# DYSON-SCHWINGER EQUATIONS

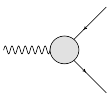
DIRK KREIMER'S LECTURE SERIES, NOTES BY LUTZ KLACZYNSKI

Contents



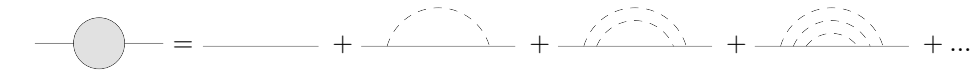
## 1. Introduction: Dyson-Schwinger equations as fixed point equations

A quantum field theory(QFT) is characterized by a family of functions known as Green's functions. These functions, often denoted by $G^r$, contain all necessary information needed to compute cross sections, decay rates and various other quantities of physical interest. The symbol $r$ parametrizes the family and stands for the *external leg structure*(or 'amplitude'). For example, let the external leg structure be given by a single photon and two fermions, then the corresponding QED vertex Green's function in momentum space,

(1.1) $G^{\wwvert}(p_1, p_2, \alpha) =$ 

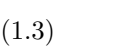
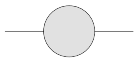
depends only on two momenta and the coupling parameter $\alpha$ which we may choose to be the fine-structure constant. Unfortunately, in most physically relevant cases like this one, these functions can only be approximated by perturbative methods (Gell-Mann-Low formula) with respect the coupling parameter(s). Moreover, many terms arising in these calculations turn out to be ill-defined due to divergences and need to be renormalized. If this cures the divergences and only a finite number of *amplitudes* need renormalization, we speak of a *renormalizable QFT* and denote the set of these amplitudes by $\mathcal{R}$. In the above case, we find that $r = \wwvert \in \mathcal{R}$ for QED.

**Self-similiarity of Feynman graph series.** Ironically, it was perturbation theory that led to *non-perturbative approaches* courtesy of the inherent *self-similiarity property* of 'blob' Feynman diagrams like on the rhs of (1.1). This property emerges on account of infinitely repeated radiative corrections as required by perturbation theory. Let us consider a simple example to make this point clear: the fermion propagator in Yukawa theory. If we content ourselves with so-called 'rainbow' corrections only at the internal fermion line, we get the perturbative series

(1.2) 

*Date*: October 11, 2013.

which defines the blob diagram as the series on the rhs (we suppress the fermionic charge arrows). If we now translate this to Feynman integrals by the prescription

(1.3) $1 =$  , $\int K =$  , $G =$ 

with some integral kernel $K$ and the Green's function $G$, we obtain

$$G = 1 + \int K + \int K \int K + \int K \int K \int K + ... = 1 + \int K(1 + \int K + \int K \int K + ...). \tag{1.4}$$

Since the term in brackets is again the perturbation series for the Green's function $G$, we can rewrite this as the integral equation

$$G = 1 + \int K\ G\ . \tag{1.5}$$

In terms of blob diagrams this reads

(1.6) 

In case we can solve the integral equation (1.5) by non-perturbative methods (we may simply guess it), we would have a *non-perturbative solution* $G$! If we denote the integral operator on the rhs of (1.5) by $\mathcal{I}$, this equation takes the form of a fixed point equation

$$G = \mathcal{I}(G). \tag{1.7}$$

Fixed point equations for Green's functions in QFT go under the name of *Dyson-Schwinger equations*(DSE).

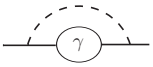
## 2. Combinatorial Dyson-Schwinger equations

We shall now change the notation slightly. We rewrite the perturbation series in (1.2) as a formal power series in a coupling parameter $\alpha$

(2.1) $X(\alpha) = 1 + \alpha$  $+ \alpha^2$  $+ \alpha^3$  $+ ...$

and assume that the Feynman rules to yield (1.4) have been altered accordingly, i.e. they are now oblivious of the coupling parameter. We view the series $X(\alpha)$ as a formal power series in $\alpha$ with Feynman graphs as coefficients. However, formal power series with all radiative corrections accounted for have in general coefficients that are, unlike our series in (2.1), *formal linear combinations* of more than one Feynman graph. This suggests that the set of Feynman graphs may be endowed with a vector space structure. And there even more to this: it turns out that this set furnishes additional algebraic structures. In fact, we shall consider a *Hopf algebra* $\mathcal{H}$ of Feynman graphs. Readers not familiar with this notion find a concise introduction to this subject in the Appendix and are recommended to digest it before continuing.

**Insertion operator.** If we define the linear insertion operator

(2.2) $B_+^{\frown}(1) :=$  , $B_+^{\frown}(\gamma) :=$ 

inserting the graph

(2.3) $\gamma \in \Big\{$  $\Big\}$

into the fermion line, we can write the DSE in (1.6) in the form

$$X(\alpha) = 1 + \alpha B_+^{\frown}(X(\alpha))\ . \tag{2.4}$$

This is a fixed point equation in the space of formal power series $\mathcal{H}[[\alpha]]$ with solution (2.1). We shall refer to equations of this type as *combinatorial Dyson-Schwinger equations*(cDSE) or simply Dyson-Schwinger equations(DSE) to distinguish them from *analytical Dyson-Schwinger equations*(aDSE) which are formulated in terms of integral operators as in (1.5). Depending on the power in which the series $X(\alpha)$ appears in the argument of the insertion operator $B_+^{\frown}$ on the rhs, cDSEs fall into two categories: linear and nonlinear DSEs. (2.4) is obviously of the linear type, whereas

$$X(\alpha) = 1 + \alpha B_+^{\gamma}(X(\alpha)^3)\ . \tag{2.5}$$

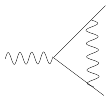
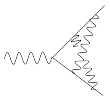
in some other context with a 'skeleton' graph $\gamma$ is not. As so often, nature is in many senses highly nonlinear. Therefore, physically relevant DSE are never linear. Non-linearity confronts the insertion operator $B_+$ with products of graphs and, on top of that, there is, unlike in (2.2), more than one insertion place for the subgraph $\gamma$. However, let's consider a real world example: the DSEs of QED take the intimidating form

$$\begin{aligned} X^{\wedge}(\alpha) &= 1 + \alpha B_+^{\gamma_1}(X^{\wedge}(\alpha)Q(\alpha)) + \alpha^2 B_+^{\gamma_2}(X^{\wedge}(\alpha)Q(\alpha)^2) + ... \\ X^{\sim}(\alpha) &= 1 - \alpha B_+^{\sim\bigcirc\sim}(X^{\sim}(\alpha)Q(\alpha)) \\ X^{-}(\alpha) &= 1 - \alpha B_+^{\frown}(X^{-}(\alpha)Q(\alpha)), \end{aligned} \tag{2.6}$$

where

$$Q(\alpha) = \frac{(X^{\wedge}(\alpha))^2}{X^{\sim}(\alpha)(X^{-}(\alpha))^2} \tag{2.7}$$

is a formal series called *invariant charge* and

$\gamma_1 =$  , $\gamma_2 =$  , ... (2.8)

are the vertex primitives. Inverses of propagator series are to be understood as formal geometric series, i.e.

$$\frac{1}{X^{\sim}} = \frac{1}{1 - \bar{X}^{\sim}} = 1 + \bar{X}^{\sim} + (\bar{X}^{\sim})^2 + ... \tag{2.9}$$

where $\bar{X}^{\sim} := 1 - X^{\sim}$. The insertion operator can then insert several graphs at different insertion places at the same time. It vanishes if it is offered more graphs than it can insert. Note that if the coupling parameter counts the number of vertices, and not, as in our case (fine-structure constant) the loop number, the formal series in denominator and numerator of the invariant charge in (2.7) may have fractional exponents. This is to be understood in terms of the binomial series

$$(X^r)^\rho = \sum_{n\geq 0} \binom{\rho}{n} (X^r - \mathbb{I})^n \qquad\qquad \rho \in \mathbb{R}. \tag{2.10}$$

We remind the reader of the definition of the binomial coefficient given by

$$\binom{\rho}{n} = \frac{\rho(\rho-1)...(\rho-1+n)}{n!}. \tag{2.11}$$

Note that this expression does *not* vanish for $n > \rho$ if $\rho \notin \mathbb{N}$ which implies that in this case the series in (2.10) is not a finite sum. Why do we only consider 3 amplitudes in QED? The reason why only 3 amplitudes are considered is that for QED, the set of amplitudes that need renormalization is

$$\mathcal{R} = \{\wedge, \sim, -\}. \tag{2.12}$$

Their self-similiarity equations need no extra amplitudes which is why the DSE system (2.6) is of particular interest.

