RGE are for Green fct's

CHAPTER 6

Renormalization Group

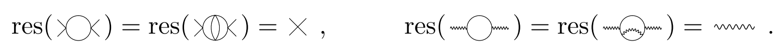
$|\Gamma| = \text{loop}\ \#$

6.1. Formal power series and Green functions

Let $\Gamma \in H_{FG}$ be a Feynman graph. The *residue* of Γ is the graph $\mathrm{res}(\Gamma)$ obtained from Γ by shrinking *all* internal edges to a single point. Instead of residue, we shall also speak of the *external leg structure*. Examples are

(6.1) 

and

(6.2) 

By $\mathcal{R}$ we denote a set of such residues of interest for a given renormalizable theory. It is generally finite. The *valence* $\mathrm{val}(r)$ of the residue $r = \mathrm{res}(\Gamma)$ is defined as the number of external legs of the corresponding graph Γ.

We consider formal power series in one variable α with coefficients in H_{FG} for example of the form

$$\Gamma^r(\alpha) = \mathbb{I} \pm \sum_{\mathrm{res}(\Gamma)=r} \frac{\alpha^{|\Gamma|}}{\mathrm{Sym}(\Gamma)} \Gamma \tag{6.3}$$

$m^2\phi^2$, $\partial\phi\partial^\mu\phi$

where the sum is over all 1PI graphs with external leg structure r and $\mathrm{Sym}(\Gamma)$ is a symmetry factor associated to the graph Γ. If $\mathrm{val}(r) = 2$, then there is a minus sign in (6.3), and a plus sign in all other cases. We formally apply a character representing some given Feynman rules and get a perturbative expansion

$$G^r(\alpha, L, \theta) := \phi(\Gamma^r(\alpha))\{L, \theta\} = 1 \pm \sum_{\mathrm{res}(\Gamma)=r} \frac{\alpha^{|\Gamma|}}{\mathrm{Sym}(\Gamma)} \phi(\Gamma)\{L, \theta\}\ , \tag{6.4}$$

of what is known as a *Green function* $G^r(\alpha, L, \theta)$ in which L and θ are external scale and angle parameters or collections of such, respectively. If $\mathrm{val}(r) = 2$, we refer to G^r as *two-point function* and if $\mathrm{val}(r) \geq 3$ as *vertex function*. Strictly speaking, this Green function is the corresponding *structure function* for the amplitude $r \in \mathcal{R}$. The textbook Green function is then given by multiplication of G^r with a form factor such as p^2 or $\not{p} = p_\mu \gamma^\mu$ for an incoming momentum $p \in \mathbb{R}^4$, well-known to readers acquainted with QFT.

6.2. Combinatorial Dyson-Schwinger equations

The formal series $X(\alpha) = \sum_{k\geq 0} \alpha^k \lambda_k \in H_\ell[[\alpha]]$ with coefficients in the ladder Hopf subalgebra satisfies the equation

$$X(\alpha) = \mathbb{I} + \alpha B_+(X(\alpha))\ , \tag{6.5}$$

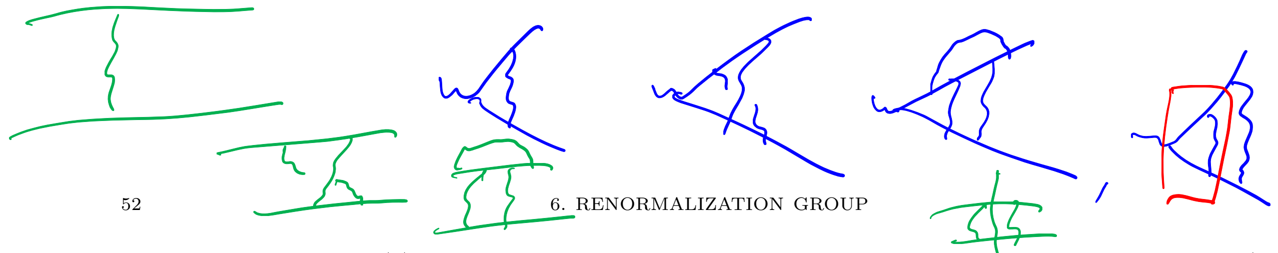
which can be easily checked since $B_+(\lambda_k) = \lambda_{k+1}$ for all $k \in \mathbb{N}$. This equation is a simple example of a *Dyson-Schwinger equation*. Such equations do also exist for series with cofficients in the Feynman graph Hopf algebra H_{FG} like in (6.3). They are systems of equations of the form

