Dyson Schwinger algebras and their applications to physics

Diploma Thesis

by

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Chapter 1

Introduction

In the last century various physicists and mathematicians tried to give a precise meaning to Quantum Field Theory. Even though they developed various successful tools for computations, most of those tools are still lacking a mathematical rigorous meaning. Especially the intuitive but mathematical ill defined path integral troubled the mathematicians in the last decades. One of the advantages of the path integral formulation is the intuitive transition between classical and quantum mechanics. This intuitive integral formulation led to various results concerning the whole Quantum Field Theory. One example is the Slavnov Taylor identities between the invariant charges of a QFT, which follow from a symmetry of the classical action. Only recently Dirk Kreimer and collaborators were able to give a mathematical definition to renormalization and the combinatorics of QFT in terms of Hopf algebras. The aim of this work is to give an introduction to the Hopf algebraic formulation of QFT and to make an attempt to formulate the transition from classical to quantum mechanics with the help of so called Dyson Schwinger algebras. In Chapter 2 we will develop the needed mathematical theory like Hopf algebras, algebraic Birkhoff decomposition and Hochschild cohomology. Chapter 3 will give an introduction to the Hopf algebraic formulation of QFT and renormalization. In 3.3 we will especially see how locality of a QFT is linked to Hochschild cohomology and the so called Dyson Schwinger equation. In Chapter 4 we will introduce Dyson Schwinger algebras and we will see how a relation among classical coupling constants leads to a relation among the corresponding invariant charges. 4.4 deals with the implication for the combinatorics of a QFT when we restrict the coupling constants to a linear subspace and how Slavnov Taylor identities naturally evolve in that formulation. Finally 4.5 is dedicated to the applications to physics.

Chapter 2

Hopf algebra

This chapter only gives an introduction to Hopf algebras. We will focus on a selection of aspects, which are important for the thesis. For further insights I refer the reader to [1] and [2].

2.1 Algebra

Let $\mathbb{K}$ be any of the fields $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{C}$. Let $\mathbf{A}$ and $\mathbf{B}$ be two vector spaces. With $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ we will denote the set of all linear maps $\Phi : \mathbf{A} \to \mathbf{B}$.

Definition 2.1.1 (Algebra)
Let $\mathbf{A}$ be a vector space, m $\in \mathrm{Hom}(\mathbf{A} \otimes \mathbf{A}, \mathbf{A})$ and u $\in \mathrm{Hom}(\mathbf{A},\mathbb{K})$.
($\mathbf{A}$,m, u), or short $\mathbf{A}$, is an (associative and unital) algebra if

1. $m \circ (id \otimes m) = m \circ (m \otimes id)$ (associativity) $\boxed{2.1.1}$
2. $m \circ (u \otimes id) = id = m \circ (id \otimes u)$ (unital property). $\boxed{2.1.2}$

The map u is called the unit map.
A morphism Φ of unital algebras $(\mathbf{A}, m_{\mathbf{A}}, u_{\mathbf{A}})$ and $(\mathbf{B}, m_{\mathbf{B}}, u_{\mathbf{B}})$ is a map $\Phi \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$ so that

1. $\Phi \circ m_{\mathbf{A}} = m_{\mathbf{B}} \circ (\Phi \otimes \Phi)$
2. $\Phi \circ u_{\mathbf{A}} = u_{\mathbf{B}}$.

Remark 2.1.1

i) We will always identify $\mathbf{A} \otimes \mathbb{K}$ with $\mathbf{A}$ ($\mathbf{A} \otimes \mathbb{K} \cong \mathbf{A}$).
ii) By $[m\circ(u\otimes id)](\lambda\otimes a) = \lambda u(1)a = \lambda a \cong \lambda\otimes a$ (for $\lambda \in \mathbb{K}, a \in \mathbf{A}$) the unit map can be identified with the neutral element of the multiplication in $\mathbf{A}$ setting $u(1) = \mathbb{1}$.

iii) We define the iterated products $m^n : \mathbf{A}^{\otimes n+1} \to \mathbf{A}$ recursively by

$$m^{(n+1)} := m \circ (m^n \otimes id) \quad \forall n \in \mathbb{N}.$$

iv) If not stated otherwise we will always consider a unital and associative algebra.

Definition 2.1.2
Let $\mathbf{A}$ be an algebra and $\mathcal{G} \subseteq \mathbf{A}$ be a linear subspace. $\mathcal{G}$ is called an ideal of an algebra if the following conditions hold.

1. $m(\mathbf{A} \otimes \mathcal{G}) \subseteq \mathcal{G} \quad m(\mathcal{G} \otimes \mathbf{A}) \subseteq \mathcal{G}$
2. $\mathrm{im}(u) \nsubseteq \mathcal{G} \Leftrightarrow (\mathbb{K}.\mathbb{1}) \nsubseteq \mathcal{G}$

Proposition 2.1.3
Let $\mathbf{A}$ be an algebra, let $\mathcal{G} \subseteq \mathbf{A}$ be an ideal of an algebra and let $\pi : \mathbf{A} \to \mathbf{A}/\mathcal{G}$ be the projection onto the quotient. There exists a unique algebra structure on $\mathbf{A}/\mathcal{G}$ so that π is a morphism of algebras.

Proof. Set $m_{\mathbf{A}/\mathcal{G}} := \pi \circ m \circ (\pi^{-1} \otimes \pi^{-1})$ and $u_{\mathbf{A}/\mathcal{G}} := \pi \circ u$. This definition is independent of the representatives since

$$\pi \circ m((a + \mathcal{G}) \otimes (b + \mathcal{G})) = \pi \circ m(a \otimes b) + \pi(\mathcal{G}) = \pi \circ m(a \otimes b).$$

We have to show that $m_{\mathbf{A}/\mathcal{G}}$ is associative and $u_{\mathbf{A}/\mathcal{G}}$ is a unit.

1. Associativity

$$\begin{aligned}
&m_{\mathbf{A}/\mathcal{G}} \circ (m_{\mathbf{A}/\mathcal{G}} \otimes id_{\mathbf{A}/\mathcal{G}}) \\
&= \pi \circ m \circ (\pi^{-1} \otimes \pi^{-1}) \circ [(\pi \circ m \circ (\pi^{-1} \otimes \pi^{-1})) \otimes (\pi \circ id_{\mathbf{A}} \circ \pi^{-1})] \\
&= \pi \circ m \circ (m \otimes id_{\mathbf{A}}) \circ [(\pi^{-1} \otimes \pi^{-1}) \otimes \pi^{-1}] \\
&\overset{(2.1.1)}{=} \pi \circ m \circ (id_{\mathbf{A}} \otimes m) \circ [\pi^{-1} \otimes (\pi^{-1} \otimes \pi^{-1})] \\
&= m_{\mathbf{A}/\mathcal{G}} \circ (id_{\mathbf{A}/\mathcal{G}} \otimes m_{\mathbf{A}/\mathcal{G}})
\end{aligned}$$

2. Unital property

$$\begin{aligned}
&m_{\mathbf{A}/\mathcal{G}} \circ (u_{\mathbf{A}/\mathcal{G}} \otimes id_{\mathbf{A}/\mathcal{G}}) = [\pi \circ m \circ (\pi^{-1} \otimes \pi^{-1})] \circ (\pi \circ u \otimes \pi \circ id_{\mathbf{A}} \circ \pi^{-1}) \\
&= \pi \circ m \circ (u \otimes id_{\mathbf{A}}) \circ (id_{\mathbb{K}} \otimes \pi^{-1}) \\
&\overset{(2.1.2)}{=} \pi \circ id_{\mathbf{A}} \circ \pi^{-1} = id_{\mathbf{A}/\mathcal{G}} = m_{\mathbf{A}/\mathcal{G}} \circ (id_{\mathbf{A}/\mathcal{G}} \otimes u_{\mathbf{A}/\mathcal{G}})
\end{aligned}$$

From the definition of $m_{\mathbf{A}/\mathcal{G}}$ and $u_{\mathbf{A}/\mathcal{G}}$ we obtain the relations

$$m_{\mathbf{A}/\mathcal{G}} \circ (\pi \otimes \pi) = \pi \circ m \qquad u_{\mathbf{A}/\mathcal{G}} = \pi \circ u$$

, which show that π is indeed a morphism of algebras.

$\square$

Even though there might be different algebra structures on the quotient we will always consider this unique algebra structure so that the canonical projection is a morphism of algebras.

Definition 2.1.4
Let $\mathbf{A}$ be an algebra and $U \subseteq \mathbf{A}$ be a subset.

1. span(U) is the smallest linear subspace of $\mathbf{A}$ that contains U.
2. $< U >$ is the smallest ideal of an algebra in $\mathbf{A}$ that contains U.

2.2 Coalgebra

Definition 2.2.1 (Coalgebra)
Let $\mathbf{C}$ be a vector space, $\Delta \in \mathrm{Hom}(\mathbf{C}, \mathbf{C} \otimes \mathbf{C})$ and $\epsilon \in \mathrm{Hom}(\mathbf{C}, \mathbb{K})$.
$(\mathbf{C}, \Delta, \epsilon)$, or short $\mathbf{C}$, is called a (coassociative and counital) coalgebra if the following conditions hold

1. $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$ (coassociativity) $\boxed{2.2.1}$
2. $(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta$ (counit property). $\boxed{2.2.2}$

The map ϵ is called the counit.
A morphism of coalgebras $(\mathbf{C}, \Delta_{\mathbf{C}}, \epsilon_{\mathbf{C}})$ and $(\mathbf{D}, \Delta_{\mathbf{D}}, \epsilon_{\mathbf{D}})$ is a map $\Phi \in \mathrm{Hom}(\mathbf{C}, \mathbf{D})$ so that

1. $\Delta_{\mathbf{D}} \circ \Phi = (\Phi \otimes \Phi) \circ \Delta_{\mathbf{C}}$
2. $\epsilon_{\mathbf{D}} \circ \Phi = \epsilon_{\mathbf{C}}$.

Example 1 Let $(\mathbf{A}, m, u)$ be a finite dimensional algebra and let $\mathbf{A}^*$ be the dual vector space. A coalgebra $(\mathbf{A}^*, \Delta, \epsilon)$ can be obtained by dualizing $(\mathbf{A}, m, u)$. Let $\mathbf{U}$ and $\mathbf{V}$ be two vector spaces and let $F : \mathbf{U} \to \mathbf{V}$ be a linear map. The conjugate $F^* : \mathbf{V}^* \to \mathbf{U}^*$ of F is defined by $(F^* \alpha_{\mathbf{V}}) = \alpha_{\mathbf{V}} \circ F$ $\forall \alpha_{\mathbf{V}} \in \mathbf{V}^*$. Conjugating m and u leads to $m^* : \mathbf{A}^* \otimes \mathbf{A}^* :\to \mathbf{A}^*$, $u^* : \mathbf{A}^* \to \mathbb{K}$ and the relations below.

$$(id \otimes m^*) \circ m^* = (m^* \otimes id) \circ m^*$$

$$(u^* \otimes m^*) \circ m^* = id = (m^* \otimes u^*) \circ m^*$$

To conclude, $(\mathbf{A}^*,\ m^*, u^*)$ is a coalgebra. We used that from $dim(\mathbf{A}) < \infty$ follows $(A \otimes A)^* = A^* \otimes A^*$.

Remark 2.2.1

i) The counit ϵ of a coalgebra $\mathbf{C}$ is unique, if it exists. This is shown by

$$\epsilon = \epsilon \circ id = \epsilon \circ (id \otimes \epsilon^{`}) \circ \Delta = (\epsilon \otimes \epsilon^{`}) \circ \Delta = \epsilon^{`} \circ (\epsilon \otimes id) \circ \Delta = \epsilon^{`}.$$

ii) The iterated coproducts $\Delta^n : \mathbf{C} \to \mathbf{C}^{\otimes(n+1)}$ are defined recursively by

$$\Delta^0 := id \quad \text{and} \quad \Delta^{n+1} := (id^{\otimes k} \otimes \Delta \otimes id^{\otimes(n-k)}) \circ \Delta^n$$

$$\forall n, k \in \mathbb{N} : 0 \leq k \leq n.$$

This definition is independent of k, due to the coassociativity condition.

iii) If $g \in \mathbf{C}$ with $\Delta(g) = g \otimes g$ we call g a grouplike element. The set of all grouplike elements of $\mathbf{C}$ is $\mathrm{Grp}(\mathbf{C})$.

iv) With equation (2.2.2) we obtain for any grouplike element g, $\epsilon(g) = 1$. R 2.2.1.iv

Lemma 2.2.2

$$\Delta^n = (id \otimes \Delta^{n-1}) \circ \Delta$$

Proof. This follows from the coassociativity of Δ and can be shown by induction.

1. $\Delta^1 = (id \otimes id) \circ \Delta = (id \otimes \Delta^0) \circ \Delta$
2. $\Delta^n = (id \otimes \Delta^{n-1}) \circ \Delta$
3. $\Delta^{n+1} = (id^{\otimes n} \otimes \Delta) \circ \Delta^n = (id \otimes id^{\otimes(n-1)} \otimes \Delta) \circ (id \otimes \Delta^{n-1}) \circ \Delta$ $(id \otimes [(id^{\otimes(n-1)} \otimes \Delta) \circ \Delta^{n-1}]) \circ \Delta = (id \otimes \Delta^n) \circ \Delta$

□

NOTE 1 The proof of the lemma only used the coassociativity of Δ.

Definition 2.2.3 (Comodule)
Let $\mathbf{C}$ be a coalgebra, let $\mathbf{N}$ be a linear space and let
$\Psi_L \in \mathrm{Hom}(\mathbf{N}, \mathbf{C} \otimes \mathbf{N}), \Psi_R \in \mathrm{Hom}(\mathbf{N}, \mathbf{N} \otimes \mathbf{C})$ be two linear maps.

1. $(\mathbf{N}, \Psi_L)$ is called a left $\mathbf{C}$-comodule if

$$(id_{\mathbf{C}} \otimes \Psi_L) \circ \Psi_L = (\Delta \otimes id_{\mathbf{N}}) \circ \Psi_L \quad \text{and} \quad (\epsilon \otimes id_{\mathbf{N}}) \circ \Psi_L = id_{\mathbf{N}}.$$

2. $(\mathbf{N}, \Psi_R)$ is called a right $\mathbf{C}$-comodule if

$$(\Psi_R \otimes id_{\mathbf{C}}) \circ \Psi_R = (id_{\mathbf{N}} \otimes \Delta) \circ \Psi_R \quad \text{and} \quad (id_{\mathbf{N}} \otimes \epsilon) \circ \Psi_R = id_{\mathbf{N}}.$$

3. $(\mathbf{N}, \Psi_L, \Psi_R)$ is called a $\mathbf{C}$-comodule if $(\mathbf{N}, \Psi_L)$ is a left $\mathbf{C}$-comodule, $(\mathbf{N}, \Psi_R)$ is a right $\mathbf{C}$-comodule and

$$(id_{\mathbf{C}} \otimes \Psi_R) \circ \Psi_L = (\Psi_L \otimes id_{\mathbf{C}}) \circ \Psi_R.$$

Example 1 Every coalgebra $\mathbf{C}$ is a $\mathbf{C}$-comodule. Just set $\Psi_L = \Delta = \Psi_R$.

Definition 2.2.4 (Coideal)
Let $\mathbf{C}$ be a coalgebra and let $\mathcal{I} \subseteq \mathbf{C}$ be a linear subspace. $\mathcal{I}$ is called a coideal $\Leftrightarrow$

1. $\Delta(\mathcal{I}) \subseteq \mathcal{I} \otimes \mathbf{C} + \mathbf{C} \otimes \mathcal{I}$
2. $\epsilon(\mathcal{I}) = 0$

Proposition 2.2.5
Let $\mathbf{C}$ be a coalgebra and let $\mathcal{I} \subseteq \mathbf{C}$ be a coideal. There exists a unique coalgebra structure on $\mathbf{C}/\mathcal{I}$ so that the canonical projection $\pi : \mathbf{C} \to \mathbf{C}/\mathcal{I}$ is a morphism of coalgebras.

Proof. Set $\Delta_{\mathbf{C}/\mathcal{I}} = (\pi \otimes \pi) \circ \Delta \circ \pi^{-1}$ and $\epsilon_{\mathbf{C}/\mathcal{I}} = \epsilon \circ \pi^{-1}$. Let $[a] \in \mathbf{C}/\mathcal{I}$ be an element of the quotient and let $a, b \in \mathbf{C}$ be two representatives of $[a] \quad \Rightarrow \quad (b - a) \in \mathcal{I}$.

$$\Delta(b-a) \in \mathbf{C} \otimes \mathcal{I} + \mathcal{I} \otimes \mathbf{C} \Rightarrow (\pi \otimes \pi) \circ \Delta(b-a) = 0 \quad \text{since} \quad \pi(\mathcal{I}) = 0$$

$$\epsilon(b-a) = 0 \quad \text{since} \quad \epsilon(\mathcal{I}) = 0$$

Thus the definition is independent of the representative. We now have to show that $\Delta_{\mathbf{C}/\mathcal{I}}$ is coassociative and $\epsilon_{\mathbf{C}/\mathcal{I}}$ is a counit.

1. Coassociativity

$$\begin{aligned}
&(id_{\mathbf{C}/\mathcal{I}} \otimes \Delta_{\mathbf{C}/\mathcal{I}}) \circ \Delta_{\mathbf{C}/\mathcal{I}} \\
&= \left([\pi \circ id_{\mathbf{C}} \circ \pi^{-1}] \otimes [(\pi \otimes \pi) \circ \Delta \circ \pi^{-1}]\right) \circ (\pi \otimes \pi) \circ \Delta \circ \pi^{-1} \\
&= (\pi \otimes \pi \otimes \pi) \circ (id_{\mathbf{C}} \otimes \Delta) \circ \Delta \circ \pi^{-1} \\
&\overset{(2.2.1)}{=} (\pi \otimes \pi \otimes \pi) \circ (\Delta \otimes id_{\mathbf{C}}) \circ \Delta \circ \pi^{-1} \\
&= (\Delta_{\mathbf{C}/\mathcal{I}} \otimes id_{\mathbf{C}/\mathcal{I}}) \circ \Delta_{\mathbf{C}/\mathcal{I}}
\end{aligned}$$

2. Counit property

$$(id_{\mathbf{C}/\mathcal{I}} \otimes \epsilon_{\mathbf{C}/\mathcal{I}}) \circ \Delta_{\mathbf{C}/\mathcal{I}} = ([\pi \circ id_{\mathbf{C}} \circ \pi^{-1}] \otimes [\epsilon \circ \pi^{-1}]) \circ (\pi \otimes \pi) \circ \Delta \circ \pi^{-1}$$
$$= \pi \circ (id_{\mathbf{C}} \otimes \epsilon) \circ \Delta \circ \pi^{-1} \stackrel{(2.2.2)}{=} \pi \circ id_{\mathbf{C}} \circ \pi^{-1} = id_{\mathbf{C}/\mathcal{I}} = (\epsilon_{\mathbf{C}/\mathcal{I}} \otimes id_{\mathbf{C}/\mathcal{I}}) \circ \Delta_{\mathbf{C}/\mathcal{I}}$$

From the definition of $\Delta_{\mathbf{C}/\mathcal{I}}$ and $\epsilon_{\mathbf{C}/\mathcal{I}}$ one can obtain

$$\Delta_{\mathbf{C}/\mathcal{I}} \circ \pi = (\pi \otimes \pi) \circ \Delta \quad \text{and} \quad \epsilon_{\mathbf{C}/\mathcal{I}} \circ \pi = \epsilon$$

, which shows that π is indeed a morphism of coalgebras. □

Even though there might be different coalgebra structures on the quotient we will always consider this unique coalgebra structure so that the canonical projection is a morphism of coalgebras.

2.3 Bialgebra

Definition 2.3.1
Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras and let $\mathbf{C}$ and $\mathbf{D}$ be two coalgebras.
Define $\sigma : \mathbf{M}_1 \otimes \mathbf{M}_2 \otimes \mathbf{M}_3 \otimes \mathbf{M}_4 \to \mathbf{M}_1 \otimes \mathbf{M}_3 \otimes \mathbf{M}_2 \otimes \mathbf{M}_4$
through $\sigma(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = x_1 \otimes x_3 \otimes x_2 \otimes x_4$.

1. $\mathbf{A} \otimes \mathbf{B}$ can be equipped with the structure of an algebra if one defines

$$m_{\mathbf{A}\otimes\mathbf{B}} := (m_{\mathbf{A}} \otimes m_{\mathbf{B}}) \circ \sigma \quad \text{and} \quad u_{\mathbf{A}\otimes\mathbf{B}} := u_{\mathbf{A}} \otimes u_{\mathbf{B}}.$$

2. $\mathbf{C} \otimes \mathbf{D}$ can be equipped with the structure of a coalgebra if one defines

$$\Delta_{\mathbf{C}\otimes\mathbf{D}} := \sigma \circ (\Delta_{\mathbf{C}} \otimes \Delta_{\mathbf{D}}) \quad \text{and} \quad \epsilon_{\mathbf{C}\otimes\mathbf{D}} := \epsilon_{\mathbf{C}} \otimes \epsilon_{\mathbf{D}}.$$

This definition can be extended to higher tensor products inductively.

Definition 2.3.2 (Bialgebra)
Let $\mathbf{B}$ be a vector space, $m \in \mathrm{Hom}(\mathbf{B} \otimes \mathbf{B}, \mathbf{B})$, $\Delta \in \mathrm{Hom}(\mathbf{B}, \mathbf{B} \otimes \mathbf{B})$, $u \in \mathrm{Hom}(\mathbb{K}, \mathbf{B})$ and $\epsilon \in \mathrm{Hom}(\mathbf{B}, \mathbb{K})$. $(\mathbf{B}, m, u, \Delta, \epsilon)$ (short $\mathbf{B}$) is called a bialgebra $\Leftrightarrow$ $(\mathbf{B}, m, u)$ is an algebra, $(\mathbf{B}, \Delta, \epsilon)$ is a coalgebra and the following conditions hold:

1. m is a morphism of coalgebras $\Leftrightarrow$ Δ is a morphism of algebras.

$$\Delta \circ m = m_{\mathbf{B}\otimes\mathbf{B}} \circ (\Delta \otimes \Delta) \quad [= (m \otimes m) \circ \sigma \circ (\Delta \otimes \Delta)]$$
$$\Delta \circ m = (m \otimes m) \circ \Delta_{\mathbf{B}\otimes\mathbf{B}} \quad [= (m \otimes m) \circ \sigma \circ (\Delta \otimes \Delta)]\,.$$

2. u is a morphism of coalgebras: $\Delta \circ u = u \otimes u \Leftrightarrow \Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$

3. ϵ is a morphism of algebras: $\epsilon \otimes \epsilon = m_{\mathbb{K}} \circ (\epsilon \otimes \epsilon) = \epsilon \circ m$. $\boxed{2.3.1}$

A morphism of bialgebras is a morphism of algebras and a morphism of coalgebras.

Remark 2.3.1

i) Sine $\mathbb{1}$ is a grouplike element it follows that $\epsilon(\mathbb{1}) = 1$ (see remark (R 2.2.1.iv)). Note the following

$$(u \circ \epsilon) : \mathbf{B} \twoheadrightarrow \mathrm{im}(u) = \mathbb{1}.\mathbb{K}, \qquad \mathrm{Ker}(u \circ \epsilon) = \mathrm{Ker}(\epsilon) \Rightarrow$$
$$(\mathbb{K}.\mathbb{1}) = \mathrm{im}(u) \stackrel{\sim}{=} \mathbf{B}/\mathrm{Ker}(\epsilon).$$

So $\mathbf{B}$ naturally decomposes into

$$\mathbf{B} = \mathbb{K}.\mathbb{1} \oplus \mathrm{Ker}(\epsilon).$$

We denote the projection onto $\mathrm{Ker}(\epsilon)$ by $\mathrm{P} := id - u \circ \epsilon := \mathbf{B} \twoheadrightarrow \mathrm{Ker}(\epsilon)$ and call $\mathrm{Ker}(\epsilon)$ the augmentation ideal.

ii) From equation (2.2.2) one can follow that $\forall x \in \mathrm{Ker}(\epsilon)$ the following holds

$$\Delta(x) = x \otimes \mathbb{1} + \mathbb{1} \otimes x + \mathrm{Ker}(\epsilon) \otimes \mathrm{Ker}(\epsilon).$$

Example 2 (Faá di Bruno)
This is a very important example. As we will see a multi-dimensional version of the Faá di Bruno algebra is strongly related to the physicist invariant charge.
Let

$$\mathrm{Diff}^0 = \{P \in \mathbb{K}[[x]] : P(x) = x + \sum_{n \geqslant 2} p_n x^n\}$$

be the set of formal diffeomorphisms tangent to the identity.
Choose $P, Q \in \mathrm{Diff}^0$ with coefficients p_n, q_n resp. We define linear maps $a_n : \mathrm{Diff}^0 \to \mathbb{K}$ with $a_n(P) = p_n$ and the set $\mathbf{B}_{FDB} := \{a_n\}_{\forall n \geqslant 1}$. Note that by definition $a_1 \equiv 1 \in \mathbb{K}$. One can now define the following

1. $u \in \mathrm{Hom}(\mathbb{K}, \mathbf{B}_{FDB}) \quad u(1) = \mathbb{1} := a_1$

2. $m(a_n \otimes a_m)(P) = a_n(P).a_m(P) = p_n.p_m \quad \forall P \in \mathrm{Diff}^0$

3. $\epsilon \in \mathrm{Hom}(\mathbf{B}_{FDB}, \mathbb{K}) \quad \epsilon(\mathbb{1}) = 1; \qquad \epsilon(a_n) = 0 \quad \forall n \geqslant 2$

4. $\Delta(a_n)(P \otimes Q) = a_n(Q \circ P) \quad \forall P, Q \in \text{Diff}^0$.
   Note: Δ is well-defined since $Q \circ P \in \text{Diff}^0$

By straightforward computation one may follow the below relation.

$$\Delta(a_N) = \sum_{1 \leq n \leq N} \sum_{m_1 + \cdots + m_n = N} a_{m_1} \cdots a_{m_n} \otimes a_n. \qquad \boxed{2.3.2}$$

Δ is extended to products so that it is a morphism of algebras.