**Hopf subalgebras.** The solution of a cDSE for an amplitude $r$ has the general form

$$X^r(\alpha) = 1 + \mathrm{sgn}(s_r) \sum_{j\geq 1} \alpha^j c_j^r \tag{2.13}$$

with coefficients $c_j^r \in \mathcal{H}$, in general linear combinations of Feynman graphs. For propagator series, the signum is $\mathrm{sgn}(s_r) = -1$, whereas for vertex series $\mathrm{sgn}(s_r) = 1$. Miraculously, the set of these coefficients up to a fixed loop order $N$, i.e. $\{c_j^r : j = 1, ..., N\}$ generate a Hopf subalgebra. Surely, it would be no surprise that one can formally generate an algebra which is a subalgebra of $\mathcal{H}$ were it not for their coproduct which is of the form

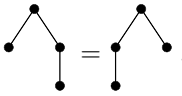
$$\Delta(c_j^r) = \sum_{k=0}^{j} \mathrm{Pol}_k(\{c_l^r\}) \otimes c_{j-k}^r, \tag{2.14}$$

where $\mathrm{Pol}_k(\{c_l^r\})$ is a homogeneous polynomial of degree $k$ in the $c_l^r$'s as variables for $l = 1, ..., j$.

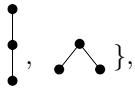
## 3. Rooted trees

The Hopf algebra of rooted trees $H$ is particularly suited to study cDSE. We quickly review the basic ingredients of this topic and introduce the corresponding notation.

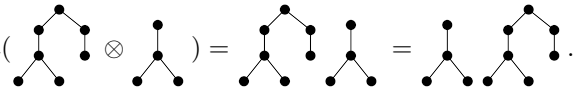
**Rooted trees.** A *rooted tree* $T$ is a connected, simply connected and non-planar graph given by edges and vertices with one distinguished vertex named *root*. All edges are oriented away from the root. We follow the convention of drawing the root as the topmost vertex, here is an example:

(3.1) $T =$  $=$ .

Note that the second equality would not hold for planar rooted trees. We denote the set of trees by $\mathcal{T}$. The cardinality of a tree $T$, denoted as $|T|$ is given by the number of its vertices. The empty tree $\mathbb{I}$ has no vertices $|\mathbb{I}| = 0$. Let $\mathcal{T}_n$ be the set of all trees with $n$ vertices, i.e.

(3.2) $\mathcal{T}_0 = \{\ \mathbb{I}\ \}, \quad \mathcal{T}_1 = \{\ \bullet\ \}, \quad \mathcal{T}_2 = \{$  $\}, \quad \mathcal{T}_3 = \{$  $\}, \ \dots\ ,$

and so on. By $m$ we denote the associative and commutative product $H \otimes H \to H$, which for two trees leads to a juxtaposition,

(3.3) $m($  $) =$ .

Products of trees are called *forests*, where a tree is also seen as a forest, albeit of only a single tree. The empty tree or empty forest $\mathbb{I}$ is the neutral element. Let $\mathcal{F}$ be the set of all forests and $\mathcal{F}_n$ the set of all forests with $n$ vertices, regardless of how many trees it contains, i.e.

(3.4) $\mathcal{F}_0 = \{\ \mathbb{I}\ \}, \quad \mathcal{F}_1 = \{\ \bullet\ \}, \quad \mathcal{F}_2 = \{\ \bullet\bullet,$  $\}, \quad \mathcal{F}_3 = \{\ \bullet\bullet\bullet,$  $\}, \ \dots\ ,$ etc.

Taking the $\mathbb{Q}$-linear span $H_n := \mathrm{span}_{\mathbb{Q}}\{\mathcal{F}_n\}$ of each such set defines a sequence of vector spaces which establishes a grading

$$H = \bigoplus_{j \geq 0} H_j \tag{3.5}$$

on the algebra $H$ of rooted trees. The product then has the grading property

$$m(H_n \otimes H_m) \subset H_{n+m}, \tag{3.6}$$

and hence the spaces $H_n$ are not subalgebras! The algebra $H$ is a *connected* algebra because its grading starts with $H^0 = \mathbb{Q}\mathbb{I}$. By slight abuse of notation, we denote the unit map $\mathbb{Q} \to \mathbb{Q}\mathbb{I}$, $\lambda \mapsto \lambda\mathbb{I}$ by $\mathbb{I}$. The subspace

$$\mathrm{Aug} = \bigoplus_{j \geq 1} H_j = H_1 \oplus H_2 \oplus H_3 \oplus \dots \tag{3.7}$$

is called *augmentation ideal*. It is indeed an ideal due to $m(H \otimes \mathrm{Aug}) \subset \mathrm{Aug}$. The linear and multiplicative map $\overline{e} : H \to \mathbb{Q}$ defined by $\overline{e}(\mathrm{Aug}) = 0$ and $\overline{e}(\mathbb{I}) = 1$ is called *counit*. The coproduct $\Delta : H \to H \otimes H$ is defined as follows. First $\Delta(\mathbb{I}) := \mathbb{I} \otimes \mathbb{I}$. Let now $T$ be a non-empty tree. A *cut* $c$ of $T$ is a subset of its edge set. If we remove these edges from the tree $T$, we obtain a forest in which one tree is still connected to the (former) root. We denote this tree by $R^c(T)$. The other trees comprise a forest which we write as $P^c(T)$. A cut $c$ is called *admissible* if the following holds true: starting from any vertex, the unique path towards the root(given by edges) crosses the edges of $c$ no more than once. Then, if we denote the set of all admissible cuts of a tree $T$ by $\mathcal{C}(T)$, the coproduct is

$$\Delta(T) = \mathbb{I} \otimes T + T \otimes \mathbb{I} + \sum_{c \in \mathcal{C}(T)} P^c(T) \otimes R^c(T). \tag{3.8}$$

Mostly, we will use a variant of *Sweedler's notation* for the coproduct, i.e.

$$\Delta(T) = \mathbb{I} \otimes T + T \otimes \mathbb{I} + \sum_{(T)} T' \otimes T'' \tag{3.9}$$

or simply $\Delta(T) = T' \otimes T''$ with or without the summation sign as long as there is no potential for confusion. The *reduced coproduct* $\Delta'$ is given by

$$\Delta'(T) := \Delta(T) - \mathbb{I} \otimes T - T \otimes \mathbb{I} \tag{3.10}$$

which vanishes only on primitive elements. Let us consider the tree $T =$  , for example. If we let $e_L$ and $e_R$ be the left-hand and right-hand side edge, respectively, the admissible cuts are $c_1 = \{e_L\}$, $c_2 = \{e_R\}$, and $c_3 = \{e_L, e_R\}$ and hence the coproduct yields

$\Delta($  $) =$  $\otimes \mathbb{I} + \mathbb{I} \otimes$  $+ 2 \bullet \otimes$  $+ \bullet\bullet \otimes \bullet.$ (3.11)

Linear maps $f, g : H \to H$ on the Hopf algebra $H$ can be subjected to a bilinear operation

$$f * g := m(f \otimes g)\Delta. \tag{3.12}$$

called *convolution product*($*$-product). Those maps preserving unity, i.e. $f(\mathbb{I}) = \mathbb{I}$ can be shown to have an inverse with respect to the $*$-product, where the neutral element is the composition of unit map and counit: $e = \mathbb{I} \circ \bar{e}$. The *antipode* $S : H \to H$ is the $*$-inverse of the identiy map id on $H$, i.e.

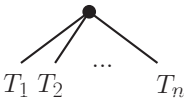
$$S * \mathrm{id} = \mathrm{id} * S = e. \tag{3.13}$$

This implies $S(\mathbb{I}) = \mathbb{I}$ and

$$S(T) = -T - \sum_{(T)} S(T')T'' \tag{3.14}$$

for a tree $T$. This follows from $(S * \mathrm{id})(T) = e(T) = 0$ and $m(S \otimes \mathrm{id})(\mathbb{I} \otimes T) = S(\mathbb{I})T = T$.

**Grafting operator.** The *grafting operator* $B_+$ is a linear map $H \to \mathrm{span}_{\mathbb{Q}}\{\mathcal{T}\}$ into the span of all trees defined by $B_+(\mathbb{I}) = \bullet$ and for a forest of trees $T_1, ..., T_n$

$B_+(T_1 \ ... \ T_n) :=$  (3.15)


mapping any forest to a single tree by attaching the roots to a single new node which then becomes the new root. A concrete example is

$B_+($  $) =$  . (3.16)

Note that the product of trees is commutative, which would cause us trouble at this point if the trees were planar. Thanks to their non-planarity, there is a unique forest $X$ for every tree $T \in \mathcal{T}$ such that $T = B_+(X)$, a fact which is somewhat obvious from the definition of the operator $B_+$. One can show that it obeys

$$\Delta B_+ = B_+ \otimes \mathbb{I} + (\mathrm{id} \otimes B_+)\Delta, \tag{3.17}$$

where $B_+ \otimes \mathbb{I}$ is to be understood as the map $T \mapsto B_+(T) \otimes \mathbb{I}$ and that this identiy qualifies $B_+$ as a non-trivial one-cocycle in a Hochschild cohomology.