$$\Gamma^r(\alpha) = \mathbb{I} + \mathrm{sgn}(s_r) B_+^r(\Gamma^r(\alpha), Q(\alpha))\ , \qquad r \in \mathcal{R}\ , \tag{6.6}$$

where $Q(\alpha)$ is the so-called *invariant charge* given by

$$Q(\alpha) = \prod_{r\in\mathcal{R}} (\Gamma^r(\alpha))^{s_r} \tag{6.7}$$

$Q \sim \frac{\Gamma^{\prec}}{\sqrt{\Gamma^{-3}}}$



with integers s_r. If $\mathrm{val}(r) = 2$ one has $s_r < 0$ and $s_r > 0$ otherwise. This ensures a minus sign in (6.6) for a propagator series. The operator $B_+^r(\cdot,\cdot)$ is defined as

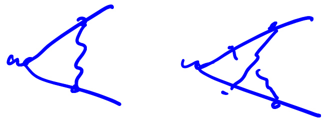
(6.8) $$B_+^r(\Gamma^r(\alpha), Q(\alpha)) = \sum_{k\geq 1} \alpha^k B_+^{k;r}(\Gamma^r(\alpha)Q(\alpha)^k)$$

with one-cocycles $B_+^{k;r}$ which themselves are defined by

(6.9) $$B_+^{k;r} = \sum_{\mathrm{res}(\gamma)=r, |\gamma|=k, \mathrm{prim.}} \frac{1}{\mathrm{Sym}(\gamma)} B_+^\gamma$$

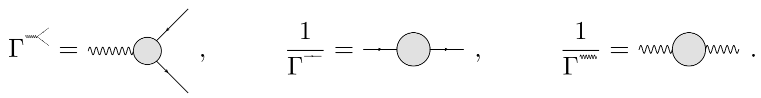
with one-cocyles B_+^γ. The sum extends over all 1PI primitive graphs γ with external leg structure r and loop number k. Recall that a graph γ is called primitive if $\Delta(\gamma) = \gamma \otimes \mathbb{I} + \mathbb{I} \otimes \gamma$. Notice that, in general, there are infinitely many primitive graphs and hence the sum in (6.8) is not finite. An example for the invariant charge $Q(\alpha)$ in QED is

(6.10) $$Q(\alpha) = \frac{\Gamma^{\prec}(\alpha)^2}{\Gamma^{\sim}(\alpha)\Gamma^{-}(\alpha)^2} \, .$$

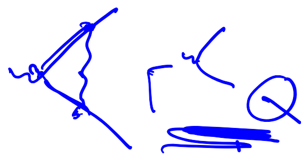


However cryptic these expressions may look, the product $\Gamma^r(\alpha)Q(\alpha)^k$ of formal power series has coefficients in H_{FG} which are exactly what one can glue into a 1PI primitive graph γ with k loops and external leg structure r. This glueing corresponds to what is known as vertex or propagator corrections in standard QFT where our formal series are generally depicted by graphs with blobs: for QED they take the form

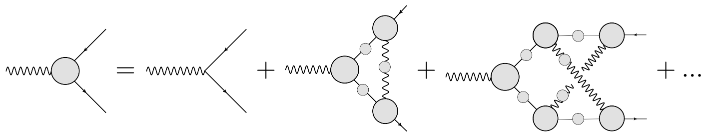
(6.11)