Lemma 2.3.3

$(\mathbf{B}_{FDB}, m, u, \Delta, \epsilon)$ is a (connected*) bialgebra.

*Connected bialgebras will be considered in the next section

Proof. The bialgebra properties follow from those of $\mathbb{K}$.

1. The associativity of m follows from the associativity of $\mathbb{K}$.
2. Since $a_1 = \mathbb{1} = 1 \in \mathbb{K}$, it follows that $m \circ (u \otimes id) = m \circ (id \otimes u) = id$.
3. $(\Delta \otimes id) \circ \Delta = (\Delta \otimes id) \circ \Delta$ can bee seen by straightforward computation.
4. By inspection of equation (2.3.2) one can observe that

$$\begin{aligned} &\Delta(a_N) = a_N \otimes \mathbb{1} + \mathbb{1} \otimes a_N + \alpha \otimes \beta \quad \alpha, \beta \in \text{Ker}(\epsilon) \\ \Rightarrow &(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta. \end{aligned}$$

5. $\Delta \circ m = (m \otimes m) \circ \sigma \circ (\Delta \otimes \Delta)$ by definition.
6. Again by inspection of equation (2.3.2) one can observe that $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$ so u is a morphism of coalgebras.
7. $\epsilon \otimes \epsilon = \epsilon \circ m$ follows from the following considerations

$$\begin{aligned} \epsilon \otimes \epsilon(\lambda \mathbb{1} \otimes \mathbb{1}) = \lambda \quad \text{otherwise} \quad \epsilon \otimes \epsilon = 0 \\ \epsilon \circ m(\lambda \mathbb{1} \otimes \mathbb{1}) = \lambda \quad \text{otherwise} \quad \epsilon \circ m = 0 \end{aligned}$$

□

Define the following formal power series

$$A(x) := \sum_{n \geqslant 1} a_n x^n \Rightarrow \Delta(A(x)) = \sum_{n \geqslant 1} A(x)^n \otimes a_n. \qquad \boxed{2.3.3}$$

$(\mathbf{B}_{FDB}, m, u, \Delta, \epsilon)$ is called the Faa di Bruno algebra.

Definition 2.3.4
Let $\mathbf{B}$ be a bialgebra and $\mathcal{I} \subseteq \mathbf{B}$ be a linear subspace. $\mathcal{I}$ is called an ideal (of a bialgebra) if $\mathcal{I}$ is an ideal of an algebra and a coideal.

NOTE 2 Since $\mathbf{B} = \mathrm{im}(u) \oplus \mathrm{Ker}(\epsilon)$, the conditions $\mathrm{im}(u) \nsubseteq \mathcal{I}$ and $\mathcal{I} \subseteq \mathrm{Ker}(\epsilon)$ are equivalent.

Example 3 The augmentation ideal $\mathrm{Ker}(\epsilon)$ is an ideal.

1. $\epsilon(\mathrm{Ker}(\epsilon)) = 0$ True
2. $0 = \Delta \circ \epsilon(\mathrm{Ker}(\epsilon)) = (\epsilon \otimes \epsilon) \circ \Delta(\mathrm{Ker}(\epsilon)) \Rightarrow$
   $\Delta(\mathrm{Ker}(\epsilon)) \subseteq \mathrm{Ker}(\epsilon) \otimes \mathbf{B} + \mathbf{B} \otimes \mathrm{Ker}(\epsilon)$
3. $\mathbb{1} \notin \mathrm{Ker}(\epsilon)$
4. From equation (2.3.1) one can obtain $\mathrm{Ker}(\epsilon).\mathbf{B} \subseteq \mathrm{Ker}(\epsilon)$ and $\mathbf{B}.\mathrm{Ker}(\epsilon) \subseteq \mathrm{Ker}(\epsilon)$.

Definition 2.3.5
On a bialgebra $\mathbf{B}$ we define the reduced coproduct $\tilde{\Delta}$.

$$\tilde{\Delta} = (\mathrm{P} \otimes \mathrm{P}) \circ \Delta$$

The space of primitive elements $\mathrm{Prim}_1(\mathbf{B})$ is

$$\mathrm{Prim}_1(\mathbf{B}) := \mathrm{Ker}(\tilde{\Delta}) \cap \mathrm{Ker}(\epsilon) = \{p \in \mathbf{B} : \Delta(p) = \mathbb{1} \otimes p + p \otimes \mathbb{1}\}.$$

Remark 2.3.2

i) Choose some $p \in \mathrm{Ker}(\epsilon)$. One obtains for the reduced coproduct of p

$$\tilde{\Delta}(p) = \Delta(p) - \mathbb{1} \otimes p - p \otimes \mathbb{1}.$$

ii) The reduced coproduct is coassociative, which allows to define an iterated product

$$\tilde{\Delta}^0 := id \quad and \quad \tilde{\Delta}^{n+1} := (id^{\otimes k} \otimes \tilde{\Delta} \otimes id^{\otimes(n-k)}) \circ \tilde{\Delta}^n$$

$$\forall n, k \in \mathbb{N} : 0 \leq k \leq n.$$

iii) From the definition of $\tilde{\Delta}$ follows $\tilde{\Delta}(\mathrm{Ker}(\epsilon)) \subseteq \mathrm{Ker}(\epsilon) \otimes \mathrm{Ker}(\epsilon)$.

iv) As in lemma (2.2.2) one can show that $\tilde{\Delta}^n = (id \otimes \tilde{\Delta}^{n-1}) \circ \tilde{\Delta}$.

Proposition 2.3.6
Let $\mathbf{B}$ be a bialgebra, let $\mathcal{I} \subseteq \mathbf{B}$ be an ideal and let $\pi : \mathbf{B} \to \mathbf{B}/\mathcal{I}$ be the canonical projection. There exists a unique bialgebra structure on $\mathbf{B}/\mathcal{I}$ so that π is a morphism of bialgebras.

Proof. Since $\mathcal{I}$ is an ideal of an algebra and a coideal, there exists a unique structure on $\mathbf{B}/\mathcal{I}$ so that π is a morphism of algebras and a morphism of coalgebras, which is precisely the definition of a morphism of bialgebras. $\square$

2.4 Filtrations and connectedness

Definition 2.4.1
A family $(\mathbf{B}^n)_{n\in\mathbb{N}}$ of growing subspaces $\mathbf{B}^n \subseteq \mathbf{B}^{n+1}$ of a bialgebra $(\mathbf{B}, m, u, \Delta, \epsilon)$ is called a filtration if following conditions hold.

1. $\mathbf{B} = \sum_{n\geqslant 0} \mathbf{B}^n$
2. $\Delta(\mathbf{B}^n) \subseteq \sum_{i+j=n} \mathbf{B}^i \otimes \mathbf{B}^j = \sum_{0\leq i\leq n} \mathbf{B}^{n-i} \otimes \mathbf{B}^i \quad \forall n \in \mathbb{N}$
3. $\mathbf{B}^n.\mathbf{B}^m = m(\mathbf{B}^n \otimes \mathbf{B}^m) \subseteq \mathbf{B}^{n+m} \quad \forall n, m \in \mathbb{N}$

Remark 2.4.1

i) From the definition of a filtration follows that $\mathbf{B}^0$ is a subbialgebra.

ii) Every grouplike element is contained in $\mathbf{B}^0$.

Definition 2.4.2 (Wedge product)
Let $\mathbf{B}$ be a bialgebra and let U and W be two linear subspaces of $\mathbf{B}$. The associative wedge product is defined by $U \wedge W := \Delta^{-1}(U \otimes \mathbf{B} + \mathbf{B} \otimes W)$ and $U^{\wedge(n+1)} = U^{\wedge n} \wedge U = (\Delta^n)^{-1}(\sum_{0\leq i\leq n} \mathbf{B}^{\otimes i} \otimes U \otimes \mathbf{B}^{\otimes(n-i)})$
NOTE 3 Choose some $p \in U \wedge W$. This is equivalent to

$$\Delta(p) \in U \otimes \mathbf{B} + \mathbf{B} \otimes W.$$

Proposition 2.4.3
Let $\mathbf{B}$ be a bialgebra and $L \subseteq \mathbf{B}$ be a subbialgebra.
There exists a filtration of $\mathbf{B}$ starting with $L \Leftrightarrow$

$$\mathbf{B} = \sum_{n\geqslant 0} L^{\wedge(n+1)}$$

A filtration is then given by $\mathbf{B}^n = L^{\wedge(n+1)}$.

Proof. A proof can be found in [1]. $\square$

Definition 2.4.4
A bialgebra $\mathbf{B}$ is called connected if there exists a filtration $\mathbf{B} = \sum_{n\geqslant 0} \mathbf{B}^n$ so that $\mathbf{B}^0 = (\mathbb{K}.\mathbb{1})$. By proposition (2.4.3) this is equivalent to

$$\mathbf{B} = \sum_{n\geqslant 0} (\mathbb{K}.\mathbb{1})^{\wedge(n+1)}.$$

Definition 2.4.5
Set $Prim_n(\mathbf{B}) := \mathrm{Ker}(\tilde{\Delta}^n) \cap \mathrm{Ker}(\epsilon) \quad \forall n \geqslant 1$.

Proposition 2.4.6
Let $\mathbf{B}$ be a bialgebra. Set $\tilde{\mathbf{B}}^n = (\mathbb{K}.\mathbb{1})^{\wedge(n+1)}$.

$$Prim_n(\mathbf{B}) = \tilde{\mathbf{B}}^n \cap \mathrm{Ker}(\epsilon) \quad \forall n \geqslant 1$$

Proof. This can be shown by induction.

1.

$$\begin{aligned}
x \in Prim_1(\mathbf{B}) \Rightarrow & \Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1} \\
& \in \tilde{\mathbf{B}}^0 \otimes \mathbf{B} + \mathbf{B} \otimes \tilde{\mathbf{B}}^0 \\
\Rightarrow & Prim_1(\mathbf{B}) \subseteq \tilde{\mathbf{B}}^1 \cap \mathrm{Ker}(\epsilon) \\
x \in \tilde{\mathbf{B}}^1 \cap \mathrm{Ker}(\epsilon) \Rightarrow & \Delta(x) \in \tilde{\mathbf{B}}^0 \otimes \mathbf{B} + \mathbf{B} \otimes \tilde{\mathbf{B}}^0 \Leftrightarrow \Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1} \\
\Rightarrow & Prim_1(\mathbf{B}) = \tilde{\mathbf{B}}^1 \cap \mathrm{Ker}(\epsilon)
\end{aligned}$$

In the second line from below we used the counit property(see equation (2.2.2)).

2. $Prim_n(\mathbf{B}) = \tilde{\mathbf{B}}^n \cap \mathrm{Ker}(\epsilon)$

3.

$$\begin{aligned}
x \in Prim_{n+1}(\mathbf{B}) \Rightarrow & \tilde{\Delta}(x) \in \mathrm{Ker}(id \otimes \tilde{\Delta}^n) \cap \mathrm{Ker}(\epsilon) \subseteq \mathbf{B} \otimes Prim_n(\mathbf{B}) = \mathbf{B} \otimes \tilde{\mathbf{B}}^n \\
\Rightarrow & \Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1} + \tilde{\Delta}(x) \in \tilde{\mathbf{B}}^0 \otimes \mathbf{B} + \mathbf{B} \otimes \tilde{\mathbf{B}}^n \\
\Rightarrow & x \in \tilde{\mathbf{B}}^{n+1} \cap \mathrm{Ker}(\epsilon) \\
x \in \tilde{\mathbf{B}}^{n+1} \cap \mathrm{Ker}(\epsilon) \Rightarrow & \Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1} + \tilde{\Delta}(x) \in \tilde{\mathbf{B}}^0 \otimes \mathbf{B} + \mathbf{B} \otimes \tilde{\mathbf{B}}^n \\
\Rightarrow & \tilde{\Delta}(x) \in \mathbf{B} \otimes \left(\tilde{\mathbf{B}}^n \cap \mathrm{Ker}(\epsilon)\right) = \mathbf{B} \otimes Prim_n(\mathbf{B}) \\
\Rightarrow & x \in Prim_{n+1}(\mathbf{B}) \\
\Rightarrow & Prim_{n+1}(\mathbf{B}) = \tilde{\mathbf{B}}^{n+1} \cap \mathrm{Ker}(\epsilon)
\end{aligned}$$

$\square$

Corollary 2.4.7
Let $\mathbf{B}$ be a connected bialgebra.

$$\mathbf{B} = \mathbb{1}.\mathbb{K} \oplus \operatorname{Ker}(\epsilon) = \mathbb{1}.\mathbb{K} \oplus \sum_{n\geqslant 0} \tilde{\mathbf{B}}^n \cap \operatorname{Ker}(\epsilon) = \mathbb{1}.\mathbb{K} \oplus \sum_{n\geqslant 1} Prim_n(\mathbf{B}) \Rightarrow$$

$$\forall x \in \operatorname{Ker}(\epsilon) \quad \exists N \geqslant 1 : \tilde{\Delta}^n(x) = 0 \quad \forall n \geq N$$

We call the minimal N so that $\tilde{\Delta}^N(x) = 0$ the augmentation degree of x and denote it with $|x|_{aug}$.

Definition 2.4.8 (Graduation)
A graduation of a bialgebra $\mathbf{B}$ is a decomposition $\mathbf{B} = \oplus_{n\in\mathbb{N}}\mathbf{B}^n$ so that for any $n, m \in \mathbb{N}$

1. $\mathbf{B}^n.\mathbf{B}^m \subseteq \mathbf{B}^{n+m}$
2. $\Delta(\mathbf{B}^n) \subseteq \oplus_{0\leq j\leq n}\mathbf{B}^j \otimes \mathbf{B}^{n-j}$.

NOTE 4 Every graduation is also a filtration, just set $\bar{\mathbf{B}}^n = \oplus_{0\leq j\leq n}\mathbf{B}^n$. Then $\bar{\mathbf{B}}^n$ is a filtration.

2.5 The convolution product

Definition 2.5.1 (Convolution Product)
Let $(\mathbf{C}, \Delta, \epsilon)$ be a coalgebra, let $(\mathbf{A}, m, u)$ be an algebra.
The convolution product $\star$ is defined for two linear maps $\phi, \psi \in \operatorname{Hom}(\mathbf{C}, \mathbf{A})$ by

$$\phi \star \psi := m \circ \phi \otimes \psi \circ \Delta.$$

From the definition of the convolution product follows $\phi \star \psi \in \operatorname{Hom}(\mathbf{C}, \mathbf{A})$.

Lemma 2.5.2
Set $e := u \circ \epsilon$.
e is the identity for the convolution product.
That means $\psi \star e = e \star \psi = \psi \quad \forall \psi \in \operatorname{Hom}(\mathbf{C}, \mathbf{A})$.

Proof.

$$\begin{aligned}\psi \star e &= m \circ (\psi \otimes u \circ \epsilon) \circ \Delta = m \circ (id \otimes u) \circ (\psi \otimes id) \circ (id \otimes \epsilon) \circ \Delta \\ &\overset{(2.1.2)}{=} (\psi \otimes id) \circ (id \otimes \epsilon) \circ \Delta \overset{(2.2.2)}{=} \psi\end{aligned}$$

The other relation can be shown identically. $\square$

Lemma 2.5.3

$$\psi_1 \star \cdots \star \psi_{n+1} = m^n \circ \psi_1 \otimes \cdots \otimes \psi_{n+1} \circ \Delta^n$$

Proof. This can be shown by induction.

1. $\psi_1 \star \psi_2 = m \circ (id \otimes id) \circ (\psi_1 \otimes \psi_2) \circ (id \otimes id) \circ \Delta =$
   $m \circ (id \otimes m^0) \circ (\psi_1 \otimes \psi_2) \circ (id \otimes \Delta^0) \circ \Delta =$
   $m^1 \circ (\psi_1 \otimes \psi_2) \circ \Delta^1$

2. $\psi_1 \star \cdots \star \psi_n = m^{n-1} \circ (\psi_1 \otimes \cdots \otimes \psi_n) \circ \Delta^{n-1}$

3. $\psi_1 \star (\psi_2 \star \cdots \star \psi_{n+1}) = m \circ (id \otimes m^{n-1}) \circ (\psi_1 \otimes \cdots \otimes \psi_{n+1}) \circ (id \otimes \Delta^{n-1}) \circ \Delta$
   $= m^n \circ (\psi_1 \otimes \cdots \otimes \psi_{n+1}) \circ \Delta^n$

□

Definition 2.5.4
Let $\mathbf{B}$ be a bialgebra and let $\mathbf{A}$ be an algebra.
Set $G_{\mathbf{A}}^{\mathbf{B}} := \{\phi \in \mathrm{Hom}(\mathbf{B}, \mathbf{A}) | \phi(\mathbb{1}_{\mathbf{B}}) = \mathbb{1}_{\mathbf{A}}\}$.
NOTE 5 Since $e(\mathbb{1}) = u_{\mathbf{A}} \circ \epsilon_{\mathbf{B}}(\mathbb{1}_{\mathbf{B}}) = u_{\mathbf{A}}(1) = \mathbb{1}_{\mathbf{A}}$, one can observe that $e \in G_{\mathbf{A}}^{\mathbf{B}}$.
Lemma 2.5.5
Let $\mathbf{B}$ be a bialgebra, let $\mathbf{A}$ be an algebra and $\phi \in G_{\mathbf{A}}^{\mathbf{B}}$.

$$(e - \phi)^{\otimes(n+1)} \circ \Delta^n = (e - \phi)^{\otimes(n+1)} \circ \tilde{\Delta}^n \quad \forall n \geqslant 1$$

Proof. Note the following.
Since $e, \phi \in G_{\mathbf{A}}^{\mathbf{B}}$ and $\Delta^n(\mathbb{1}) = \mathbb{1}^{\otimes(n+1)}$, one can obtain

$$(e - \phi)^{\otimes(n+1)} \circ \Delta^n(\mathbb{1}) = 0 = (e - \phi)^{\otimes(n+1)} \circ \tilde{\Delta}^n(\mathbb{1}) \quad \forall n \geqslant 1.$$

The relation can be shown by induction. Choose any $x \in \mathbf{B}$.

1. $(e - \phi)^{\otimes 2} \circ \Delta^1(x) = (e - \phi) \otimes (e - \phi) \circ (\tilde{\Delta}(x) - \mathbb{1} \otimes x - x \otimes \mathbb{1}) =$
   $(e - \phi) \otimes (e - \phi) \circ \tilde{\Delta}(x)$

2. $(e - \phi)^{\otimes n} \circ \Delta^{n-1}(x) = (e - \phi)^{\otimes n} \circ \tilde{\Delta}^{n-1}(x)$

3. $(e - \phi)^{\otimes(n+1)} \circ \Delta^n(x) = (e - \phi) \otimes (e - \phi)^{\otimes n} \circ (id \otimes \Delta^{n-1}) \circ \Delta(x)$
   $= (e - \phi) \otimes [(e - \phi)^{\otimes n} \circ \Delta^{n-1}] \circ (\tilde{\Delta}(x) - \mathbb{1} \otimes x - x \otimes \mathbb{1}))$
   $= (e - \phi) \otimes [(e - \phi)^{\otimes n} \circ \Delta^{n-1}] \circ \tilde{\Delta}(x)$
   $= (e - \phi) \otimes [(e - \phi)^{\otimes n} \circ \tilde{\Delta}^{n-1}] \circ \tilde{\Delta}(x) = (e - \phi)^{\otimes(n+1)} \circ \tilde{\Delta}^n(x)$

□

Theorem 2.5.6
Let $\mathbf{B}$ be a connected bialgebra and let $\mathbf{A}$ be an algebra. Then $(G_{\mathbf{A}}^{\mathbf{B}}, \star)$ is a group.

Proof. Choose some $\phi, \psi \in G_{\mathbf{A}}^{\mathbf{B}}$

1. $\psi \star \phi(\mathbb{1}) = m \circ \psi \otimes \phi \circ \Delta(\mathbb{1}) = \psi(\mathbb{1}).\phi(\mathbb{1}) = \mathbb{1}$

2. Define $\psi^{\star 0} = e$, $\psi^{\star 1} = \psi$ and with these definitions set

$$\phi^{\star -1} := \sum_{n \geqslant 0} (e - \phi)^{\star n}.$$

Let x be an element of $\mathbf{B}$. From the connectedness of $\mathbf{B}$ follows that $\tilde{\Delta}^n(x) = 0 \quad \forall n \geq |x|_{aug} < \infty$. Together with lemma (2.5.5) one can observe that $\phi^{\star -1}(x) = \sum_{n < |x|_{aug}} (e - \phi)^{\star n}(x)$ is a finite sum $\forall x \in \mathbf{B}$. Further, note that since $(e - \phi)^{\star n}(\mathbb{1}) = 0 \; \forall n \geqslant 1$ one obtains $\phi^{\star -1}(\mathbb{1}) = e(\mathbb{1}) = \mathbb{1}$.
This leads to the conclusion that $\phi^{\star -1} \in G_{\mathbf{A}}^{\mathbf{B}}$.
Choose some $x \in \mathrm{Ker}(\epsilon)$

$$\begin{aligned}
&- \phi \star \phi^{\star -1}(x) = (e - \phi) \star \phi^{\star -1}(x) - e \star \phi^{\star -1}(x) = \\
&\sum_{n < |x|_{aug}} (e - \phi)^{\star (n+1)}(x) - \sum_{n < |x|_{aug}} (e - \phi)^{\star n}(x) = \\
&[(e - \phi)^{\star (|x|_{aug})} - (e - \phi)^{\star 0}](x) = -e(x) \quad \text{since} \quad \tilde{\Delta}^{|x|_{aug}}(x) = 0 \\
&\Rightarrow \phi \star \phi^{\star -1} = e \quad \text{and} \quad \phi^{\star -1} \star (\phi \star \phi^{\star -1}) = \phi^{\star -1} \star e = \phi^{\star -1} \\
&\Leftrightarrow \phi \star \phi^{\star -1} = e \quad \text{and} \quad \phi^{\star -1} \star \phi = e
\end{aligned}$$

The last line follows since e is unique.

□

2.6 Algebraic Birkhoff decomposition

Definition 2.6.1
Let $\mathbf{B}$ be a bialgebra and let $\mathbf{A} = \mathbf{A}_- \oplus \mathbf{A}_+$ be an algebra, decomposed into the direct sum of two linear spaces $\mathbf{A}_-$ and $\mathbf{A}_+$.
A Birkhoff decomposition of a $\phi \in G_{\mathbf{A}}^{\mathbf{B}}$ is a pair $\phi_+, \phi_- \in G_{\mathbf{A}}^{\mathbf{B}}$ so that

$$\phi = \phi_-^{\star -1} \star \phi_+ \quad \text{and} \quad \phi_\pm(\mathrm{Ker}(\epsilon)) \subseteq \mathbf{A}_\pm$$

Theorem 2.6.2
Let $\mathbf{B}$ be a connected bialgebra, let $\mathbf{A} = \mathbf{A}_- \oplus \mathbf{A}_+$ be an algebra decomposed into two linear spaces $\mathbf{A}_\pm$ and let $R : \mathbf{A} \to \mathbf{A}_-$ be the projection induced by the decomposition of the algebra $\mathbf{A}$.
For every $\phi \in G_{\mathbf{A}}^{\mathbf{B}}$ there exists a unique Birkhoff decomposition. It can be computed recursively through

$$\phi_-(x) = -R \circ \bar{\phi}(x) \quad \text{and} \quad \phi_+(x) = (id - R) \circ \bar{\phi}(x) \quad \forall x \in \mathrm{Ker}(\epsilon) \qquad \text{(i)}$$
$$\bar{\phi} = \phi + m \circ (\phi_- \otimes \phi) \circ \tilde{\Delta}. \qquad \text{(ii)}$$

The map $\bar{\phi}$ is called the Bogoliubov map.

Proof. The proof consists of three steps.

1. Let $\phi_\pm$ be some Birkhoff decomposition of some Birkhoff decomposable ϕ.
From equation (ii) one can follow that $\bar{\phi} = \phi_+ - \phi_-$.
Since $\phi_\pm(\mathrm{Ker}(\epsilon)) \in \mathbf{A}_\pm$ and $R(\mathbf{A}_+) = 0$, one can conclude that
$R \circ \bar{\phi}(x) = R \circ \phi_+(x) - R \circ \phi_-(x) = -\phi_-(x)$ and $(id - R) \circ \bar{\phi}(x) = \phi_+(x)$.
This shows that every Birkhoff decomposition, if it exists, can be computed with the help of equation (i) and (ii).

2. Let $\phi_\pm$ be some Birkhoff decomposition of some ϕ. Taking any connected filtration $\tilde{\mathbf{B}}^n$ of $\mathbf{B}$ one can conclude the below.

   (a) $\phi_-(\mathbb{1}) = \mathbb{1}$.
   (b) Since $\tilde{\Delta}(\tilde{\mathbf{B}}^{N+1}) \subseteq \sum_{1 \le k \le N} \tilde{\mathbf{B}}^k \otimes \tilde{\mathbf{B}}^{N+1-k}$, one can observe that $\bar{\phi}(\tilde{\mathbf{B}}^{N+1})$ and thus $\phi_\pm(\tilde{\mathbf{B}}^{N+1})$ are already completely determined by the values $\bar{\phi}(\tilde{\mathbf{B}}^n) \quad \forall n < N + 1$.

   One may conclude that every Birkhoff decomposition satisfies equation (i) and (ii) and every ϕ_- which satisfies equation (i) and (ii) is uniquely determined by the condition $\phi_-(\mathbb{1}) = \mathbb{1}$. That shows that the Birkhoff decomposition is unique if it exists.