**Ladder trees.** Ladders, denoted $\lambda_k$ and defined by $\lambda_0 := \mathbb{I}$ and $\lambda_{k+1} := B_+(\lambda_k)$, take the form

$\lambda_k =$  $k$-*times* . (3.18)

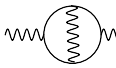
Their coproduct is $\Delta(\lambda_k) = \sum_{j=0}^{k} \lambda_j \otimes \lambda_{k-j}$. They therefore give rise to a Hopf subalgebra $H_\ell \subset H$. The solution of the linear cDSE

$$X(\alpha) = 1 + \alpha B_+(X(\alpha)) \tag{3.19}$$

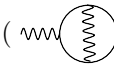
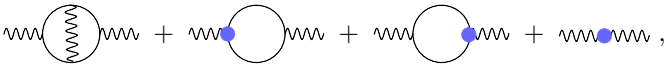
in fact lies in $H_\ell[[\alpha]] \subset H[[\alpha]]$ and is given by $X(\alpha) = 1 + \sum_{j\geq 1} \alpha^j \lambda_j$, which the reader is invited to check. cDSE and their solutions in $H[[\alpha]]$ have been classified by Loic Foissy[Foi]. An interesting question is which classes may describe relevant physics.

## 4. Ward identity in QED

We view Feynman rules as characterized by a linear and multiplicative map $\phi : \mathcal{H} \to \mathcal{A}$ taking a Feynman graph to an element of some target algebra $\mathcal{A}$ of, say, smooth functions depending on external momenta, etc. A map evaluating Feynman graphs in this fashion is referred to as a *character*. In the case of regularized Feynman rules in dimensional or analytical regularization, a Feynman graph is evaluated to a Laurent series in the regulator with smooth momentum dependent functions as coefficients. Consider the QED 2-loop graph

(4.1) $\Gamma_1 =$ 

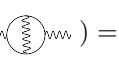
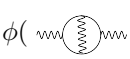
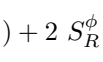
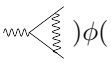
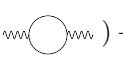
To renormalize it, one needs two subtractions:

(4.2) $($  $)_R =$ 

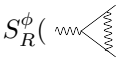
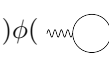
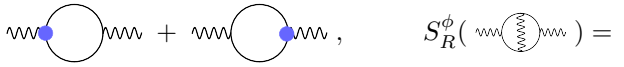
where the second and third terms cure the vertex subdivergence given by the subgraph

(4.3) $\gamma =$ 

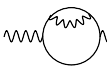
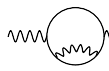
and the last term deals with the overall divergence. Within the framework of Hopf-algebraic renormalization, this reads in terms of characters

(4.4) $\phi_R($  $) = \phi($  $) + 2\ S_R^\phi($  $)\phi($  $) + S_R^\phi($  $)$

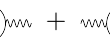
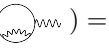
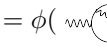
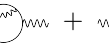
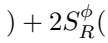
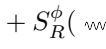
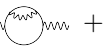
where the character $S_R^\phi$ is called *counterterm*. If we apply it to the series $X^r$ we essentially obtain what is known in quantum field theory as the renormalization Z-factor $Z_r$. We identify

(4.5) $2\ S_R^\phi($  $)\phi($  $) =$  $, \qquad S_R^\phi($  $) =$ 

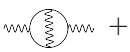
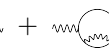
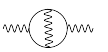
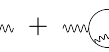
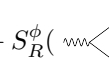
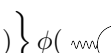
In the first expresssion, the counterterm may put the photon momentum to zero and evaluate all fermion momenta at the renormalization point $\mu$ which yields the momentum independent coefficient for the subdivergence subtraction. On the 2-loop level in perturbation theory, we have the two additional graphs

(4.6) $\Gamma_2 =$  $, \qquad \Gamma_3 =$ 

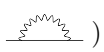
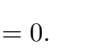
We treat them in one go and get

(4.7) $\phi_R($  $+$  $) = \phi($  $+$  $) + 2S_R^\phi($  $)\phi($  $)$
$+ S_R^\phi($  $+$  $)$

Putting all graphs together, we find

(4.8) $\phi_R($  $+$  $+$  $)$
$= \phi($  $+$  $+$  $) + S_R^\phi($  $+$  $+$  $)$
$+ 2\left\{ S_R^\phi($  $) + S_R^\phi($  $)\right\}\phi($  $)$

Thanks to what is known as *Ward identity*, we can choose a renormalization scheme in which the term in curly brackets vanishes:

(4.9) $S_R^\phi($  $) + S_R^\phi($  $) = 0.$

This simplyfies the renormalization procedure significantly: the sum of the three graphs need only one subtraction for the overall divergence, it behaves like a primitive element(i.e. a divergent graph void of subdivergences). Although the individual counterterms are needed for curing the subdivergences, their

service becomes obsolete when we take the sum of the three graphs. Moreover, it is possible to achieve this for every loop order for these two amplitudes:

$$S_R^\phi(X^{\wedge}(\alpha)) = S_R^\phi(X^{\sim}(\alpha)), \tag{4.10}$$

which follows from $S_R^\phi(c_j^{\wedge}) = -S_R^\phi(c_j^{\sim})$ for every $j \geq 1$, where the minus sign is a feature of the propagator series. (4.10) is very well known in physics and is usually written in terms of the renormalization $Z$-factors $Z_1(\alpha) = Z_2(\alpha)$, (see [ItZu80] section 7.1.3). This allows us to identify the two series and establish an *equivalence relation* on the Hopf algebra $\mathcal{H}$ of QED Feynman graphs [Sui06] where

$$c_j^{\wedge} + c_j^{\sim} \sim 0 \tag{4.11}$$

for the coefficients of the two series $X^{\wedge}(\alpha)$ and $X^{\sim}(\alpha)$. In this sense, we may, by virtue of (4.10), write

$$\frac{X^{\wedge}}{X^{\sim}} = 1 \tag{4.12}$$

since, as we have seen, the denominator is the natural place for a propagator series. The identity (4.10) can be used to rewrite the DSEs and *decouple* the DSE of the photon series

$$X^{\sim}(\alpha) = 1 - \sum_{j \geq 1} \alpha^j c_j^{\sim} \tag{4.13}$$

from the other two equations in (2.6). In *Quantum Chromodynamics*(QCD), things are much more involved and the corresponding identifications are

$$\frac{X^{\wedge}}{X^{\sim}} = \frac{X^{\wedge}}{X^{\wedge}} = \frac{X^{\wedge}}{X^{\sim}} = \frac{X^{\wedge}}{X^{\sim}} \tag{4.14}$$

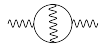
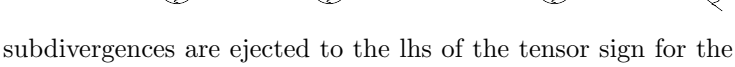
which originate in what is known as *Slavnov-Taylor identities*. However, we should be more precise at this point. To understand what is going on here, we come back to the graph $\Gamma_1$ in (4.1). The Hopf-algebraic machinery that leads to the renormalized value of the graph in (4.4) works as follows. Two linear maps $\psi, \zeta$ from the Hopf algebra $\mathcal{H}$ of Feynman graphs to some target algebra $\mathcal{A}$ of interest can be combined to yield another linear map by a bilinear operation known as convolution product:

$$(\psi * \zeta)(\Gamma) = m_{\mathcal{A}}(\psi \otimes \zeta)\Delta(\Gamma) = \psi(\mathbb{I})\zeta(\Gamma) + \psi(\Gamma)\zeta(\mathbb{I}) + \sum_{(\Gamma)} \psi(\Gamma')\zeta(\Gamma'') \tag{4.15}$$

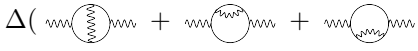
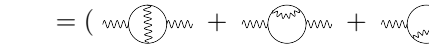
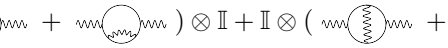
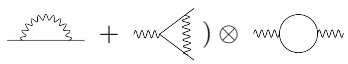
where $\Gamma$ is some connected Feynman graph. Now here is how the renormalized Feynman rules $\phi_R$, the counterterm $S_R^\phi$ and the unrenormalized Feynman rules $\phi$ are related:

$$\phi_R(\Gamma) = (S_R^\phi * \phi)(\Gamma) = m_{\mathcal{A}}(S_R^\phi \otimes \phi)\Delta(\Gamma) = \phi(\Gamma) + S_R^\phi(\Gamma) + \sum_{(\Gamma)} S_R^\phi(\Gamma')\phi(\Gamma''), \tag{4.16}$$

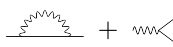
where $S_R^\phi(\mathbb{I}) = 1$ and $\phi(\mathbb{I}) = 1$ has been used. The coproduct takes $\Gamma$, tailors it to pieces and thereby *prepares* the ground for the maps $S_R^\phi$ and $\phi$ to do their job: in our example, the coproduct gives