The Dyson-Schwinger equation for the QED vertex reads in this notation

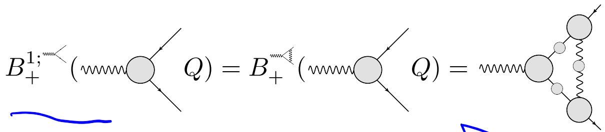


(6.12)



where the tree-level graph $\prec = \mathbb{I}$ is what we count as an empty graph. To understand the action of the one-cocycles, consider the second term on the rhs of (6.12): it can be written as

(6.13)



and has the following meaning: the growth operator $B_+^{\prec}$ uses the vertex series $\Gamma^{\prec} = \prec$ to provide for all radiative corrections at one vertex, say the leftmost one of the superscript skeleton graph $\gamma = \prec$. Then, it takes the invariant charge Q to glue in additional graphs so as to guarantee that every propagator is fully dressed and the remaining vertices are fully corrected. For the higher loop primitives, higher powers of Q are needed to dress all propagators and vertices which come with additional loops.

However, we come back to the general case and rewrite (6.6) into

(6.14) $$\Gamma^r(\alpha) = \mathbb{I} + \mathrm{sgn}(s_r) \sum_{k\geq 1} \alpha^k B_+^{k;r}(\Gamma^r(\alpha)Q(\alpha)) \, , \qquad r \in \mathcal{R}$$

whose solution exists and may be written in the form

(6.15) $$\Gamma^r(\alpha) = \mathbb{I} + \mathrm{sgn}(s_r) \sum_{k=1}^{\infty} \alpha^k c_k^r \, , \qquad r \in \mathcal{R} \, ,$$

where $c_k^r \in H_{FG}$ is a linear combination of 1PI graphs with k loops and external leg structure r. These coefficients generate a Hopf subalgebra with coproduct

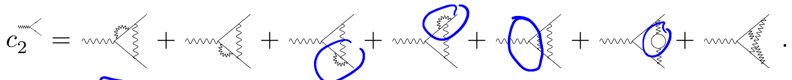
(6.16) $$\Delta(c_k^r) = \sum_{j=0}^{k} P_{k,j}^r \otimes c_{k-j}^r \, ,$$

c_8 $c_3 \otimes c_5$

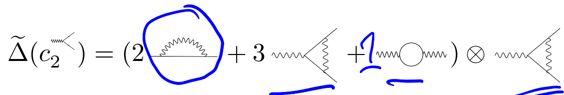
where $P^r_{k,j}$ is a polynomial in these generators(see also [**KrY06**]). For example, in QED one has

(6.17) 

and

(6.18) 

The reduced coproduct of the latter is

(6.19) 

which is, in terms of the generators,

(6.20) $$\widetilde{\Delta}(c_2) = (2\, c_1 + 3\, c_1 + c_1) \otimes c_1 = P_{2,1} \otimes c_1 \ .$$

The other polynomials are $P_{2,0} = \mathbb{I}$ and $P_{2,2} = c_2$ for the trivial part of the coproduct.

6.3. The structure of Green functions

If we apply the renormalized Feynman rules ϕ_R to (6.15) as in (6.4), the corresponding Green function reads

(6.21) $$G^r_R(\alpha, L, \theta) = \phi_R(\Gamma^r(\alpha))\{L, \theta\} = 1 + \mathrm{sgn}(s_r) \sum_{k=1}^{\infty} \alpha^k \phi_R(c^r_k)\{L, \theta\} \ .$$

The individual coefficients $\phi_R(c^r_k)$ are polynomials in the external scale parameter L which is why we can rewrite (6.21) to obtain