3. As above we define ϕ_- recursively by

   (a) $\phi_-(\mathbb{1}) = \mathbb{1}$
   (b) $\phi_-(x) = -R \circ \bar{\phi}(x), \quad \bar{\phi} = \phi + m \circ (\phi_- \otimes \phi) \circ \tilde{\Delta} \quad \forall x \in \mathrm{Ker}(\epsilon)$

   It follows that $\phi_-(\mathrm{Ker}(\epsilon)) \subseteq \mathrm{im} R \equiv \mathbf{A}_-$.
   Set $\phi_+ := \phi_- \star \phi \quad \Rightarrow \phi_+(\mathbb{1}) = \mathbb{1}$ with Theorem (2.5.6) and
   $\phi_+|_{\mathrm{Ker}(\epsilon)} = \phi_- \star \phi|_{\mathrm{Ker}(\epsilon)} = [\phi_- + \bar{\phi}]_{\mathrm{Ker}(\epsilon)} = [(id - R) \circ \bar{\phi}]_{\mathrm{Ker}(\epsilon)}$.
   So $\phi_+(\mathrm{Ker}(\epsilon)) \subseteq \mathbf{A}_+$ and $\phi = \phi_-^{\star -1} \star \phi_+$, which completes the proof.

$\square$

Example 4 Consider a connected bialgebra $\mathbf{B}$ with the target algebra $\mathbf{A} = \mathbf{B}$ decomposed into $\mathbf{A}_- = \mathbf{B}$ and $\mathbf{A}_+ = \{0\}$, hence $R = id$. Then for $\phi \in G^{\mathbf{B}}_{\mathbf{B}}$ its Birkhoff decomposition satisfies $\phi_+(\mathbb{1}) = \mathbb{1}$ and $\phi_+(\mathrm{Ker}(\epsilon)) = 0$. One may conclude that $\phi_+ = e = u \circ \epsilon$ so that

$$\phi = \phi_-^{\star-1} \star \phi_+ = \phi_-^{\star-1} \star e = \phi_-^{\star-1} \Leftrightarrow \phi_- = \phi^{\star-1}$$

This gives a recursive relation for the inverse of a $\phi \in G^{\mathbf{B}}_{\mathbf{B}}$.

$$\begin{aligned} \forall x \in \mathrm{Ker}(\epsilon) : \quad & \phi^{\star-1}(x) = \phi_-(x) = -R \circ [\phi + m \circ \phi_- \otimes \phi \circ \tilde{\Delta}](x) \Leftrightarrow \\ & \phi^{\star-1}(x) = -\phi(x) - m \circ \phi^{\star-1} \otimes \phi \circ \tilde{\Delta}(x) \end{aligned}$$

2.7 Hopf algebra

Definition 2.7.1 (Hopf algebra)
Let $\mathbf{H}$ be a bialgebra and set $End^{\times}_{\star}(\mathbf{H}) := \{\phi \in End(\mathbf{H}) | \exists \psi \in End(\mathbf{H}) : \phi \star \psi = e = \psi \star \phi\}$.
$\mathbf{H}$ is called a Hopf algebra if $id \in End^{\times}_{\star}(\mathbf{H})$. The unique inverse $S := id^{\star-1}$ is called the antipode.
A morphism $\phi : \mathbf{H}_1 \to \mathbf{H}_2$ of Hopf algebras is a morphism of bialgebras so that

$$S_1 \circ \phi = \phi \circ S_2.$$

Definition 2.7.2
Let $\mathbf{H}$ be a Hopf algebra and $\mathcal{I} \subseteq \mathbf{H}$ be a linear subspace.
$\mathcal{I}$ is called a Hopf ideal if it is an ideal of a bialgebra and

$$S(\mathcal{I}) \subseteq \mathcal{I}.$$

Proposition 2.7.3
Let $\mathbf{H}$ be a Hopf algebra, let $\mathcal{I} \subseteq \mathbf{H}$ be a Hopf ideal and let $\pi : \mathbf{H} \to \mathbf{H}/\mathcal{I}$ the canonical projection.
There exists a unique Hopf algebra structure on $\mathbf{H}/\mathcal{I}$ so that π is a morphism of Hopf algebras.

Proof. Since $\mathbf{H}$ is a bialgebra, we already know from proposition (2.3.6) that there exists a unique bialgebra structure on $\mathbf{H}/\mathcal{I}$ so that π is a morphism of bialgebras. Set $S_{\mathbf{H}/\mathcal{I}} := \pi \circ S_{\mathbf{H}} \circ \pi^{-1}$. This definition is independent of the choice of a representative since $S_{\mathbf{H}}(\mathcal{I}) \subseteq \mathcal{I}$. We have to check that $S_{\mathbf{H}/\mathcal{I}}$ is the convolution inverse of $id_{\mathbf{H}/\mathcal{I}}$.

Recall that $m_{\mathbf{H}/\mathcal{I}} = \pi \circ m_{\mathbf{H}} \circ \pi^{-1} \otimes \pi^{-1}$ and $\Delta_{\mathbf{H}/\mathcal{I}} = \pi \otimes \pi \circ \Delta_{\mathbf{H}} \circ \pi^{-1}$ as in proposition (2.1.3) and (2.2.5).

$$
\begin{aligned}
&S_{\mathbf{H}/\mathcal{I}} \star_{\mathbf{H}/\mathcal{I}} id_{\mathbf{H}/\mathcal{I}} = m_{\mathbf{H}/\mathcal{I}} \circ S_{\mathbf{H}/\mathcal{I}} \otimes id_{\mathbf{H}/\mathcal{I}} \circ \Delta_{\mathbf{H}/\mathcal{I}} = \\
&\pi \circ m_{\mathbf{H}} \circ \pi^{-1} \otimes \pi^{-1} \circ (\pi \circ S_{\mathbf{H}} \circ \pi^{-1} \otimes \pi \circ id_{\mathbf{H}} \circ \pi^{-1}) \circ \pi \otimes \pi \circ \Delta_{\mathbf{H}} \circ \pi^{-1} = \\
&\pi \circ m_{\mathbf{H}} \circ S_{\mathbf{H}} \otimes id_{\mathbf{H}} \circ \Delta_{\mathbf{H}} \circ \pi^{-1} = \pi \circ u_{\mathbf{H}} \circ \epsilon_{\mathbf{H}} \circ \pi^{-1} = \\
&u_{\mathbf{H}/\mathcal{I}} \circ \epsilon_{\mathbf{H}/\mathcal{I}} = e_{\mathbf{H}/\mathcal{I}}
\end{aligned}
$$

So $S_{\mathbf{H}/\mathcal{I}}$ is indeed the convolution inverse of the identity in $\mathbf{H}/\mathcal{I}$. From the definition it follows that $\pi \circ S_{\mathbf{H}} = S_{\mathbf{H}/\mathcal{I}} \circ \pi$. □

Corollary 2.7.4
Every connected bialgebra is a Hopf algebra.

Proof. Follows from theorem (2.5.6) and $id \in G^{\mathbf{B}}_{\mathbf{B}}$. □

Proposition 2.7.5
Let $\mathbf{B}$ be a connected bialgebra and let $\mathcal{I} \subseteq \mathbf{B}$ be an ideal of a bialgebra. $\mathbf{B}/\mathcal{I}$ is a connected bialgebra and the canonical projection π is a Hopf algebra morphism.

Proof. Let $\mathbf{B} = (\mathbb{K}.\mathbb{1}_{\mathbf{B}}) \oplus \sum_{n \geqslant 1} \mathbf{B}_n$ be a connected filtration of $\mathbf{B}$. Set $\mathcal{I}_n = \pi(\mathbf{B}_n)$. Since π is a morphism of algebras, it follows that $\pi(\mathbb{K}.\mathbb{1}_{\mathbf{B}}) = \mathbb{K}.\mathbb{1}_{\mathbf{B}/\mathcal{I}}$. Since π is surjective, one obtains $\mathbf{B}/\mathcal{I} = (\mathbb{K}.\mathbb{1}_{\mathbf{B}/\mathcal{I}}) \oplus \sum_{n \geqslant 1} \mathcal{I}_n$. From the bialgebra morphism property of π follows that $\mathcal{I}_n$ is indeed a connected filtration of $\mathbf{B}/\mathcal{I}$. From corollary (2.7.4) we can follow that $\mathbf{B}$ and $\mathbf{B}/\mathcal{I}$ are Hopf algebras. In the proof of proposition (2.7.3) we saw that by setting $S_{\mathbf{B}/\mathcal{I}} := \pi \circ S_{\mathbf{H}} \circ \pi^{-1}$ one obtains an antipode in $\mathbf{B}/\mathcal{I}$ so that π is a Hopf algebra morphism. But that completes the proof since the antipode of a Hopf algebra is unique. □

There are some interesting features about the antipode that I would like to mention without a proof. The proof can be found in [1] or in [2].

Proposition 2.7.6
Let $\mathbf{H}$ be a Hopf algebra with antipode S.

1. $S \circ u = u \Leftrightarrow S(\mathbb{1}) = \mathbb{1}$
2. $\epsilon \circ S = \epsilon \Rightarrow S(\mathrm{Ker}(\epsilon)) \subseteq \mathrm{Ker}(\epsilon)$
3. $S \circ m = m \circ \tau \circ (S \otimes S)$ with $\tau(x_1 \otimes x_2) = x_2 \otimes x_1$.
4. $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$ with $\tau(x_1 \otimes x_2) = x_2 \otimes x_1$.

2.8 Characters and their decomposition

Definition 2.8.1
Let $\mathbf{B}$ be a bialgebra and let $\mathbf{A}$ be an algebra.
The set of characters is the set of all algebra morphisms $\phi : \mathbf{B} \to \mathbf{A}$.

$$\bar{G}_{\mathbf{A}}^{\mathbf{B}} := \{\phi \in G_{\mathbf{A}}^{\mathbf{B}} | \phi \circ m_{\mathbf{B}} = m_{\mathbf{A}} \circ \phi \otimes \phi\}$$

Proposition 2.8.2
Let $\mathbf{H}$ be a Hopf algebra and let $\mathbf{A}$ be a commutative algebra.
$(\bar{G}_{\mathbf{A}}^{\mathbf{B}}, \star)$ is a group and one can compute the inverse of $\phi \in \bar{G}_{\mathbf{A}}^{\mathbf{B}}$ by $\phi^{\star -1} = \phi \circ S$.

Proof. Choose some $\phi \in \bar{G}_{\mathbf{A}}^{\mathbf{B}}$.

$$(\phi \circ S) \star \phi = m_{\mathbf{A}} \circ [(\phi \circ S) \otimes \phi] \circ \Delta = m_{\mathbf{A}} \circ (\phi \otimes \phi) \circ (S \otimes id) \circ \Delta =$$
$$\phi \circ m \circ (S \otimes id) \circ \Delta = \phi \circ (S \star id) = \phi \circ u \circ \epsilon = u_{\mathbf{A}} \circ \epsilon = e$$

Further, $\phi \star (\phi \circ S) = e$ can be proven analogously.
We have to prove that $\phi^{\star -1} = \phi \circ S \in \bar{G}_{\mathbf{A}}^{\mathbf{B}}$.
$\phi^{\star -1}(\mathbb{1}) = \phi \circ S(\mathbb{1}) = \phi(\mathbb{1}) = \mathbb{1}$ and

$$\phi^{\star -1} \circ m = \phi \circ S \circ m = \phi \circ m \circ \tau \circ (S \otimes S) = m_{\mathbf{A}} \circ (\phi \otimes \phi) \circ \tau \circ (S \otimes S) =$$
$$m_{\mathbf{A}} \circ \tau \circ [(\phi \circ S) \otimes (\phi \circ S)] = m_{\mathbf{A}} \circ (\phi^{\star -1} \otimes \phi^{\star -1})$$

We used the commutativity of $\mathbf{A}$ above.
Choose any two $\psi, \phi \in \bar{G}_{\mathbf{A}}^{\mathbf{B}}$.

$$\begin{aligned}(\phi \star \psi) \circ m &= m_{\mathbf{A}} \circ \phi \otimes \psi \circ \Delta \circ m = m_{\mathbf{A}} \circ \phi \otimes \psi \circ (m \otimes m) \circ \sigma \circ (\Delta \otimes \Delta)\\ &= m_{\mathbf{A}} \circ [(\phi \circ m) \otimes (\psi \circ m)] \circ \sigma \circ \Delta \otimes \Delta\\ &= m_{\mathbf{A}} \circ (m_{\mathbf{A}} \otimes m_{\mathbf{A}}) \circ [(\phi \otimes \phi) \otimes (\psi \otimes \psi)] \circ \sigma \circ (\Delta \otimes \Delta)\\ &= m_{\mathbf{A}} \circ (m_{\mathbf{A}} \otimes m_{\mathbf{A}}) \circ \sigma \circ [(\phi \otimes \psi) \otimes (\phi \otimes \psi)] \circ (\Delta \otimes \Delta)\\ &= m_{\mathbf{A}} \circ [(m_{\mathbf{A}} \circ (\phi \otimes \psi) \circ \Delta] \otimes [(m_{\mathbf{A}} \circ (\phi \otimes \psi) \circ \Delta]\\ &= m_{\mathbf{A}} \circ (\phi \star \psi) \otimes (\phi \star \psi)\end{aligned}$$

We used the commutativity of $\mathbf{A}$ again. Further, note $\phi \star \psi(\mathbb{1}) = \phi(\mathbb{1}).\psi(\mathbb{1}) = \mathbb{1}$. This proves that $\phi \star \psi \in \bar{G}_{\mathbf{A}}^{\mathbf{B}}$. □

Lemma 2.8.3 (Rota-Baxter equation)
Let $\mathbf{A}$ be an algebra and $R : \mathbf{A} \to \mathrm{im}(R)$ be a projection so that im(R) and Ker(R) are subalgebras (they do not need to be unital).
Then the following equality holds.

$$m_{\mathbf{A}} \circ R \otimes R = R \circ m_{\mathbf{A}} \circ [R \otimes id + id \otimes R - id \otimes id]. \tag{i}$$

Proof. Note that the Rota-Baxter equation is equivalent to

$$R(xy) + R(x)R(y) = R[R(x)y + xR(y)] \quad \forall x, y \in \mathbf{A} \tag*{\boxed{2.8.1}}$$

1. $x, y \in \mathrm{Ker}(R) \Rightarrow xy \in \mathrm{Ker}(R)$ since Kern(R) is a subalgebra. Then both sides of equation (i) vanish.
2. $x, y \in \mathrm{im}(R) \Rightarrow xy \in \mathrm{im}(R)$ since im(R) is a subalgebra. Together with $R|_{\mathrm{im}R} = id$ both sides reduce to $2R(xy) = 2xy$.
3. $x \in \mathrm{Ker}(R)$ and $y \in \mathrm{im}(R)$, then equation (i) reduces to $R(xy) = R[xR(y)] = R(xy)$ again because $R|_{\mathrm{im}R} = id$.
   Analogously for $y \in \mathrm{Ker}(R)$ and $x \in \mathrm{im}(R)$.

□

Proposition 2.8.4
Let $\mathbf{B}$ be a connected bialgebra and let $\mathbf{A} = \mathbf{A}_+ \oplus \mathbf{A}_-$ be a commutative algebra decomposed into two subalgebras. They do not have to be unital.

$$\phi \in \bar{G}^{\mathbf{B}}_{\mathbf{A}} \Rightarrow \phi_+, \phi_- \in \bar{G}^{\mathbf{B}}_{\mathbf{A}}$$

Proof. We have already shown that under the assumptions above the convolution product of two characters is a character. Since $\phi_+ = \phi_- \star \phi$, we only need to show that ϕ_- is a character. This can be proven with the help of the Rota-Baxter equation.
Note the following.

$$\begin{aligned}
&\tilde{\Delta}(xy) = \Delta(xy) - \mathbb{1} \otimes xy - xy \otimes \mathbb{1} = \Delta(x)\Delta(y) - \mathbb{1} \otimes xy - xy \otimes \mathbb{1} \\
&= [\tilde{\Delta}(x) + \mathbb{1} \otimes x + x \otimes \mathbb{1}][\tilde{\Delta}(y) + \mathbb{1} \otimes y + y \otimes \mathbb{1}] - \mathbb{1} \otimes xy - xy \otimes \mathbb{1} \\
&= [\tilde{\Delta}(x) + \mathbb{1} \otimes x][\tilde{\Delta}(y) + \mathbb{1} \otimes y] + [\tilde{\Delta}(x) + \mathbb{1} \otimes x].y \otimes \mathbb{1} \\
&+ x \otimes \mathbb{1}.[\tilde{\Delta}(y) + \mathbb{1} \otimes y] - \mathbb{1} \otimes xy
\end{aligned}$$

With the help of the commutativity of $\mathbf{A}$ one can obtain the following.

$$\begin{aligned}
&m_{\mathbf{A}} \circ \phi_- \otimes \phi \circ m_{\mathbf{B}\otimes\mathbf{B}} \circ [\tilde{\Delta}(x) + \mathbb{1} \otimes x] \otimes [\tilde{\Delta}(y) + \mathbb{1} \otimes y] \\
&= m_{\mathbf{A}} \circ \phi_- \otimes \phi \circ m_{\mathbf{B}} \otimes m_{\mathbf{B}} \circ \sigma \circ [\tilde{\Delta}(x) + \mathbb{1} \otimes x] \otimes [\tilde{\Delta}(y) + \mathbb{1} \otimes y] \\
&= m_{\mathbf{A}} \circ (m_{\mathbf{A}} \otimes m_{\mathbf{A}}) \circ \sigma \circ (\phi_- \otimes \phi) \otimes (\phi_- \otimes \phi) \circ [\tilde{\Delta}(x) + \mathbb{1} \otimes x] \otimes [\tilde{\Delta}(y) + \mathbb{1} \otimes y] \\
&= m_{\mathbf{A}} \circ (m_{\mathbf{A}} \otimes m_{\mathbf{A}}) \circ (\phi_- \otimes \phi) \otimes (\phi_- \otimes \phi) \circ [\tilde{\Delta}(x) + \mathbb{1} \otimes x] \otimes [\tilde{\Delta}(y) + \mathbb{1} \otimes y] \\
&= \bar{\phi}(x).\bar{\phi}(y)
\end{aligned}$$

Using both equations, the analogue of the last equation for the other terms and the commutativity of $\mathbf{A}$ yields the below.

$$\begin{aligned}
&\bar{\phi}(xy) = \phi(xy) + m_{\mathbf{A}} \circ \phi_- \otimes \phi \circ \{[\tilde{\Delta}(x) + \mathbb{1} \otimes x][\tilde{\Delta}(y) + \mathbb{1} \otimes y] \\
&+ [\tilde{\Delta}(x) + \mathbb{1} \otimes x].y \otimes \mathbb{1} + x \otimes \mathbb{1}.[\tilde{\Delta}(y) + \mathbb{1} \otimes y] - \mathbb{1} \otimes xy\} \\
&= \bar{\phi}(x)\bar{\phi}(y) + \bar{\phi}(x)\phi_-(y) + \phi_-(x)\bar{\phi}(y) \\
&= \bar{\phi}(x)\bar{\phi}(y) - \bar{\phi}(x)R(\bar{\phi}(y)) - R(\bar{\phi}(x))\bar{\phi}(y) \\
&= -m_{\mathbf{A}} \circ [R \otimes id + id \otimes R - id \otimes id](\bar{\phi}(x) \otimes \bar{\phi}(y)) \quad \Rightarrow \\
&\phi_-(xy) = -R \circ \bar{\phi}(xy) = \phi_-(x)\phi_-(y)
\end{aligned}$$

The last line follows by using the Rota-Baxter equation (2.8.1).

□

Proposition 2.8.5
Let $\mathbf{B}$ be a connected bialgebra graded as an algebra and let $\mathbf{A}$ be some commutative algebra. Let $T : \mathbb{N} \to End(\mathbf{A})$ be a family of endomorphism so that

$$m_{\mathbf{A}} \circ T_i \otimes T_j = T_{i+j} \circ m_{\mathbf{A}} \circ [T_i \otimes id + id \otimes T_j - id \otimes id] \quad \forall n, m \in \mathbb{N}.$$

Choose some $\phi \in \bar{G}^{\mathbf{B}}_{\mathbf{A}}$ and consider the Birkhoff decomposition defined recursively by

$$\phi_-(\mathbb{1}) = \mathbb{1} \quad \phi_-(x) = -T_{|x|} \circ \bar{\phi}(x) \quad \forall x \in \mathrm{Ker}(\epsilon) \Rightarrow$$

ϕ_- and ϕ_+ are both characters.

Proof. The proof is the same as for proposition (2.8.4).

$$\begin{aligned}
&\phi_-(xy) = -T_{|xy|} \circ [\phi(xy) + m \circ \phi_- \otimes \phi \circ \tilde{\Delta}(xy)] = \\
&T_{|x|+|y|}[\bar{\phi}(x)T_{|y|}(\bar{\phi}(y)) + T_{|x|}(\bar{\phi}(x))\bar{\phi}(y) - \bar{\phi}(x)\bar{\phi}(y)] = \\
&\phi_-(x)\phi_-(y)
\end{aligned}$$

□

2.9 Hochschild cohomology

Definition 2.9.1 (Hochschild cochain complex)
Let **C** be a coalgebra and let **N** be a C-bicomodul with left and right comodule structure Ψ_L and Ψ_R.
Define cochains

$$HC^k(\mathbf{N}) := \mathrm{Hom}(\mathbf{N}, \mathbf{C}^{\otimes k}) \quad \forall k \in \mathbb{N}$$

and coboundary maps $d^k : HC^k(\mathbf{N}) \to HC^{k+1}(\mathbf{N})$ by

$$d^k := \sum_{0\leqslant i\leqslant k+1} (-1)^i d_i^k \quad ; d_i^k(L) := \begin{cases} (id \otimes L) \circ \psi_L & i = 0 \\ [id^{\otimes(i-1)} \otimes \Delta \otimes id^{\otimes(k-i)}] \circ L & 1 \leq i \leq k \\ (L \otimes id) \circ \psi_R & i = k+1. \end{cases}$$

Lemma 2.9.2
$(HC(\mathbf{N}), d)$ is a cochain complex, meaning $d \circ d = 0$.

Proof. I refer the reader to [9] for a more detailed proof. Let $L \in HC^k(\mathbf{N})$, then one can prove that

$$d_j^{k+1} \circ d_i^k = d_i^{k+1} \circ d_{j-1}^k \quad \forall 0 \leqslant i < j \leqslant k+2$$

just by computing both sides and comparing them. From that relation one can prove the assertion.

$$\begin{aligned} d^{k+1} \circ d^k &= \sum_{0\leqslant j\leqslant k+2} (-1)^j d_j^{k+1} \circ \sum_{0\leqslant i\leqslant k+1} (-1)^i d_i^{k+1} \\ &= \sum_{0\leqslant j\leqslant i\leqslant k+1} (-1)^{i+j} d_j^{k+1} \circ d_i^k + \sum_{0\leqslant j-1\leqslant i\leqslant k+1} (-1)^{i+j} d_j^{k+1} \circ d_i^k = 0 \end{aligned}$$

In the last step the indices were relabelled $j-1 \to i$, $i \to j$. □

Remark 2.9.1

i) Consider a bialgebra **B**.
Then **B** can be equipped with the structure of a **B**- bicomodul. In the following we will always consider the bicomodulstructure $\psi_L = \Delta$ and $\psi_R = id \otimes \mathbb{1}$.

ii) The maps $L \in HZ^1(\mathbf{B}) := HC^1(\mathbf{B}) \cap \mathrm{Ker}(d)$ are called Hochschild - 1 -cocycles.
If one uses our choice of bicomodul structure on a bialgebra **B**, then the defining equation for a Hochschild - 1 - cocycle is

$$\Delta \circ L = L \otimes \mathbb{1} + (id \otimes L) \circ \Delta.$$

iii) In this text we will mostly consider Hochschild- 1 - cocycles L so that $L(\mathbb{1}) \neq 0$. We will denote such Hochschild-1-cocycles just by cocycles.

Lemma 2.9.3
Let $\mathbf{B}$ be some bialgebra and let L be a Hochschild-1-cocycles.

$$im(L) \subseteq \operatorname{Ker}(\epsilon)$$

Proof. Choose some $x \in \operatorname{Ker}(\epsilon)$.

$$\begin{aligned}\epsilon \circ id \circ L(x) = (\epsilon \otimes \epsilon) \circ \Delta \circ L(x) &= (\epsilon \otimes \epsilon) \circ [L \otimes \mathbb{1} + id \otimes L \circ \Delta](x) \\ &= \epsilon \circ L(x) + (\epsilon \otimes \epsilon \circ L) \circ \Delta(x) \\ &= \epsilon \circ L(x) + (\epsilon \otimes \epsilon \circ L) \circ [\mathbb{1} \otimes x + x \otimes \mathbb{1} + \tilde{\Delta}(x)] \\ &= \epsilon \circ L(x) + \epsilon \circ L(x) = 2\epsilon \circ L(x)\end{aligned}$$

We used that $\tilde{\Delta}(x) \in \operatorname{Ker}(\epsilon) \otimes \operatorname{Ker}(\epsilon)$.
Analogously, one can show that $\epsilon \circ L(\mathbb{1}) = 2\epsilon \circ L(\mathbb{1})$. So in total we obtain $\epsilon \circ L = 0$, which proves the assertion. □

Proposition 2.9.4
Let $\mathbf{B}$ be a connected bialgebra and let L be a cocycle. As one can follow from Proposition (2.4.6) $Prim_n(\mathbf{B})$ is a connected filtration of $\mathbf{B}$.

$$x \in Prim_n(\mathbf{B}) \Rightarrow L(x) \in Prim_{n+1}(\mathbf{B}) \quad \forall n \geq 0$$

Proof. The property can be shown by induction.