$\Delta($  $) =$  $,$ (4.17)

where the subdivergences are ejected to the lhs of the tensor sign for the counterterm $S_R^\phi$ to act on. In physics, (4.16) is known as *forest formula*. The coproduct of the sum of all 2-loop graphs is

$\Delta($  $)$
$= ($  $) \otimes \mathbb{I} + \mathbb{I} \otimes ($  $)$ (4.18)
$+ 2($  $) \otimes$ 

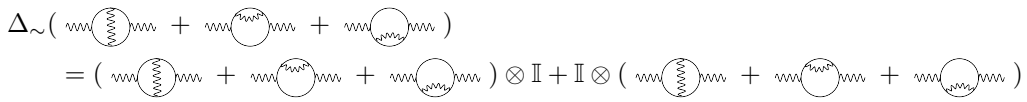
Because the counterterm maps the sum

$$c_1^{\wedge} + c_1^{\sim} = \text{  } \tag{4.19}$$

of the two 1-loop contributions to zero, we might as well declare them to be equivalent to zero when they appear on the lhs of the tensor sign, i.e. when they play the role of subdivergences cut out for the

purpose of renormalization. As a result, the coproduct $\Delta_\sim$ knowing about the equivalence of (4.19) to zero simplifies to

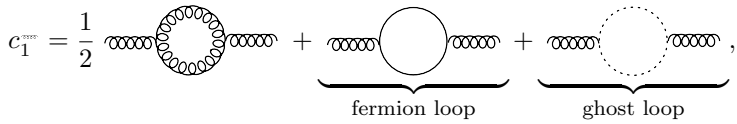
(4.20)



## 5. Slavnov-Taylor identities and Quantum Gravity

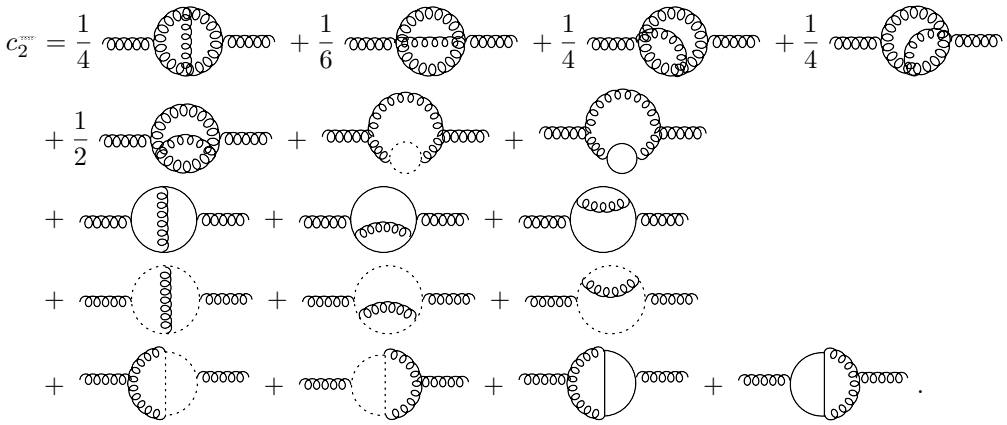
We come across this cancellation of counterterms in a similiar manner in QCD, where the Slavnov-Taylor identities can be employed in much the same way. Let us consider the 1-loop contributions to the gluon propagator

(5.1)




where we ignore self-loops for the moment and suppress charge flow arrows. However, the coefficient in (5.1) is primitive and needs only one subtraction. On the 2-loop level, we encounter the first minor combinatorial explosion:

(5.2)



Now, by means of the Slavnov-Taylor identities (4.14), the coproduct of this coefficient reads

$$\Delta(c_2^{\wwwww}) = c_2^{\wwwww} \otimes \mathbb{I} + \mathbb{I} \otimes c_2^{\wwwww} + (\textstyle\sum_r c_1^r) \otimes c_1^{\wwwww}, \tag{5.3}$$

where the sum is over the 1-loop coefficients of the appropriate amplitudes. This is what happens here: the elements ejected to the lhs of the tensor sign by the action of the coproduct cancel in such a way that $c_1^{\wwwww}$ factors out, just like in QED due to the Ward identity(see also [Krei08]).

**Quantum Gravity.** The problem with quantum gravity is that the number of amplitudes crying for renormalization is infinite, i.e. quantum gravity is hence by definition non-renormalizable. However, suppose we have found a renormalization scheme in which the counterterm map gave rise to an infinite sequence of Slavnov-Taylor type identities of the form

$$\frac{X^{n+2}}{X^{n+1}} = \frac{X^{n+1}}{X^n} \qquad\qquad n = 2, 3, 4, ..., \tag{5.4}$$

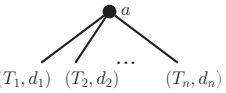
where the superscript counts the number of external graviton legs. This equivalence relation in the Hopf algebra of quantum gravity Feynman graphs would then render the number of amplitudes in need of renormalization finite (you get all the others for free) and *quantum gravity would hence be renormalizable* [Krei08a]. However, the existence of a renormalization scheme and the underlying symmetry giving rise to (5.4) remains an open question.

## 6. Symmetries and Hopf Ideals

We shall now introduce the Hopf algebra $H_D$ of *decorated rooted trees*. Let $D$ be a countable set. A *decorated rooted tree* is a pair $(T, d)$ consisting of a rooted tree $T$ and a map $d : T^{[0]} \to D$ assigning an

element of $D$ to each vertex, where $T^{[0]}$ denotes the set of vertices of the tree $T$. For each decoration $a \in D$, we define a grafting operator $B_+^a(\mathbb{I}) = \bullet_a$ and for a forest of decorated trees $(T_1, d_1), ..., (T_n, d_n)$

$$B_+^a(T_1 \ ... \ T_n) := \tag{6.1}$$

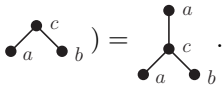



just as in section 3 with the subtle difference that $B_+^a$ attaches a root decorated by $a$. If $D = \{a, b, c\}$ is the decoration set, then an example of a decorated rooted tree and how we draw it is given by

$$(T, d) = \tag{6.2}$$


where the decoration map $d$ assigns the symbol $a$ to the root, $c$ to the roots child, and so on. This tree can be generated by the 3 grafting operators $B_+^a, B_+^b$ and $B_+^c$:

$$B_+^a(B_+^c(B_+^a(\mathbb{I})B_+^b(\mathbb{I}))) = B_+^a(B_+^c(\bullet_a \bullet_b)) = B_+^a( \ ) = \tag{6.3}$$




The Hopf algebra $H_D$ is the Hopf algebra generated by these decorated trees. Note that $(T, d) \neq (T, d')$ if the decoration maps $d$ and $d'$ differ. The coproduct acts on the generators of this Hopf algebra in the same way as it does in the case of the Hopf algebra of (undecorated) rooted trees $H$.

Let now $D = \{a, b, c, d\}$ be the decoration set(the decoration map will be implicit henceforth). Consider a system of DSEs for 3 series in $H_D[[\alpha]]$ given by

$$\begin{aligned} X_1 &= \mathbb{I} + \alpha B_+^a(X_1 X_3) + \alpha B_+^b(X_2^2) \\ X_2 &= \mathbb{I} + \alpha B_+^c(X_2^2) \\ X_3 &= \mathbb{I} + \alpha B_+^d(X_3^2), \end{aligned} \tag{6.4}$$

where we have suppressed the parameter $\alpha$ in some places. The last two equations are decoupled from the first and can be solved iteratively. Moreover, the series $X_2(\alpha)$ and $X_3(\alpha)$ have coefficients with homogeneously decorated forests, i.e. the coefficients $X_2(\alpha)$ are all decorated by $c$ and those of $X_3(\alpha)$ by $d$. In contrast to that, the first series $X_1(\alpha)$ has coefficients decorated with all elements in $D$. Let us compute the first few coefficients of $X_2$ and $X_3$. If we write our ansatz as

$$X_j(\alpha) = \mathbb{I} + c_{j,1}\alpha + c_{j,2}\alpha^2 + c_{j,3}\alpha^3 + ... \tag{6.5}$$

for $j = 2, 3$, then

$$X_j(\alpha)^2 = \mathbb{I} + 2c_{j,1}\alpha + (2c_{j,2} + c_{j,1}^2)\alpha^2 + (2c_{j,3} + 2c_{j,1}c_{j,2})\alpha^3 + ... \tag{6.6}$$