(6.22) $$G^r_R(\alpha, L, \theta) = 1 + \sum_j \gamma^r_j(\alpha, \theta) L^j \ ,$$

where j may be a multi-index and $\gamma^r_j(\alpha, \theta)$ is a function of the loop parameter α and the angle parameter θ. In a very simple linear case, where $Q(\alpha) = \mathbb{I}$ and the operators in (6.9) are simplified significantly to yield the analogon of (6.5) for $H_{FG}[[\alpha]]$, the two-point Green function in (6.22) takes the form

(6.23) $$G(\alpha, L) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \gamma(\alpha)^j L^j = \exp(-\gamma(\alpha) L) \ ,$$?

i.e. $\gamma_j(\alpha) = (-1)^j \gamma(\alpha)^j / j!$, where $\gamma(\alpha)$ is known as the anomalous dimension.

The Dyson-Schwinger equations in (6.6) for the Hopf algebra of Feynman graphs H_{FG}, henceforth abbreviated by DSE, correspond to a system of integral equations for the Green functions in the target algebra $\mathcal{A}$ of the Feynman rules. This is on account of the universal property of graded Hopf algebras with Hochschild one-cocycles according to which the operators B^γ_+ in (6.9) translate to integral operators on the target algebra of the Feynman rules. This may take the form

(6.24) $$(\phi \circ B^\gamma_+)(X)\{q\} = \int d_\gamma(k, q)\ \phi(X)\{k, q\}$$

for a graph X with some integration measure $d_\gamma(k, q)$ associated to the graph γ. The renormalized version of (6.24) is

(6.25) $$(\phi_R \circ B^\gamma_+)(X)\{q\} = \int d_\gamma(k, q)\ (\phi(X)\{k, q\} - \phi(X)\{k, q_0\})$$

where q_0 is an external momentum such that $q_0^2 = \mu^2$, with μ being the renormalization point. To distinguish between these two different types of DSE we refer to the system of integral equations as *analytic* DSE and those in (6.6) as *combinatorial* DSE.

Infinitesimal characters. There is an interesting way to obtain the coefficient functions $\gamma_j^r(\alpha)$ in (6.22), where we suppress the angle-dependence in the notation for the moment. First we define a linear map $Y^{-1} : H_{FG} \to H_{FG}$ by $Y^{-1}(\mathbb{I}) = 0$ and

$Y(\gamma) = |\gamma|\gamma$

$$Y^{-1}(\gamma) = \frac{1}{|\gamma|}\gamma \tag{6.26}$$

for a product of Feynman graphs $\gamma = \prod_j \gamma_j$, where $|\gamma| := \sum_j |\gamma_j|$ counts the loops. This choice of notation is justified as Y^{-1} really is the inverse of the grading operator Y on the augmentation ideal Aug. Next, we introduce a family of linear maps $\sigma_n : H_{FG} \to \mathbb{C}$ by

$$\sigma_1 := \partial_L \phi_R Y^{-1}(S * Y)|_{L=0} \tag{6.27}$$

and

$$\sigma_n := \frac{1}{n!}\sigma_1^{*n} := \frac{1}{n!}\underbrace{\sigma_1 * \ldots * \sigma_1}_{n-\text{times}} = \frac{1}{n!}m^{n-1}\sigma_1^{\otimes n}\Delta^{n-1} \tag{6.28}$$

for $n \geq 2$, where m is the usual multiplication in $\mathbb{C}$ and $*$ is the convolution product

$$\sigma_1 * \sigma_1 = m(\sigma_1 \otimes \sigma_1)\Delta\,. \tag{6.29}$$

$\phi_R \in G^{H_{FG}}_{\mathbb{C}}$

Note that the map σ_1 is a so-called *infinitesimal character* on H_{FG} which means

$$\sigma_1(xy) = \sigma_1(x)\hat{\mathbb{I}}(y) + \hat{\mathbb{I}}(x)\sigma_1(y) \tag{6.30}$$

for all $x, y \in H_{FG}$. This implies $\sigma_1(\mathbb{I}) = 0$ and that it vanishes on nontrivial products, i.e.

$$\sigma_1(h) = 0 \tag{6.31}$$

if $h = h_1 h_2$ with $h_1, h_2 \in \text{Aug}$.