1. $\Delta \circ L(\mathbb{1}) = L(\mathbb{1}) \otimes \mathbb{1} + (id \otimes L) \circ \Delta(\mathbb{1}) = L(\mathbb{1}) \otimes \mathbb{1} + \mathbb{1} \otimes L(\mathbb{1}) \Rightarrow L(\mathbb{1}) \in Prim_1(\mathbf{B})$

2. $x \in Prim_{n-1}(\mathbf{B}) \Rightarrow L(x) \in Prim_n(\mathbf{B})$ for some $n \in \mathbb{N}$.

3. Choose some $x \in Prim_n(\mathbf{B})$. Since $Prim_n(\mathbf{B})$ is a filtration, we know that $\tilde{\Delta}(x) \in Prim_{n-1} \otimes Prim_{n-1}$.

   $$\tilde{\Delta} \circ L(x) = x \otimes L(\mathbb{1}) + (id \otimes L) \circ \tilde{\Delta}(x) \subseteq Prim_n(\mathbf{B}) \otimes Prim_n(\mathbf{B}) \Rightarrow$$
   $$\tilde{\Delta}^{n+1} \circ L(x) = (\tilde{\Delta}^n \otimes id) \circ \tilde{\Delta} \circ L(x) = 0 \Rightarrow L(x) \in Prim_{n+1}(\mathbf{B})$$

□

Lemma 2.9.5
The space $HZ^1(\mathbf{B})$ of Hochschild-1-cocycles is a linear subspace of $End(\mathbf{B})$.

Proof. Choose any two $L_1, L_2 \in HZ^1(\mathbf{B})$ and some $\alpha \in \mathbb{K}$.

$$\Delta \circ (L_1 + \alpha L_2) = L_1 \otimes \mathbb{1} + id \otimes L_1 \circ \Delta + \alpha.(L_2 \otimes \mathbb{1} + id \otimes L_2 \circ \Delta)$$
$$= (L_1 + \alpha L_2) \otimes \mathbb{1} + id \otimes (L_1 + \alpha L_2) \circ \Delta$$

□

Chapter 3

Feynman graphs and Hopf algebraic renormalization

3.1 The Hopf algebra of Feynman graphs

Feynman graphs are graphs which are built by a set of vertices $\bar{R}_v$, a set of edges R_e and an index set $\mathcal{I}_v$ for every vertex $v \in \bar{R}_v$. Let us further define the set $R_v := \{v_i | v \in \bar{R}_v, i \in \mathcal{I}_v\}$, which is just the set of all labelled vertices. Edges are connected to vertices so that the type of the vertex and the type of the edges are compatible. Edges which are connected to two vertices are called internal and edges which are connected to only one vertex are called external. A Feynman graph is called one particle irreducible (1PI) if it is connected and cannot be disconnected by removing a single line. Graphs which are contained in $\bar{R}_v$ and R_e are not defined to be 1PI. In this text we will neglect the external structure of a graph, meaning the labelling of the external edges with momenta or space-time points. All the concepts that are being described in this section can be generalized to Feynman graphs with external structure with the help of distributions. I refer the reader to [3].
For example in ϕ^3 theory, $\bar{R}_v$ consists of the trivalent and the bivalent vertex, R_e just contains one straight edge. The index set for the trivalent vertex would be just a set with one element and the index set of the bivalent vertex would be a set with two elements. The index is needed to distinguish between the monomial ϕ^2 and $(\partial_\mu \phi \partial^\mu \phi)$ in the Lagrangian.

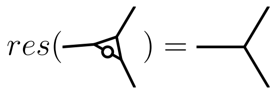
Definition 3.1.1
H is the algebra spanned as a linear space by $\mathbf{D}_{1PI}$, the set of all one particle irreducible graphs, with the commutative product defined through the disjoint union. Let Γ_1 and Γ_2 be some elements of **H**.

$$m(\Gamma_1 \otimes \Gamma_2) = \Gamma_1.\Gamma_2 = \Gamma_1 \cup \Gamma_2$$

The unit is given by $u(1) = \emptyset = \mathbb{1}$.

Definition 3.1.2
Let Γ be some element in $\mathbf{H}$. The residue of a graph Γ, in character $res(\Gamma)$, is the graph which remains after shrinking all internal structures to a point.

Example 5



Definition 3.1.3
Let Γ be some graph in $\mathbf{H}$ and let Γ_{in} be the set of internal lines of Γ. γ is called a subgraph of Γ, in character $\gamma \subset \Gamma$, if the following conditions hold.

1. γ is a non trivial subset $\gamma \subset \Gamma_{in}$. Non trivial means in this case that γ or the complement of γ are not the empty graph.
2. Every connected component of γ is 1PI.
3. $res(\gamma) \in \bar{R}_v$

Definition 3.1.4
Let Γ be some graph in $\mathbf{H}$ and let $\gamma \subset \Gamma$ be a subgraph.
Γ/γ_i is the graph that is obtained if one replaces γ with $res(\gamma)_i$ and $i \in \mathcal{I}_{res(\gamma)}$.

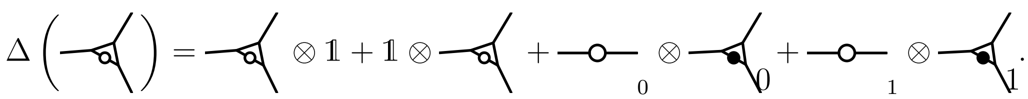
At this stage we need to include labelled graphs and graphs with labelled vertices into our algebra $\mathbf{H}$. If $res(\Gamma) = v \in \bar{R}_v$ we can label this graph by the index set $\mathcal{I}_v$. A labelled graph resp. a labelled vertex is treated algebraically the same way as the unlabelled graph resp. unlabelled vertex. We will need these labelled graphs and vertices in a moment for the Hopf algebraic renormalization procedure.

We can define a coproduct on the algebra $\mathbf{H}$. Let $\Gamma \in \mathbf{H}$ be a connected graph. Set

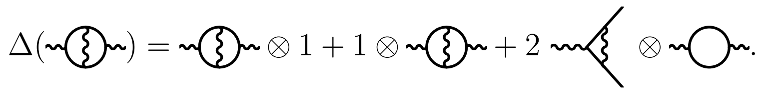
$$\Delta(\Gamma) = \Gamma \otimes \mathbb{1} + \mathbb{1} \otimes \Gamma + \sum_{v \in \bar{R}_v} \sum_{i \in \mathcal{I}_v} \sum_{\substack{\gamma \subset \Gamma \\ res(\gamma) = v}} \gamma_i \otimes \Gamma/\gamma_i$$

and extend the coproduct on products of graphs so that Δ is a morphism of algebras. Further, set $\epsilon(\mathbb{1}) = 1$ and $\epsilon(\Gamma) = 0$ if $\Gamma \neq \mathbb{1}$.

Example 6 An example of a coproduct in ϕ^3 theory would be



An example of a coproduct in QED would be



Note: Every index set in QED only consists of one element. Thus we don't need to label the graphs.

Theorem 3.1.5
$(\mathbf{H}, m, u, \Delta, \epsilon)$ is a connected bialgebra.

Proof. See [3] for a proof. □

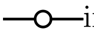
NOTE 6 By definition of the coproduct on graphs the set of all primitive elements $Prim_1(\mathbf{H})$ consists of all graphs which have no subdivergences. Thus the term primitive graph used here and the term primitive graph used by physicists are the same.

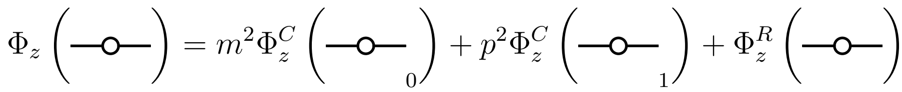
3.2 The renormalization procedure

Let us recapitulate the renormalization procedure and try to identify the famous BPHZ procedure as a Hopf algebraic Birkhoff decomposition (see [8]).
The first step of renormalization is to choose a regularisation scheme and the corresponding Feynman rules. Let us choose a scheme which depends on a complex regularisation parameter z so that we obtain the physical limit as $z \to 0$. What is a Feynman rule? A Feynman rule is a map Φ_z which sends a graph into the algebra of Laurent series in z. Let us denote this algebra by $\mathbf{A}$. This map is extended linearly to a linear combination of connected graphs meaning $\Phi_z(\sum_n \alpha_n \Gamma_n) = \sum_n \alpha_n \Phi_z(\Gamma_n)$ with $\alpha_n \in \mathbb{K}$ and $\Gamma_n \in \mathbf{H}$.
A product of graphs is being sent to the product of Laurent series, meaning $\Phi_z(\Gamma_1.\Gamma_2) = \Phi_z(\Gamma_1).\Phi_z(\Gamma_1)$ and by definition $\Phi_z(\mathbb{1}) = 1$.
To conclude, a Feynman rule is an element of the character group $\bar{\mathbf{G}}^{\mathbf{H}}_{\mathbf{A}}$.
The second step of renormalization is to choose a renormalization scheme. That is a rule how to obtain finite values for $\Phi_z(\Gamma)$ as $z \to 0$ for a primitive graph $\Gamma \in Prim_1(\mathbf{H})$. One example would be the famous minimal subtraction scheme. Let us denote it by T_{MS}. The map T_{MS} projects a Laurent series to its principle part. So for example $T_{MS}(\sum_{-5 \leq n \leq \infty} a_n z^n) = \sum_{-5 \leq n \leq -1} a_n z^n$. We can obtain finite values for primitive graphs by $\Phi^R_z(\Gamma) = \Phi_z(\Gamma) - T_{MS} \circ \Phi_Z(\Gamma) = [id - T_{MS}] \circ \Phi_z(\Gamma)$.
As a result, a renormalization scheme is a projection onto a subspace $\mathbf{A}_-$ of $\mathbf{A}$.

The third step of renormalization is using the famous BPHZ procedure to obtain renormalized values for graphs which contain subdivergences. We define the so called counter term Φ_z^C recursively through the following.

$$\begin{aligned}
\Phi_z^C(\Gamma) &= -T \circ [\Phi_z(\Gamma) + \sum_{v \in \bar{R}_v} \sum_{i \in \mathcal{I}_v} \sum_{\substack{\gamma \subset \Gamma \\ res(\gamma)=v}} \Phi_z^C(\gamma_i).\Phi_z(\Gamma/\gamma_i)] \\
&= -T \circ [\Phi_z + m \circ \Phi_z^C \otimes \Phi_z \circ \tilde{\Delta}](\Gamma) = -T \circ \bar{\Phi}_z(\Gamma)
\end{aligned}$$

$\Phi_z^C(\gamma_i)$ returns the counter term which is proportional to the corresponding monomial in the Lagrangian and $\Phi_z(\Gamma/\gamma_i)$ indicates to use the right Feynman rule at the vertex labelled with i. Recall that every labelled vertex corresponds to a monomial in the Lagrangian. Thus we obtain e.g. for the graph  in Φ^3 theory the following relation.



One can then obtain the renormalized value by

$$\begin{aligned}
\Phi_z^R(\Gamma) &= \Phi_z(\Gamma) + \sum_{v \in R_v} \sum_{i \in \mathcal{I}_v} \sum_{\substack{\gamma \subset \Gamma \\ res(\gamma)=v}} \Phi_z^C(\gamma_i).\Phi_z(\Gamma/\gamma_i) + \Phi_z^C(\Gamma) \\
&= [\bar{\Phi}_z + \Phi_z^C](\Gamma) = [id - T] \circ \bar{\Phi}_z(\Gamma).
\end{aligned}$$

We are now able to summarize the Hopf algebraic renormalization procedure.

1. Choose a regularization scheme and a character $\Phi_z \in \bar{G}_{\mathbf{A}}^{\mathbf{H}}$ which depends on the regularisation parameter so that one obtains the physical limit as $z \to 0$.

2. Choose a decomposition of the target algebra $\mathbf{A}$ into two subspaces $\mathbf{A} = \mathbf{A}_- \oplus \mathbf{A}_+$ so that the divergent parts in the complex regularisation parameter are contained in $\mathbf{A}_-$.

3. Calculate the Birkhoff decomposition $\Phi_z = (\Phi_z^C)^{\star -1} \star \Phi_z^R$. The renormalized values can then be obtained with the help of the map Φ_z^R.

This gives a precise definition of the renormalization procedure in terms of Hopf algebraic Birkhoff decomposition.

Every QFT is equipped with a special grading which indicates the order of divergence of a graph, the superficial degree of divergence (sdd). One can lower the sdd by subtracting the Taylor polynomial in the external variables up to the order of the sdd to render the integral finite. For example consider the integral

$$\int_0^\infty g(x,t)dx = \int_0^\infty \frac{x}{x+t}.$$

The integrand is of the order x^0 for large values of x. Thus it does not converge. If one subtracts the Taylor polynomial in t up to the first order one obtains

$$g(x,t) - g(x,t_0) - \partial_t|_{t=t_0} g(x,t) = \frac{(t-t_0)^2 x}{(x+t)(x+t_0)^2}.$$

This integrand is of the order x^{-2} for large values of x and is thus convergent. The parameter t_0 is arbitrary and is the so called renormalization point.

Notation 3.2.1

Let $\mathcal{I}$ be some finite set and let $f \in \mathbb{N}^{\times \mathcal{I}}$ be some multi index indexed with the set $\mathcal{I}$. Let further $\{\zeta^{(i)}\}_{i\in\mathcal{I}}$ be a set of object with a commutative product. One then defines

$$\zeta^f := \prod_{i\in\mathcal{I}} (\zeta^{(i)})^{f_i}.$$

Further, set $|f| := \sum_{i\in\mathcal{I}} f_i$, $f! := \prod_{i\in\mathcal{I}} f_i!$ and for two multi indices $f, g \in \mathbb{N}^{\times \mathcal{I}}$ we set $f \le g \Leftrightarrow f_i \le g_i \forall i \in \mathcal{I}$.

We are now ready to introduce the momentum scheme, which is a renormalization scheme with interesting properties. For simplicity we will set the renormalization point to 0. This will simplify the optic of the calculations but will not change the outcome. Set $\partial^i_{o,k} := \partial^i_{x_k}|_{(x_k=0)}$.

Definition 3.2.2

On the algebra of smooth functions $C^\infty(\mathbb{K}^n)$ we define the Taylor operator $T \in End(C^\infty(\mathbb{K}^n))$ by

$$T_k(f)(x) := \sum_{|K|\le k} \frac{x^K}{K!} \partial_0^K f(x).$$

Proposition 3.2.3

The Taylor Operator satisfies the following property.

$$m \circ T_k \otimes T_l = T_{k+l} \circ m \circ [T_k \otimes id + id \otimes T_l - id \otimes id] \quad \forall k, l \in \mathbb{N}.$$

Proof. With the help of the Leibniz rule $\partial \circ m = m \circ (\partial \otimes id + id \otimes \partial)$ one can obtain the following relation by induction.

$$\partial_0^J \circ m = \sum_{K \leq J} \binom{J}{K} \partial_0^K \otimes \partial_0^{J-K} (\star).$$

Note the following

$$\partial_0^J (T_k f)(x) = \partial_0^J \sum_{|K| \leq k} \frac{x^K}{K!} \partial_0^K f(x) = \begin{cases} \partial_0^J f(x) & |J| \leq k \\ 0 & \text{else} \end{cases} (\star\star)$$

Since $T_k f$ is a polynomial in the variables $x^{(k)}$ and thus linear, it is sufficient to check that the relation below holds.

$$\partial_0^J [(T_k f)g + f(T_l g) - fg] = \partial_0^J [(T_k f)(T_l g)] \quad \forall |J| \leq k + l$$

This relation can be obtained with the help of $(\star)$ and $(\star\star)$.

$$\partial_0^J [(T_k f)g + f(T_l g) - fg] = \sum_{K \leq J} \binom{J}{K} m \circ (\partial_0^K \otimes \partial_0^{J-K})[(T_k f) \otimes g + f \otimes (T_l g) - f \otimes g]$$

$$= \sum_{K \leq J} \binom{J}{K} [[\partial_0^K T_k f][\partial_0^{J-K} g] + [\partial_0^K f][\partial_0^{J-K} T_l g] - [\partial_0^K f][\partial_0^{J-K} g]]$$

1. $|K| \leq k, |J - K| \leq l$:

   $$[\partial_0^K T_k f][\partial_0^{J-K} g] + [\partial_0^K f][\partial_0^{J-K} T_l g] - [\partial_0^K f][\partial_0^{J-K} g] = [\partial_0^K T_k f][\partial_0^{J-K} T_l g]$$

2. $|K| > k$ and $|J - K| \leq l$:

   $$[\partial_0^K T_k f][\partial_0^{J-K} g] + [\partial_0^K f][\partial_0^{J-K} T_l g] - [\partial_0^K f][\partial_0^{J-K} g] = 0 = [\partial_0^K T_k f][\partial_0^{J-K} T_l g]$$

3. $|K| \leq k$ and $|J - K| > l$: analogously to the case $|K| > k$ and $|J - K| \leq l$.

4. $|K| > k$ and $|J - K| > l$
   Note $|J - K| = |J| - |K| > l \Leftrightarrow |J| > k + l$ (since $|K| \leq |J|$) and that case is not needed by the definition of the Taylor operator.

If we summarize the above we obtain the assertion.

$$\partial_0^J [(T_k f)g + f(T_l g) - fg] = \sum_{K \leq J} \binom{J}{K} [\partial_0^K T_k f][\partial_0^{J-K} T_l g] = \partial_0^J [(T_k f)(T_l g)].$$

□

Let Φ_z be some Feynman rule. One can define the so called momentum scheme which is induced by the Taylor operator. Set

$$\Phi_z^C(\Gamma) := -T_{|\Gamma|_{sdd}} \circ \bar{\Phi}_z(\Gamma) \quad \Phi_z^R(\Gamma) := [id - T_{|\Gamma|_{sdd}}] \circ \bar{\Phi}_z(\Gamma)$$

for any $\Gamma \in \mathrm{Ker}(\epsilon)$ and $\Phi_z^{C,R}(\mathbb{1}) = \mathbb{1}$.

Theorem 3.2.4
There exists a renormalization scheme so that the Birkhoff decomposition maps $\Phi_z^{R,C}$ are characters.

Proof. With the help of proposition (2.8.5) it is easy to notice that the momentum scheme leads to such a Birkhoff decomposition. □

NOTE 7 If the momentum scheme leads to finite renormalized Feynman rules, then the QFT is local and vice versa.

3.3 Combinatorial Dyson Schwinger equation

A QFT is completely determined by the so called 1PI vertex functions and the 1PI propagators. Let us define those in terms of the Hopf algebra of graphs. As mentioned before, a QFT is built by a set of vertices R_v and a set of edges R_e. For every labelled vertex $v \in R_v$ there is a coupling constant $g^{(v)}$.

Definition 3.3.1
Let $\Gamma \in \mathbf{H}$ be some graph. Define $N(\Gamma) = (N_{v_1}(\Gamma), \cdots, N_{v_n}(\Gamma)) \in \mathbb{N}^{\times R_v}$ where $N_v(\Gamma)$ counts the number of vertices of type v in Γ. Set

$$X^{(e)}(g) := \mathbb{1} - \sum_{\substack{\Gamma \in \mathbf{D}_{1PI} \\ res(\Gamma)=e}} \frac{1}{Sym(\Gamma)} g^{N(\Gamma)} \Gamma \quad \forall e \in R_e$$

$$X^{(v)}(g) := g^{(v)} \mathbb{1} + \sum_{\substack{\Gamma \in \mathbf{D}_{1PI} \\ res(\Gamma)=v}} \frac{1}{Sym(\Gamma)} g^{N(\Gamma)} \Gamma \quad \forall v \in R_v$$

Further, let $n_e(\Gamma)$ be the number of external edges of type e in Γ. We then define the so called invariant charges

$$Q^{(v)}(g) := \frac{X^{(v)}(g)}{\prod_{e \in R_e} (X^{(e)}(g))^{n_e(v)/2}} \quad \forall v \in R_v. \tag{3.3.1}$$

Expressions like $\frac{1}{X_e}$ are understood as the corresponding power series. We will treat those expressions in the next chapter more explicitly. $Sym(\Gamma)$ is just the well known symmetry factor of a graph which is the rank of the automorphism group of Γ. Elements of the set $\{X^{(e)}\}_{e\in R_e}$ are called propagators, elements of the set $\{X^{(v)}\}_{v\in R_v}$ are called vertex functions and elements of the set $\{X^{(r)}\}_{r\in R_e\cup R_v}$ are called graph functions. The term proportional to $\mathbb{1}$ corresponds to the first order contribution which is the graph contained in R_e resp. R_v, which we defined not to be 1PI. That is why the first term of the vertex function is proportional to the corresponding coupling constant. For example the first order contribution to the vertex function in Φ^3 theory would be the graph  which is already proportional to the coupling constant g and thus this graph corresponds to $g\mathbb{1}$ since we defined this graph not to be 1PI. The first order contribution to the propagator is the graph — which is just $\mathbb{1}$ since we defined this graph not to be 1PI. As one can compute $Q^{(v)}(g)$ is a power series in g and the only first order term is $g^{(v)}\mathbb{1}$. As we will see in chapter 3, the invariant charge $Q^{(v)}$ is the quantum mechanical generalization of the classical coupling constant $g^{(v)}$.

As we will see, graph functions are generated by so called insertion operators, which turn out to be cocycles. Before we can define those insertion operators and discuss their properties we will have to make some definitions first. The properties of the insertion operators and their connection to graph functions play a key role in the Hopf algebraic analysis of QFT. The definitions below are being introduced and more explicitly discussed in [5].

Definition 3.3.2
Let Γ be some element in $\mathbf{H}$. Define $|\Gamma|_V$ to be the number of distinct elements in $\mathbf{H}$ which are equal after removing all external edges. Those elements in $\mathbf{H}$ can be obtained from each other by permutation of the external edges.

Each graph Γ consists of internal edges Γ_{in} and vertices Γ_v. Those edges and vertices and subsets of them are called places of Γ. Every place of Γ has adjacent edges. If the place is a vertex then the edges attached to it are adjacent to the place. If the place is an edge which corresponds to a point on that edge, the two edges attached to that point define the adjacent edges.

Definition 3.3.3
Let Γ be some connected 1PI graph and let $X \in \mathbf{H}$ be some element.

1. $\Gamma|X$ is the number of insertion places of Γ so that X can be inserted at those places.

2. $\mathbf{bij}(\gamma, X, \Gamma)$ is the number of bijections between the external edges of X and the adjacent edges of insertion places p in γ so that Γ is obtained.

3. maxf(Γ) is the number of maximal forests of a graph Γ that is the number of ways to shrink subdivergences to a point so that the resulting graph is primitive.

NOTE 8 maxf(Γ) can be calculated the following way. If x is some connected graph in $\mathbf{H}$ we can define $\hat{x}$ to be the graph without any scalars in front. Then set $\Delta(\Gamma) = \sum c(\Gamma_1, \Gamma_2)\hat{\Gamma}_1 \otimes \hat{\Gamma}_2$. From this one obtains
maxf(Γ) $= \sum_{\gamma \in Prim_1(\mathbf{H})} \sum c(\Gamma_1, \Gamma_2)\delta_\gamma(\Gamma_2)$ with

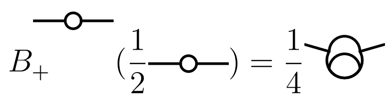
$$\delta_\gamma \in \mathbf{H}' \quad \delta_\gamma(\Gamma) = \begin{cases} 1 & \Leftrightarrow \gamma = \Gamma \\ 0 & else \end{cases}$$

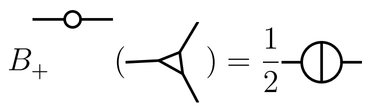
Definition 3.3.4 (Insertion operators)
For every $\gamma \in Prim_1(\mathbf{H})$ and for every $X \in \mathbf{H}$ we set

$$B_+^\gamma(X) := \sum_{\Gamma \in \mathbf{D}_{1PI}} \frac{\mathbf{bij}(\gamma, X, \Gamma)}{Sym(\gamma)} \frac{1}{|X|_V \mathrm{maxf}(\Gamma)\{\gamma|\Gamma\}} \Gamma$$

NOTE 9 The insertion Operator B_+^γ inserts the argument X into the primitive graph γ so that the resulting graph is divided by its symmetry factor multiplied by the symmetry factor of the argument X.

Example 7





I refer the reader to [4] and [5] for a proof of the following theorem.

Theorem 3.3.5 (Hochschild Theorem)
Let B_+^γ be the insertion operator defined above. Set $\Lambda^{(v)}(g) := \prod_{e\in R_e} X^{(e)}(g)^{n_e(v)/2}$.

1. The insertion operator B_+^γ is a cocycle $\forall \gamma \in Prim_1(\mathbf{H})$.
2. The graph functions fulfil the following system of equations.

$$X^{(e)}(g) = \mathbb{1} - \sum_{|N|\geqslant 1} \sum_{\substack{|\gamma|=1 \\ N(\gamma)=N \\ res(\gamma)=e}} B_+^\gamma(Q(g)^N X^{(e)}(g)) \quad \forall e \in R_e$$

$$X^{(v)}(g) = g^{(v)}\mathbb{1} + \sum_{|N|\geqslant 2} \sum_{\substack{|\gamma|=1 \\ N(\gamma)=N \\ res(\gamma)=v}} B_+^\gamma(Q(g)^N \Lambda^{(v)}(g)) \quad \forall v \in R_v$$

Remark 3.3.1

i) Since the set of Hochschild-1-cocycles is a linear space, we can rewrite the above equations in terms of new cocycles.

$$X^{(e)} = \mathbb{1} + \sum_{|N|\geqslant 1} L_e^{(N)}(Q(g)^N X^{(e)}(g)) \quad \forall e \in R_e$$

$$X^{(v)} = g^{(v)}\mathbb{1} + \sum_{|N|\geqslant 2} L_v^{(N)}(Q(g)^N \Lambda^{(v)}(g)) \quad \forall v \in R_v$$

ii) A system of equations of the type above is called combinatorial Dyson Schwinger equation (DSE).

iii) Note that every graph that is contained in a graph function is in the image of cocycles.