Plugging this into the two DSEs for $X_2$ and $X_3$ in (6.4) yields for $j = 2$

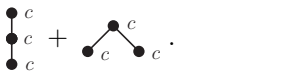
$$\begin{aligned} X_2 &= \mathbb{I} + \alpha B_+^c(\mathbb{I} + 2c_{2,1}\alpha + (2c_{2,2} + c_{2,1}^2)\alpha^2 + ...) \\ &= \mathbb{I} + \alpha B_+^c(\mathbb{I}) + 2B_+^c(c_{2,1})\alpha^2 + B_+^c(2c_{2,2} + c_{2,1}^2)\alpha^3 + ... \\ &= \mathbb{I} + \bullet_c \ \alpha + 2B_+^c(c_{2,1})\alpha^2 + [2B_+^c(c_{2,2}) + B_+^c(c_{2,1}^2)]\alpha^3 + ... \end{aligned} \tag{6.7}$$

which implies $c_{2,1} = \bullet_c$ and thus

$$c_{2,2} = 2B_+^c(c_{2,1}) = 2 \tag{6.8}$$


and consequently

$$c_{2,3} = 2B_+^c(c_{2,2}) + B_+^c(c_{2,1}^2) = 4 \ + \ . \tag{6.9}$$




Notice that we would not get the tree with side branchings like in the second term if the DSE for $X_2$ was *linear*. However, it is not, and the general recursion formula reads

$$c_{2,j+1} = \sum_{l=0}^{j} B_+^c(c_{2,l}c_{2,j-l}). \tag{6.10}$$

This is obtained by inserting

$$X_2^2 = (\sum_{j\geq 0} c_{2,j}\alpha^j)^2 = \sum_{j\geq 0}(\sum_{l=0}^{j} c_{2,l}c_{2,j-l})\alpha^j = \mathbb{I} + 2 \bullet_c \alpha + (\bullet_c \bullet_c + 4 \ )\alpha^2 + ... \tag{6.11}$$

into the DSE for $X_2$ in (6.4), where $c_{2,0} := \mathbb{I}$. For the coefficients of $X_3$ we just have to replace the decoration $c$ by $d$, and then the same recursion applies. We may now go on to tackle $X_1$ and take the view that we have solved the two DSEs for $X_2$ and $X_3$, the solutions given recursively by (6.10). To this end, we have to insert (6.11) and

$$X_1X_3 = \sum_{j\geq 0}(\sum_{l=0}^{j} c_{1,l}c_{3,j-l})\alpha^j = \mathbb{I} + (c_{1,1} + \bullet_d)\alpha + ( \; + c_{1,1}\bullet_d + c_{1,2})\alpha^2 + ... \tag{6.12}$$

into (6.4). This yields

$$X_1(\alpha) = \mathbb{I} + (\bullet_a + \bullet_b)\alpha + ( \; + \; + \; + 2 \; )\alpha^2 + ... \tag{6.13}$$

If we take the coproduct of the second coefficient $c_{1,2}$ we get

$$\Delta'(c_{1,2}) = (\bullet_a + \bullet_b + \bullet_d) \otimes \bullet_a + 2\,\bullet_c \otimes \bullet_b = (c_{1,1} + c_{3,1}) \otimes \bullet_a + 2c_{2,1} \otimes \bullet_b \tag{6.14}$$

for the reduced part. For the coefficients $c_{2,2}$ and $c_{3,2}$ it yields

$$\Delta'(c_{2,2}) = 2\,\bullet_c \otimes \bullet_c = 2c_{2,1} \otimes c_{2,1}, \qquad \text{and} \qquad \Delta'(c_{3,2}) = 2\,\bullet_d \otimes \bullet_d = 2c_{3,1} \otimes c_{3,1}, \tag{6.15}$$

where in fact, one can show by virtue of the recursion (6.10) and

$$\Delta B_+^r = B_+^r \otimes \mathbb{I} + (\mathrm{id} \otimes B_+^r)\Delta \tag{6.16}$$

for $r = c, d$ that each series' coefficients $c_{2,l}$ and $c_{3,l}$ generate their own Hopf subalgebras in $H_D$, respectively. We can actually put them together to make one joint Hopf subalgebra. For the coefficients of $X_1$ this is obviously not the case, (6.14) being testimony to this: neither $\bullet_a$ nor $\bullet_b$ are by themselves coefficients of any of the involved series. If we could, for some reason, view $c_{1,1} + c_{3,1}$ as equivalent to $2c_{2,1}$ and therefore equate them in a quotient Hopf algebra, we would get

$$\Delta'(c_{1,2}) = (c_{1,1} + c_{3,1}) \otimes \bullet_a + 2c_{2,1} \otimes \bullet_b = 2c_{2,1} \otimes (\bullet_a + \bullet_b) = 2c_{2,1} \otimes c_{1,1}. \tag{6.17}$$

Equating these two guys surely does not solve the problem that we have in trying to establish a Hopf subalgebra generated by the coefficients of all three series in the DSE system (6.4). However, there is a systematic way of achieving our goal. The equivalence $c_{1,1} + c_{3,1} \sim 2c_{2,1}$ is the equivalence of the coefficients in (6.11) and (6.12) up to first order in $\alpha$. If we set

$$X_1X_3 = X_2^2, \tag{6.18}$$

having this equality to all orders, then one can prove that all coefficients of the three series generate a Hopf subalgebra. This equality means for the so-called *invariant charges* $Q_a$ and $Q_b$ defined by

$$Q_a := X_3 \qquad \text{and} \qquad Q_b := \frac{X_2^2}{X_1} \tag{6.19}$$

that they are equal: $Q_a = Q_b$. The indices $a, b$ refer to the two grafting operators in the first line of the DSE system in (6.4). These charges, their name inspired by physics, play the following role. We reformulate our DSE system by associating an invariant charge to each equation and their grafting operators: the DSE system reads in terms of these charges

$$\begin{aligned} X_1 &= \mathbb{I} + \alpha B_+^a(Q_aX_1) + \alpha B_+^b(Q_bX_1) \\ X_2 &= \mathbb{I} + \alpha B_+^c(Q_cX_2) \\ X_3 &= \mathbb{I} + \alpha B_+^d(Q_dX_3), \end{aligned} \tag{6.20}$$

where $Q_d = X_3 = Q_a$. The reader is asked to figure out the (notional) logic behind these definitions.

**Hopf ideal.** The equivalence relation $Q_a = Q_b$, understood as an equivalence of coefficients order by order, enables us to generate more than just a Hopf subalgebra. Consider, prior to establishing this equivalence relation, the *ideal* $I \subset H$ generated by the coefficients of the series $Q_a - Q_b$, or, equivalently by those of $X_1X_3 - X_2^2$. Readers not familiar with ideals are advised at this point to take some time to get aquainted with this concept by reading Appendix section C. One can show that

$$\Delta(I) = H \otimes I + I \otimes H \tag{6.21}$$

which defines a *coideal* in $H$. Since for the antipode $S$, one has $S(I) \subset I$, the subspace $I$ is not just an ideal and a coideal but in fact also a *Hopf ideal*. *Feynman rules* are implemented on $H_D$ as Hopf characters, that is, multiplicative and linear maps $\phi : H_D \to A$ with some target algebra $A$. When we apply the equivalence relation, we obtain the quotient Hopf algebra $H/I$. As the ideal $I$ will shrink to zero

in $H/I$, only characters with $\phi(I) = 0$ have well-defined representatives on $H/I$. This may for example be true for the counterterm in a specific renormalization scheme, courtesy, say, of a *gauge symmetry* like in QED giving rise to the Ward-Takahashi identity(section 4, see also [Sui06]). The Hopf ideal $I$ may actually be defined by the property $\ker\phi = I$. Those characters not vanishing on $I$ do not have induced maps on $H/I$. However, this poses no problem whatsoever. We can easily define appropriate characters on the quotient Hopf algebra that serve our purposes as we please. Now here is the essential message of this section: *symmetries correspond Hopf ideals*(see also [KrSui09]).