$\ln \frac{\rho^2 + \rho^2 + 3\rho q}{m^2}$

$\ln \frac{3q^2 + 5\rho^2 + \frac{1}{2}\rho\cdot q}{m^2}$

Lemma 6.3.1. *$S * Y$ is an infinitesimal character.*

Proof. Let $x, y \in \text{Aug}$. Then

$$\begin{aligned}
(S * Y)(xy) &= \sum_{(x)}\sum_{(y)} S(x'y')Y(x''y'') \\
&= \sum_{(x)}\sum_{(y)}[S(x')S(y')Y(x'')y'' + S(x')S(y')x''Y(y'')] \\
&= \sum_{(x)} S(x')Y(x'')\sum_{(y)} S(y')y'' + \sum_{(x)} S(x')x''\sum_{(y)} S(y')Y(y'') = 0
\end{aligned} \tag{6.32}$$

on account of $\sum_{(x)} S(x')x'' = (\text{id} * S)(x) = (S * \text{id})(x) = 0$ which holds by definition of the antipode S. □

The next assertion makes clear why these maps are of particular interest to us.

Proposition 6.3.2. *The linear map σ_n evaluates a graph Γ to its n-th order coefficient of $\phi_R(\Gamma)$ with respect to the variable L, i.e.*

$$\sigma_n(\Gamma) = \frac{1}{n!}\frac{\partial^n}{\partial L^n}\phi_R(\Gamma)\{L\}\Big|_{L=0}\,. \tag{6.33}$$

Proof. We have to use the fact that the set $\mathfrak{g}$ of infinitesimal characters is the Lie algebra generating the Lie group of characters G on H_{FG} in the sense that $G = \exp_*(\mathfrak{g})$, i.e. for every character ϕ, there exists an infinitesimal character $\sigma \in \mathfrak{g}$ such that

$$\phi = \exp_*(\sigma) := \sum_{n=0}^{\infty}\frac{\sigma^{*n}}{n!} \tag{6.34}$$

and vice versa with $\sigma_1^{*0} := \hat{\mathbb{I}}$ being the neutral element of the convolution product $*$. The inverse map of $\exp_*$ is given by

$$\log_*(\phi) = -\sum_{n=1}^{\infty}\frac{1}{n}(\hat{\mathbb{I}} - \phi)^{*n} = \sigma\,. \tag{6.35}$$

For more on this, see Appendix section A.3 or [**Man06**]. This is but a small step away from realizing that $\exp_*(L\mathfrak{g})$ for a variable L is the character group with target algebra $\mathbb{C}[L]$, i.e. for our character ϕ_R we have

$$\phi_R = \exp_*(L\sigma_R) \tag{6.36}$$

with some generator σ_R(see Appendix A.3). Then, clearly, we find

$$\partial_L \phi_R = \sigma_R * \phi_R \qquad \Rightarrow \qquad \partial_L \phi_R|_{L=0} = \sigma_R \ . \tag{6.37}$$

To prove (6.33) it suffices to show that $\sigma_R = \sigma_1$. To this end, we take a Feynman graph Γ and first calculate

$$\begin{aligned}
\phi_R Y^{-1}(S * Y)(\Gamma) &= \left(\hat{\mathbb{I}} + L\sigma_R + \frac{L^2}{2!}(\sigma_R * \sigma_R) + \ldots\right) Y^{-1}(S * Y)(\Gamma) \\
&= L\sigma_R Y^{-1}(S * Y)(\Gamma) + \mathcal{O}(L^2) = \frac{L}{|\Gamma|}\sigma_R \left(\sum_j S(\Gamma'_j)Y(\Gamma''_j)\right) + \mathcal{O}(L^2) \\
&= \frac{L}{|\Gamma|}\sigma_R(S(\mathbb{I})Y(\Gamma)) + \mathcal{O}(L^2) = L\sigma_R(\Gamma) + \mathcal{O}(L^2) \ .
\end{aligned} \tag{6.38}$$

