Z_{\Gamma_1} \otimes Z_{\Gamma_2} - Z_{\Gamma_2} \otimes Z_{\Gamma_1}, \Delta(\delta_\Gamma) \rangle. \quad (14)$$

In the above, we distinguished carefully between graphs Γ as generators of the Lie algebra, denoted by Z_Γ , and graphs Γ as generators of the Hopf algebra, denoted by δ_Γ . The Lie algebra of graphs exponentiates to the character group of the Hopf algebra as explained below, eventually leading to Birkhoff factorization in that group.

The space of primitives of $\mathcal{U}(\mathcal{L})$ is in one-to-one correspondence with the set $\text{Indec}(\mathcal{H}_F)$ of indecomposables of \mathcal{H}_F , which is the linear span of its generators.

\mathcal{H}_F is a connected graded commutative Hopf algebra which describes renormalization theory. It operates on the superficially divergent 1PI Feynman graphs of the theory. The residues of these graphs are in one-to-one correspondence with the terms in the Lagrangian of a given theory. Often it is the case that several terms in a Lagrangian correspond to graphs with the same number and type of external legs, but match to different form-factor projections of the graph. In such cases, the above approach can be easily adopted. Below in Section 3 we give an example for QED, incorporating its form-factor decomposition into our approach.

2.2 Bogoliubov's recursive subtraction formula

The above algebra structures are available once one has decided on the set of 1PI graphs of interest. Those one-particle irreducible graphs Γ provide the generators δ_Γ of the Hopf algebra $\mathcal{H}_F = \bigoplus_{i=0}^{\infty} \mathcal{H}_{(i)}$, where $\mathcal{H}_{F,\text{lin}} := \text{span}(\delta_\Gamma)$, with their disjoint union providing the commutative product, which we denote by juxtaposition.

Let Γ be a 1PI graph. The Hopf algebra \mathcal{H}_F described above comes out to have a coproduct $\Delta : \mathcal{H}_F \rightarrow \mathcal{H}_F \otimes \mathcal{H}_F$:

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma, \quad (15)$$

where the sum is over all unions of 1PI superficially divergent proper subgraphs, and we extend this definition to products of graphs, $\Delta(\Gamma_1 \Gamma_2) = \Delta(\Gamma_1) \Delta(\Gamma_2)$, so that we get a bialgebra.

While the pre-Lie product respectively the Lie bracket inserted graphs into each other, dually the coproduct disentangles them. This is precisely what we make use of in renormalization theory: we have to render each subgraph finite before we can construct a local counterterm. Having a coproduct, two further structure maps of \mathcal{H}_F are immediate, the counit and the antipode. The counit \bar{e} vanishes on any non-trivial Hopf algebra element, $\bar{e}(X) = 0$, $X \neq \mathbb{I}$, but $\bar{e}(\mathbb{I}) = 1$. The antipode $S : \mathcal{H}_F \rightarrow \mathcal{H}_F$ is given by

$$S(\Gamma) = -\Gamma - \sum_{\gamma \subset \Gamma} S(\gamma) \Gamma/\gamma, \quad (16)$$

for $\Gamma \in \ker \bar{e}$, and $S(\mathbb{I}) = \mathbb{I}$. We can work out examples for the coproduct of a graph:

$$\Delta(\text{wavy line with a loop}) = \text{wavy line with a loop} \otimes \mathbb{I} + \mathbb{I} \otimes \text{wavy line with a loop} + 2 \text{wavy line with a tadpole} \otimes \text{wavy line with a loop} . \quad (17)$$

And an antipode:

$$S(\text{diagram}) = - \text{diagram} + \text{diagram} . \quad (18)$$

We note in passing that the gluing operation underlying the pre-Lie insertion of graphs relies on gluing data which can be reconstructed from the subgraphs γ and cographs Γ/γ in the above coproduct. This is crucial in the proof of locality of counterterms upon studying the Hochschild cohomology of this Hopf algebra [81].

We have by now obtained a Hopf algebra generated by combinatorial elements, 1PI Feynman graphs. Its existence is indeed automatic once one has chosen interactions and free fields.

As disjoint scattering processes give rise to independent (divergent) amplitudes one is led to the study of regularized characters of the Hopf algebra, to say A -valued maps $\phi : \mathcal{H}_F \rightarrow A$ such that $\phi \circ m = m_A \circ (\phi \otimes \phi)$. Here A denotes a unital commutative algebra.

Usually, Hopf algebra characters, i.e., linear multiplicative maps, assign to any element in the Hopf algebra an element in the base field, and form a group under convolution, denoted by \mathcal{G} . Motivated by the need for regularizing our theory, due to ultraviolet (UV) divergencies showing up in higher loop calculations, we take here a slightly more general point of view, replacing the base field as target space, say \mathbb{C} , by a suitable commutative and unital algebra, A , of -regularized- Feynman amplitudes. The group of regularized, or A -valued, Hopf algebra characters is denoted by \mathcal{G}_A , and the group law is given by the convolution product

$$\phi_1 \star \phi_2 := m_A \circ (\phi_1 \otimes \phi_2) \circ \Delta, \quad (19)$$

so that the coproduct, counit and coinverse (the antipode) give the product, unit and inverse of this group, as befits a Hopf algebra.

The study of tree-level amplitudes in lowest order perturbation theory justifies to assign to each edge a propagator and to each elementary scattering process a vertex which define the Feynman rules $\phi(\text{res}(\Gamma))$ and the underlying Lagrangian, on the level of residues of these very graphs. With the Feynman rules providing a canonical character ϕ , we will have to make one further choice: a renormalization scheme. The need for such a choice is no surprise: after all we are eliminating short-distance singularities in the graphs which renders their remaining finite part ambiguous, albeit in a most interesting manner.

We choose a \mathbb{K} -linear map $R : A \rightarrow A$, from which we obviously demand that it does not modify the UV-singular structure, i.e., $R^2 = R$, and furthermore that it obeys

$$R(x)R(y) + R(xy) = R(R(x)y) + R(xR(y)), \quad (20)$$

an equation which guarantees the multiplicativity of renormalization and lies at the heart of the Birkhoff decomposition which emerges below: it tells us that elements in A split into two parallel subalgebras given by the image and kernel of R . Algebras for which such a map exists fall into the class of Rota–Baxter algebras, well-known in mathematics. See below in Section 4 (and Appendix A) for more details .

Let us take a shortcut for the moment and see how all the above structure comes together in renormalization theory. Starting with a regularized Feynman rules character ϕ , we define a further character S_R^ϕ which deforms $\phi \circ S$, the inverse of the ϕ , slightly and delivers the counterterm for Γ in the renormalization scheme R :

$$S_R^\phi(\Gamma) = -R[m_A(S_R^\phi \otimes \phi \circ P)\Delta(\Gamma)] = -R[\phi(\Gamma)] - R\left[\sum_{\gamma \subset \Gamma} S_R^\phi(\gamma)\phi(\Gamma/\gamma)\right], \quad (21)$$

for Γ in $\ker \bar{e}$. Comparing with the undeformed inverse of ϕ

$$\phi \circ S(\Gamma) = m_A(\phi \circ S \otimes \phi \circ P)\Delta(\Gamma) = -\phi(\Gamma) - \sum_{\gamma \subset \Gamma} \phi \circ S(\gamma)\phi(\Gamma/\gamma) \quad (22)$$

allows to easily understand finiteness of renormalized quantities, thanks to the independence of counterterms on kinematical variables. Later, in Section 4 we will fully derive Equation (21) and the results below from a more mathematical point of view from the fact that R is a Rota–Baxter operator.