Let Φ_z be some Feynman rule. We then can define the unrenormalized Greens function through

$$G_r(\{g\},\{p\}) := \Phi_z(X^{(r)})(\{g\},\{p\}) \quad \forall r \in R_v \cup R_e$$

in which $\{p\}$ is the set of external momenta and $\{g\}$ is the set of coupling constants. The structure of the DSE assures that the physical limit for the renormalized Greens functions exists order by order. Fix the momentum scheme. Let γ be some primitive graph. Set $\Phi_z(\gamma) = \int \mu_z^\gamma$. This defines a rational function μ_z^γ. We assume that $\lim_{z\to 0} \Phi_z^R(\gamma) = \lim_{z\to 0}[\Phi_z(\gamma) - T \circ \Phi_z(\gamma)]$ exists for any primitive graph. From the structure of the Feynman rules it follows that $\Phi_z \circ B_+^\gamma(\Gamma) = \int \mu_z^\gamma \Phi_z(\Gamma)$, since $B_+^\gamma(\Gamma)$ inserts the graph Γ into γ in a suitable manner.

Theorem 3.3.6
Let Γ be some element in $\mathbf{H}$.
If $\lim_{z\to 0}\Phi_z^R(\gamma) = \lim_{z\to 0}[\Phi_z(\gamma) - T\circ\Phi_z(\gamma)]$ exists for any primitive graph, the physical limit $\lim_{z\to 0}\Phi_z^R(\Gamma)$ will exist.

Proof. Since Φ_z^R is a morphism of algebras for every $z \neq 0$, it follows that if $\lim_{z\to 0}\Phi_z^R(\Gamma)$ exists it is a morphism of algebras. This means it is sufficient to check the above for any connected graph Γ. From the Hochschild theorem we know that every connected graph in $\mathbf{H}$ is in the image of insertion Operators. The induction over the augmentation degree starts trivial for a primitive graph since we have assumed that $\lim_{z\to 0}\Phi_z^R(\gamma)$ exists for every primitive graph γ. Let P be the projection onto the augmentation ideal $\mathrm{Ker}(\epsilon)$ and choose some $X \in \mathbf{H}$ and $\gamma \in Prim_1(\mathbf{H})$. Now obverse the following.

$$
\begin{aligned}
&\bar{\Phi}_z(B_+^\gamma(X)) = [\Phi_z + m\circ\Phi_z^C\otimes\Phi_z\circ\tilde{\Delta}](B_+^\gamma(X))\\
&= [\Phi_z + m\circ\Phi_z^C\otimes\Phi\circ P\otimes P\circ\Delta](B_+^\gamma(X))\\
&= [\Phi_z + m\circ(\Phi_z^C\circ P\otimes\Phi\circ P)\circ\Delta](B_+^\gamma(X))\\
&= \Phi_z(B_+^\gamma(X)) + m\circ(\Phi_z^C\circ P\otimes\Phi\circ P)(B_+^\gamma(X)\otimes\mathbb{1} + (id\otimes B_+^\gamma)\circ\Delta(X))\\
&= \Phi_z(B_+^\gamma(X)) + m\circ(\Phi_z^C\circ P\otimes\Phi)(id\otimes B_+^\gamma)\circ\Delta(X)\\
&= m\circ\Phi_z^C\otimes\Phi_z(\mathbb{1}\otimes B_+^\gamma(X)) + m\circ(\Phi_z^C\circ P\otimes\Phi\circ B_+^\gamma)\circ\Delta(X)\\
&= m\circ\Phi_z^C\otimes\Phi_z\circ B_+^\gamma\circ\Delta(X) = \int\mu_z^\gamma\{m\circ\Phi_z^C\otimes\Phi_z\circ\Delta(X)\} = \int\mu_z^\gamma\Phi_z^C\star\Phi_z(X)\Rightarrow\\
&\bar{\Phi}_z(B_+^\gamma(X)) = \int\mu_z^\gamma\Phi_z^R(X)
\end{aligned}
$$

We used that B_+^γ is a cocycle and so $im(B_+^\gamma)\subseteq\mathrm{Ker}(\epsilon)$. So in total we obtain

$$\lim_{z\to 0}\Phi_z^R(B_+^\gamma(X)) = \lim_{z\to 0}[id - T]\circ\int\mu_z^\gamma\Phi_z^R(X).$$

In general the integrands, which appear in the above line, are of the form which are being discussed in [7] and thus one may conclude that the integral converges. This completes the proof. □

For a detailed example consider [6] or for a detailed analysis of a toy model consider [9].

We are now in the position to summarize the main ingredients for a local and renormalizable QFT.

1. The Feynman rules for primitive graphs have to lead to local and thus renormalizable expressions.

2. The Hopf algebra of graphs has to be “generated” from cocycles which shall mean that every connected graph is in the image of cocycles. This assures locality for higher order terms.

Actually, we will see that there is a Hopf subalgebra which is in mathematical terms generated by cocycles, which governs the renormalization of the QFT. These Hopf subalgebras are the topic of the next chapter (see [4]).

Chapter 4

Dyson Schwinger algebras

4.1 Faá di Bruno

In this section we will describe the multi dimensional Faá di Bruno algebra.

Definition 4.1.1
Let L be some finite set.
We denote by W_L the set of non empty words with letters in L.
Let $u \in W_L$ be some word. We denote by $\|u\|$ the length of the word and by $\underline{u}_j$ the j-th letter in the word u.
For some $x \in \mathbb{K}^d$ and some $u \in W_{\{1,\cdots d\}}$ set

$$x^u := \prod_{j=1}^{\|u\|} x^{(\underline{u}_j)}$$

In the remainder of this section we will always consider the set $L := \{1, \cdots, d\}$ where d is the dimension of the Faá di Bruno algebra, which is defined below.

Let $D^d := \{P \in \mathbb{K}[[x_1, \cdots, x_d]] | P(0) = 0 \wedge DP|_{x=0} = id_{\mathbb{K}}\}$ be the set of formal diffeomorphisms tangent to the identity. Let $P \in D^d$ be some element with $P^{(j)}(x) = \sum_{v \in W_L} p_v^{(j)} x^v$ where $P^{(j)}$ is the j-th component of $P(x) \in \mathbb{K}^d$. One can define functionals $a_v^{(j)} : D^d \to \mathbb{K}$ through $a_v^{(j)}(P) = p_v^{(j)}$ for any $j \in L$ and $v \in W_L$. Note the following. By definition $a_i^{(j)} = \delta_{i,j}$ for the one letter word $i \in W_L$. Set $\mathbb{1} \equiv a_j^{(j)} \quad \forall j \in L$.
The product on the field $\mathbb{K}$ induces a product on the set of functionals $a_v^{(j)}$.

$$[m \circ (a_v^{(j)} \otimes a_w^{(i)})](P) = p_v^{(j)} p_w^{(i)}$$

Note that $\mathbb{1}$ is the unit.

Definition 4.1.2 (Faá di Bruno)
The unital and commutative algebra $\mathbf{A}_{FdB}$, which is generated by the functionals $a_v^{(j)}$ with the product m defined above and the unit $\mathbb{1}$, is called the d dimensional Faá di Bruno algebra.

Lemma 4.1.3
Let $P, Q \in D^d$ be some elements. Then $Q \circ P \in D^d$ and

$$(Q \circ P)^{(j)} = \sum_{u \in W_L} x^u \sum_{\substack{\|w\| \leq \|u\| \\ w \in W_L}} \sum_{\substack{v_1 \cdots v_{\|w\|} = u \\ v_i \in W_L}} (\prod_k p_{v_k}^{(\underline{w}_k)}) q_w^{(j)} \quad (\star).$$

Proof. If $(\star)$ is true then $Q \circ P \in D^d$ since $q_i^{(j)} = \delta_{i,j}$ and $p_i^{(j)} = \delta_{i,j}$.

$$\begin{aligned}
(Q \circ P)^{(j)}(x) &= \sum_{w \in W_L} q_w^{(j)} \prod_k (\sum_{v \in W_L} p_{v_k}^{(\underline{w}_k)} x^v) \\
&= \sum_{w \in W_L} q_w^{(j)} \sum_{v_1, \cdots, v_{\|w\|} \in W_L} (\prod_k p_{v_k}^{(\underline{w}_k)}) x^{v_1 \cdots v_{\|w\|}} \\
&= \sum_{u \in W_L} x^u \sum_{w \in W_L} \sum_{\substack{v_1 \cdots v_{\|w\|} = u \\ v_i \in W_L}} (\prod_k p_{v_k}^{(\underline{w}_k)}) q_w^{(j)} \\
&= \sum_{u \in W_L} x^u \sum_{\substack{\|w\| \leq \|u\| \\ w \in W_L}} \sum_{\substack{v_1 \cdots v_{\|w\|} = u \\ v_i \in W_L}} (\prod_k p_{v_k}^{(\underline{w}_k)}) q_w^{(j)}
\end{aligned}$$

The last line follows since $\|v_i\| \geq 1$ by definition, so every time $\|w\| > \|u\|$ the sum over v_i is empty. □

We define a coproduct on $\mathbf{A}_{FdB}$ with the help of the composition.

$$\Delta(a_v^{(j)})(P \otimes Q) = a_v^{(j)}(Q \circ P)$$

The counit on $\mathbf{A}_{FdB}$ is defined by $\epsilon(\mathbb{1}) = 1$ and 0 otherwise. The coproduct is extended to a product so that it is a morphism of algebras and thus the product m is a morphism of coalgebras.

Lemma 4.1.4

$$\Delta(a_u^{(j)}) = \sum_{\substack{\|w\| \leq \|u\| \\ w \in W_L}} \sum_{\substack{v_1 \cdots v_{\|w\|} = u \\ v_i \in W_L}} (\prod_k a_{v_k}^{(\underline{w}_k)}) \otimes a_w^{(j)}$$

Proof. Follows from lemma (4.1.3). □

Theorem 4.1.5
$(\mathbf{A}_{FdB}, m, \mathbb{1}, \Delta, \epsilon)$ is a connected bialgebra and thus a Hopf algebra.

Proof. The coproduct is defined so that it is a morphism of algebras. The unital and associative property follows from that of $\mathbb{K}$. So we only need to check
$(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta$ and that there exists a connected filtration of $\mathbf{A}_{FdB}$.

1. From lemma (4.1.4) one obtains the following.

$$(\epsilon \otimes id) \circ \Delta(a_u^{(j)}) = \sum_{\|w\| \leq \|u\|} \sum_{v_1 \cdots v_{\|w\|} = u} \underbrace{(\prod_k \epsilon(a_{v_k}^{(w_k)}))}_{\delta_{w,u}} a_w^{(j)} = a_u^{(j)}$$

$$(id \otimes \epsilon) \circ \Delta(a_u^{(j)}) = \sum_{\|w\| \leq \|u\|} \sum_{v_1 \cdots v_{\|w\|} = u} (\prod_k a_{v_k}^{(w_k)}) \underbrace{\epsilon(a_w^{(j)})}_{\delta_{w,j}} = a_u^{(j)}$$

2. Set $\mathbf{A}_{FdB}^{lin} := \{a_u^{(j)}\}_{j \in L, u \in W_L}$.
Define a degree on $\mathbf{A}_{FdB}$ through $|a_u^{(j)}| := \|u\| - 1$ and $|a_{v_1}^{(j_1)} \cdots a_{v_k}^{(j_k)}| := |a_{v_1}^{(j_1)}| + \cdots + |a_{v_k}^{(j_k)}|$ for any $a_{v_1}^{(j_1)}, \cdots, a_{v_k}^{(j_k)} \in \mathbf{A}_{FdB}^{lin}$.
Set $\mathbf{A}_{FdB}^n := \mathrm{span}(\{\zeta \in \mathbf{A}_{FdB} | \zeta = \prod_h \zeta_h \quad \zeta_h \in \mathbf{A}_{FdB}^{lin} \quad \wedge |\zeta| = n\})$.
Note $\mathbf{A}_{FdB}^0 = (\mathbb{K}.\mathbb{1})$ and $\mathbf{A}_{FdB} = \oplus_{n \in \mathbb{N}} \mathbf{A}_{FdB}^n$.

$$\begin{aligned}
\Delta(a_u^{(j)}) &= \sum_{1 \leq a \leq \|u\|} \sum_{\|w\| = a} \sum_{v_1 \cdots v_a = u} (\prod_{k=1}^{a} a_{v_k}^{(w_k)}) \otimes a_w^{(j)} \\
&= \sum_{\{0 \leq (a-1) \leq (\|u\|-1)\}} \sum_{\{\|w\|-1 = (a-1)\}} \sum_{\{v_1 \cdots v_a = u\}} (\prod_{k=1}^{a} a_{v_k}^{(w_k)}) \otimes a_w^{(j)} \\
&= \sum_{\{0 \leq b \leq \|u\|-1\}} \sum_{\{\|w\|-1 = b\}} \sum_{\{v_1 \cdots v_{b+1} = u\}} (\prod_{k=1}^{b+1} a_{v_k}^{(w_k)}) \otimes a_w^{(j)} \\
&\in \sum_{0 \leq b \leq |a_u^{(j)}|} \mathbf{A}_{FdB}^{|a_u^{(j)}| - b} \otimes \mathbf{A}_{FdB}^b
\end{aligned}$$

In the last line we used the following.

$$|\prod_{k=1}^{b+1} a_{v_k}^{(w_k)}| = \sum_{1 \leq k \leq b+1} (\|v_k\| - 1) = \|u\| - 1 - b = |a_u^{(j)}| - b$$

The relation

$$\mathbf{A}^a_{FdB}.\mathbf{A}^b_{FdB} \subseteq \mathbf{A}^{a+b}_{FdB}$$

follows from the definition of the degree.

□

Definition 4.1.6
In analogy to the example we were discussing in the first chapter we define the following power series.

$$A^{(j)}(x) := \sum_{w \in W_L} a^{(j)}_w x^w$$

Notation 4.1.7
By A^J we will denote the set $\{A^{(j)}\}_{j \in J}$.

Proposition 4.1.8
The coproduct for the elements of A^J computes to

$$\Delta(A^{(j)}(x)) = \sum_{w \in W_L} A(x)^w \otimes a^{(j)}_w. \tag{4.1.1}$$

Proof. This is a straightforward computation.

$$\begin{aligned}
\Delta(A^{(j)}(x)) &= \sum_{u \in W_L} \Delta(a^{(j)}_u) x^u \\
&= \sum_{u \in W_L} \{ \sum_{\|w\| \leq \|u\|} \sum_{v_1 \cdots v_{\|w\|} = u} a^{\underline{w}_1}_{v_1} \cdots a^{\underline{w}_{\|w\|}}_{v_{\|w\|}} x^{v_1} \cdots x^{v_{\|w\|}} \otimes a^{(j)}_w \} \\
&= \sum_{w \in W_L} (\sum_{v_1 \in W_L} a^{\underline{w}_1}_{v_1} x^{v_1}) \cdots (\sum_{v_{\|w\|} \in W_L} a^{\underline{w}_{\|w\|}}_{v_{\|w\|}} x^{v_{\|w\|}}) \otimes a^{(j)}_w \\
&= \sum_{w \in W_L} \left(\prod_k A^{(\underline{w}_k)}(x) \right) \otimes a^{(j)}_w
\end{aligned}$$

□

Remark 4.1.1

i) We can reobtain the coproduct for $a_u^{(j)}$ from the power series $A^{(j)}(x)$ by projecting on the u-th coefficient. R 4.1.1.i

$$\begin{aligned}
\sum_{u\in W_L} x^u \Delta(a_u^{(j)}) &= \Delta(A^{(j)}(x)) = \sum_{w\in W_L} (\prod_k A^{(\underline{w}_k)}(x)) \otimes a_w^{(j)} \\
&= \sum_{w\in W_L} (\prod_k \sum_{v\in W_L} a_v^{(\underline{w}_k)} x^v) \otimes a_w^{(j)} \\
&= \sum_{w\in W_L} \sum_{v_1,\cdots v_{\|w\|}\in W_L} x^{v_1\cdots v_{\|w\|}} (\prod_k a_v^{(\underline{w}_k)}) \otimes a_w^{(j)} \\
&= \sum_{u\in W_L} x^u \sum_{\|w\|\leq\|u\|} \sum_{v_1,\cdots v_{\|w\|}=u} (\prod_k a_v^{(\underline{w}_k)}) \otimes a_w^{(j)}
\end{aligned}$$

ii) We say that A^J generates the Faá di Bruno algebra.

4.2 Dyson Schwinger algebra

In the next three sections we will determine the underlying structure of the DSE defined in chapter 2. For that purpose it is convenient to start with a more abstract definition of a so called Dyson Schwinger algebra (DSA). The element $a_u^{(p)}$ defined below will turn out to be the u-th order of the invariant charge $Q^{(p)}$ of the QFT considered where as the element $b_u^{(q)}$ defined below will turn out to be the u-th order of the propagator $X^{(q)}$ of the QFT considered.

Definition 4.2.1

Let L be a finite set. Let $\oslash$ denote the empty word. One then defines the set of words with letters in L by $W_L^0 := \{\oslash\} \cup W_L$ and one further defines $x^\oslash = 1$.

Definition 4.2.2 (Dyson Schwinger algebra)

Let P and Q be some finite sets with $|P| = p$ and $|Q| = q$. $\wp$ is called a (p,q)-dimensional Dyson Schwinger algebra if the following conditions hold

1. $\wp$ is a commutative bialgebra with unit $\mathbb{1}$ and counit ϵ.

2. $\wp$ is generated as an algebra by some elements $\{a_u^{(p)}\}_{p\in P,u\in W_P}$ and $\{b_u^{(q)}\}_{q\in Q,u\in W_P^0}$ so that $a_{p'}^{(p)} = \mathbb{1}\delta_{p,p'}$ $\forall p' \in P$ and $b_\oslash^{(q)} = \mathbb{1}$ are the only elements proportional to $\mathbb{1}$ and the other elements are all distinct from each other.

3. The coproducts of $\{a_u^{(p)}\}_{p\in P,u\in W_P}$ and $\{b_u^{(q)}\}_{q\in Q,u\in W_P^0}$ read

$$\Delta(a_u^{(p)}) = \sum_{\|w\|\leq\|u\|}\sum_{v_1\cdots v_{\|w\|}=u}(\prod_k a_{v_k}^{(\underline{w}_k)})\otimes a_w^{(p)}$$

$$\Delta(b_u^{(q)}) = \sum_{\|w\|\leq\|u\|}\sum_{v_0\cdots v_{\|w\|}=u} b_{v_0}^{(q)}(\prod_k a_{v_k}^{(\underline{w}_k)})\otimes b_w^{(q)}$$

Where the sum is over words in W_P^0 and we set $a_\oslash^{(p)}\equiv 0$ for notational convenience.

Remark 4.2.1

i) If we talk about a Dyson Schwinger algebra in this chapter, we will always consider a fixed DSA $\wp$ with sets P and Q.

ii) Every (p,q)-dimensional DSA contains a p dimensional Faá di Bruno algebra.

Theorem 4.2.3
Every Dyson Schwinger algebra $\wp$ is connected.

Proof. Set $\wp_{lin} := \{a_u^{(p)}\}_{p\in P,u\in W_P}\cup\{b_u^{(q)}\}_{q\in Q,u\in W_P^0}$
Define a degree on $\wp$ through $|a_u^{(p)}| = \|u\|-1$ and $|b_u^{(q)}| = \|u\|$. For any $\zeta_1,\cdots,\zeta_n\in\wp_{lin}$ set $|\zeta_1\cdots\zeta_n| := |\zeta_1|+\cdots+|\zeta_n|$.
Further, set $\wp^n := \mathrm{span}(\{\zeta\in\wp|\zeta=\prod_h\zeta_h \quad \zeta_h\in\wp_{lin} \quad \wedge|\zeta|=n\})$.
Note: $\wp=\oplus_{n\geq 0}\wp^n$ and $\wp^0=(\mathbb{K}\mathbb{1})$. The relation $\wp^n.\wp^m=\wp^{n+m}$ follows from the definition of the degree. In the proof of theorem (4.1.5) we have already proven that

$$\Delta(a_u^{(p)})\in\sum_{1\leq k\leq|a_u^{(p)}|}\wp^{|a_u^{(p)}|-k}\otimes\wp^k.$$

Let u be some word in W_P^0.

$$\begin{aligned}\Delta(b_u^{(q)}) &= \sum_{0\leq j\leq\|u\|}\sum_{\|w\|=j}\sum_{v_0\cdots v_j=u} b_{v_0}^{(q)}(\prod_{k=1}^{j} a_{v_k}^{(\underline{w}_k)})\otimes b_w^{(q)}\\ &\in \sum_{1\leq j\leq|b_u^{(q)}|}\wp^{|b_u^{(q)}|-j}\otimes\wp^j\end{aligned}$$

In the last line we used the relation below.

$$
\begin{aligned}
|b_{v_0}^{(q)}(\prod_{k=1}^{j} a_{v_k}^{(\underline{w}_k)})| &= |b_{v_0}^{(q)}| + \sum_{k=1}^{j} |a_{v_k}^{(\underline{w}_k)}| \\
&= \|v_0\| + \sum_{k=1}^{j} (\|v_k\| - 1) = \|u\| - j = |b_u^{(q)}| - j
\end{aligned}
$$

□

Definition 4.2.4
Define the following formal power series.

$$A^{(p)}(x) := \sum_{u \in W_P} a_u^{(p)} x^u \quad \forall p \in P$$

$$B^{(q)}(x) := \sum_{u \in W_P^0} b_u^{(q)} x^u \quad \forall q \in Q$$

Lemma 4.2.5

$$\Delta(A^{(p)}(x)) = \sum_{u \in W_P} A(x)^u \otimes a_u^{(p)} \quad \forall p \in P$$

$$\Delta(B^{(q)}(x)) = \sum_{u \in W_P^0} B^{(q)}(x) A(x)^u \otimes b_u^{(q)} \quad \forall q \in Q$$

Proof. The first equation has already been proven in the last section. See equation (4.1.1).

$$
\begin{aligned}
\Delta(B^{(q)}(x)) &= \sum_{\|u\| \geq 0} \Delta(b_u^{(q)}) x^u \\
&= \sum_{\|u\| \geq 0} x^u \sum_{\|w\| \leq \|u\|} \sum_{v_0 \cdots v_{\|w\|} = u} b_{v_0}^{(q)} (\prod_k a_{v_k}^{(\underline{w}_k)}) \otimes b_w^{(q)} \\
&= \sum_{\|u\| \geq 0} \sum_{\|w\| \leq \|u\|} \sum_{v_0 \cdots v_{\|w\|} = u} b_{v_0}^{(q)} x^{v_0} (\prod_k a_{v_k}^{(\underline{w}_k)} x^{v_k}) \otimes b_w^{(q)} \\
&= \sum_{\|w\| \geq 0} (\sum_{\|v_0\| \geq 0} b_{v_0}^{(q)} x^{v_0}) (\prod_k \sum_{\|v_k\| \geq 1} a_{v_k}^{(\underline{w}_k)} x^{v_k}) \otimes b_w^{(q)} \\
&= \sum_{\|w\| \geq 0} B^{(q)}(x) A(x)^w \otimes b_w^{(q)}
\end{aligned}
$$

□

Notation 4.2.6
Let L be a finite set and let $\Gamma(x) = \sum_{u \in W_L} x^u \gamma_u$ be a power series in some variables $x^{(i)}$.
By $[\Gamma]_u$ we denote the projection onto the u-th coefficient, so $[\Gamma]_u := \gamma_u$.

Lemma 4.2.7
The coproducts of $a_u^{(p)}$ and $b_u^{(q)}$ can be reobtained from $A^{(p)}$ and $B^{(q)}$ resp. by projecting onto the u-th coefficient.

Proof. We have already shown in the last section that $\Delta([A^{(p)}]_u) = [\Delta(A^{(p)}(x))]_u$ (remark (R 4.1.1.i)).

$$\sum_{\|u\| \geq 0} x^u \Delta(b_u^{(q)}) = \Delta(B^{(q)}(x))$$
$$= \sum_{\|u\| \geq 0} x^u \sum_{\|w\| \leq \|u\|} \sum_{v_0 \cdots v_{\|w\|} = u} b_{v_0}^{(q)} (\prod_k a_{v_k}^{(\underline{w}_k)}) \otimes b_w^{(q)} \quad \Rightarrow$$
$$[\Delta(B^{(q)}(x)]_u = \sum_{\|w\| \leq \|u\|} \sum_{v_0 \cdots v_{\|w\|} = u} b_{v_0}^{(q)} (\prod_k a_{v_k}^{(\underline{w}_k)}) \otimes b_w^{(q)} = \Delta([B^{(q)}(x)]_u)$$

□

Definition 4.2.8
Since we can obtain the whole bialgebra structure of the Dyson Schwinger algebra $\wp$ from $\{A^{(p)}(x)\}_{p \in P}$ and $\{B^{(q)}(x)\}_{q \in Q}$, we say that the pair (A^P, B^Q) generates the Dyson Schwinger algebra. The elements of A^P and B^Q are called generators.