## 7. Insertion Operators as Hochschild One-Cocycles

Our next goal is to define an insertion operator $B_+^\gamma : \mathcal{H} \to \mathcal{H}$ for a Feynman graph $\gamma$ such that it is a Hochschild one-cocycle, i.e.

$$\Delta B_+^\gamma = B_+^\gamma \otimes \mathbb{I} + (\mathrm{id} \otimes B_+^\gamma)\Delta \tag{7.1}$$

on some subspace of $\mathcal{H}$. This property is necessary to ensure what is known as *locality* of the counterterms of the corresponding quantum field theory. We first note that it implies for $\gamma$ to be a primitive graph: if we plug in the empty graph $\mathbb{I}$, we see that

$$\Delta B_+^\gamma(\mathbb{I}) = B_+^\gamma(\mathbb{I}) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+^\gamma)\Delta(\mathbb{I}) = \gamma \otimes \mathbb{I} + \mathbb{I} \otimes \gamma, \tag{7.2}$$

where $B_+^\gamma(\mathbb{I}) := \gamma$ is mandatory, since, inserting nothing$(= \mathbb{I})$ must yield the graph $\gamma$. Consider a simple QED example: what is the graph

$$\Gamma = B_+^{\text{○}}(\ \text{}\ )\ ? \tag{7.3}$$

If we apply the coproduct and use (7.1), we find

$$\begin{aligned}\Delta(\Gamma) = \Delta B_+^{\text{○}}(\ \text{}\ ) &= B_+^{\text{○}}(\ \text{}\ ) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+^{\text{○}})\Delta(\ \text{}\ ) \\ &= B_+^{\text{○}}(\ \text{}\ ) \otimes \mathbb{I} + \ \text{}\ \otimes B_+^{\text{○}}(\mathbb{I}) + \mathbb{I} \otimes B_+^{\text{○}}(\ \text{}\ ) \\ &= \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \ \text{}\ \otimes\ \text{}\ .\end{aligned} \tag{7.4}$$

If we compare this to the coproduct of the graph $\ \text{}\ $, we see that

$$\Delta(\ \text{}\ ) = \ \text{}\ \otimes \mathbb{I} + \mathbb{I} \otimes\ \text{}\ + 2\ \text{}\ \otimes\ \text{} \tag{7.5}$$

and, dividing this by 2, we find

$$\Gamma = B_+^{\text{○}}(\ \text{}\ ) = \frac{1}{2}\ \text{} \tag{7.6}$$

and conclude that *prefactors do matter*. Moreover, we need to define the insertion operator also on a product of graphs. To tackle the general case, we need some definitions. In what follows, we restrict ourselves to Quantum Chromodynamics(QCD). Because the Feynman graphs of QED and $\phi^4$-theory can combinatorially be seen as special cases of QCD graphs, all assertions of this section also hold for these two simpler theories.

**Residues.** First, the *residue* of a graph $\Gamma$ is the graph $\mathrm{res}(\Gamma)$ obtained from $\Gamma$ by shrinking *all* internal edges to a single point. Instead of residue, we shall also speak of the *external leg structure*. Examples are

$$\mathrm{res}(\ \text{}\ ) = \mathrm{res}(\ \text{}\ ) = \ \text{}\ , \qquad \mathrm{res}(\ \text{}\ ) = \mathrm{res}(\ \text{}\ ) = \ \text{} \tag{7.7}$$

and

$$\mathrm{res}(\ \text{}\ ) = \mathrm{res}(\ \text{}\ ) = \times\ , \qquad \mathrm{res}(\ \text{}\ ) = \mathrm{res}(\ \text{}\ ) = \ \text{}\ . \tag{7.8}$$

By $\mathcal{R}$ we denote the set of such residues of interest[1] for a given renormalizable theory. It is generally finite (by definition of renormalizability). In the case of QED, it has only 3 elements, simply the vertex $\curlywedge$ and the two edges $\sim, -$ for the photon and the fermion, respectively. The case of QCD is a bit richer where

$$\mathcal{R} = \{\ \text{}\ \} \tag{7.9}$$

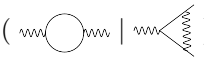
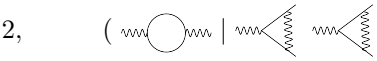
[1]It is 'of interest' if the corresponding amplitude needs renormalization.

is the residue set. We split this set into two subsets: the set of vertex residues $\mathcal{R}_V$ and that of the edge residues $\mathcal{R}_E$, i.e.

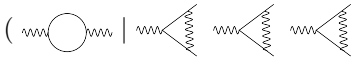
(7.10) $\mathcal{R}_V = \{$  $\}$ and $\mathcal{R}_E = \{$  $\}$,

where  is the gauge boson's line,  that of the ghost and  the fermion's line, i.e. for a quark. These sets contain the edge and vertex types that characterize the elementary building blocks of QCD's Feynman graphs.

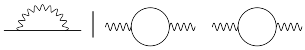
**Insertion places.** If we want to insert one graph into another we have to specify where and how. An *insertion place* of a graph $\Gamma$ is a vertex or an internal edge. These vertices and edges will be replaced by the connected graph to be inserted. In order for this graph to fit in, it needs to have the appropriate external leg structure, i.e. the proper residue. Let $\gamma = \prod_j \gamma_j$ be a product of graphs. We denote the number of possible insertions of $\gamma$ into $\Gamma$ by $(\Gamma|\gamma)$. Consider the graph $\Gamma =$ . Let us write $V($  $) = \{v_L, v_R\}$ for the vertex set(left and right vertex) and $E($  $) = \{e_t, e_b\}$ for the set of internal edges(top and bottom edge). Then, we have, for example

(7.11) $($  $) = |\{\{v_L\}, \{v_R\}\}| = 2,$ $($  $) = |\{\{v_L, v_R\}\}| = 1$

and

(7.12) $($  $) = |\emptyset| = 0.$

This latter example illustrates one reason why an insertion operator can only obey (7.1) on a *sub*space of the Hopf algebra: there is no way to incorporate 3 vertex graphs into $\Gamma =$  which has only 2 vertices. This problem, however, does not arise with propagator graphs:

(7.13) $($  $) = 1,$

the reason being that we can put an arbitrary number of photon self-energy graphs arranged in a series into the photon line. The other reason why (7.1) may be violated is that the insertion operator vanishes if the superscript graph $\gamma$ has no insertion place for the graph in the argument. However, there is a very simple way to compute the number of insertions. To this end, we assign a variable $a_e$ to each edge $e \in E(\Gamma)$ of a connected graph $\Gamma$ and identify $a_e = a_{e'}$ if $e$ and $e'$ are of the same edge type in $\mathcal{R}_E$. Then, given $\mathcal{R}_E = \{$ , ,  $\}$, we can have at most $|\mathcal{R}_E| = 3$ edge variables for $\Gamma$. Let $n_{\gamma,s}$ be the number of connected components $\gamma_j$ in $\gamma = \prod_j \gamma_j$ with residue $\mathrm{res}(\gamma_j) = s$ and $m_{\Gamma,s}$ be the number of edges or vertices in $E(\Gamma)$ and $V(\Gamma)$ of type $s \in \mathcal{R}$. Then, the coefficients of the series

(7.14) $$\frac{1}{(1-a_e)^{m_{\Gamma,e}}} = \sum_{n\geq 0} d_n a_e^n$$

for an edge $e \in E(\Gamma)$ tell us how many insertion places there are in $\Gamma$ for a product of propagator graphs with external leg structure $e$. More precisely, there are $d_n$ possibilities for $n$ propagator graphs with residue $e$ to be inserted into the edges of $\Gamma$ of type $e$, regardless of the order within one insertion place(edge). The number of insertions is given by

(7.15) $$(\Gamma|\gamma) = \prod_{v\in\mathcal{R}_V} \binom{m_{\Gamma,s}}{n_{\gamma,s}} \prod_{e\in\mathcal{R}_E} \frac{1}{n_{\gamma,e}!}\, \partial_e^{n_{\gamma,e}} \frac{1}{(1-a_e)^{m_{\Gamma,e}}}\bigg|_{a=0},$$

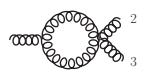
where $a = 0$ is shorthand for setting all edge variables to zero and

(7.16) $$\partial_e := \frac{\partial}{\partial a_e}.$$

Note that we set

(7.17) $$\binom{m_{\Gamma,s}}{n_{\gamma,s}} = 0$$

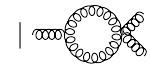
if $m_{\Gamma,s} < n_{\gamma,s}$ for a vertex type $s$, i.e. if there are more vertex graphs of external leg structure $s$ in $\gamma$ than there are vertices of this type in $\Gamma$. The reader is invited to check the validity of (7.15) for some examples and maybe ponder over it for a while. Next, we take a graph $\gamma$ and consider all graphs that we obtain upon permuting the external edges and define $|\gamma|_\vee$ to be the number of these graphs. An example is the graph

(7.18) 

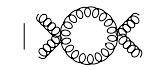
with external edges 1,2 and 3. By permuting the external edges we get the two other graphs

(7.19) 

All $3! = 6$ permutations yield only 3 different graphs. For example, interchanging legs 2 and 3 in (7.18) does not change the graph, the corresponding two momenta $p_2$ and $p_3$ still 'share' the same vertex. The number we get is thus

(7.20) 

Another example is

(7.21) 

**Insertion bijections.** Inserting a graph $\gamma_1$ (or a product of graphs) into another graph $\gamma_2$ corresponds to glueing the external edges of $\gamma_2$ to those of the insertion place $p \in E(\gamma_1) \cup V(\gamma_1)$ in $\gamma_1$. These glueing instructions can be viewed as a bijection: every external leg of $\gamma_2$ is assigned a leg of the insertion place to which it is glued. Let $\mathrm{top}_p(\gamma_1, \gamma_2, \Gamma)$ be the number of such bijections for one insertion place $p$ such that $\Gamma$ results from inserting $\gamma_2$ into $\gamma_1$(at $p$). However, this number does not depend on the insertion place but rather on the type of insertion place which is already sufficiently characterized by the external leg structure $\mathrm{res}(\gamma_2)$. We can therefore drop the index $p$. If we define the *ramification index* $\mathrm{ram}(\gamma_1, \gamma_2, \Gamma)$ as the number of insertion places for $\gamma_2$ into $\gamma_1$ so as to obtain $\Gamma$, we get the total number of insertion bijections for this given by

$$\mathrm{bij}(\gamma_1, \gamma_2, \Gamma) = \mathrm{top}(\gamma_1, \gamma_2, \Gamma)\mathrm{ram}(\gamma_1, \gamma_2, \Gamma). \tag{7.22}$$

To get a feel for this, the reader may peek into [Krei08] to find some illustrative examples.