We conclude that S_R^ϕ is an element of the group of regularized characters, \mathcal{G}_A , of the Hopf algebra, $S_R^\phi \in \text{Spec}(\mathcal{G}_A)$. We now have determined the renormalized Lagrangian:

$$Z^r = S_R^\phi(\Gamma^r). \quad (23)$$

The standard results of renormalization theory follow immediately using the group of regularized characters: the renormalization of a graph Γ is obtained by the application of a renormalized character, $S_R^\phi \star \phi$

$$S_R^\phi \star \phi(\Gamma) = m_A(S_R^\phi \otimes \phi)\Delta(\Gamma) \quad (24)$$

for $\Gamma \in \ker \bar{e}$, and Bogoliubov's \bar{R} -operation is obtained as

$$\bar{R}(\Gamma) = m_A(S_R^\phi \otimes \phi)(\text{id} \otimes P)\Delta(\Gamma) = \phi(\Gamma) + \sum_{\gamma \subset \Gamma} S_R^\phi(\gamma)\phi(\Gamma/\gamma). \quad (25)$$

In the following we write $\bar{R}(\Gamma) =: \bar{\phi}(\Gamma)$, so that we have

$$S_R^\phi \star \phi(\Gamma) = \bar{\phi}(\Gamma) + S_R^\phi(\Gamma). \quad (26)$$

$S_R^\phi \star \phi$ is an element in the group of regularized characters, \mathcal{G}_A , of the Hopf algebra. This Lie group has indeed the previous Lie algebra \mathcal{L} of graph insertions as its Lie algebra: \mathcal{L} exponentiates to \mathcal{G}_A .

What we have achieved at this moment is a local renormalization of quantum field theory. Let $m_{\underline{r}}$ be a monomial in the Lagrangian L of degree $\omega_{\underline{r}}$,

$$m_{\underline{r}} = D_{\underline{r}}\{\varphi\}, \quad (27)$$

for some suitable derivation $D_{\underline{r}}$ on the fields φ . Then one can prove using the Hochschild cohomology and induction over the augmentation degree in \mathcal{H}_F :

Theorem 2.2. (*locality*)

$$Z^r D_r \{\varphi\} = D_r Z^r \{\varphi\}, \quad (28)$$

renormalization commutes with infinitesimal space-time variations of the fields.

Let us finally give the renormalization of a Feynman graph, say $\Gamma = \text{loop with wavy lines}$.

$$\begin{aligned} \Delta(\text{loop with wavy lines}) &= \text{loop with wavy lines} \otimes \mathbb{I} + \mathbb{I} \otimes \text{loop with wavy lines} + 2 \text{triangle with wavy lines} \otimes \text{circle with wavy line} \\ \bar{\phi}(\text{loop with wavy lines}) &= \phi(\text{loop with wavy lines}) + 2S_R^\phi(\text{triangle with wavy lines})\phi(\text{circle with wavy line}) \\ &= \phi(\text{loop with wavy lines}) - 2R\left[\phi(\text{triangle with wavy lines})\right]\phi(\text{circle with wavy line}) \\ S_R^\phi(\text{loop with wavy lines}) &= -R\left[\bar{\phi}(\text{loop with wavy lines})\right] \\ \phi_R(\text{loop with wavy lines}) &\equiv S_R^\phi \star \phi(\text{loop with wavy lines}) = [\text{id} - R] \circ \left[\bar{\phi}(\text{loop with wavy lines})\right]. \end{aligned} \quad (29)$$

3 Example: QED

The QED Lagrangian (density) for an electron coupled to the electromagnetic field in coordinate space reads (we skip the $1/2(\partial \cdot A)^2$ term)

$$L_{QED}(\alpha, m) = i\bar{\psi}\not{\partial}\psi + \bar{\psi}e\not{A}\psi + m\bar{\psi}\psi + \frac{1}{4}F^2, \quad (30)$$

with the electromagnetic field tensor $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^2 := F_{\mu\nu}F^{\mu\nu}$. The Euler-Lagrange equations for this term give the Maxwell equations. For the first and third term they give the Dirac equation. We use units defined by $c = \hbar = 1$, for which the elementary charge squared, $e^2 = 4\pi\alpha_{QED}$. It is a dimensionless quantity depending on the fine-structure constant $\alpha := \alpha_{QED} \simeq \frac{1}{137}$. The first term in (30) describes the kinetic energy of the electron, and together with the mass term, $m\bar{\psi}\psi$, it constitutes the free Lagrangian density for an electron. The second term in (30) describes the minimal coupling of the electron with the electromagnetic field. The first two terms are dictated by local gauge invariance of the QED Lagrangian, i.e., invariance with respect to multiplying the electron field ψ by a position dependent phase factor. Let us introduce for every term in L_{QED} a 1PI Green's function $G^r(p^2, \alpha, m, \mu)$, with

$$r \in \left\{ i\bar{\psi}\not{\partial}\psi, \bar{\psi}e\not{A}\psi, m\bar{\psi}\psi, \frac{1}{4}F^2 \right\}, \quad (31)$$

all of which transform as scalars under the Lorentz group.

Usually, 1PI Green's functions of QED are given in standard notation by the expressions

1. $\Gamma_\nu(p_1, p_2, m, \alpha, \mu)$ the vertex function;

2. $S_F^{-1}(p, m, \alpha, \mu)$ the inverse fermion propagator;
3. $P_{\nu\tau}^{-1}(p, m, \alpha, \mu)$ the inverse photon propagator,

all depending on the bare parameters mass m and coupling constant α , and 't Hooft's parameter μ which sets the scale for the one-parameter groups of automorphisms of the Lie algebra which run the renormalization group [34]. An extra parameter, such as 't Hooft's unit mass μ , enters naturally in the context of regularization, and is needed for dimensional reasons [32]. From symmetry considerations we find the following form factor decompositions, where we now assume the vertex at zero momentum transfer for simplicity:

1. $\Gamma_\nu(p, p, m, \alpha, \mu) = eG^{\bar{\psi}A\psi}(p^2, m, \alpha, \mu)\gamma_\nu + eH^{\bar{\psi}A\psi}(p^2, m, \alpha, \mu)\frac{\not{p}\not{p}_\nu}{p^2};$
2. $S^{-1}(p, m, \alpha, \mu) = G^{\bar{\psi}\not{p}\psi}(p^2, m, \alpha, \mu)\not{p} + G^{m\bar{\psi}\psi}(p^2, m, \alpha, \mu)m\mathbb{I};$
3. $P_{\nu\tau}^{-1}(p, m, \alpha, \mu) = \Pi_{\text{tr}}^{\frac{1}{4}F^2}(p^2, m, \alpha, \mu)\{g_{\nu\tau}p^2 - p_\nu p_\tau\} + \Pi_{\text{long}}^{\frac{1}{2}(\partial \cdot A)^2}(p^2, m, \alpha, \mu)p_\nu p_\tau,$

reintroducing a longitudinal term for completeness. Let us introduce a graphical notation for the monomials of the QED Lagrangian (30), which will form the building blocks of our graphical Hopf algebra of Feynman graphs.

$$\bar{\psi}\not{\partial}\psi \longleftrightarrow \overrightarrow{\partial} \quad m\bar{\psi}\psi \longleftrightarrow \overrightarrow{m} \quad \bar{\psi}eA\psi \longleftrightarrow \text{wavy line with vertex}$$

The first two graphs on the left represent the electron propagator parts, corresponding to the derivation and mass contribution in (30), respectively. The third graph is the QED vertex, representing the interaction of the electromagnetic field with fermions in (30). To ease the notation we suppressed spinorial indices.

Next we have the transversal respectively longitudinal parts of the bosonic photon propagator,

$$\frac{1}{4}F^2 \longleftrightarrow \text{wavy line with } t \quad 1/2(\partial \cdot A)^2 \longleftrightarrow \text{wavy line with long}$$

In the following, we will work with a transversal photon propagator for concreteness. Once we have the graphical notation we define the -coordinate space- QED Feynman rules $\hat{\phi}$ such that for any $\underline{r} \in \mathfrak{R}_{QED}$

$$\mathfrak{R}_{QED} := \left\{ \overrightarrow{\partial}, \overrightarrow{m}, \text{wavy line with vertex}, \text{wavy line with } t \right\} \quad (32)$$

we get back the corresponding coordinate space QED Lagrange monomial r in (31)

$$\hat{\phi}(\underline{r}) = r. \quad (33)$$

We write the QED Lagrangian (30) pictorially

$$L_{QED}(\alpha, m) = \sum_{\underline{r} \in \mathfrak{R}_{QED}} \hat{\phi}(\underline{r}). \quad (34)$$

With the QED free propagators and vertex in \mathfrak{R}_{QED} at hand we have available the one-particle irreducible Feynman diagrams which provide amplitudes corresponding to these propagations and interaction, as before. We can now introduce partitions of unity for the form-factor decomposition of any Green's function we are interested in. For example, if we do want to decompose the self-energy of the fermion into its kinetic energy and mass part

$$G^{\bar{\psi}\not{p}\psi}(p^2, m, \alpha, \mu) = \frac{1}{p^2} \text{Tr}(\not{p} S^{-1}(p, m, \alpha, \mu)) \quad (35)$$

$$G^{m\bar{\psi}\psi}(p^2, m, \alpha, \mu) = \frac{1}{m} \text{Tr}(S^{-1}(p, m, \alpha, \mu)). \quad (36)$$

We can easily incorporate this by promoting our Hopf algebra to pairs (Γ, σ) [33], where σ indicates the desired form-factor obtained by composing the Feynman rules with a suitable projector as above.