NOTE 10 In general we obtain for a product of power series the following.

$$\Delta(\Gamma_1(x) \cdots \Gamma_n(x)) = \sum_{u_1, \cdots, u_n} x^{u_1} \cdots x^{u_n} \Delta(\gamma_1 \cdots \gamma_n)$$
$$= \sum_{u_1, \cdots, u_n} x^{u_1} \cdots x^{u_n} \Delta(\gamma_1) \cdots \Delta(\gamma_n)$$
$$= \Delta(\Gamma_1(x)) \cdots \Delta(\Gamma_n(x))$$

Lemma 4.2.9

$$\Delta(A(x)^u) = \sum_{\|w\| \geq \|u\|} A(x)^w \otimes [A^u]_w \quad \forall p \in P$$
$$\Delta(B^{(q)}(x) A(x)^u) = \sum_{\|w\| \geq \|u\|} B^{(q)}(x) A(x)^w \otimes [B^{(q)} A^u]_w \quad \forall q \in Q, p \in P$$

Proof. We will only show the relation for $\Delta(A(x)^u)$. The other relation can be obtained analogously.

$$\begin{aligned}
\Delta(A(x)^u) &= \Delta(\prod_k A^{(\underline{u}_k)}(x)) \\
&= \prod_k \Delta(A^{(\underline{u}_k)}(x)) = \prod_k \sum_{\|v_k\|\geq 1} A(x)^{v_k} \otimes [A^{(\underline{u}_k)}]_{v_k} \\
&= \sum_{\|w\|\geq\|u\|} A(x)^w \otimes \sum_{v_1\cdots v_{\|u\|}=w} \prod_k [A^{(\underline{u}_k)}]_{v_k} \\
&= \sum_{\|w\|\geq\|u\|} A(x)^w \otimes [A^u]_w
\end{aligned}$$

We used the following in the last line.

$$\begin{aligned}
A(x)^u &= \prod_{k=1}^{\|u\|} \sum_{\|v_k\|\geq 1} x^{v_k} a_{v_k}^{(\underline{u}_k)} \\
&= \sum_{\|w\|\geq\|u\|} x^w \sum_{v_1\cdots v_{\|u\|}=w} \prod_{k=1}^{\|u\|} a_{v_k}^{(\underline{u}_k)} \Rightarrow \\
[A^u]_w &= \sum_{v_1\cdots v_{\|u\|}=w} \prod_{k=1}^{\|u\|} a_{v_k}^{(\underline{u}_k)}
\end{aligned}$$

□

Theorem 4.2.10
Let **B** be some bialgebra. Let P and Q be two finite sets.

1. Let $\{H_u^{(p)}\}_{p\in P, u\in W_P, \|u\|\geq 2} \cup \{H_u^{(q)}\}_{q\in q, u\in W_P \|u\|\geq 1}$ be a set of cocycles on **B**. Consider the Dyson Schwinger equation (DSE) below.

$$\begin{aligned}
A^{(p)}(x) &= x^{(p)} + \sum_{u\in W_P, \|u\|\geq 2} H_u^{(p)}(A(x)^u) \quad \forall p \in P \\
B^{(q)}(x) &= \mathbb{1} + \sum_{u\in W_P, \|u\|\geq 1} H_u^{(q)}(A(x)^u B^{(q)}(x)) \quad \forall q \in Q
\end{aligned}$$

   The DSE has a unique solution so that

$$[A^{(p)}]_{\tilde{p}} = \delta_{p,\tilde{p}}\mathbb{1} \quad [B^{(q)}]_{\emptyset} = \mathbb{1}$$

   are the only terms proportional to $\mathbb{1}$ and the solution (A^P, B^Q) generates a Dyson Schwinger algebra $\wp \subseteq \mathbf{B}$. We shall denote this type of DSE as the standard one.

2. Let $\wp \subseteq \mathbf{B}$ be a Dyson Schwinger algebra generated by (A^P, B^Q). There exists a set of cocycles $\{H_u^{(p)}\}_{p\in P, u\in W_P, \|u\|\geq 2} \cup \{H_u^{(q)}\}_{q\in q, u\in W_P \|u\|\geq 1}$ so that A^P and B^Q satisfy the DSE above.

Proof. Let us start with the first assertion.

1. Note that for any set of power series $\underline{A}^P \cup \underline{B}^Q$ so that

$$\underline{A}^{(p)}(x) = x^{(p)} + \sum_{\|u\|\geq 2} \underline{a}_u^{(p)} x^u$$

and

$$\underline{B}^{(q)}(x) = \mathbb{1} + \sum_{\|u\|\geq 1} \underline{b}_u^{(q)} x^u$$

one may obtain the following.

$$\underline{A}(x)^u = \sum_{\|w\|\geq\|u\|} x^w \sum_{v_1\cdots v_{\|u\|}=w} \prod_{k=1}^{\|u\|} \underline{a}_{v_k}^{(u_k)} \Rightarrow$$
$$|[\underline{A}^u]_w| = |\prod_{k=1}^{\|u\|} \underline{a}_{v_k}^{(u_k)}| = \sum_{k=1}^{\|u\|}(\|v_k\|-1) = \|w\| - \|u\| < \|w\| \quad \forall \|u\| \geq 2$$
$$\underline{A}(x)^u \underline{B}^{(q)}(x) = \sum_{\|w\|\geq\|u\|} x^w \sum_{v_0\cdots v_{\|u\|}=w} \prod_{k=1}^{\|u\|} \underline{b}_{v_0}^{(q)} \underline{a}_{v_k}^{(u_k)} \Rightarrow$$
$$|[\underline{A}^u \underline{B}^{(q)}]_w| = |\underline{b}_{v_0}^{(q)}| + |\prod_{k=1}^{\|u\|} \underline{a}_{v_k}^{(u_k)}| = \|v_o\| + \sum_{k=1}^{\|u\|}(\|v_k\|-1)$$
$$= \|w\| - \|u\| < \|w\| \quad \forall \|u\| \geq 1$$

As a result one obtains for the solution of the DSE if it exists, the following.

$$A^{(p)}(x) = x^{(p)} + \sum_{\|u\|\geq 2} H_u^{(p)}(A(x)^u) = x^{(p)} + \sum_{\|u\|\geq 2} \sum_{\|w\|\geq\|u\|} x^w H_u^{(p)}([A^u]_w) \Rightarrow$$
$$[A^{(p)}]_{p'} = \delta_{p,p'}\mathbb{1} \quad [A^{(p)}]_w = \sum_{2\leq\|u\|\leq\|w\|} H_u^{(p)}([A^u]_w)$$
$$B^{(q)}(x) = \mathbb{1} + \sum_{\|u\|\geq 1} H_u^{(q)}(A(x)^u B^{(q)}(x)) = \mathbb{1} + \sum_{\|u\|\geq 1} \sum_{\|w\|\geq\|u\|} x^w H_u^{(q)}([A^u B^{(q)}]_w) \Rightarrow$$
$$[B^{(q)}]_\oslash = \mathbb{1} \quad [B^{(q)}]_w = \sum_{1\leq\|u\|\leq\|w\|} H_u^{(p)}([A^u B^{(q)}]_w)$$

So in total we obtain that $[A^{(p)}]_w$ and $[B^{(q)}]_w$ only depend on terms which are lower in degree. Thus the DSE has a unique solution. Further since $im(H) \subseteq \mathrm{Ker}(\epsilon)$, one can follow that $[A^{(p)}]_{\tilde{p}}$ and $[B^{(q)}]_{\oslash}$ are the only terms proportional to $\mathbb{1}$.
We will now show that

$$\Delta(A^{(p)}(x)) = \sum_{u \in W_P} A(x)^u \otimes a_u^{(p)} \quad \forall p \in P \quad (\star)$$

$$\Delta(B^{(q)}(x)) = \sum_{u \in W_P^0} B^{(q)}(x) A(x)^u \otimes b_u^{(q)} \quad \forall q \in Q \quad (\star\star)$$

and thus (A^P, B^Q) generate a Dyson Schwinger algebra.

$$\begin{aligned}
\Delta(A^{(p)}(x)) &= x^{(p)}\mathbb{1} \otimes \mathbb{1} + \sum_{\|u\| \geq 2} \Delta \circ H_u^{(p)}(A(x)^u) \\
&= x^{(p)}\mathbb{1} \otimes \mathbb{1} + \sum_{\|u\| \geq 2} \{H_u^{(p)} \otimes \mathbb{1} + id \otimes H_u^{(p)} \circ \Delta\}(A(x)^u) \\
&= \{x^{(p)} + \sum_{\|u\| \geq 2} H_u^{(p)}(A(x)^u)\} \otimes \mathbb{1} + \sum_{\|u\| \geq 2} id \otimes H_u^{(p)} \circ \Delta(A(x)^u) \\
&= A^{(p)}(x) \otimes \mathbb{1} + \sum_{\|u\| \geq 2} id \otimes H_u^{(p)} \circ \Delta(A(x)^u) \\
\Delta(B^{(q)}(x)) &= \mathbb{1} \otimes \mathbb{1} + \sum_{\|u\| \geq 1} \Delta \circ H_u^{(p)}(A(x)^u B^{(q)}(x)) \\
&= \mathbb{1} \otimes \mathbb{1} + \sum_{\|u\| \geq 1} \{H_u^{(p)} \otimes \mathbb{1} + id \otimes H_u^{(p)} \circ \Delta\}(A(x)^u B^{(q)}(x)) \\
&= \{\mathbb{1} + \sum_{\|u\| \geq 1} H_u^{(p)}(A(x)^u B^{(q)}(x))\} \otimes \mathbb{1} + \sum_{\|u\| \geq 1} id \otimes H_u^{(p)} \circ \Delta(A(x)^u B^{(q)}(x)) \\
&= B^{(q)}(x) \otimes \mathbb{1} + \sum_{\|u\| \geq 1} id \otimes H_u^{(p)} \circ \Delta(A(x)^u B^{(q)}(x))
\end{aligned}$$

We will prove the coproduct formulas by showing that $(\star)$ and $(\star\star)$ solve the ansatz above. We will make use of lemma (4.2.9).

$$\sum_{\|u\| \geq 2} id \otimes H_u^{(p)} \circ \Delta(A(x)^u) = \sum_{\|u\| \geq 2} id \otimes H_u^{(p)}\Big(\sum_{\|w\| \geq \|u\|} A(x)^w \otimes [A^u]_w\Big)$$
$$= \sum_{\|u\| \geq 2} \sum_{\|w\| \geq \|u\|} A(x)^w \otimes H_u^{(p)}([A^u]_w) = \sum_{\|w\| \geq 2} A(x)^w \otimes \sum_{2 \leq \|u\| \leq \|w\|} H_u^{(p)}([A^u]_w)$$

$$= \sum_{\|w\|\geq 2} A(x)^w \otimes [A^{(p)}]_w = \sum_{\|w\|\geq 1} A(x)^w \otimes [A^{(p)}]_w - A^{(p)}(x) \otimes \mathbb{1}$$

and

$$\sum_{\|u\|\geq 1} id \otimes H_u^{(q)} \circ \Delta(A(x)^u B^{(q)}(x))$$
$$= \sum_{\|u\|\geq 1} id \otimes H_u^{(q)}(\sum_{\|w\|\geq\|u\|} A(x)^w B^{(q)}(x) \otimes [A^u B^{(q)}]_w)$$
$$= \sum_{\|u\|\geq 1} \sum_{\|w\|\geq\|u\|} A(x)^w B^{(q)} \otimes H_u^{(q)}([A^u B^{(q)}]_w)$$
$$= \sum_{\|w\|\geq 1} A(x)^w B^{(q)} \otimes \sum_{1\leq\|u\|\leq\|w\|} H_u^{(q)}([A^u B^{(q)}]_w)$$
$$= \sum_{\|w\|\geq 1} A(x)^w B^{(q)}(x) \otimes [B^{(q)}]_w = \sum_{\|w\|\geq 0} A(x)^w B^{(q)}(x) \otimes [B^{(q)}]_w - B^{(q)}(x) \otimes \mathbb{1}$$

This shows that (A^P, B^Q) generates a Dyson Schwinger algebra $\wp \subseteq \mathbf{B}$.

2. Now let $\wp \subseteq \mathbf{B}$ be a Dyson Schwinger algebra generated by (A^P, B^Q). Choose linear maps $\{H_u^{(p)}\}_{p\in P, u\in W_P, \|u\|\geq 2} \cup \{H_u^{(q)}\}_{q\in q, u\in W_P \|u\|\geq 1}$ so that the following holds.

$$a_w^{(p)} = \sum_{2\leq\|u\|\leq\|w\|} H_u^{(p)}([A^u]_w) \quad \forall \|w\| \geq 2$$
$$b_w^{(q)} = \sum_{1\leq\|u\|\leq\|w\|} H_u^{(p)}([A^u B^{(q)}]_w) \quad \forall \|w\| \geq 1$$

This can be done by linear extension for example. Since the set of all Hochschild-1-cocycles is a linear subset of $End(\mathbf{B})$, one can decompose every linear map into a Hochschild-1-cocycle and an element of the composite, so for $H \in End(\mathbf{B})$ we obtain a decomposition $H = H_\epsilon + H^\perp$ with $H_\epsilon \in HZ^1(\mathbf{B})$ and $H^\perp \in (HZ^1(\mathbf{B}))^\perp$.

$$\sum_{2\leq\|u\|\leq\|w\|} \Delta \circ H_u^{(p)}([A^u]_w) = \Delta(a_w^{(p)}) = a^{(p)} \otimes \mathbb{1} + \sum_{2\leq\|v\|\leq\|w\|} [A^v]_w \otimes a_v^{(p)}$$
$$= \sum_{2\leq\|u\|\leq\|w\|} H_u^{(p)}([A^u]_w) \otimes \mathbb{1} + \sum_{2\leq\|v\|\leq\|w\|} [A^v]_w \otimes \sum_{2\leq\|u\|\leq\|v\|} H_u^{(p)}([A^u]_v)$$
$$= \sum_{2\leq\|u\|\leq\|w\|} \{H_u^{(p)} \otimes \mathbb{1} + id \otimes H_u^{(p)} \circ \Delta\}([A^u]_w)$$

We used the following.

$$\Delta([A^u]_w) = \sum_{\|v\| \leq \|w\|} [A^v]_w \otimes [A^u]_v$$

So in total we obtain the relation below.

$$\sum_{2 \leq \|u\| \leq \|w\|} (H_u^{(p)})^{\perp}([A^u]_w) = 0 \Rightarrow a_w^{(p)} = \sum_{2 \leq \|u\| \leq \|w\|} (H_u^{(p)})_{\epsilon}([A^u]_w).$$

Analogously, one can compute the following.

$$\sum_{1 \leq \|u\| \leq \|w\|} (H_u^{(p)})^{\perp}([A^u B^{(q)}]_w) = 0 \Rightarrow b_w^{(q)} = \sum_{1 \leq \|u\| \leq \|w\|} (H_u^{(p)})_{\epsilon}([A^u B^{(q)}]_w).$$

As a result, we obtain the assertion.

$$A^{(p)}(x) = \sum_{1 \leq \|u\|} a_u^{(p)} x^u = x^{(p)} + \sum_{u \in W_P, \|u\| \geq 2} (H_u^{(p)})_{\epsilon}(A(x)^u) \quad \forall p \in P$$

$$B^{(q)}(x) = \sum_{0 \leq \|u\|} b_u^{(q)} x^u = \mathbb{1} + \sum_{u \in W_P, \|u\| \geq 1} (H_u^{(q)})_{\epsilon}(A(x)^u B^{(q)}(x)) \quad \forall q \in Q.$$

□

The DSE above and the DSE defined in chapter 2 are still somewhat different as one can notice by comparing both expressions. In the next section we will derive that the DSE used above is equivalent to the one defined in chapter 2. Let us anticipate the result of the next section for a moment. The DSE defined above describes the propagators B^Q and invariant charges A^P of a QFT. In the last theorem we saw that the DSE and the coproduct formulas are equivalent. The coproduct formulas are needed to compute the renormalized values since computing them includes computing the coproduct. Let us consider a QFT which consists of only one vertex with the coupling constant g and the invariant charge $Q(g)$. QED would be a good example of a theory like that. The coproduct of the invariant charge is

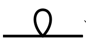
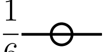
$$\Delta\left(Q(g)\right) = \sum_{b \geq 1} Q(g)^b \otimes [Q]_b.$$

This leads to the following.

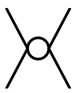
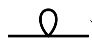
$$\Phi^R\left(Q(g)\right) = \Phi^C \star \Phi\left(Q(g)\right) = \Phi\left(Q(\Phi^C(Q))\right)$$

This result shows the well known fact for a renormalizable QFT that one obtains the renormalized value of the invariant charge by substituting the coupling constant g with the counter term of the invariant charge $\Phi^C(Q)$ and then applying the Feynman rules on $Q(\Phi^C(Q))$. Since the bare Lagrangian is obtained if one replaces the coupling constants with the counter terms of the corresponding invariant charges, one is led to the somewhat imprecise formulation that the invariant charge is the quantum mechanical generalization of the classical coupling constant. Remember, one needs the coproduct formulas for renormalization, which is an essential part of QFT. The DSE shows the explicit dependence on the coupling constant. Thus the theorem above describes the transition of the interaction from classical to quantum mechanics. In the next two sections we will analyse this transition. We will see how quantum mechanics "changes" if we impose linear restrictions at the "classical level". As it will turn out, locality expressed through the use of cocycles imposes such a strong restriction on quantum mechanics that the classical structure of a QFT already determines the combinatorics of the QFT. (In this text we will only analyse linear dependencies among the coupling constants. But one can show that also non linear dependencies on the classical level like those in gauge theories already determine the combinatorial structure of the QFT.)

At this point I would like to make an important remark. Since the propagators and invariant charges are power series in the coupling constants, the elements of the DSA are not single graphs but they are the sum of all graphs with the corresponding symmetry factors for a given order. For example in Φ^4 theory the first three orders to the propagator would be

$b_0 = \mathbb{1} \quad b_1 = (\frac{1}{2}$  $) \quad b_2 = (\frac{1}{6}$  $+ \frac{1}{4}$  $).$

and the first two orders to the invariant charge would be

$a_1 = \mathbb{1} \quad a_2 = (\frac{3}{2}$  $-$  $).$

In order to compute the invariant charge in terms of graphs one has to use equation (3.3.1).

4.3 Diffeomorphisms of generators

In this section we will make the connection between the DSE defined above and the one defined in chapter 2. As wee will see, the underlying DSAs are the same and the transition between the two DSEs is just a "change of basis". We will clarify this terminology below. In definition (4.2.2) we defined a

Dyson Schwinger algebra, as a bialgebra which is generated as an algebra by some elements with certain properties. But since we only required existence of such elements, we still have the freedom of choosing different elements to generate the DSA. In the following we will consider new elements which result from a diffeomorphism of the generators A^P and B^Q.

Notation 4.3.1
In the remainder of the text we will always denote by A^P and B^Q the generators discussed above.

Definition 4.3.2
Let $\Gamma(x)$ be some power series with coefficients in some bialgebra. The dimension of Γ is the dimension of the variable x. Define $dim(\Gamma)$ to be the dimension of $\Gamma(x)$.

Notation 4.3.3
If Γ^Z is a set of power series so that $dim(\Gamma^{z_1}) = dim(\Gamma^{z_2}) = n \quad \forall z_1, z_2 \in Z$ we just say Γ^Z has dimension n or in formula $dim(\Gamma^Z) = n$.

Definition 4.3.4
Let $\mathbf{B}$ be a connected bialgebra, let Z and Z' be two finite sets and let Γ^Z and $\Gamma^{Z'}$ be two power series with coefficients in $\mathbf{B}$ so that $dim(\Gamma^Z)$ and $dim(\Gamma^{Z'})$ are defined. We say Γ^Z and $\Gamma^{Z'}$ are diffeomorph if the following conditions hold.

1. There exists a diffeomorphism $F : \mathbb{K}^{\times Z} \to \mathbb{K}^{\times Z'}$ so that $F(\Gamma^Z) = \Gamma^{Z'}$.
2. $[F^{-1}(\Gamma^{Z'})]_w \in \mathbf{B}$
3. $dim(\Gamma^Z) = dim(\Gamma^{Z'})$

NOTE 11 If Γ^Z and $\Gamma^{Z'}$ are diffeomorph, then $|Z| = |Z'|$ and $[\Gamma^z]_u = [(F^{(-1)})^{(z)}(\Gamma^{Z'})]_u$

Proposition 4.3.5
Let $\wp$ be a DSA and let C^Z be a set of power series. If C^Z is diffeomorph to $A^P \cup B^Q$, then C^Z generates $\wp$.

Proof. Follows from $[A^{(p)}(x)]_u = [(F^{-1})^{(p)}(C^Z)]_u = a_u^{(p)}$ and $[B^{(q)}(x)]_u = [(F^{-1})^{(q)}(C^Z)]_u = b_u^{(q)}$. □

Choose an automorphism $\Omega \in Aut(\mathbb{K}^{\times P})$ and define the following diffeomorphism.

$$F : \mathbb{K}^{\times P} \times \mathbb{K}^{\times Q} \to \mathbb{K}^{\times P} \times \mathbb{K}^{\times Q}$$

$$[F(x,y)]^P = (\Omega.x)^P \quad [F(x,y)]^Q = y^Q$$

Set the following.

$$D^P := (\Omega.A)^P$$

where we defined

$$(\Omega.A)^{(p)}(x) = \sum_{\tilde{p}\in P} \omega_{\tilde{p}}^{p} A^{(\tilde{p})}(x).$$

This leads to a new set of generators (D^P, B^Q).

NOTE 12 From the definition of D^P it follows that only the terms $[D^P]_P$ are proportional to $\mathbb{1}$ and $[D^{(p)}]_{\tilde{p}} = \omega_{\tilde{p}}^{p}\mathbb{1}$.

Lemma 4.3.6

$$\Delta(D^{(p)}(x)) = \sum_{\|u\|\geq 1} (\Omega^{-1}.D)(x)^u \otimes [D^{(p)}]_u \quad \forall p \in P$$

$$\Delta(B^{(q)}(x)) = \sum_{\|u\|\geq 0} B^{(q)}(x)(\Omega^{-1}.D)(x)^u \otimes [B^{(q)}]_u \quad \forall q \in Q$$

Proof.

$$\begin{aligned}
\Delta(D^{(p)}(x)) &= \Delta((\Omega.A)^{(p)}(x)) = \sum_{\tilde{p}\in P} \omega_{\tilde{p}}^{p} \Delta(A^{(\tilde{p})}(x)) \\
&= \sum_{\tilde{p}\in P} \omega_{\tilde{p}}^{p} \sum_{\|u\|\geq 1} A(x)^u \otimes [A^{(\tilde{p})}]_u = \sum_{\|u\|\geq 1} A(x)^u \otimes [D^{(p)}]_u \\
&= \sum_{\|u\|\geq 1} (\Omega^{-1}.D)(x)^u \otimes [D^{(p)}]_u
\end{aligned}$$

$$\begin{aligned}
\Delta(B^{(q)}(x)) &= \sum_{\|u\|\geq 0} B^{(q)}(x) A(x)^u \otimes [B^{(q)}]_u \\
&= \sum_{\|u\|\geq 0} B^{(q)}(x)(\Omega^{-1}.D)(x)^u \otimes [B^{(q)}]_u
\end{aligned}$$

□

Since (D^P, B^Q) generates the DSA $\wp$, we are free to add new generators to the set. Let T be a finite set and choose some $\Theta \in \mathrm{Hom}(\mathbb{K}^{\times P}, \mathbb{K}^{\times T})$. Set $E^{(t)}(x) := (\Theta.D)^{(t)}(x) \Rightarrow (D^P, B^Q, E^T)$ generates the DSA $\wp$.

Theorem 4.3.7
Let $\mathbf{B}$ be some bialgebra. Let $\Omega \in Aut(\mathbb{K}^{\times P})$ and $\Theta \in \mathrm{Hom}(\mathbb{K}^{\times P}, \mathbb{K}^{\times T})$ be two linear maps, $x \in \mathbb{K}^{\times P}$

1. Let $\{H_u^{(y)}\}_{y\in(T\cup P\cup Q), u\in W_P}$ be a set of cocycles on $\mathbf{B}$. Consider the following DSE.

$$\begin{aligned}
B^{(q)}(x) &= \mathbb{1} + \sum_{u\in W_P, \|u\|\geq 1} H_u^{(q)}\left(D(x)^u B^{(q)}(x)\right) \quad \forall q \in Q \\
D^{(p)}(x) &= (\Omega.x)^{(p)} + \sum_{u\in W_P, \|u\|\geq 2} H_u^{(p)}\left(D(x)^u\right) \quad \forall p \in P \\
E^{(t)}(x) &= (\Theta.\Omega.x)^{(t)} + \sum_{u\in W_P, \|u\|\geq 2} H_u^{(t)}\left(D(x)^u\right) \quad \forall t \in T
\end{aligned}$$

The DSE has a unique solution so that only the following terms are proportional to $\mathbb{1}$.

$$\begin{aligned}
&[B^{(q)}]_{\oslash} = \mathbb{1} \quad \forall q \in Q \\
&[D^{(p)}]_{\tilde{p}} = \omega_{\tilde{p}}^{p}\mathbb{1} \quad \forall p, \tilde{p} \in P \\
&[E^{(t)}]_{\tilde{p}} = (\Theta.\Omega)_{\tilde{p}}^{t}\mathbb{1} \quad \forall t \in T; \tilde{p} \in P
\end{aligned}$$

Further, the following coproduct formulas shall hold.

$$\begin{aligned}
\Delta\left(B^{(q)}(x)\right) &= \sum_{u\in W_P, \|u\|\geq 0} B^{(q)}(x)(\Omega^{-1}.D)(x)^u \otimes [B^{(q)}]_u \quad \forall q \in Q \\
\Delta\left(D^{(p)}(x)\right) &= \sum_{u\in W_P, \|u\|\geq 1} (\Omega^{-1}.D)(x)^u \otimes [D^{(p)}]_u \quad \forall p \in P(\star) \\
E^{(t)}(x) &= (\Theta.D)^{(t)}(x) \quad \forall t \in T.
\end{aligned}$$

Thus (D^P, B^Q, E^T) generates a $(|P|, |Q|)$ dimensional DSA $\wp \in \mathbf{B}$.