**Symmetry factor.** Let $H(\Gamma)$ be the set of internal half-edges of a graph $\Gamma$. We choose $h \in H(\Gamma)$ and denote the vertex it is attached to by $v(h) \in V(\Gamma)$. The edge it partakes of is denoted by $e(h) \in E(\Gamma)$. An *edge and vertex preserving* bijection is a bijection $\sigma : H(\Gamma) \to H(\Gamma)$ such that

$$e(h) = e(h') \Leftrightarrow e(\sigma(h)) = e(\sigma(h')) \quad \text{and} \quad v(h) = v(h') \Leftrightarrow v(\sigma(h)) = v(\sigma(h')) \tag{7.23}$$

for all $h, h' \in H(\Gamma)$. We call such bijections *automorphisms* of $\Gamma$. They are a group $\mathrm{Aut}(\Gamma)$ with respect to the composition $\circ$ as group operation. The bijections $\sigma_1 = \mathrm{id}$ and $\sigma_2$ given by

(7.24) 

are edge and vertex preserving, whereas

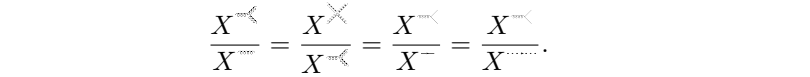
(7.25) 

are not. The *symmetry factor* $\mathrm{Sym}(\Gamma) := |\mathrm{Aut}(\Gamma)|$ of a graph $\Gamma$ is the number of edge and vertex preserving bijections on the set $H(\Gamma)$ of half-edges, i.e. the rank of the automorphism group $\mathrm{Aut}(\Gamma)$.

**Insertion operators.** Let $\gamma$ be a primitive connected graph with $\mathrm{res}(\gamma) = r \in \mathcal{R}$ and $X$ be a graph for which $(\gamma|X) > 0$, i.e. $X$ can be inserted into $\gamma$. We define a linear insertion operator for $\gamma$ by $B_+^\gamma(\mathbb{I}) = \gamma$ and for $X$ by

$$B_+^\gamma(X) = \frac{1}{(\gamma|X)|X|_\vee} \sum_{\Gamma \in \mathcal{G}} \frac{\mathrm{bij}(\gamma, X, \Gamma)}{\mathrm{maxf}(\Gamma)} \Gamma, \tag{7.26}$$

where $\mathcal{G}$ is the set of divergent connected 1PI graphs, i.e. the canonical generator set of the Hopf algebra $\mathcal{H}$ and $\mathrm{maxf}(\Gamma)$ is the number of subgraphs $\gamma \subsetneq \Gamma$ such that the cograph $\Gamma/\gamma$ is primitive. For the next theorem, we denote the set of primitive elements in $\mathcal{H}$ by $\mathrm{Prim}(\mathcal{H})$ and recall the Slavnov-Taylor identities

(7.27) 

Let for a fixed loop number $k > 0$ and an amplitude $r \in \mathcal{R}$ the set of all graphs $\Gamma \in \mathcal{G}$ with $\mathrm{res}(\Gamma) = r$ and loop number $|\Gamma| = k$ be denoted by $M_k^r$. Then, we are ready to state the next

**Theorem 7.1.** *If we define linear operators*

$$B_+^{k;r} := \sum_{\gamma \in M_k^r \cap \mathrm{Prim}(\mathcal{H})} \frac{1}{\mathrm{Sym}(\gamma)} B_+^\gamma \tag{7.28}$$

*and implement the Slavnov-Taylor identities (7.27), the series* $X^r(\alpha) = \mathbb{I} + \mathrm{sgn}(s_r) \sum_{j\geq 1} c_j^r \alpha^j$ *with coefficients in* $\mathcal{H}$ *given by*

$$c_j^r = \sum_{\Gamma \in M_j^r} \frac{1}{\mathrm{Sym}(\Gamma)} \Gamma \tag{7.29}$$

*solve the DSE system*

$$X^r = \mathbb{I} + \mathrm{sgn}(s_r) \sum_{k\geq 1} \alpha^k B_+^{r;k}(X^r Q^k), \qquad r \in \mathcal{R} \tag{7.30}$$

*where* $Q = \prod_{j\in\mathcal{R}} (X^j)^{s_j}$ *is the invariant charge. Finally, we have*

$$\Delta B_+^{k;r}(X^r Q^k) = B_+^{k;r}(X^r Q^k) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+^{k;r}) \Delta(X^r Q^k). \tag{7.31}$$

*Proof.* - □

What (7.31) tells us is this: the coefficients of the series

$$X^r(\alpha) Q(\alpha)^k = \sum_{j\geq 0} \tau_{k,j}^r \alpha^j = \mathbb{I} + \sum_{j\geq 1} \tau_{k,j}^r \alpha^j \tag{7.32}$$

span a subspace $\langle \tau_{k,j}^r \in \mathcal{H} : j \geq 0 \rangle_{\mathbb{Q}}$ on which the linear operator $B_+^{k;r}$ satisfies the Hochschild one-cocycle property (7.1) we started this section with. The point is, however, for QCD and hence also for QED, we need in general a *linear combination* of insertion operators $B_+^\gamma$ with primitive superscript 'skeleton' graph $\gamma$ to obtain a Hochschild one-cocycle on a suitable subspace.

**Locality.** To see how (7.31) garantees *locality* in the sense of renormalization in momentum scheme, we introduce the necessary notions quickly. Let $\Gamma$ be a propagator graph with external euclidean four-momentum $q \in \mathbb{R}^4$. In dimensional regularization, the corresponding regularized Feynman rules are given by a *character*: an algebra morphism $\phi_z(\cdot, q) : \mathcal{H} \to \mathcal{A}$ which evaluates the graph $\Gamma$ to a Laurent series

$$\phi_z(\Gamma, q) = \sum_{j\in\mathbb{Z}} u_j(L_q) z^j \tag{7.33}$$

with a finite number of poles and coefficients in the set $\mathbb{C}[L_q]$, i.e. polynomials $u_j(L_q)$ in the momentum variable $L_q = \ln(q^2/\mu^2)$, where $\mu > 0$ is the renormalization point. The target algebra is therefore given by $\mathcal{A} = \mathbb{C}[L_q][z^{-1}, z]]$. The renormalized and pole-free value of the graph $\Gamma$ can be written as a sum

$$\phi_{R,z}(\Gamma, q) = \bar{\phi}_z(\Gamma, q) + S_{R,z}^\phi(\Gamma, q), \tag{7.34}$$

where the second term is the *counterterm*. It accounts for the last renormalization subtraction needed to yield a finite value of $\phi_{R,z}(\Gamma, q)$ for the limit $z \to 0$. The character $\bar{\phi}_z(\cdot, q)$ is called *Bogolubov map* and provides the value of $\Gamma$ purged of all subdivergences and is related to the counterterm by

$$S_{R,z}^\phi(\Gamma, q) = -R\bar{\phi}_z(\Gamma, q), \tag{7.35}$$

$R : \mathcal{A} \to \mathbb{C}[z^{-1}, z]]$ being the linear evaluation map setting $q^2 = \mu^2$(or $L_q = 0$). For the graph $\Gamma$ this yields

$$S_{R,z}^\phi(\Gamma) = -R\bar{\phi}_z(\Gamma, q) = -\sum_{j\in\mathbb{Z}} v_j(0) z^j, \tag{7.36}$$

if we let $\bar{\phi}_z(\Gamma, q) = \sum_{j\in\mathbb{Z}} v_j(L_q) z^j$ be the value of the Bogolubov map for $\Gamma$. The renormalized value then reads

$$\phi_{R,z}(\Gamma, q) = (\mathrm{id} - R)\bar{\phi}_z(\Gamma, q) = \sum_{j\in\mathbb{Z}} [v_j(L_q) - v_j(0)] z^j, \tag{7.37}$$

which, in order to be pole-free, requires $v_j(L_q) = v_j(0)$ for $j < 0$. Hence: the pole-term coefficients must be constants and are not allowed to be dependent on $L_q$ in which case we refer to the pole as *local*, i.e. if

its Laurent series coefficient is constant. Furthermore, we say that the Bogolubov map is *local* for $\Gamma$, if all of its poles are local, i.e. if the limit

$$\lim_{z\to 0}\frac{\partial}{\partial L_q}\bar{\phi}_z(\Gamma,q)=v_0'(L_q) \tag{7.38}$$

exists. Now let us see how locality is related to the Hochschild cohomology property of the insertion operators $B_+^{k;r}$. We assume for simplicity that all graphs are *single-scale* in the sense that their renormalized value depends only on one momentum. Then, for a connected Feynman graph $\gamma$, one can write

$$\phi_z(\Gamma,q)=\phi_z(B_+^\gamma(W),q)=\int d^4p\ K_z^\gamma(q,p)\phi_z(W,p) \tag{7.39}$$

where $W\in\mathcal{H}$ such that $\Gamma=B^\gamma(W)$ and $K_z^\gamma(q,\cdot)$ is the corresponding integral kernel of the regularized Feynman integral for $\gamma$ given by

$$\phi_z(\gamma,q)=\phi_z(B_+^\gamma(\mathbb{I}),q)=\int d^4p\ K_z^\gamma(q,p)\underbrace{\phi_z(\mathbb{I},p)}_{=1}=\int d^4p\ K_z^\gamma(q,p). \tag{7.40}$$