The sum over all projectors defines a partition of unity

$$\text{id} = \sum_{\sigma} P_{\sigma}. \quad (37)$$


This structure can be easily incorporated on the level of Hopf algebras, generalizing the study of external structures by setting for the pairs (Γ, σ)

$$\Delta(\Gamma, \sigma) = \sum (\Gamma', 1) \otimes (\Gamma'', \sigma). \quad (38)$$

Note that should we wish we can partition the unity on the left hand side above,

$$(\Gamma', 1) \rightarrow (\Gamma', \sigma_R) \quad (39)$$

if we want to use information that only particular form-factors σ_R need renormalization. Under the Feynman rules these pair of graphs then evaluate to the amplitudes corresponding to the structure functions defined by the projectors signified by the indicated external leg structures. The resulting Hopf algebras for such pairs (Γ, σ) are decorated versions of the ones for graphs only, and define graph-like structures very similar to the tree-like structures of Turaev for the Hopf algebra of rooted trees [109]. Examples can be found in [83, 20].

As an example, we might wish to renormalize the mass part of , using the knowledge that the photon self-energy is transversal. We hence work out the coproduct

$$\Delta((\text{loop with photon}, \sigma_m)) = (\text{loop with photon}, \sigma_m) \otimes \mathbb{I} + \mathbb{I} \otimes (\text{loop with photon}, \sigma_m) + (\text{loop with photon}, \sigma_{\text{trans}}) \otimes (\text{loop with photon}, \sigma_m). \quad (40)$$

Under the Feynman rules, we evaluate using the corresponding projectors and obtain the expected Lorentz scalar structure functions and counterterms following the routine as outlined in Eqs. (29).

4 Renormalization as a factorization problem

As we have seen the notion of connected graded commutative Hopf algebra appears naturally in the context of perturbative renormalization of 1PI Feynman graphs. Both, composing Feynman graphs in terms of the pre-Lie insertion product, where we replace vertices by Feynman graphs with compatible external leg structure, as well as their decomposition by eliminating subgraphs, i.e., replacing non-trivial 1PI subgraphs by their residues, look very familiar when inspecting the subtraction procedure encoded in the original BPHZ prescription [17, 18, 28, 32]. The latter was invented to extract the finite part of the Feynman integral corresponding to a Feynman graph via a regularized Feynman rules character, while maintaining fundamental physical principles, such as locality, unitarity, and Lorentz invariance.

The commutative Hopf algebra of Feynman graphs, \mathcal{H}_F , and its graded dual, $\mathcal{H}_F^* = \text{Hom}(\mathcal{H}_F, \mathbb{C})$ are intimately related by the Milnor–Moore theorem. The space \mathcal{H}_F^* together with the convolution product and the counit map $\bar{e} : \mathcal{H}_F \rightarrow \mathbb{C}$ as unit forms a unital, associative and non-commutative \mathbb{C} -algebra, which contains the group of characters, $\mathcal{G} := \text{char}(\mathcal{H}_F, \mathbb{C})$, i.e., linear functionals $\phi \in \mathcal{H}_F^*$ from \mathcal{H}_F to \mathbb{C} respecting multiplication, $\phi(\Gamma_1\Gamma_2) = \phi(\Gamma_1)\phi(\Gamma_2)$, $\Gamma_1, \Gamma_2 \in \mathcal{H}_F$. This group of multiplicative maps possesses a corresponding Lie algebra, $\mathcal{L} = \partial \text{char}(\mathcal{H}_F, \mathbb{C}) \subset \mathcal{H}_F^*$, of derivations, or infinitesimal characters, i.e., linear maps $Z \in \mathcal{H}_F^*$, satisfying Leibniz’ rule

$$Z(\Gamma_1\Gamma_2) = Z(\Gamma_1)\bar{e}(\Gamma_2) + \bar{e}(\Gamma_1)Z(\Gamma_2)$$

for all $\Gamma_1, \Gamma_2 \in \mathcal{H}_F$. The grading of \mathcal{H}_F implies a decreasing filtration on \mathcal{H}_F^* , which allows us to introduce a metric, and therefore a distance map. \mathcal{H}_F^* is complete with respect to the induced topology. The exponential map \exp^* gives a bijection between the Lie algebra \mathcal{L} and its corresponding group \mathcal{G} .

Using QED as an example we have seen that in general Feynman rules for any perturbative QFT form a subclass of characters. Also, we had to face the severe problem that the associated Feynman integrals for graphs beyond the tree level suffer from ultraviolet divergencies in the limit of large momenta, or equivalently small distances. Therefore, one is forced to invoke a regularization of such integrals, or more generally the Feynman rules themselves. Actually, there is no specific selection rule for such a regularization, indeed one must assure that the final physical result is independent of such an unphysical intermediate step. At the same time it is of vital importance that the regularization prescription used in calculations respects as many physical properties of the underlying theory as possible, such as gauge symmetries. We will ignore such subtleties and take the following stance. In the above Hopf algebraic setting, the regularization of our theory is achieved by replacing the base field \mathbb{C} as target space of maps in \mathcal{H}_F^* by an unital algebra A , of which we demand commutativity, and the existence of a linear map R satisfying the Rota–Baxter relation

$$R(x)R(y) + R(xy) = R(R(x)y) + (xR(y)) \quad (41)$$

for all $x, y \in A$. For R being such a Rota–Baxter map, $\tilde{R} := \text{id}_A - R$ also satisfies relation (41). Such algebras are well-known in mathematics under the name Rota–Baxter

algebra (see Appendix A for more details). As a principal example we mention here dimensional regularization, where the image of Feynman rules lives in the field of Laurent series, $A = \mathbb{C}[\varepsilon^{-1}, \varepsilon]$, with the pole part projection $R := R_{ms}$ as Rota–Baxter map. In the examples one encounters in QED renormalization calculations, where the regularization could have been for instance a simple cut-off of the momentum integrals, the map R is given in terms of an evaluation at a specific fixed momentum (on-shell scheme), $R_q(f)(p) := f(q)$, and trivially satisfies relation (41), as it is an idempotent algebra homomorphism, i.e., the zeroth order in the Taylor expansion of the map f at point q .

The space $\mathcal{A} := \text{Hom}(\mathcal{H}_F, (A, R))$ of A -valued linear functionals, together with the convolution product (19) and unit $e := u_A \circ \bar{e}$, $e(\mathbb{I}) = 1_A$, forms an associative non-commutative algebra, containing the group of regularized characters, $\mathcal{G}_A := \text{char}(\mathcal{H}_F, A)$, and its corresponding Lie algebra, $\mathcal{L}_A := \partial \text{char}(\mathcal{H}_F, A)$, of regularized derivations. The linearity of the Rota–Baxter map R on the regularization target space A , gives rise to a Rota–Baxter algebra structure on $\text{Hom}(\mathcal{H}_F, (A, R))$, induced in terms of the linear map \mathcal{R} , which is defined for any $f \in \mathcal{A}$, by $\mathcal{R}(f) := R \circ f \in \mathcal{A}$. As before, we can equip \mathcal{A} with a decreasing filtration of Rota–Baxter ideals

$$\mathcal{A} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_n \supset \cdots$$

making it a complete filtered non-commutative Rota–Baxter algebra with convolution product as composition, $(\mathcal{A}, \mathcal{R}, \{\mathcal{A}_n\}_{n \geq 1})$, since $\mathcal{R}(\mathcal{A}_n) \subset \mathcal{A}_n$ for all n . Here we have \mathcal{L}_A as a Lie subalgebra of \mathcal{A}_1 , and \mathcal{G}_A is a subgroup of $\widehat{\mathcal{G}} := e + \mathcal{A}_1$, such that

$$\exp^* : \mathcal{A}_1 \rightarrow e + \mathcal{A}_1, \quad \exp^*(Z) := \sum_{n=0}^{\infty} \frac{Z^{*n}}{n!}, \quad (42)$$

$$\log^* : e + \mathcal{A}_1 \rightarrow \mathcal{A}_1, \quad \log^*(e + Z) := - \sum_{n=1}^{\infty} \frac{(-Z)^{*n}}{n} \quad (43)$$

are well-defined with respect to convolution and inverse to each other. Furthermore \exp^* restricts to a bijection between \mathcal{L}_A and \mathcal{G}_A .