2. Let $\wp \subseteq \mathbf{B}$ be a DSA generated by (D^P, B^Q, E^T) with the following properties

$$\begin{aligned}
&[B^{(q)}]_{\oslash} = \mathbb{1} \quad \forall q \in Q \\
&[D^{(p)}]_{\tilde{p}} = \omega_{\tilde{p}}^{p}\mathbb{1} \quad \forall p, \tilde{p} \in P \\
&[E^{(t)}]_{\tilde{p}} = (\Theta.\Omega)_{\tilde{p}}^{t}\mathbb{1} \quad \forall t \in T; \tilde{p} \in P
\end{aligned}$$

and

$$\Delta\left(B^{(q)}(x)\right) = \sum_{u\in W_P, \|u\|\geq 0} B^{(q)}(x)(\Omega^{-1}.D)(x)^u \otimes [B^{(q)}]_u \quad \forall q\in Q$$
$$\Delta\left(D^{(p)}(x)\right) = \sum_{u\in W_P, \|u\|\geq 1} (\Omega^{-1}.D)(x)^u \otimes [D^{(p)}]_u \quad \forall p\in P$$
$$E^{(t)}(x) = (\Theta.D)^{(t)}(x) \quad \forall t\in T.$$

Then there exists a set of cocycles $\{H_u^{(y)}\}_{y\in(T\cup P\cup Q), u\in W_P}$ so that the generators satisfy the DSE below.

$$B^{(q)}(x) = \mathbb{1} + \sum_{u\in W_P, \|u\|\geq 1} H_u^{(q)}\left(D(x)^u B^{(q)}(x)\right) \quad \forall q\in Q$$
$$D^{(p)}(x) = (\Omega.x)^{(p)} + \sum_{u\in W_P, \|u\|\geq 2} H_u^{(p)}\left(D(x)^u\right) \quad \forall p\in P$$
$$E^{(t)}(x) = (\Theta.\Omega.x)^{(t)} + \sum_{u\in W_P, \|u\|\geq 2} H_u^{(t)}\left(D(x)^u\right) \quad \forall t\in T$$

Proof. The proof is very similar to the prove of theorem (4.2.10). Since E^T only depends on (D^P, B^Q), we only need to proof existence and uniqueness for (D^P, B^Q).

1. Consider:

$$B^{(q)}(x) = \mathbb{1} + \sum_{\|u\|\geq 1} H_u^{(q)}\left(D(x)^u B^{(q)}(x)\right) \Leftrightarrow$$
$$[B^{(q)}]_\oslash = \mathbb{1} \quad [B^{(q)}]_u = \sum_{1\leq\|v\|\leq\|u\|} H_v^{(q)}([D^v B^{(q)}]_u)$$
$$D^{(p)}(x) = (\Omega.x)^{(p)} + \sum_{\|u\|\geq 2} H_u^{(p)}\left(D(x)^u\right) \Leftrightarrow$$
$$[D^{(p)}]_{\tilde{p}} = \omega_{\tilde{p}}^p \mathbb{1} \quad [D^{(p)}]_u = \sum_{2\leq\|v\|\leq\|u\|} H_v^{(p)}([D^v]_u)$$

Since $im(H)\subseteq \mathrm{Ker}(\epsilon)$, the only coefficients which are proportional to $\mathbb{1}$ are indeed the first order terms. Note that this is also true for E^T. As in the proof of theorem (4.2.10), $[B^{(q)}]_u$ and $[D^{(p)}]_u$ only depend on terms which are lower in degree. Thus the DSE has a unique solution. We will now proof $(\star)$ by showing that $(\star)$ solves the following ansatz.

$$\Delta\left(D^{(p)}(x)\right) = D^{(p)}(x)\otimes\mathbb{1} + \sum_{\|u\|\geq 2} (id\otimes H_u^{(p)})\circ\Delta\left(D(x)^u\right)$$

The other equations can be shown analogously. Note that for E^T we would need to use the identity

$$E^T = (\Theta.D)^T \Leftrightarrow \Delta(E^T) = \sum_{\|v\|\geq 1} (\Omega^{-1}.D)(x)^v \otimes [E^{(T)}]_v$$

, which follows from the assumption that $(\star)$ is true and the fact that only the terms $[E^T]_P$ are proportional to $\mathbb{1}$ with $(id \otimes \epsilon) \circ \Delta = id$.

$$\Delta\left(D(x)^u\right) = \sum_{\|v\|\geq\|u\|} (\Omega^{-1}.D)(x)^v \otimes [D^u]_v \Rightarrow$$

$$\begin{aligned}
\Delta\left(D^{(p)}(x)\right) &= D^{(p)}(x) \otimes \mathbb{1} + \sum_{\|u\|\geq 2} (id \otimes H_u^{(p)}) \left(\sum_{\|v\|\geq\|u\|} (\Omega^{-1}.D)(x)^v \otimes [D^u]_v\right) \\
&= D^{(p)}(x) \otimes \mathbb{1} + \sum_{\|u\|\geq 2} \sum_{\|v\|\geq\|u\|} (\Omega^{-1}.D)(x)^v \otimes H_u^{(p)}([D^u]_v) \\
&= D^{(p)}(x) \otimes \mathbb{1} + \sum_{\|v\|\geq 2} (\Omega^{-1}.D)(x)^v \otimes [D^{(p)}]_v \\
&= \sum_{\|v\|\geq 1} (\Omega^{-1}.D)(x)^v \otimes [D^{(p)}]_v
\end{aligned}$$

We used the following identity in the last line.

$$\begin{aligned}
D^{(p)}(x) &= (\Omega.\Omega^{-1}.D)^{(p)}(x) \\
&= \sum_{\tilde{p}\in P} \omega_{\tilde{p}}^{p} (\Omega^{-1}.D)^{(\tilde{p})}(x) = \sum_{\tilde{p}\in P} [D^p]_{\tilde{p}} (\Omega^{-1}.D)^{(\tilde{p})}(x)
\end{aligned}$$

And with the help of lemma (4.3.6) we conclude that (D^P, B^Q, D^T) generate a $(|P|, |Q|)$ dimensional DSA.

2. Let $\wp \subseteq \mathbf{B}$ be a DSA generated by (D^P, B^Q, E^T) where the generators have the properties described in the theorem. Choose linear maps $\{H_u^{(y)}\}_{y\in(T\cup P\cup Q), u\in W_P}$ so that the following holds.

$$\begin{aligned}
[D^{(p)}]_w &= \sum_{2\leq\|v\|\leq\|w\|} H_v^{(p)}([D^v]_w) \quad \forall \|w\| \geq 2 \\
[B^{(q)}]_w &= \sum_{1\leq\|v\|\leq\|w\|} H_v^{(q)}([D^v B^{(q)}]_w) \quad \forall \|w\| \geq 1 \\
[E^{(t)}]_w &= \sum_{2\leq\|v\|\leq\|w\|} H_v^{(t)}([D^v]_w) \quad \forall \|w\| \geq 2.
\end{aligned}$$

As in theorem (4.2.10) one shows that

$$\sum_{2\leq\|v\|\leq\|w\|}(H_v^{(r)})^{\perp}([D^v]_w)=0\quad\forall r\in T\cup P$$
$$\sum_{1\leq\|v\|\leq\|w\|}(H_v^{(q)})^{\perp}([D^vB^{(q)}]_w)=0\quad\forall q\in Q$$

from which the assertion follows.

□

Consider our standard DSE and transform the variables by an automorphism $\Omega\in Aut(\mathbb{K}^{\times P})$.

$$A^{(p)}(\Omega.x)=(\Omega.x)^{(p)}+\sum_{u\in W_P,\|u\|\geq 2}H_u^{(p)}(A(\Omega.x)^u)\quad\forall p\in P$$
$$B^{(q)}(\Omega.x)=\mathbb{1}+\sum_{u\in W_P,\|u\|\geq 1}H_u^{(q)}(A(\Omega.x)^uB^{(q)}(\Omega.x))\quad\forall q\in Q$$

With the help of theorem (4.3.7) we can conclude that the coefficients of $A^{(p)}(\Omega.x)$ and $B^{(q)}(\Omega.x)$ generate a $(|P|,|Q|)$ dimensional DSA $\wp_\Omega\subseteq\wp$ so that

$$\Delta((\Omega^{-1}.A.\Omega)^{(p)}(x))=\sum_{u\in W_P,\|u\|\geq 1}(\Omega^{-1}.A.\Omega)(x)^u\otimes[(\Omega^{-1}A.\Omega)^{(p)}]_u$$
$$\Delta((B.\Omega)^{(q)}(x))=\sum_{u\in W_P,\|u\|\geq 0}(B.\Omega)^{(q)}(x)(\Omega^{-1}.A.\Omega)(x)^u\otimes[(B.\Omega)^{(q)}]_u$$

and the only terms proportional to $\mathbb{1}$ are

$$[(\Omega^{-1}.A.\Omega)^{(p)}]_{\tilde{p}}=\delta_{p,\tilde{p}}\mathbb{1}\quad[(B.\Omega)^{(q)}(x)]_{\oslash}=\mathbb{1}$$

where we defined the right action by $(\Gamma.\Omega)(x)=\Gamma(\Omega.x)$. This leads to the following proposition.

Proposition 4.3.8
Every automorphism $\Omega\in Aut(\mathbb{K}^{\times P})$ induces an automorphism $\pi_\Omega\in Aut(\wp)$ by

$$\pi_\Omega(B^{(q)}(x))=(B.\Omega)^{(q)}(x)\quad\pi_\Omega(A^{(p)}(x))=(\Omega^{-1}.A.\Omega)^{(p)}(x)$$

The above expressions are equivalent to

$$\pi_\Omega([B^{(q)}]_u)=[(B.\Omega)^{(q)}]_u\quad\pi_\Omega([A^{(p)}]_u)=[(\Omega^{-1}.A.\Omega)^{(p)}]_u\quad\forall u\in W_P.$$

Proposition 4.3.9
Let G be a Lie group and let ρ be a representation on $Aut(\mathbb{K}^{\times P}) \Rightarrow$
$\Psi : G \to Aut(\wp)$ defined by $\Psi(g) = \pi_{\rho(g)}$ is a representation of G on $\wp$.

Proof. Can be concluded from the following.

$$\begin{aligned}
&\pi_\Omega \circ \pi_\Xi \left(A^{(p)}(x)\right) \\
&\qquad = \pi_\Omega \left(\sum_{p' \in P} (\Xi^{-1})^p_{p'} A^{(p')}(\Xi.x) \right) = \sum_{p' \in P} (\Xi^{-1})^p_{p'} (\Omega^{-1}.A)^{(p')}(\Omega.\Xi.x) \\
&\qquad = (\Xi^{-1}.\Omega^{-1}.A)^{(p)}(\Omega.\Xi.x) = ((\Omega.\Xi)^{-1}.A)^{(p)}(\Omega.\Xi.x) \\
&\qquad = \pi_{\Omega.\Xi}(A^{(p)}(x))
\end{aligned}$$

Analogously, one may show the relation below.

$$\pi_\Omega \circ \pi_\Xi(B^{(q)}(x)) = \pi_{\Omega.\Xi}(B^{(q)}(x))$$

□

NOTE 13 This is especially true for the Lie group $Aut(\mathbb{K}^{\times P})$.

Let $s \in \mathbb{R}$ be any number, we want to understand expressions like $\Delta(B^{(q)}(x)^s)$.

Definition 4.3.10 (generalized binomial coefficient)
Choose some $s \in \mathbb{R}$ and $k \in \mathbb{Z}$. Set

$$\binom{s}{k} = \begin{cases} \frac{s(s-1)\cdots(s-(k-1))}{k!} & k > 0 \\ 1 & k = 0 \\ 0 & k < 0 \end{cases}$$

We state without a proof.

Lemma 4.3.11

1. $$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c} \quad b, c \geq 0, a \in \mathbb{R}$$

2. $$(1+x)^s = \sum_{b \geq 0} \binom{a}{b} x^b \quad |x| < 1$$

Definition 4.3.12
Let s be any real number. Let Γ be any power series with coefficients that are elements of some bialgebra so that $\Gamma(x) = \mathbb{1} + \sum_{\|u\|\geq 1} x^u[\Gamma]_u$. Set

$$\Gamma(x)^s = (\mathbb{1} + \hat{\Gamma}(x))^s := \sum_{k\geq 0} \binom{s}{k} \hat{\Gamma}(x)^k.$$

Lemma 4.3.13
Let s be any real number. Let Γ be any power series with coefficients that are elements of some bialgebra so that $\Gamma(x) = \mathbb{1} + \sum_{\|u\|\geq 1} x^u[\Gamma]_u$.

$$\Rightarrow \Delta\left(\Gamma(x)^s\right) = \Delta\left(\Gamma(x)\right)^s$$

Proof.

$$\Delta\left(\Gamma(x)^s\right) = \Delta\left(\sum_k \binom{s}{k}\hat{\Gamma}(x)^k\right) = \sum_k \binom{s}{k} \Delta\left(\hat{\Gamma}(x)\right)^k$$
$$= \left(\mathbb{1}\otimes\mathbb{1} + \Delta\left(\hat{\Gamma}(x)\right)\right)^s = \Delta\left(\Gamma(x)\right)^s$$

□

Lemma 4.3.14

$$\Delta(B^{(q)}(x)^s) = \sum_{\|u\|\geq 0} \left(B^{(q)}(x)\right)^s A(x)^u \otimes [(B^{(q)})^s]_u \quad \forall q \in Q$$

Proof. Note the following identities.

$$B^{(q)}(x)^s = \left(\mathbb{1} + \hat{B}^{(q)}(x)\right)^s = \sum_{l\geq 0} \binom{s}{l} \hat{B}^{(q)}(x)^l$$
$$= \sum_{l\geq 0} \binom{s}{l} \sum_{\|w\|\geq l} x^w [(\hat{B}^{(q)})^l]_w = \sum_{\|w\|\geq 0} x^w \sum_{l\leq \|w\|} \binom{s}{l} [(\hat{B}^{(q)})^l]_w \Rightarrow$$
$$[(B^{(q)})^s]_w = \sum_{l\leq\|w\|} \binom{s}{l} [(\hat{B}^{(q)})^l]_w \quad \forall s \in \mathbb{R}$$

$$\hat{B}^{(q)}(x)^l = \left(\sum_{\|u\|\geq 1} x^u b_u^{(q)}\right)^l = \sum_{\|w\|\geq l} x^w \sum_{\substack{u_1\cdots u_l = w\\ \|u_i\|\geq 1}} \prod_{1\leq m\leq l} b_{u_m}^{(q)} \Rightarrow$$
$$[(\hat{B}^{(q)})^l]_w = \sum_{\substack{u_1\cdots u_l = w\\ \|u_i\|\geq 1}} \prod_{1\leq m\leq l} b_{u_m}^{(q)} \quad \forall n \in \mathbb{N}$$

We can now prove the lemma with the help of the above identities and lemma (4.3.11).

$$\Delta\left(B^{(q)}(x)^s\right)=\Delta\left(B^{(q)}(x)\right)^s=\left(\sum_{0\le\|u\|}B^{(q)}(x)A(x)^u\otimes b_u^{(q)}\right)^s$$

$$=\left(B^{(q)}(x)\otimes\mathbb{1}+\sum_{1\le\|u\|}B^{(q)}(x)A(x)^u\otimes b_u^{(q)}\right)^s$$

$$=\left(\mathbb{1}\otimes\mathbb{1}+\hat{B}^{(q)}(x)\otimes\mathbb{1}+\sum_{1\le\|u\|}B^{(q)}(x)A(x)^u\otimes b_u^{(q)}\right)^s$$

$$=\sum_{k\ge0}\binom{s}{k}\left(\hat{B}^{(q)}(x)\otimes\mathbb{1}+\sum_{\|u\|\ge1}B^{(q)}(x)A(x)^u\otimes b_u^{(q)}\right)^k$$

$$=\sum_{k\ge0}\binom{s}{k}\sum_{l\le k}\binom{k}{l}\left(\hat{B}^{(q)}(x)\otimes\mathbb{1}\right)^{k-l}\cdot\left(\sum_{\|u\|\ge1}B^{(q)}(x)A(x)^u\otimes b_u^{(q)}\right)^l$$

$$=\sum_{k\ge0}\sum_{l\le k}\binom{s}{k}\binom{k}{l}\sum_{\substack{u_1,\cdots,u_l\\\|u_i\|\ge1}}\left(\hat{B}^{(q)}(x)\right)^{k-l}\left(B^{(q)}(x)\right)^lA(x)^{u_1\cdots u_l}\otimes\prod_{1\le m\le l}b_{u_m}^{(q)}$$

$$=\sum_{l\ge0}\sum_{0\le k-l}\sum_{\|w\|\ge l}\binom{s}{l}\binom{s-l}{k-l}\left(\hat{B}^{(q)}(x)\right)^{k-l}\left(B^{(q)}(x)\right)^lA(x)^w\otimes\underbrace{\left(\sum_{\substack{u_1\cdots u_l=w\\\|u_i\|\ge1}}\prod_{1\le m\le l}b_{u_m}^{(q)}\right)}_{[(\hat{B}^{(q)})^l]_w}$$

$$=\sum_{l\ge0}\sum_{\|w\|\ge l}\binom{s}{l}\left(\mathbb{1}+\hat{B}^{(q)}(x)\right)^{s-l}\left(B^{(q)}(x)\right)^lA(x)^w\otimes[(\hat{B}^{(q)})^l]_w$$

$$=\sum_{\|w\|\ge0}\left(B^{(q)}(x)\right)^sA(x)^w\otimes\underbrace{\left(\sum_{l\le\|w\|}\binom{s}{l}[(\hat{B}^{(q)})^l]_w\right)}_{[(B^{(q)})^s]_w}$$

$$=\sum_{\|w\|\ge0}\left(B^{(q)}(x)\right)^sA(x)^w\otimes[(B^{(q)})^s]_w$$

$\square$

We will now consider a special diffeomorphism. This will lead us to the connection between Dyson Schwinger algebras and the vertex functions defined in the last chapter. Choose some real numbers $s_{(p,q)} \in \mathbb{R} \quad \forall p \in P; q \in Q$, choose a linear automorphism $\Omega \in Aut(\mathbb{K}^{|P|})$ and define the following diffeomorphism.

$$F : \mathbb{K}^P \times \mathbb{K}^Q \to \mathbb{K}^P \times \mathbb{K}^Q$$
$$F^{(p)}(x, y) := \{\prod_{q \in Q} (y^{(q)})^{s_{(p,q)}}\}(\Omega.x)^{(p)} \quad F^{(q)}(x, y) = y^{(q)}$$

This leads to

$$F^{(p)}(A^P, B^Q) = \{\prod_{q \in Q} \left(B^{(q)}(x)\right)^{s_{(p,q)}}\}(\Omega.A(x))^{(p)} \quad F^{(q)}(A^P, B^Q) = B^{(q)}(x).$$

For notational convenience we set $\Lambda^{(p)}(x) := \{\prod_{q \in Q} \left(B^{(q)}(x)\right)^{s_{(p,q)}}\}$ and $C^{(p)}(x) := \Lambda^{(p)}(x)(\Omega.A(x))^{(p)}$.

Lemma 4.3.15
(A^P, B^Q) and (C^P, B^Q) are diffeomorph.

Proof. The only thing which remains to be proven is $[(F^{-1})^{(p)}(C^P, B^Q)]_u \in \wp$.

$$(F^{-1})^{(p)}(C^P, B^Q) = \left(\Omega^{-1}.\{C.\Lambda^{-1}\}\right)^{(p)}(x) = \sum_{\tilde{p} \in P} (\omega^{-1})^p_{\tilde{p}} C^{(\tilde{p})}(x)(\Lambda^{(\tilde{p})}(x))^{-1}$$

$$\Rightarrow [(F^{-1})^{(p)}(C^P, B^Q)]_u = \sum_{\tilde{p} \in P} (\omega^{-1})^p_{\tilde{p}} [C^{(\tilde{p})}(\Lambda^{\tilde{p}})^{-1}]_u \in \wp$$

The last line follows since the sum on the right hand side only consists of finitely many terms, which are all elements of $\wp$. □

Proposition 4.3.16
(C^P, B^Q) generates the Dyson Schwinger algebra $\wp$.

Proof. Follows from lemma (4.3.15) together with note (11) since $a_u^{(p)} = [(F^{-1})^{(p)}(C^P, B^Q)]_u = \sum_{\tilde{p} \in P} (\omega^{-1})^p_{\tilde{p}} [C^{(\tilde{p})}(\Lambda^{\tilde{p}})^{-1}]_u$ □

Notation 4.3.17
In the remainder of the text we will always denote by C^P the power series discussed above.

Lemma 4.3.18

$$\Delta\left(C^{(p)}(x)\right) = \sum_{\|u\| \geq 1} \Lambda^{(p)}(x) A(x)^u \otimes [C^{(p)}]_u$$

Proof. With the help of lemma (4.3.14) one obtains

$$\Delta(\Lambda^{(p)}(x)) = \sum_{\|u\|\geq 0} \Lambda^{(p)}(x)A(x)^u \otimes [\Lambda^{(p)}]_u$$

With the help of the above relation we can derive the coproduct for $C^{(p)}(x)$.

$$\begin{aligned}
&\Delta\left(C^{(p)}(x)\right) = \Delta\left(\Lambda^{(p)}(x)\right).\{\sum_{\tilde{p}\in P}\omega^p_{\tilde{p}}\Delta\left(A^{(\tilde{p})}(x)\right)\}\\
&= \left(\sum_{\|w\|\geq 0} \Lambda^{(p)}(x)A(x)^w \otimes [\Lambda^{(p)}]_w\right) \cdot \left(\sum_{\tilde{p}\in P}\omega^p_{\tilde{p}}\sum_{\|v\|\geq 1} A(x)^v \otimes [A^{(\tilde{p})}]_v\right)\\
&= \left(\sum_{\|w\|\geq 0} \Lambda^{(p)}(x)A(x)^w \otimes [\Lambda^{(p)}]_w\right) \cdot \left(\sum_{\|v\|\geq 1} A(x)^v \otimes [(\Omega.A)^{(p)}]_v\right)\\
&= \sum_{\|u\|\geq 1} \Lambda^{(p)}(x)A(x)^u \otimes \sum_{wv=u}[\Lambda^{(p)}]_w[(\Omega.A)^{(p)}]_v = \sum_{\|u\|\geq 1} \Lambda^{(p)}(x)A(x)^u \otimes [\Lambda^{(p)}(\Omega.A)^{(p)}]_u\\
&= \sum_{\|u\|\geq 1} \Lambda^{(p)}(x)A(x)^u \otimes [C^{(p)}]_u
\end{aligned}$$

□

NOTE 14 Since we changed the generators of the DSA, $A^{(p)}(x)$ has to be interpreted as $A^{(p)}(x) = (\Omega^{-1}.\{C.\Lambda^{-1}\})^{(p)}(x)$.

Now consider the Dyson Schwinger equation from chapter 2.

$$\begin{aligned}
B^{(q)}(x) &= \mathbb{1} + \sum_{u\in W_P,\|u\|\geq 1} L^{(q)}_u(\{C.\Lambda^{-1}\}(x)^u B^{(q)}(x)) \quad \forall q \in Q\\
C^{(p)}(x) &= x^{(p)} + \sum_{u\in W_P,\|u\|\geq 2} L^{(p)}_u(\{C.\Lambda^{-1}\}(x)^u \Lambda^{(p)}(x)) \quad \forall p \in P
\end{aligned}$$

with

$$\Lambda^{(p)}(x) = \prod_{q\in Q} B^{(q)}(x)^{s_{(q,p)}}$$

for some real numbers $s_{(q,p)} \in \mathbb{R}$. As in theorem (4.2.10) one can prove that the above DSE has a unique solution so that

$$[B^q]_\oslash = \mathbb{1} \quad [C^{(p)}]_{\tilde{p}} = \delta^p_{\tilde{p}}\mathbb{1}$$

are the only terms proportional to $\mathbb{1}$,

$$\Delta(B^{(q)}(x)) = \sum_{\|u\|\geq 0} B^{(q)}(x)(C\Lambda^{-1})(x)^u \otimes [B^{(q)}]_u$$
$$\Delta(C^{(p)}(x)) = \sum_{\|u\|\geq 1} \Lambda^{(p)}(x)(C\Lambda^{-1})(x)^u \otimes [C^{(p)}]_u$$

and (C^P, B^Q) generates a $(|P|, |Q|)$ dimensional DSA so that

$$A^{(p)}(x) = (C\Lambda^{-1})^{(p)}(x).$$

So instead of considering the above DSE we can consider the standard DSE

$$A^{(p)}(x) = x^{(p)} + \sum_{u\in W_P, \|u\|\geq 2} H_u^{(p)}(A(x)^u) \quad \forall p \in P$$
$$B^{(q)}(x) = \mathbb{1} + \sum_{u\in W_P, \|u\|\geq 1} H_u^{(q)}(A(x)^u B^{(q)}(x)) \quad \forall q \in Q$$

with

$$C^{(p)}(x) = \Lambda^{(p)}(x) A^{(p)}(x).$$

This formalism opens the possibility for further investigation. An interesting question would be, if there exists a diffeomorphism so that the DSA is nothing else than a Faá di Bruno algebra.

4.4 Conditional Dyson Schwinger algebras

Consider the following SO(2) invariant Lagrangian.

$$\mathcal{L} = (\partial\phi)^2 + (\partial\psi^2) + \underbrace{\frac{g}{4!}(\phi^2 + \psi^2)^2}_{\frac{g^{(1)}}{4!}\phi^4 + \frac{g^{(2)}}{4}\phi^2\pi^2 + \frac{g^{(2)}}{4!}\pi^4}$$

The only thing which indicates that this Lagrangian is SO(2) invariant is the relation among the coupling constants. In this section we will investigate the question how the combinatorics of the QFT changes if we impose relations like those above to the coupling constants.