□

A nice consequence is the following

Corollary 6.3.3. *The coefficient functions of the Green function G^r are given by*

$$\gamma_j^r(\alpha) = \sigma_j(\Gamma^r(\alpha)) \qquad \text{and} \qquad G_R^r(\alpha, L) = \exp_*(L\sigma_1)(\Gamma^r(\alpha)) \ , \tag{6.39}$$

where the $*$-exponential is defined as in (6.34).

6.4. Renormalization Group Equation

The coefficient functions γ_k^r of the Green function G^r satisfy

$$\gamma_k^r(\alpha) = \frac{1}{k}\left(\gamma_1^r(\alpha) + \sum_{u \in \mathcal{R}} s_u \gamma_1^u(\alpha)\alpha\partial_\alpha\right)\gamma_{k-1}^r(\alpha) \ , \qquad r \in \mathcal{R} \ , \tag{6.40}$$

which is a consequence of

$$(\mathcal{P}_{lin} \otimes \mathcal{P}_{lin})\Delta(\Gamma^r(\alpha)) = \mathcal{P}_{lin}\Gamma^r(\alpha) \otimes \mathcal{P}_{lin}\Gamma^r(\alpha) + \mathcal{P}_{lin}Q(\alpha) \otimes \alpha\partial_\alpha \mathcal{P}_{lin}\Gamma^r(\alpha) \ , \tag{6.41}$$

where $\mathcal{P}_{lin}$ is the projector onto the linear span of the Hopf algebra's generators, i.e. the Feynman graphs, but excluding $\mathbb{I}$. It is a fairly easy exercise to derive the so-called *renormalization group equation*

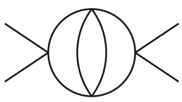
$$\left(-\frac{\partial}{\partial L} + \alpha\beta(\alpha)\frac{\partial}{\partial \alpha} + \gamma_1^r(\alpha)\right) G^r(\alpha, L) = 0 \tag{6.42}$$

from (6.40) with $G^r(\alpha, L) = 1 + \sum_{k=1}^{\infty} \gamma_k^r(\alpha) L^k$ and the function

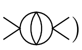
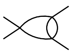
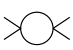
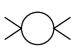
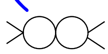
$$\beta(\alpha) := \partial_L \phi_R(Q(\alpha))|_{L=0} = \ldots = \sum_{u \in \mathcal{R}} s_u \gamma_1^u(\alpha) \tag{6.43}$$

known as β-function of the corresponding theory. A proof of both (6.40) and (6.42) can be found in Appendix section A.5, where the reader will also be introduced to a slightly stronger version of (6.41) and see how to fill the void ... in (6.43). Further relevant references are [**KrSui06**] and [**Y11**].

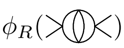
Example: a scalar 3-loop graph. Consider the graph

(6.44) $\Gamma =$ 

with reduced coproduct

(6.45) $\tilde{\Delta}($  $) = 2$  $\otimes$  $+$  $\otimes$  .

Say, the physical limit of some renormalized Feynman rules ϕ_R is

(6.46) $\phi_R($  $) = c_1 L + c_2 L^2 + c_3 L^3 \ ,$

QED:

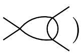
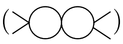
where $L = \ln(q^2/\mu^2)$ with $q := q_1 + q_2 = q_3 + q_4$, by momentum conservation. Given that we have

$$\phi_R(X)\{L\} = \sum_{j=1}^{\mathrm{cor}(X)} \sigma_j(X) L^j = \sum_{j=1}^{\mathrm{cor}(X)} \frac{1}{j!} \sigma_1^{*j}(X) L^j \tag{6.47}$$