Atkinson’s [7] (see Appendix A) factorization theorem for associative Rota–Baxter algebras implies in the above setting, that for a fixed $\phi = e + Z \in \mathcal{G}_A$ the solutions $X \in e + \mathcal{R}(\mathcal{A}_1)$, $Y \in e + \tilde{\mathcal{R}}(\mathcal{A}_1)$ of the equations

$$X = e - \mathcal{R}(X \star Z) \quad \text{resp.} \quad Y = e - \tilde{\mathcal{R}}(Z \star Y) \quad (44)$$

solve the factorization problem

$$e + Z = \phi = X^{-1} \star Y^{-1}, \quad (45)$$

which can be easily checked. If the Rota–Baxter map R is idempotent, the decomposition in (45) is unique. In the following we denote $\phi_- := X$ and $\phi_+ := Y^{-1}$. Spitzer’s classical identity [106] for commutative Rota–Baxter algebras can be generalized to non-commutative Rota–Baxter algebras, thereby implying one of the main results of the Hopf algebraic approach to renormalization in QFT [33], to wit the algebraic Birkhoff decomposition of Connes and Kreimer, which we formulate as a theorem.

Theorem 4.1. [46] *Let \mathcal{H}_F be a connected graded Hopf algebra of Feynman graphs associated with a perturbatively treated renormalizable QFT. Let A be a commutative unital Rota–Baxter algebra with an idempotent Rota–Baxter operator R . Let \mathcal{A} be the complete filtered algebra $\text{Hom}(\mathcal{H}_F, A)$.*

1. *$(\mathcal{A}, \mathcal{R}, \{\mathcal{A}_n\}_{n \geq 1})$ is a complete filtered Rota–Baxter algebra with the idempotent operator $\mathcal{R}(f) := R \circ f$.*
2. *For $\phi = e + Z \in \mathcal{G}_A$, there is unique $\phi_- \in e + \mathcal{R}(\mathcal{A}_1)$ and $\phi_+ \in e + \tilde{\mathcal{R}}(\mathcal{A}_1)$ such that*

$$\phi = \phi_-^{-1} \star \phi_+. \quad (46)$$

3. *The elements $\phi_- = e - \mathcal{R}(\phi_- \star (\phi - e))$ and $\phi_+ = e - \tilde{\mathcal{R}}(\phi_+ \star (\phi^{-1} - e))$ solving Eq. (46) take the following form for $\Gamma \in \ker \bar{e}$:*

$$\phi_-(\Gamma) = -R(\phi(\Gamma) + \sum_{(\Gamma)} \phi_-(\Gamma')\phi(\Gamma'')), \quad (47)$$

$$\phi_+(\Gamma) = \tilde{R}(\phi(\Gamma) + \sum_{(\Gamma)} \phi_-(\Gamma')\phi(\Gamma'')). \quad (48)$$

4. *Spitzer’s identity for non-commutative complete Rota–Baxter algebras implies that the linear maps ϕ_- and ϕ_+ can be written as*

$$\phi_- = \exp^* \left(-\mathcal{R}(\chi(Z)) \right) \text{ resp. } \phi_+ = \exp^* \left(\tilde{\mathcal{R}}(\chi(Z)) \right) \quad (49)$$

and naturally give algebra homomorphisms.

Equation (48) follows by general arguments for Rota–Baxter algebras, as we can write for ϕ_+ the following equation

$$\phi_+ = e + \tilde{\mathcal{R}}(\phi_+ \star (e - \phi^{-1})) = e + \tilde{\mathcal{R}}(\phi_- \star (\phi - e)).$$

The unique map $\chi : \mathcal{A}_1 \rightarrow \mathcal{A}_1$ in (49) is the key-result for the generalization of Spitzer’s identity to non-commutative Rota–Baxter algebras, and satisfies the equation

$$\chi(Z) = Z - BCH\left(\mathcal{R}(\chi(Z)), \tilde{\mathcal{R}}(\chi(Z))\right), \quad (50)$$

where $BCH(x, y)$ denotes the Baker–Campbell–Hausdorff relation

$$\exp(x) \exp(y) = \exp(x + y + BCH(x, y)).$$

The non-linear map χ was introduced in [45, 46], and was called the *BCH*-recursion. The reader may find it helpful to consult [48] for more details, and [89] for a more conceptual proof in the context of Lie algebras.

As a proposition to this theorem, we mention without giving further details the fact that Bogoliubov's \bar{R} -operation (25) can be written as an exponential using the double Rota–Baxter convolution product, $\star_{\mathcal{R}}$, on $(\mathcal{A}, \mathcal{R}, \{\mathcal{A}_n\}_{n \geq 1})$ (see Eq. (60) in Appendix A):

$$\bar{R}(\Gamma) = \bar{\phi}(\Gamma) = \phi_- \star (\phi - e)(\Gamma) = -\exp^{\star_{\mathcal{R}}}(-\chi(Z))(\Gamma)$$

for $\Gamma \in \ker \bar{e}$. Finally let us mention that the above notion of complete Rota–Baxter algebra and Theorem 4.1 becomes very transparent for uni- and nilpotent upper (or lower) triangular matrices with entries in a commutative Rota–Baxter algebra [48, 49].

The above theorem presents a purely algebraic setting for the formulation of renormalization as a factorization problem in the group of regularized Hopf algebra characters, situated in the theory of non-commutative Rota–Baxter algebras with idempotent Rota–Baxter map. The formulae for the counterterm (47) and renormalized character (48) are completely dictated by a general decomposition structure, which characterizes Rota–Baxter algebras [7]. The additional property of R being a projector implies a direct decomposition of the algebra, hence the uniqueness of the factorization in (46). We would like to emphasize the necessary freedom in the choice of the regularization prescription, encoded in the particular structure of the commutative Rota–Baxter algebra A as target space of linear Hopf algebra functionals in $\text{Hom}(\mathcal{H}_F, A)$.

Specializing the target space Rota–Baxter algebra A in the above theorem to the field of Laurent series, i.e., using dimensional regularization, we recover the original setting in [33], opening a hitherto hidden geometric viewpoint on perturbative renormalization in terms of a correspondence to the Riemann–Hilbert problem. This approach was further extended in [37, 38, 39].

5 Diffeomorphisms of physical parameters

In the above, we obtained a unique Birkhoff decomposition of Feynman rules $\phi \in \text{Spec}(\mathcal{G}_A)$ into two characters $\phi_- =: S_R^\phi \in \text{Spec}(\mathcal{G}_A)$ and $\phi_+ = S_R^\phi \star \phi \in \text{Spec}(\mathcal{G}_A)$, for any idempotent Rota–Baxter map R . Thanks to Atkinson's theorem this is possible for any renormalization scheme R . For the minimal subtraction scheme it amounts to the decomposition of the Laurent series $\phi(\Gamma)(\varepsilon)$, which has poles of finite order in the regulator ε , into a part holomorphic at the origin and a part holomorphic at complex infinity. This has a geometric interpretation upon considering the Birkhoff decomposition of a loop around the origin, providing the clutching data for the two half-spheres defined by that very loop, which is central in the work of Connes and Kreimer [34, 35]. The geometric interpretation leads to motivic Galois theory upon studying the equisingularity of the corresponding connection in the Riemann–Hilbert correspondence [37, 38], itself a result of the Hochschild cohomology of these Hopf algebras [10, 11].