Let $\Gamma(x) = \sum_{\|u\|\geq 1} x^u [\Gamma]_u$ be some kind of power series with $x \in \mathbb{K}^{\times P}$ for some finite set P and coefficients in some commutative algebra. Let further C be a linear subspace of $\mathbb{K}^{\times P}$. We can then restrict $\Gamma(x)$ to the linear subspace C, which we will do the following way. Choose a disjoint decomposition

$P = I \uplus J$ so that $|I| = dim(C)$. We can then parametrize C by I, which leads to a relation between the variables $x^{(\cdot)}$.

$$x^{(j)} = \sum_{i \in I} \theta_i^j x^{(i)} = (\Theta.x)^{(j)} \quad \forall j \in J \quad \Theta \in \mathrm{Hom}(\mathbb{K}^{\times I}, \mathbb{K}^{\times J})$$

By $C = (x^{(j)} = \sum_{i \in I} \theta_i^j x^{(i)} | \quad \forall j \in J)$ we will denote the linear subspace C together with the parametrization defined within the brackets, which we will call a condition.

Notation 4.4.1

Let $C = \left(x^{(j)}(x) = \sum_{i \in I} \theta_i^j x^{(i)} | \forall j \in J\right)$ be a condition. Further, choose a word $u \in W_P$. Define numbers $z_\zeta^{(u)}$ so that

$$x^u|_C = \sum_{\zeta \in W_I} z_\zeta^{(u)} x^\zeta.$$

With the notation above we can follow

$$\Gamma(x)|_C = \sum_{\zeta \in W_I} x^\zeta \sum_{w \in W_J^0} \sum_{\substack{u,v \in W_I^0 \\ uv = \zeta}} z_v^{(w)} [\Gamma]_{uw}.$$

NOTE 15 The sum above is finite in every order. Thus we can write

$$\Gamma(x)^C = \Gamma(x)|_C = \sum_{\zeta \in W_I} x^\zeta [\Gamma]_\zeta^C$$

where the coefficients $[\Gamma]_{W_I}^C$ are elements in the same algebra as the $[\Gamma]_{W_P}$.

Lemma 4.4.2

Keep the definitions made above. We obtain the coproduct of the coefficient $[\Gamma]_\zeta^C$ by imposing the condition C on $\Delta(\Gamma(x))$ and then projecting on the ζ th coefficient.

$$\Delta([\Gamma]_\zeta^C) = [\Delta(\Gamma)]_\zeta^C$$

Proof.

$$\Delta(\Gamma(x))|_C = \sum_{s \in W_P} x^s|_C \Delta([\Gamma]_s) = \sum_{\zeta \in W_I} x^\zeta \sum_{w \in W_J^0} \sum_{\substack{u,v \in W_I^0 \\ uv = \zeta}} z_v^{(w)} \Delta([\Gamma]_{uw})$$

$$= \sum_{\zeta W_I} x^\zeta \Delta \left(\sum_{w \in W_J^0} \sum_{\substack{u,v \in W_I^0 \\ uv = \zeta}} z_v^{(w)} [\Gamma]_{uw} \right)$$

$$= \sum_{\zeta W_I} x^\zeta \Delta([\Gamma]_\zeta^C) \Leftrightarrow \Delta([\Gamma]_\zeta^C) = [\Delta(\Gamma)]_\zeta^C$$

$\square$

We are now trying to find out what happens if we restrict the generators of a DSA to a linear subspace. Consider the following DSE with a disjoint decomposition $P = I \uplus J$.

$$\begin{aligned}
B^{(q)}(x) &= \mathbb{1} + \sum_{u \in W_P} H_u^{(q)}(D(x)^u B^{(q)}(x)) \quad \forall q \in Q \\
D^{(i)}(x) &= x^{(i)} + \sum_{u \in W_P, \|u\| \geq 2} H_u^{(i)}(D(x)^u) \quad \forall i \in I \\
D^{(j)}(x) &= x^{(j)} + \sum_{u \in W_P, \|u\| \geq 2} H_u^{(j)}(D(x)^u) \quad \forall j \in J
\end{aligned}$$

If we now impose the condition $C = (x^{(j)} = (\Theta.x)^{(j)} | \Theta \in \mathrm{Hom}(\mathbb{K}^{\times I}, \mathbb{K}^{\times J}))$ and set $y^{(i)} = x^{(i)} \Rightarrow y \in \mathbb{K}^{\times I}$ for transparency we obtain the conditional DSE.

$$\begin{aligned}
B^{(q)}(y) &= \mathbb{1} + \sum_{u \in W_P} H_u^{(q)}(D(y)^u B^{(q)}(y)) \quad \forall q \in Q \\
D^{(i)}(y) &= y^{(i)} + \sum_{u \in W_P, \|u\| \geq 2} H_u^{(i)}(D(y)^u) \quad \forall i \in I \\
D^{(j)}(y) &= (\Theta.y)^{(j)} + \sum_{u \in W_P, \|u\| \geq 2} H_u^{(j)}(D(y)^u) \quad \forall j \in J
\end{aligned}$$

Proposition 4.4.3
Let $P = I \uplus J$ be a decomposition into two disjoint sets. Consider the condition $C = \left(x^{(j)}(x) = \sum_{i \in I} \theta_i^j x^{(i)} = \sum_{i \in I} \theta_i^j y^{(i)} | \forall j \in J\right)$ on the linear space $\mathbb{K}^{\times P}$. Let Ω be an automorphism of $\mathbb{K}^{\times I}$ and let

$$\begin{aligned}
B^{(q)}(y) &= \mathbb{1} + \sum_{u \in W_P} H_u^{(q)}(D(y)^u B^{(q)}(y)) \quad \forall q \in Q \\
D^{(i)}(y) &= (\Omega.y)^{(i)} + \sum_{u \in W_P, \|u\| \geq 2} H_u^{(i)}(D(y)^u) \quad \forall i \in I \\
D^{(j)}(y) &= (\Theta.\Omega.y)^{(j)} + \sum_{u \in W_P, \|u\| \geq 2} H_u^{(j)}(D(y)^u) \quad \forall j \in J
\end{aligned}$$

be a conditional DSE.

There exist cocycles $\{H_\zeta^{(Q),C}, H_\zeta^{(I),C}, H_\zeta^{(J),C}\}_{\zeta \in W_I}$ so that the below holds.

$$\begin{aligned}
B^{(q)}(y) &= \mathbb{1} + \sum_{\zeta \in W_I} H_\zeta^{(q),C}(D(y)^\zeta B^{(q)}(y)) \quad \forall q \in Q \\
D^{(i)}(y) &= (\Omega.y)^{(i)} + \sum_{\zeta \in W_I, \|\zeta\| \geq 2} H_\zeta^{(i),C}(D(y)^\zeta) \quad \forall i \in I \\
D^{(j)}(y) &= (\Theta.\Omega.y)^{(j)} + \sum_{\zeta \in W_I, \|\zeta\| \geq 2} H_\zeta^{(j),C}(D(y)^\zeta) \quad \forall j \in J
\end{aligned}$$

Proof. Define new cocycles

$$H_\zeta^{(g),C} = \sum_{w \in W_J^0} \sum_{\substack{u,v \in W_I^0 \\ uv = \zeta}} m_v^{(w)} H_{uw}^{(g)} \quad \forall g \in P \cup Q$$

with numbers $m_v^{(w)}$ so that

$$x^w = \prod_k x^{(w_k)} = \prod_k \left(\sum_{i \in I} \theta_i^{(w_k)} y^{(i)} \right) = \sum_{v \in W_I^0} m_v^{(w)} y^v \quad \forall w \in W_J^0.$$

Consider the DSE below.

$$\begin{aligned}
\underline{B}^{(q)}(y) &= \mathbb{1} + \sum_{\zeta \in W_I} H_\zeta^{(q),C}(\underline{D}(y)^\zeta \underline{B}^{(q)}(y)) \quad \forall q \in Q \\
\underline{D}^{(i)}(y) &= (\Omega.y)^{(i)} + \sum_{\zeta \in W_I, \|\zeta\| \geq 2} H_\zeta^{(i),C}(\underline{D}(y)^\zeta) \quad \forall i \in I \\
\underline{D}^{(j)}(y) &= (\Theta.\Omega.y)^{(j)} + \sum_{\zeta \in W_I, \|\zeta\| \geq 2} H_\zeta^{(j),C}(\underline{D}(y)^\zeta) \quad \forall j \in J
\end{aligned}$$

We have already proven in theorem (4.3.7) that this system of equations has a unique solution so that

$$(\Theta.\underline{D})^{(j)}(y) = \underline{D}(y)^{(j)} \quad \forall j \in J.$$

So in total we obtain the below.

$$
\begin{aligned}
\underline{B}^{(q)}(y) &= \mathbb{1} + \sum_{\zeta \in W_I} H_\zeta^{(q),C}(\underline{D}(y)^\zeta \underline{B}^{(q)}(y)) \\
&= \mathbb{1} + \sum_{\zeta \in W_I} \sum_{w \in W_J^0} \sum_{\substack{u,v \in W_I^0 \\ uv=\zeta}} m_v^{(w)} H_{uw}^{(q)}(\underline{D}(y)^\zeta \underline{B}^{(q)}(y)) \\
&= \mathbb{1} + \sum_{\zeta \in W_I} \sum_{w \in W_J^0} \sum_{\substack{u,v \in W_I^0 \\ uv=\zeta}} m_v^{(w)} H_{uw}^{(q)}(\underline{D}(y)^u \underline{D}(y)^v \underline{B}^{(q)}(y)) \\
&= \mathbb{1} + \sum_{w \in W_J^0} \sum_{u \in W_I^0} H_{uw}^{(q)} \left(\underline{D}(y)^u \left(\sum_{v \in W_I^0} m_v^{(w)} \underline{D}(y)^v \right) \underline{B}^{(q)}(y) \right) \\
&= \mathbb{1} + \sum_{w \in W_J^0} \sum_{u \in W_I^0} H_{uw}^{(q)} \left(\underline{D}(y)^u \underline{D}(y)^w \underline{B}^{(q)}(y) \right) \\
&= \mathbb{1} + \sum_{b \in W_P} H_b^{(q)} \left(\underline{D}(y)^b \underline{B}^{(q)}(y) \right)
\end{aligned}
$$

All others analogously. This completes the proof since the solution of a DSE is unique. □

NOTE 16 Let $C = \left(x^{(j)}(x) = \sum_{i \in I} \theta_i^j x^{(i)} | \forall j \in J \right)$ be a condition. With the help of theorem (4.2.10) one can conclude that

$$A^{C,(j)}(x) = (\Theta.A^C)^{(j)}(x) \quad \forall j \in J.$$

This proves that a linear relation ($x^{(j)} = (\Theta.y)^{(j)}$) among the coupling constants leads to the same relation among the corresponding invariant charges ($A^{C,(j)}(x) = (\Theta.A^C)^{(j)}(x)$).

Recall the definition (2.1.4) .

Theorem 4.4.4

Let $\wp$ be a DSA with generators (A^P, B^Q). Let $P = I \uplus J$ be a disjoint decomposition and let $C = (x^{(j)} = \sum_{i \in I} \theta_i^j x^{(i)} | \forall j \in J)$ be a condition. Set

$$\mathcal{I}_C = < [A^{(j)} - \sum_{i \in I} \theta_i^j A^{(i)}]_\zeta^C | \zeta \in W_I \quad j \in J > \Rightarrow$$

$\mathcal{I}_C$ is a coideal of $\wp$ and thus an ideal of a bialgebra.

Proof.

$$\Delta\left(A^{(j)}(x)-\sum_{i\in I}\theta_i^j A^{(i)}(x)\right)=\sum_{u\in W_I^0;v\in W_J^0} A(x)^u A(x)^v\otimes[A^{(j)}-\sum_{i\in I}\theta_i^j A^{(i)}]_{uv}$$

$$=\sum_{u\in W_I^0;v\in W_J^0} A(x)^u\left\{\prod_{1\le k\le\|v\|}\left[\left(A^{(\underline{v}_k)}(x)-\sum_{i\in I}\theta_i^{\underline{v}_k}A^{(i)}(x)\right)+\sum_{i\in I}\theta_i^{\underline{v}_k}A^{(i)}(x)\right]\right\}\otimes\cdots$$

$$=\sum_{u\in W_I^0;v\in W_J^0} A(x)^u\left\{\sum_{\substack{a,b\in W_J^0\\ ab=v}}\left[\prod_{1\le\alpha\le a}\left(A^{(\underline{a}_\alpha)}(x)-\sum_{i\in I}\theta_i^{\underline{a}_\alpha}A^{(i)}(x)\right)\right]\left[\prod_{1\le\beta\le b}\left(\sum_{i\in I}\theta_i^{\underline{b}_\beta}A^{(i)}(x)\right)\right]\right\}\otimes\cdot$$

$$=\underbrace{\cdots\left\{A^{(J)}(x)-\sum_{i\in I}\theta_i^J A^{(i)}(x)\right\}\cdots\otimes\cdots}_{a\neq\oslash}+\underbrace{\sum_{u\in W_I^0;v\in W_J^0} A(x)^u\prod_{1\le k\le\|v\|}\left(\sum_{i\in I}\theta_i^{\underline{v}_k}A^{(i)}(x)\right)\otimes\cdots}_{a=\oslash}$$

With the convention

$$\prod_{\emptyset}=1$$

The $a\neq\oslash$ term is, when restricted to C, of the needed form to be an element of $\mathcal{I}_C$.
Consider the term for $a=\oslash$. Compare the structure of $\prod_{1\le k\le\|v\|}\left(\sum_{i\in I}\theta_i^{\underline{v}_k}A^{C,(i)}(x)\right)$ with the structures discussed earlier in lemma (4.4.2) for example. Just exchange $A^C(x)^v$ with $x^v|_C$!

$$\sum_{u\in W_I^0;v\in W_J^0} A^C(x)^u\prod_{1\le k\le\|v\|}\left(\sum_{i\in I}\theta_i^{\underline{v}_k}A^{C,(i)}(x)\right)\otimes[A^{(j)}-\sum_{i\in I}\theta_i^j A^{(i)}]_{uv}$$
$$=\sum_{\zeta\in W_I} A^C(x)^\zeta\otimes\sum_{w\in W_J^0}\sum_{\substack{u,v\in W_I^0\\ uv=\zeta}} z_v^{(w)}[A^{(j)}-\sum_{i\in I}\theta_i^j A^{(i)}]_{uv}$$
$$=\sum_{\zeta\in W_I} A^C(x)^\zeta\otimes[A^{(j)}-\sum_{i\in I}\theta_i^j A^{(i)}]_\zeta^C$$

So in total one obtains with the help of lemma (4.4.2)

$$\Delta\left([A^{(j)}-\sum_{i\in I}\theta_i^j A^{(i)}]_\zeta^C\right)\in\mathcal{I}_C\otimes\wp+\wp\otimes\mathcal{I}_C$$

, which proves the theorem. $\square$

Notation 4.4.5
Let $\wp$ be a DSA generated by (A^P, B^Q) and consider a condition C. With the help of theorem (4.3.7) and proposition (4.4.3) we can conclude that the coefficients of $A^P(X)|_C$ and $B^Q(X)|_C$ generate a $(dim(C), |Q|)$ dimensional DSA, which we will denote as $\wp_C \subseteq \wp$.

Theorem 4.4.6

$$\wp/\mathcal{I}_C \cong \wp_C$$

Proof. Let $\wp$ be generated by (A^P, B^Q), let $P = I \uplus J$ be a decomposition into two disjoint sets and let $C = (x^{(j)} = (\Theta.x)^{(j)} | \forall j \in J)$ be a condition with some $\Theta \in \mathrm{Hom}(\mathbb{K}^{\times I}, \mathbb{K}^{\times J})$. Consider the projection onto the subbialgebra $\wp_C$.

$$\pi_C : \wp \to \wp_C$$

Note: π_C is a morphism of bialgebras. We then obtain an isomorphism between $\wp/\mathrm{Ker}(\pi_C)$ and $\wp_C$. But since $\pi_C([A^{(j)}]_u^C - [(\Theta.A)^{(j)}]_u^C) = 0 \quad \forall j \in J, u \in W_I$, we obtain $\mathrm{Ker}(\pi_C) = \mathcal{I}_C$. Together with proposition (2.7.5) we can follow the assertion.

$$\wp/\mathcal{I}_C \cong \wp_C$$

□

Since every (p,q)-dim DSA contains every (u,q)-dim DSA with $u < p$, one can conclude that there exists a distinct DSA $\wp_{core}$ which contains every DSA with q propagators. This DSA $\wp_{core}$ is generated by the QFT which has vertices with arbitrary high valances. The Hopf algebra of Feynman graphs generated by this QFT is called the core algebra and was introduced in [10] and further discussed in [11].

As we saw in this section linear relation among coupling constants translate into the same relation among the invariant charges and the underlying DSA changes in such a way that the resulting DSA can be obtained by dividing the original DSA with the ideal defined above. Compare this result to example 9 in the following section. There we use this identity to derive the Feynman rules for the conditional QFT from the original QFT and we show the connection to the corresponding Slavnov Taylor identities.

For further analyses of DSE and the corresponding Hopf algebras from a somewhat different perspective I can recommend [12] and [13].

4.5 Applications to physics

Example 8 Linear transformation of scalar fields.
Consider the following Lagrangian with two scalar fields.

$$\mathcal{L} = (\partial\psi)^2 + (\partial\phi)^2 + \frac{g}{n!m!}\psi^n\phi^m \quad (\star)$$

We can then carry out a linear transformation on the field ϕ e.g.

$$\phi = \chi + v \Rightarrow \mathcal{L} = (\partial\psi)^2 + (\partial\chi)^2 + \sum_{1\leq k\leq m} \frac{g}{n!k!}\frac{v^{m-k}}{(m-k)!}\psi^n\chi^m(\star)$$

Compare this to the general Lagrangian

$$\mathcal{L} = (\partial\psi)^2 + (\partial\chi)^2 + \sum_{1\leq k\leq m} \frac{g^{(n,k)}}{n!k!}\psi^n\chi^k.$$

This Lagrangian would lead to invariant charges

$$Q^{(n,m)}(\{g^{(n,m)}\}).$$

We can reobtain the Lagrangian $(\star)$ by enforcing the following condition on the space of coupling constants.

$$C = (g^{(n,k)} = \frac{v^{m-k}}{(m-k)!}g^{(n,m)}|\forall 1 \leq k \leq (m-1))$$

So for the QFT generated by the Lagrangian $(\star)$ we obtain the following Slavnov Taylor identities.

$$[Q^{(n,k)} - \frac{v^{m-k}}{(m-k)!}Q^{(n,m)}]^C_u = 0 \quad \forall u \in \mathbb{N}$$

The resulting DSA underlying the above QFT is thus (2,1) dimensional. We conclude that the DSA underlying the QFT before and after the linear transformation are isomorph. This shows that the underlying DSA before and after the spontaneous symmetry breaking are isomorph since a spontaneous broken QFT can be obtained from the unbroken one by a linear transformation.
Example 9 SO(2) invariant QFT

Consider the following Lagrangian.

$$\mathcal{L} = (\partial\psi)^2 + (\partial\phi)^2 + \frac{g^{(1)}}{4!}\phi^4 + \frac{g^{(2)}}{4}\phi^2\pi^2 + \frac{g^{(3)}}{4!}\pi^4(\star\star)$$

We obtain a SO(2) invariant Lagrangian if we set
$g^{(3)} = g^{(1)}$ and $g^{(2)} = \frac{1}{3}g^{(1)}$. We will denote this condition with $C_{SO(2)}$. Let Φ be a Feynman rule defined on the DSA $\wp$ generated by $(\star\star)$. We can obtain the corresponding Feynman rule on $\wp_{C_{SO(2)}}$ by projecting onto the sub-DSA.

$$\Phi_{C_{SO(2)}} := \Phi \circ \pi_{C_{SO(2)}}$$

Since

$$\pi_{C_{SO(2)}}([Q^{(2)} - Q^{(1)}]_u^{C_{SO(2)}}) = 0 \quad \pi_{C_{SO(2)}}([Q^{(3)} - \frac{1}{3}Q^{(1)}]_u^{C_{SO(2)}}) = 0$$

, we obtain the Slavnov Taylor identities below.

$$\Phi_{C_{SO(2)}}([Q^{(2)} - Q^{(1)}]_u^{C_{SO(2)}}) = 0 \quad \Phi_{C_{SO(2)}}([Q^{(3)} - \frac{1}{3}Q^{(1)}]_u^{C_{SO(2)}}) = 0$$

Example 10 Counterterm renormalization
If a QFT is renormalizable one can obtain the renormalized Greens functions by computing the unrenomalized Greens functions of the bare Lagrangian. For example consider the Lagrangian below.

$$\mathcal{L} = (\partial\psi)^2 + (\partial\phi)^2 + \frac{g}{n!m!}\psi^n\phi^m$$

The bare Lagrangian is thus

$$\mathcal{L}_{Bare} = \Phi^C(X^\psi(g))(\partial\psi)^2 + \Phi^C(X^\phi(g))(\partial\phi)^2 + \frac{\Phi^C(X^{(v)}(g))}{n!m!}\psi^n\phi^m.$$

Redefining $\phi \to \frac{1}{\Phi^C(X^\phi(g))^{1/2}}\phi$ and $\psi \to \frac{1}{\Phi^C(X^\psi(g))^{1/2}}\psi$ leads to

$$\mathcal{L}_{Bare} = (\partial\psi)^2 + (\partial\phi)^2 + \frac{\Phi^C(Q^{(v)}(g))}{n!m!}\psi^n\phi^m$$

and thus

$$\Phi^R(X^{(v)}(g)) = \Phi^C(\Lambda^{(v)}(g))\Phi\left(X^{(v)}\left(\Phi^C(Q)\right)\right).$$

This is precisely the same result we would obtain with the help of the DSA.

From lemma (4.3.18) we obtain in general the below.

$$\begin{aligned}
\Delta(X^{(v)}(g)) &= \sum_{\|u\|\geq 1} \Lambda^{(v)}(g)Q(g)^u \otimes [X^{(v)}]_u \Rightarrow \\
\Phi^R(X^{(v)}(g)) = (\Phi^C \star \Phi)(X^{(v)}(g)) &= \sum_{\|u\|\geq 1} \Phi^C(\Lambda^{(v)}(g)Q(g)^u)\Phi([X^{(v)}]_u) \\
&= \Phi^C(\Lambda^{(v)}(g))\Phi\left(\sum_{\|u\|\geq 1} \Phi^C(Q(g))^u[X^{(v)}]_u\right) \\
&= \Phi^C(\Lambda^{(v)}(g))\Phi\left(X^{(v)}\left(\Phi^C(Q)\right)\right)
\end{aligned}$$

Example 11 Bare Lagrangian
Consider the scalar QFT with the Lagrangian

$$\mathcal{L} = (\partial\psi)^2 + (\partial\phi)^2 + \frac{g^{(1)}}{4!}\phi^4 + \frac{g^{(2)}}{4}\phi^2\pi^2 + \frac{g^{(2)}}{4!}\pi^4.$$

The bare Lagrangian computes to

$$\mathcal{L}_{Bare} = (\partial\psi)^2 + (\partial\phi)^2 + \frac{\Phi^C(Q^{(1)}(g))}{4!}\phi^4 + \frac{\Phi^C(Q^{(2)}(g))}{4}\phi^2\pi^2 + \frac{\Phi^C(Q^{(3)}(g))}{4!}\pi^4.$$

We now impose the condition $g^{(3)} = g^{(1)} = q$ and $g^{(2)} = \frac{1}{3}g^{(1)} = \frac{1}{3}q$. This changes the Lagrangian into the following.

$$\mathcal{L} = (\partial\psi)^2 + (\partial\phi)^2 + \frac{q}{4!}(\phi^2 + \psi^2)^2$$

If the same relation for the invariant charges were not hold, we would not be able to absorb the counter term into the Lagrangian and the QFT would not be renormalizable. But as we have learned the condition imposes the following conditions on the invariant charges $Q^{(3)} = Q^{(1)} = Q$ and $Q^{(2)} = \frac{1}{3}Q^{(1)} = \frac{1}{3}Q$, which leads to the bare Lagrangian below.

$$\mathcal{L} = (\partial\psi)^2 + (\partial\phi)^2 + \frac{\Phi^C(Q(q))}{4!}(\phi^2 + \psi^2)^2$$

Thus the QFT considered above is renormalizable.

Chapter 5

Conclusion

As we saw in chapter 3, DSE leads to local and thus renormalizable QFT. Thus the results formulated in this text are implications of locality. The order by order contributions of the propagators and invariant charges of a QFT form a Hopf subalgebra of the Hopf algebra of Feynman graphs. They form a DSA to be precise. We learned that a DSE in the form discussed in the text always leads to a DSA and vice versa, which means that every local QFT has a corresponding DSA on which we can define Feynman rules and renormalize the QFT. If we were given the Feynman rules on the DSA, we would not need to formulate the QFT in terms of Feynman graphs and Feynman rules upon them. This shows that the physical important contributions to the propagators and invariant charges are not the Feynman graphs but rather the order by order contributions. Feynman graphs are only needed to compute the Feynman rules for a given order. In chapter 4 we further learned that the invariant charges of a QFT are the quantum mechanical generalization of the coupling constants and we studied the transition between classical and quantum mechanics in terms of conditional DSAs. We saw that locality imposes such a strong condition on the QFT that a relation among coupling constants, which corresponds to the classical level translates to the same relation among the invariant charges. This is an important result because it allows us to absorb the counter terms into the Lagrangian if we have more interaction monomials than coupling constants. In this sense it is thus legitimate to say that the classical level already determines the combinatorial structure of a local QFT. We further studied the implications for the underlying DSA if we restrict the space of coupling constants to a linear subspace respectively if we preform a linear automorphism of the coupling constants. We learned that a linear automorphism corresponds to an automorphism of the DSA and a restriction corresponds to a transition to a quotient, which is isomorph to a sub-DSA. Since a diffeomorphism of fields leads to a transformation of coupling constants, we are now in the position to study the change of Feynman rules under this transformation, at least for cases which lead to linear relation among the coupling constants. As we saw in example 9 this leads automatically to the Slavnov Taylor identities of a QFT.

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