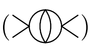
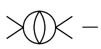
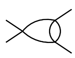
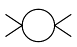
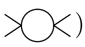
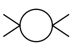
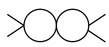
for a Feynman graph X and the infinitesimal characters $\sigma_j : H_{FG} \to \mathbb{C}$ introduced in the previous section, we want to see how the coefficients c_1, c_2 and c_3 relate to those of its subgraphs. The coradical degree of a graph X is defined by

$$\mathrm{cor}(X) = \min\{\ n \mid \mathcal{P}_{lin}^{(n+1)}(X) = 0\ \}\ , \tag{6.48}$$

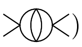
with $\mathcal{P}_{lin}^{(n+1)} := \mathcal{P}_{lin}^{\otimes n+1} \Delta^n$, analogous to the definitions for the coradical filtration of the Hopf algebra of rooted trees H in section 3.5. Let now for the subgraphs

(6.49) $\phi_R($$) = e_1 L + e_2 L^2$, $\phi_R($$) = d_1 L$, $\phi_R($$) = d_1^2 L^2$

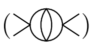
be the case. The infinitesimal character $Y^{-1}(S * Y)$ yields

(6.50) $Y^{-1}(S*Y)($$) =$  $- \frac{2}{3}$   $+ \frac{2}{3}($$)^3 - \frac{2}{3}$  

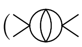
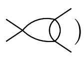
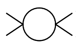
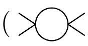
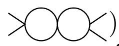
which evaluates to

(6.51) $\phi_R(Y^{-1}(S*Y)($$)) = c_1 L + \frac{1}{3}(3c_2 - 2e_1 d_1) L^2 + \frac{1}{3}(3c_3 - 2e_2 d_1) L^3$.

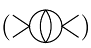
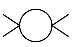
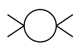
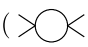
Not surprisingly, the map σ_1 picks out the term

(6.52) $\sigma_1($$) = c_1$.

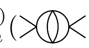
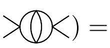
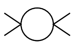
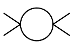
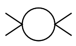
The next map $\sigma_2 = (\sigma_1 * \sigma_1)/2!$ yields

(6.53) $\sigma_2($$) = \frac{2}{2!}\sigma_1($  $)\sigma_1($  $) + \frac{1}{2!}\sigma_1($  $)\, \sigma_1($  $)_{=0} = e_1 d_1$.

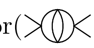
For the third coefficient we have

(6.54) $\sigma_3($$) = \frac{2}{3!}\sigma_1($  $)\sigma_1($  $)\sigma_1($  $) = \frac{1}{3} d_1^3$

since

(6.55) $P_{lin}^{(3)}($$) = P_{lin}^{\otimes 3}(\Delta \otimes \mathrm{id})\Delta($$) = 2$  $\otimes$  $\otimes$  .

All higher σ_n for $n \geq 4$ evaluate to zero, which is no suprise as the coradical degree of Γ is

(6.56) $\mathrm{cor}($$) = 3$.

We conclude that the leading log coefficient $c_3 = d_1^3/3$ and the next-to-leading log coefficient $c_2 = e_1 d_1$ of $\phi_R(\Gamma)$ are determined by the value of σ_1 on the subgraphs of Γ. This is not surprising if we write ϕ_R as $*$-exponential

$$\phi_R = \exp_*(L\sigma_1) = \hat{\mathbb{I}} + L\sigma_1 + \frac{L^2}{2!}\sigma_1 * \sigma_1 + \frac{L^3}{3!}\sigma_1 * \sigma_1 * \sigma_1 + ... \tag{6.57}$$

with infinitesimal character σ_1: all terms of higher order than $k = 1$ contain only values of σ_1 on *proper subgraphs* and *cographs* of Γ since the trivial part of the coproduct of Γ evaluates to zero on account of $\sigma_1(\mathbb{I}) = 0$:

$$(\sigma_1 \otimes \sigma_1)(\mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I}) = \sigma_1(\mathbb{I})\sigma_1(\Gamma) + \sigma_1(\Gamma)\sigma_1(\mathbb{I}) = 0\ . \tag{6.58}$$