Our understanding of each term in the perturbative expansion and its renormalization have found hence satisfying mathematical interpretations. The character group \mathcal{G}_A is a poorly understood object though, it is far too big. Fortunately renormalization can be

captured by the study of diffeomorphisms of physical parameters, as by the very definition the range of allowed modifications in renormalization theory is the variation of the coefficients of monomials $\hat{\phi}(\underline{r})$ of the underlying Lagrangian

$$L = \sum_{\underline{r} \in \mathfrak{R}} Z^{\underline{r}} \hat{\phi}(\underline{r}). \quad (51)$$

We can now eliminate the use of \mathcal{G}_A as one can regain the Birkhoff decomposition at the level of diffeomorphisms of the coupling constants.

One proceeds by using that renormalized couplings provide a formal diffeomorphism

$$g_{\text{new}} = g_{\text{old}} Z^g, \quad (52)$$

where

$$Z^g = \frac{Z^v}{\prod_{e \in \text{res}(v)_{\text{ext}}^{[1]}} \sqrt{Z^e}}, \quad (53)$$

for some vertex v , which obtains the new coupling in terms of a diffeomorphism of the old. This formula provides indeed a Hopf algebra homomorphism from the Hopf algebra of diffeomorphisms to the Hopf algebra of Feynman graphs, regarding Z^g , a series over counterterms for all 1PI graphs with the external leg structure corresponding to the coupling g , in two different ways: it is at the same time a formal diffeomorphism in the coupling constant g_{old} and a formal series in Feynman graphs. As a consequence, there are two competing coproducts acting on Z^g . Their consistency defines the required homomorphism, which transposes to a homomorphism from the largely unknown group of regularized characters of \mathcal{H}_F to the one-dimensional diffeomorphisms of this coupling. Hence one concludes [35]:

Theorem 5.1. *Let the unrenormalized effective coupling constant $g_{\text{eff}}(\varepsilon)$ viewed as a formal power series in g be considered as a loop of formal diffeomorphisms and let $g_{\text{eff}}(\varepsilon) = (g_{\text{eff}_-})^{-1}(\varepsilon) g_{\text{eff}_+}(\varepsilon)$ be its Birkhoff decomposition in the group of formal diffeomorphisms. Then the loop $g_{\text{eff}_-}(\varepsilon)$ is the bare coupling constant and $g_{\text{eff}_+}(0)$ is the renormalized effective coupling.*

6 The role of Hochschild cohomology

The Hochschild cohomology of the Hopf algebras of 1PI graphs illuminates the structure of 1PI Green's functions in various ways:

- it gives a coherent proof of locality of counterterms –the very fact that

$$[Z^{\underline{r}}, D_{\underline{r}}] = 0, \quad (54)$$

the coefficients in the Lagrangian remain independent of momenta, and hence the Lagrangian a polynomial expression in fields and their derivatives; [11, 81, 84]

- the quantum equation of motions take a very succinct form identifying the Dyson kernels with the primitives of the Hopf algebra [11, 81, 84], and hence replacing a sum over all graphs by a sum over all primitive graphs;
- sub-Hopf algebras emerge from the study of the Hochschild cohomology which connect the representation theory of these Hopf algebras to the structure of theories with internal symmetries leading to the Slavnov–Taylor identities for the couplings [84];
- these Hopf algebras are intimately connected to the structure of transcendental functions like the generalized polylogarithms which play a prominent role these days ranging from applied particle physics to recent developments in mathematics, in particular the primitive graphs which provide the Dyson kernels allow for a motivic interpretation [16].

For more information, we refer the reader to the literature indicated.

A Basic facts about general Rota–Baxter algebras

For the reader’s convenience we collect some basic notions of more mathematical nature concerning mainly Rota–Baxter operators, in the hope that from the above presentation they become redundant. For more details we refer the reader to the standard literature, e.g. [7, 61, 98]. Rota–Baxter operators (also known as Baxter operators in older mathematical references) were an active field of mathematical research in the late 1960s and early 1970s. After an almost three decades long period of dormancy they reappeared, as if on cue, in the mathematical literature in the context of dendriform algebras [2, 43, 85], number theory [62], generalizations of shuffle products [47, 60], and Hopf algebras [6], as well as in theoretical physics in the seminal work of Kreimer and collaborators on the Hopf algebra of renormalization [33, 45, 46, 73].

In the following \mathbb{K} denotes the base field of characteristics zero, over which all algebraic structures are defined. In general an algebra always means an associative unital \mathbb{K} -algebra, not necessarily commutative. The algebra unit is simply denoted by 1.

In Section 4 we encountered Rota–Baxter operators respectively the Rota–Baxter relation in the context of renormalization schemes, i.e. subtraction operators for the BPHZ method. Let A be an algebra together with a linear endomorphism $R : A \rightarrow A$. We call the tuple (A, R) a Rota–Baxter algebra of weight $\theta \in \mathbb{K}$, if the map R fulfills the Rota–Baxter relation (of weight θ)

$$R(x)R(y) + \theta R(xy) = R(R(x)y + xR(y)) \quad (55)$$

for all $x, y \in A$. Without proof we state the fact that the operator $\tilde{R} := \theta \text{id}_A - R$ is a Rota–Baxter operator of weight θ , too, such that the mixed relation

$$R(x)\tilde{R}(y) = \tilde{R}(R(x)y) + R(x\tilde{R}(y)) \quad (56)$$

is satisfied for all $x, y \in A$. The map $B := \theta \text{id}_A - 2R$ satisfies the modified Rota–Baxter relation

$$B(x)B(y) + \theta^2 xy = B(B(x)y + xB(y)). \quad (57)$$

For $\theta \neq 0$, the normalized map $\theta^{-1}R$ is a Rota–Baxter operator of weight one. Therefore without loss of generality we may suppose in the following the canonical weight one case. A Rota–Baxter (left-) right-ideal I is a (left-) right-ideal I of A such that $R(I) \subseteq I$. A Rota–Baxter ideal is a Rota–Baxter left- and right-ideal.

The American mathematician Glen Baxter introduced this relation 1960 in his probability studies in fluctuation theory [9]. Later, the Italian born American mathematician Gian-Carlo Rota [98, 99, 100, 101], and others [29, 70], notably F.V. Atkinson [7], explored in detail Baxter’s work from different perspectives in analysis, algebra and combinatorics. The case $\theta = 0$ corresponds to the integration by parts property of the usual Riemann integral $I : \mathcal{F} \rightarrow \mathcal{F}$, $I[f](x) := \int_0^x f(t) dt$ in the algebra \mathcal{F} of continues functions on \mathbb{R} , to wit,

$$I[f_1] I[f_2] = I[I[f_1]f_2 + f_1 I[f_2]], \quad (58)$$

for $f_1, f_2 \in \mathcal{F}$. We already encountered the pole part projection R_{ms} in dimensional regularization as an example of an idempotent Rota–Baxter map of weight one. The images of R as well as \tilde{R} form subalgebras in A . Let R be a projector on A . For R to satisfy the Rota–Baxter relation is equivalent to a direct decomposition of $A = R(A) \oplus \tilde{R}(A)$. This is just the special case of Atkinson’s additive decomposition theorem [7], characterizing a general Rota–Baxter algebra (A, R) as a subdirect difference of the images of R and \tilde{R} .

The Lie algebra associated to (A, R) , with standard commutator bracket forms a Rota–Baxter Lie algebra, (\mathcal{L}_A, R) , with R fulfilling

$$[R(x), R(y)] + \theta R([x, y]) = R([R(x), y] + [x, R(y)]), \quad (59)$$

better known as (operator) classical Yang–Baxter equation. Let us mention here that it was rediscovered in this form in the early 1980ies by some Russian physicists in the context of classical integrable systems (see e.g. [8, 105] for references and more details). This curious coincidence of Baxter and Baxter just happens to reveal the connections of Rota–Baxter operators with many areas of mathematics and physics.

The vector space underlying A , equipped with the new product

$$x \star_R^{(1)} y := xR(y) + R(x)y - xy \quad (60)$$

is again a Rota–Baxter algebra with Rota–Baxter map R , which we denote (A_1, R) . Hence, all Rota–Baxter algebras (A, R) , associative or non-associative, come with a whole hierarchy of so-called double Rota–Baxter algebras, (A_n, R) , $n \in \mathbb{N}$. Relation (55) naturally implies that $R(x \star_R^{(1)} y) = R(x)R(y)$, i.e. R is an algebra homomorphism from A_1 to A , or more generally from A_n to A_{n-1} .

Baxter’s original motivation for his work was to prove that for a commutative algebra A together with a linear map R satisfying relation (55), that later bore his name, the following

identity for fixed $a \in A$

$$\exp\left(-\frac{1}{\theta}R(\log(1-\theta at))\right) = \sum_{n=0}^{\infty} t^n \underbrace{R(R(R(\cdots R(a)a \cdots)a)a)}_{n\text{-times}} \quad (61)$$

holds in the formal power series ring $\mathcal{A} := A[[t]]$, with t being a commuting parameter. This famous relation is called Spitzer's classical identity, and appeared in Frank Spitzer's 1956 paper [106]. In [45] this was generalized to non-commutative Rota–Baxter algebras, based on a BCH-type recursion formula, the key result we used earlier, see Eq. (50). A more general setting in which the above factorization may be regarded, and of which the algebra $A[[t]]$ is just a special case, is that of complete filtered Rota–Baxter algebras [46, 48, 49].

The right hand side of (61) is the unique solution of the recursive equation

$$X = 1 + tR(Xa). \quad (62)$$

This is a natural generalization of the recursion $f = 1 + I[gf]$ corresponding to the differential equation $f' = gf$, $f(0) = 1$, solved by $\exp(I[g])$, in (\mathcal{F}, I) where I is the Riemann integral. It was again Atkinson [7] in 1963, who observed that for any Rota–Baxter algebra, not necessarily commutative, a solution to Eq. (62), and its companion equation for \tilde{R}

$$Y = 1 + t\tilde{R}(aY) \quad (63)$$

in $A[[t]]$, solve the multiplicative decomposition problem

$$(1 - \theta a) = X^{-1}Y^{-1}, \quad (64)$$

for any element $a \in A$. R being a projector implies a unique decomposition of the element $(1 - \theta a) \in A$. This genuine factorization property of Rota–Baxter algebras is further analyzed in [48].

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References

- [1] E. Abe, *Hopf Algebras*, Cambridge University Press, Cambridge, (1980).
- [2] M. Aguiar, *Prepoisson algebras*, Lett. Math. Phys., **54**, no. 4, 263–277, (2000).
- [3] M. Aguiar, N. Bergeron, F. Sottile, *Combinatorial Hopf algebras and generalized Dehn–Sommerville relations*, to appear in *Compositio Mathematica*. [arXiv:math.CO/0310016](https://arxiv.org/abs/math.CO/0310016)

- [4] M. Aguiar, F. Sottile, *Structure of the Loday–Ronco Hopf algebra of trees*, to appear in Journal of Algebra. [arXiv:math.CO/0409022](#).
- [5] M. Aguiar, F. Sottile, *Cocommutative Hopf algebras of permutations and trees*, to appear in Journal of Algebraic Combinatorics. [arXiv:math.QA/0403101](#)
- [6] G. E. Andrews, L. Guo, W. Keigher, K. Ono, *Baxter Algebras and Hopf Algebras*, Trans. Amer. Math. Soc., **355**, 4639–4656, (2003). [arXiv:math.RA/0407181](#)
- [7] F. V. Atkinson, *Some aspects of Baxter’s functional equation*, J. Math. Anal. Appl., **7**, 1–30, (1963).
- [8] O. Babelon, D. Bernard, M. Talon, *Introduction to classical integrable systems*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, (2003).
- [9] G. Baxter, *An analytic problem whose solution follows from a simple algebraic identity*, Pacific J. Math., **10**, 731–742, (1960).
- [10] C. Bergbauer, D. Kreimer, *The Hopf algebra of rooted trees in Epstein–Glaser renormalization*, Ann. Inst. Henri Poincaré, **6**, 343–367, (2005). [arXiv:hep-th/0403207](#)
- [11] C. Bergbauer, D. Kreimer, *Hopf Algebras in Renormalization Theory: Locality and Dyson–Schwinger Equations from Hochschild Cohomology*, preprint: June 2005, [arXiv:hep-th/0506190](#)
- [12] H. Bethe, *The electromagnetic shift of energy levels*, Phys. Rev., **72**, 339–341, (1947).
- [13] I. Bierenbaum, R. Kreckel, D. Kreimer, *On the invariance of residues of Feynman graphs*, J. Math. Phys., **43**, 4721–4740, (2002). [arXiv:hep-th/0111192](#)
- [14] I. Bierenbaum, S. Weinzierl, *The massless two-loop two-point function*, Eur. Phys. J. C, **32**, 67–78, (2003). [arXiv:hep-ph/0308311](#)
- [15] A. S. Blaer, K. Young, *Field Theory Renormalization using Callan–Symanzik Equation*, Nucl. Phys. B, **83**, 493–514, (1974).
- [16] S. Bloch, H. Esnault, D. Kreimer, *Motives associated to graph polynomials*, preprint: May 2005, [arXiv:math.ag/0510011](#) .
- [17] N. N. Bogoliubov, O. S. Parasiuk, *On the multiplication of causal functions in the quantum theory of fields*. Acta Math., **97**, 227–266, (1957).
- [18] N. N. Bogoliubov, D. V. Shrikov, *Introduction to the theory of Quantized Fields*, Intersci. Monogr. Phys. Astron., **3**, 1–720, (1959).
- [19] L. Boutet de Monvel, *Hopf algebra of Feynman diagrams, renormalization and Wiener–Hopf factorization (following A. Connes and D. Kreimer)*, Séminaire Bourbaki, Vol. 2001/2002. Astrisque No. 290, Exp. No. **900**, viii, 149–165, (2003).
- [20] D. J. Broadhurst, D. Kreimer, *Renormalization automated by Hopf algebra*, J. Symb. Comput. **27**, 581–600, (1999). [arXiv:hep-th/9810087](#)
- [21] D. J. Broadhurst, D. Kreimer, *Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Pade–Borel resummation*, Phys. Lett. B, **475**, 63–70, (2000). [arXiv:hep-th/9912093](#)
- [22] D. J. Broadhurst, D. Kreimer, *Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality*, Nucl. Phys. B, **600**, 403–422, (2001). [arXiv:hep-th/0012146](#)

- [23] D. J. Broadhurst, D. Kreimer, *Towards cohomology of renormalization: bigrading the combinatorial Hopf algebra of rooted trees*, Comm. Math. Phys., **215**, 217–236, (2000). [arXiv:hep-th/0001202](#)
- [24] D. J. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Phys. Lett. B, **426**, no. 3-4, 339–346, (1998). [arXiv:hep-th/9609128](#)
- [25] Ch. Brouder, *Runge–Kutta Methods and Renormalization*, Euro. Phys. J. C, **12**, 512–534, (2000). [arXiv:hep-th/9904014](#)
- [26] L. Brown (ed.), *Renormalization. From Lorentz to Landau (and beyond)*, Springer-Verlag, New York, (1993).
- [27] J. C. Butcher, *An algebraic theory of integration methods*, Mathematics of Computation, Vol. **26**, 117–106, (1972).
- [28] W. E. Caswell, A. D. Kennedy, *A Simple approach to renormalization theory*, Phys. Rev. D, **25**, 392–408, (1982).
- [29] P. Cartier, *On the structure of free Baxter algebras*, Advances in Math., **9**, 253–265, (1972).
- [30] F. Chapoton, M. Livernet, *Pre-Lie algebras and the rooted trees operad*, Internat. Math. Res. Notices, no. **8**, 395–408, (2001). [arXiv:math.QA/0002069](#)
- [31] C. Chrysomalakos, H. Quevedo, M. Rosenbaum, J. D. Vergara, *Normal coordinates and primitive elements in the Hopf algebra of renormalization*, Comm. in Math. Phys., **225**, no. 3, 465–485, (2002). [arXiv:hep-th/0105252](#)
- [32] J. C. Collins, *Renormalization*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, (1984).
- [33] A. Connes, D. Kreimer, *Hopf algebras, Renormalization and Noncommutative Geometry*, Comm. in Math. Phys., **199**, 203–242, (1998). [arXiv:hep-th/9808042](#)
- [34] A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann–Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*, Comm. in Math. Phys., **210**, 249–273, (2000). [arXiv:hep-th/9912092](#)
- [35] A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann–Hilbert problem. II. The β -function, diffeomorphisms and the renormalization group*, Comm. in Math. Phys., **216**, 215–241, (2001). [arXiv:hep-th/0003188](#)
- [36] A. Connes, D. Kreimer, *Insertion and elimination: the doubly infinite Lie algebra of Feynman graphs*, Ann. Inst. Henri Poincaré, **3**, no. 3, 411–433, (2002). [arXiv:hep-th/0201157](#)
- [37] A. Connes, M. Marcolli, *From Physics to Number Theory via Noncommutative Geometry, Part II: Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory*, to appear in the volume "Frontiers in Number Theory, Physics, and Geometry", preprint: Nov. 2004, [arXiv:hep-th/0411114](#).
- [38] A. Connes, M. Marcolli, *Renormalization and motivic Galois theory*, Int. Math. Res. Not., no. 76, 4073–4091, (2004). [arXiv:math.NT/0409306](#)
- [39] A. Connes, M. Marcolli, *Quantum Fields and Motives*, J. Geom. Phys., **56**, 55–85, (2006). [arXiv:hep-th/0504085](#)