6.5. Renormalization Group Flow

We define be a family of derivations $\{\theta_t\}_{t\geq 0}$ on H_{FG} by setting $\theta_t(\Gamma) := e^{|\Gamma|t}\Gamma$ for a Feynman graph Γ, which is related to the grading operator Y according to

$$Y(\Gamma) = \frac{d}{dt}\theta_t(\Gamma)\bigg|_{t=0} \; . \tag{6.59}$$

Both Y and θ can also be defined as maps acting on linear maps $\psi : H_{FG} \to \mathbb{C}$ through

$$(Y\psi)(\Gamma) := \psi(Y(\Gamma)) \; , \qquad (\theta_t\psi)(\Gamma) := \psi(\theta_t(\Gamma)) \; . \tag{6.60}$$

Recall that regularized Feynman rules ϕ yield parameter-dependent functions $\phi(\Gamma)\{z, \mu\}$, where $z \in \mathbb{C}$ and $\mu > 0$ are the regulator and the renormalization scale parameter, respectively. In the following, we consider Feynman rules ϕ on H_{FG} such that

$$\theta_{tz}\phi(\Gamma)\{z, \mu\} = \phi(\Gamma)\{z, \mu e^t\} \; . \tag{6.61}$$

This is for example the case if the graph Γ is mapped to terms proportional to factors like

$$\left(\frac{q^2}{\mu^2}\right)^{-z|\Gamma|/2} = e^{-z|\Gamma|L/2} \; . \tag{6.62}$$

Each choice of $\mu > 0$ corresponds to a fixed renormalization scheme. Continuously changing it by $t \mapsto \mu e^t$ amounts to 'flowing' through this set of renormalization schemes. We are interested in the map

$$t \mapsto S_R^\phi * \theta_{tz}(S_R^\phi)^{*-1} \tag{6.63}$$

and, in particular, in the limit

$$F_t = \lim_{z\to 0} S_R^\phi * \theta_{tz}(S_R^\phi)^{*-1} \; . \tag{6.64}$$

It can be shown to exist and moreover, $F_{t+s} = F_t * F_s$ establishes a semi-group structure[**CoKr01**]. The map

$$\beta = \partial_t F_t|_{t=0} \tag{6.65}$$

turns out to be the β-function (of the corresponding theory) in physics(see next section). Now, note that infinitesimal characters $\psi : H_{FG} \to \mathbb{C}$ define a Lie algebra $\mathfrak{g}$ with bracket

$$[\psi, \psi']_* = \psi * \psi' - \psi' * \psi \qquad \psi, \psi' \in \mathfrak{g} \; . \tag{6.66}$$

Let $Z_0 \in \mathfrak{g}$ be a map of this type defined by

$$[Z_0, \psi]_* = Y\psi \tag{6.67}$$

for all $\psi \in \mathfrak{g}$. Then we have the interesting 'scattering' formula[**CoKr01**]

$$S_R^\phi = \lim_{t\to\infty} \exp_*(-t(\beta/z + Z_0)) \exp_*(tZ_0) \; , \tag{6.68}$$

where we remind the reader that $\exp_*$ is the $*$-convolution exponential[1] given by

$$\exp_*(\sigma) = \sum_{k=0}^{\infty} \frac{\sigma^{*n}}{n!} \; , \tag{6.69}$$

for an infinitesimal character $\sigma \in \mathfrak{g}$, where $\sigma^{*0} = \hat{\mathbb{I}}$. This exponential always evaluates to a finite sum on any element in H_{FG}, on account of $\sigma(\mathbb{I}) = 0$ and the coradical filtration.

[1] Some authors omit the $*$-sign altogether, as it is generally clear from the context.