- [40] B. Delamotte, *A hint of renormalization*, Am. J. Phys., **72**, 170–184, (2004). [arXiv:hep-th/0212049](#)
- [41] F. Dyson, *The S Matrix in Quantum Electrodynamics*, Phys. Rev., **75**, 1736–1755, (1949).
- [42] F. Dyson, *The Radiation Theories of Tomonaga, Schwinger, and Feynman*, Phys. Rev., **75**, 486–502, (1949).
- [43] K. Ebrahimi-Fard, *Loday-type algebras and the Rota-Baxter relation*, Lett. Math. Phys., **61**, no. 2, 139–147, (2002).
- [44] K. Ebrahimi-Fard, I. Mencattini, D. Kreimer, *On the Insertion-Elimination Lie algebra of Feynman graphs*, prepared for 5th International Workshop on Lie Theory and Its Applications in Physics, Varna, Bulgaria, 16-22 Jun 2003. Published in *Varna 2003, Lie theory and its applications in physics*, 124–134, (2003).
- [45] K. Ebrahimi-Fard, L. Guo, D. Kreimer, *Integrable Renormalization II: the General case*, Ann. Inst. Henri Poincaré, **6**, 369–395, (2005). [arXiv:hep-th/0403118](#)
- [46] K. Ebrahimi-Fard, L. Guo, D. Kreimer, *Spitzer’s identity and the Algebraic Birkhoff Decomposition in pQFT*, J. Phys. A: Math. Gen., **37**, 11037–11052, (2004). [arXiv:hep-th/0407082](#)
- [47] K. Ebrahimi-Fard, L. Guo, *Mixable Shuffles, Quasi-shuffles and Hopf Algebras*, accepted for publication in Journal of Algebraic Combinatorics. [arXiv:hep-th/0506418](#)
- [48] K. Ebrahimi-Fard, L. Guo, *Matrix Representation of Renormalization in Perturbative Quantum Field Theory*, submitted, preprint: August 2005, [arXiv:hep-th/0508155](#).
- [49] K. Ebrahimi-Fard, J. Gracia-Bondía, L. Guo, J. C. Várilly, *Combinatorics of renormalization as matrix calculus*, Phys. Lett. B. in press [arXiv:hep-th/0508154](#)
- [50] R. Ehrenborg, *On posets and Hopf algebras*, Adv. Math., **119**, 1–25, (1996).
- [51] H. Epstein, V. Glaser, *The role of locality in perturbation theory*, Ann. Inst. Henri Poincaré Sect. A, **XIXA**, 211–295, (1973).
- [52] H. Figueroa, J. M. Gracia-Bondía, J. C. Várilly, *Elements of Noncommutative Geometry*, Birkhäuser, (2001).
- [53] H. Figueroa, J. M. Gracia-Bondía, *On the antipode of Kreimer’s Hopf algebra*, Mod. Phys. Lett., **A16**, 1427–1434, (2001). [arXiv:hep-th/9912170](#)
- [54] H. Figueroa, J. M. Gracia-Bondía, *The uses of Connes and Kreimer’s algebraic formulation of renormalization theory*, Int. J. Mod. Phys. A, **19**, no. 16, 2739–2754, (2004). [arXiv:hep-th/0301015](#)
- [55] H. Figueroa, J. M. Gracia-Bondía, *Combinatorial Hopf algebras in quantum field theory I*, Reviews of Mathematical Physics, **17**, 881–976, (2005). [arXiv:hep-th/0408145](#)
- [56] L. Foissy, *Les algèbres de Hopf des arbres enracinés décorés I, II*, Bull. Sci. Math., **126**, no. 3, 193–239, (2002); *ibid.*, **126**, no. 4, 249–288, (2002).
- [57] F. Girelli, T. Krajewski, P. Martinetti, *Wave-Function renormalization and the Hopf algebra of Connes and Kreimer*, Mod. Phys. Lett., **A16**, 299–303, (2001). [arXiv:math-ph/0103020](#)
- [58] F. Girelli, T. Krajewski, P. Martinetti, *An algebraic Birkhoff decomposition for the continuous renormalization group*, J. Math. Phys., **45**, 4679–4697, (2004). [arXiv:hep-th/0401157](#)

- [59] R. Grossman, R. G. Larson, *Hopf-algebraic structure of families of trees*, J. of Algebra, **126**, 184–210, (1989).
- [60] L. Guo, W. Keigher, *Baxter algebras and shuffle products*, Advances in Math., **150**, no. 1, 117–149, (2000). [arXiv:math.RA/0407155](#)
- [61] L. Guo, *Baxter algebras and differential algebras*, in "Differential algebra and related topics", (Newark, NJ, 2000), World Sci. Publishing, River Edge, NJ, 281–305, (2002). [arXiv:math.RA/0407180](#)
- [62] L. Guo, *Baxter Algebras, Stirling Numbers, and Partitions*, J. Algebra Appl., **4**, no. 2, 153–164, (2005). [arXiv:math.AC/0402348](#)
- [63] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, Berlin, Germany: Springer (1992)
- [64] K. Hepp, *Théorie de la renormalisation*, Springer, Berlin, (1969).
- [65] M. Hoffman, *Combinatorics of rooted trees and Hopf algebras*, Trans. Amer. Math. Soc., **355**, 3795–3811, (2003). [arXiv:math.CO/0201253](#)
- [66] R. Holtkamp, *Comparison of Hopf algebras on trees*, Arch. Math. (Basel), **80**, 368–383, (2003).
- [67] S. A. Joni, G.-C. Rota, *Coalgebras and bialgebras in combinatorics*, Studies in Applied Mathematics, **61**, 93–139, (1979). Reprinted in: "Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries", J.P.S. Kung Ed., Contemp. Mathematicians, Birkhäuser Boston, Boston, MA, 1995.
- [68] C. Kassel, *Quantum Groups*, Springer-Verlag, New York, (1995).
- [69] D. Kastler, *Connes–Moscovici–Kreimer Hopf Algebras*, Proceedings of the conference on "Mathematical Physics and in Mathematics and Physics", Siena, 20–25 June 2000. Fields Inst. Commun., **30**, 219–248, Amer. Math. Soc., Providence, RI, 2001. [arXiv:hep-th/0104017](#)
- [70] J. F. C. Kingman, *Spitzer's identity and its use in probability theory*, J. London Math. Soc., **37**, 309–316, (1962).
- [71] T. Krajewski, R. Wulkenhaar, *On Kreimer's Hopf algebra structure of Feynman graphs*, Euro. Phys. J. C, **7**, 697–708, (1998). [arXiv:hep-th/9805098](#)
- [72] D. Kreimer, *On the Hopf algebra structure of perturbative quantum field theories*, Adv. Theor. Math. Phys., **2**, 303–334, (1998). [arXiv:q-alg/9707029](#)
- [73] D. Kreimer, *Chen's iterated integral represents the operator product expansion*, Adv. Theor. Math. Phys., **3**, no. 3, 627–670, (1999). [arXiv:hep-th/9901099](#)
- [74] D. Kreimer, *Factorization in quantum field theory: An exercise in Hopf algebras and local singularities*, contributed to Les Houches School of Physics: *Frontiers in Number Theory, Physics and Geometry*, Les Houches, France, 9–21, March 2003. [arXiv:hep-th/0306020](#)
- [75] D. Kreimer, *Shuffling quantum field theory*, Lett. Math. Phys., **51**, 179–191, (2000). [arXiv:hep-th/9912290](#)
- [76] D. Kreimer, *Combinatorics of (perturbative) Quantum Field Theory*, Phys. Rep., **363**, 387–424, (2002). [arXiv:hep-th/00110059](#)
- [77] D. Kreimer, *On Overlapping Divergencies*, Comm. in Math. Phys. **204**, 669–689, (1999). [arXiv:hep-th/9810022](#)

- [78] D. Kreimer, *What is the trouble with Dyson–Schwinger equations?*, contributed to the proceedings “Loops and Legs 2004”, April 2004, Zinnowitz, Germany. [arXiv:hep-th/0407016](#)
- [79] D. Kreimer, *The Residues of Quantum Field Theory – Numbers we should know*, contributed to the proceedings of the *Workshop on Noncommutative Geometry and Number Theory*, August 18-22 2003, Max Planck Institut für Mathematik, Bonn. [arXiv:hep-th/0407016](#)
- [80] D. Kreimer, *Knots and Feynman Diagrams*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, (2000).
- [81] D. Kreimer, *New Mathematical Structures in Renormalizable Quantum Field Theory*, Annals of Phys., **303**, 2003, 179–202.
- [82] D. Kreimer, *Structures in Feynman graphs: Hopf algebras and symmetries*, *Dennisfest Proceedings*, Stony Brook, June 2001. [arXiv:hep-th/0202110](#)
- [83] D. Kreimer, R. Delbourgo, *Using the Hopf algebra structure of QFT in calculations* Phys. Rev. D, **60**, 105025, (1999). [arXiv:hep-th/9903249](#)
- [84] D. Kreimer, *Anatomy of a gauge theory*, preprint: Sept. 2005, [arXiv:hep-th/0509135](#).
- [85] J.-L. Loday, *Dialgebras*, in “Dialgebras and related operads” Springer Lecture Notes in Mathematics **1763**, 7–66, (2001). [arXiv:math.QA/0102053](#)
- [86] J.-L. Loday, M. O. Ronco, *Hopf algebra of the planar binary trees*, Advances in Math., **139**, 293–309, (1998).
- [87] D. Malyshev, *Hopf algebra of ribbon graphs and renormalization*, J. High Energy Phys., no. 5, no. 13, 28 pp, (2002).
- [88] D. Malyshev, *Leading RG logs in ϕ^4 theory* Phys. Lett. B, **578**, no. 1-2, 231–234, (2004).
- [89] D. Manchon, *Hopf algebras, from basics to applications to renormalization*, Comptes-rendus des Rencontres mathématiques de Glanon 2001. [arXiv:math.QA/0408405](#)
- [90] D. Manchon, S. Paycha, *Shuffle relations for regularised integrals of symbols*, preprint: October 2005, [arXiv:math-ph/0510067](#).
- [91] I. Mencattini, D. Kreimer, *Insertion and elimination Lie algebra: the ladder case*, Lett. in Math. Phys., **67**, 61–74, (2004). [arXiv:math.QA/0308042](#)
- [92] I. Mencattini, D. Kreimer, *The structure of the ladder Insertion-Elimination Lie algebra*, Comm. in Math. Phys., **259**, 413–432, (2005). [arXiv:math-ph/0408053](#)
- [93] J. W. Milnor, J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math., **81**, 211–264, (1965).
- [94] S. Moch, P. Uwer, S. Weinzierl, *Nested sums, expansion of transcendental functions and multiple-loop integrals*, J. Math. Phys., **43**, no. 6, 3363–3386, (2002). [arXiv:hep-ph/0110083](#)
- [95] F. Panaite, *Relating the Connes-Kreimer and Grossmann-Larson Hopf algebras built on rooted trees* Lett. Math. Phys., **51**, 211–219, (2000). [arXiv:math.QA/0003074](#)
- [96] G. Pinter, *The Hopf algebra structure of Connes and Kreimer in Epstein–Glaser renormalization* Lett. Math. Phys., **54**, 227–233, (2000). [arXiv:hep-th/0012057](#)

- [97] M. Rosenbaum, J. D. Vergara, *Hopf algebra of renormalization, normal coordinates and Kontsevich deformation quantization*, J. Phys. A: Math. Gen., **37**, 7939–7953, (2004). [arXiv:hep-th/0404233](#)
- [98] G.-C. Rota, D. Smith, *Fluctuation theory and Baxter algebras*, Istituto Nazionale di Alta Matematica, **IX**, 179–201, (1972). Reprinted in: “Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries”, J.P.S. Kung Ed., Contemp. Mathematicians, Birkhäuser Boston, Boston, MA, 1995.
- [99] G.-C. Rota, *Baxter algebras and combinatorial identities. I, II.*, Bull. Amer. Math. Soc. **75**, 325–329, (1969); *ibid.* **75**, 330–334, (1969). Reprinted in: “Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries”, J.P.S. Kung Ed., Contemp. Mathematicians, Birkhäuser Boston, Boston, MA, 1995.
- [100] G.-C. Rota, *Ten mathematics problems I will never solve*, Mitt. Dtsch. Math.-Ver., **2**, 45–52, (1998).
- [101] G.-C. Rota, *Baxter operators, an introduction*, In: “Gian-Carlo Rota on Combinatorics, Introductory papers and commentaries”, J.P.S. Kung Ed., Contemp. Mathematicians, Birkhäuser Boston, Boston, MA, 1995.
- [102] G.-C. Rota, *Hopf algebras in combinatorics*, in “Hopf algebra methods in combinatorics. Problèmes combinatoires et thorie des graphes”, (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), 363–365, Colloq. Internat. CNRS, 260, CNRS, Paris, 1978. Reprinted in: “Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries”, J.P.S. Kung Ed., Contemp. Mathematicians, Birkhäuser Boston, Boston, MA, 1995.
- [103] M. Sakakibara, *On the Differential equations of the characters for the Renormalization group*, Mod. Phys. Lett. A, **19**, 1453–1456, (2004). [arXiv:math-ph/0401048](#)
- [104] W. Schmitt, *Incidence Hopf Algebras*, J. of Pure and Appl. Alg., **96**, 299–330, (1994).
- [105] M. A. Semenov-Tian-Shansky, *What is a classical r -matrix?*, Funct. Ana. Appl., **17**, no.4., 254–272, (1983).
- [106] F. Spitzer, *A combinatorial lemma and its application to probability theory*, Trans. Amer. Math. Soc., **82**, 323–339, (1956).
- [107] M. Sweedler, *Hopf Algebras*, W.A. Benjamin, Inc., New York, (1969).
- [108] G. ’t Hooft, M. J. G Veltman, *Diagrammar*, Louvain 1973, Particle Interactions At Very High Energies, Part B*, New York 1973, 177–322 and CERN Geneva - CERN 73-9 (73,REC.OCT).
- [109] V. Turaev, *Loops on surfaces, Feynman diagrams, and trees*, J. Geom. Phys., **53**, no. 4, 461–482, (2005). [arXiv:hep-th/0403266](#)
- [110] V. Turaev, *Coalgebras of words and phrases*, preprint: Aug. 2004, [arXiv:math.QA/0408258](#).
- [111] A. S. Wightman, R. Streeter, *PCT, Spin and Statistics, and all that*, Redwood City, USA: Addison-Wesley (1989)
- [112] A. S. Wightman, “*Orientation*”, in “Renormalization Theory”, proceedings of the Int. School of Math. Phys. “Ettore Majorana”, Erice, Italy, 1975, eds. G. Velo and A. S. Wightman, D. Reidel, Dordrecht, Holland, 1–24, (1976).
- [113] W. Zimmermann, *Convergence of Bogoliubov’s method of renormalization in momentum space*, Comm. in Math. Phys., **15**, 208–234, (1969).