Then (7.28) suggests that we may define the integral kernel

$$K_z^{k;r}(q,p):=\sum_{\gamma\in M_k^r\cap\mathrm{Prim}(\mathcal{H})}\frac{1}{\mathrm{Sym}(\gamma)}K_z^\gamma(q,p) \tag{7.41}$$

for which, by linearity of $\phi_z(\cdot,q)$, we get

$$\phi_z(B_+^{k;r}(\cdot),q)=\int d^4p\ K_z^{k;r}(q,p)\phi_z(\cdot,p). \tag{7.42}$$

To prove locality of the Bogolubov map, we will use

$$\overline{\phi}_z=S_{R,z}^\phi*\phi_zP, \tag{7.43}$$

where $P:\mathcal{H}\to\mathrm{Aug}$ projects onto the augmentation ideal $\mathrm{Aug}\subset\mathcal{H}$. In simple terms, $P$ is the projection operator on $\mathcal{H}$ with one-dimensional kernel $\ker P=\mathbb{Q}\mathbb{I}$ (this defines Aug). Let now $W\in\mathcal{H}$ be such that

$$\Delta B_+^{k,r}(W)=B_+^{k,r}(W)\otimes\mathbb{I}+(\mathrm{id}\otimes B_+^{k,r})\Delta(W)=B_+^{k,r}(W)\otimes\mathbb{I}+W'\otimes B_+^{k,r}(W'') \tag{7.44}$$

where $\Delta(W)=W'\otimes W''$ is a shorthand for the corresponding sum. Then, we have

$$\begin{aligned}\overline{\phi}_z(B_+^{k,r}(W),q)&=(S_{R,z}^\phi*\phi_zP)(B_+^{k,r}(W),q)=(S_{R,z}^\phi\otimes\phi_zP)(\Delta B_+^{k,r}(W),q)\\&=(S_{R,z}^\phi\otimes\phi_zP)(B_+^{k,r}(W)\otimes\mathbb{I}+W'\otimes B_+^{k,r}(W''),q)\\&=S_{R,z}^\phi(B_+^{k,r}(W))\phi_zP(\mathbb{I},q)+S_{R,z}^\phi(W')\phi_zP(B_+^{k,r}(W''),q)\\&=S_{R,z}^\phi(W')\phi_z(B_+^{k,r}(W''),q)=S_{R,z}^\phi(W')\int d^4p\ K_z^{k;r}(q,p)\phi_z(W'',p).\end{aligned} \tag{7.45}$$

Because the counterterm does not depend on any momentum and only of the renormalization point $\mu$, we can push it inside the integral and get

$$\overline{\phi}_z(B_+^{k,r}(W),q)=\int d^4p\ K_z^{k;r}(q,p)S_{R,z}^\phi(W')\phi_z(W'',p)=\int d^4p\ K_z^{k;r}(q,p)\phi_{R,z}(W,p) \tag{7.46}$$

where we have used $\phi_{R,z}=S_{R,z}^\phi*\phi_z$. Now, taking the derivative with respect to the momentum parameter $L_q$,

$$\frac{\partial}{\partial L_q}\overline{\phi}_z(B_+^{k,r}(W),q)=q^2\frac{\partial}{\partial q^2}\overline{\phi}_z(B_+^{k,r}(W),q)=\int d^4p\ q^2\frac{\partial}{\partial q^2}K_z^{k;r}(q,p)\phi_{R,z}(W,p) \tag{7.47}$$

we see that locality demands

$$\lim_{z\to 0}\int d^4p\ |q^2\frac{\partial}{\partial q^2}K_z^{k;r}(q,p)\phi_{R,z}(W,p)|<\infty. \tag{7.48}$$

This is satisfied in particular if

$$\int d^4p\ |\partial_{q^2}K_0^{k;r}(q,p)|<\infty. \tag{7.49}$$

The reason is that $\phi_{R,0}(W,p)$ yields a polynomial in $L_p = \ln(p^2/\mu^2)$ which does not alter the convergence behaviour of the integral $\int d^4p\ \partial_{q^2} K_0^{k;r}(q,p)$. Suppose this integral converges (which it does), then we see that the Bogolubov map is local on the coefficients of the perturbation series

$$X^r = \mathbb{I} + \mathrm{sgn}(s_r) \sum_{k\geq 0} c_k^r \alpha^k = \mathbb{I} + \mathrm{sgn}(s_r) \sum_{k\geq 1} \alpha^k B_+^{r;k}(X^r Q^k), \tag{7.50}$$

if we employ (7.46) with $W = X^r Q^k$, that is,

$$\overline{\phi}_z(X^r, q) = 1 + \mathrm{sgn}(s_r) \sum_{k\geq 1} \alpha^k \int d^4p\ K_z^{k;r}(q,p) \phi_{R,z}(X^r Q^k, p) \tag{7.51}$$

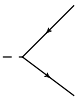
is local in every order of $\alpha$.

## 8. Radiative corrections in Yukawa theory

In this section, we consider an example of a linear DSE for the vertex series in massless *Yukawa theory*, which is given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + i\overline{\psi}\partial\!\!\!/\psi - g\varphi\overline{\psi}\psi \tag{8.1}$$

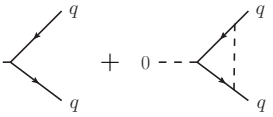
describing interacting massless spin 1/2 fermions and scalar mesons represented by the spinor field $\psi$ and the scalar field $\varphi$, respectively. The Feynman rules in momentum space are

(8.2)  $= -ig$ ,  $= \dfrac{i}{\not{p} + i\epsilon}$ ,  $= \dfrac{i}{p^2 + i\epsilon}$

accompanied with the corresponding integration directives and terms for external incoming and outgoing particles.

### 8.1. Vertex series.

The first two terms of the 1PI perturbation series for the vertex are

(8.3)  $= -ig\left(1 + (-ig)^2 \int \dfrac{d^4k}{(2\pi)^4} \dfrac{i}{\not{k} + i\epsilon} \dfrac{i}{(k-q)^2 + i\epsilon} \dfrac{i}{\not{k} + i\epsilon} + ...\right)$

where we have set the external boson's momentum to zero for simplicity. To evaluate the integral, we first perform what is known as a Wick rotation: the $k_0$-integration contour is rotated counterclockwise in the complex plane by $\pi/2$, then parametrized by $k_0 = ik_4$, whereas the zeroth component $q_0$ of the external momentum is rotated clockwise by this angle to yield $q_4 = -iq_0$, i.e.

$$(k-q)^2 = (k_0 - q_0)^2 - (\mathbf{k}-\mathbf{q})^2 = (ik_4 - iq_4)^2 - (\mathbf{k}-\mathbf{q})^2 = -[(k_4-q_4)^2 + (\mathbf{k}-\mathbf{q})^2] = -(k_E - q_E)^2 \tag{8.4}$$

is what happens to the Minkowski product in the process. The index 'E' stands for *Euclidean*. Then follows

$$(-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not{k} + i\epsilon} \frac{i}{(k-q)^2 + i\epsilon} \frac{i}{\not{k} + i\epsilon} = -g^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 (k_E - q_E)^2} \tag{8.5}$$

for the integral, where the Feynman prescription '$i\epsilon$' has been dropped. This Euclidean integral is logarithmically divergent and needs to be regularized. To this end, we introduce the convergence factor

$$(k_E^2)^{-\rho} \tag{8.6}$$

with regulator $\rho \in \mathbb{C}$ and use

$$\int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(k_E^2)^r ((k_E - q_E)^2)^s} = (q_E^2)^{-(r+s-2)} \frac{1}{(4\pi)^2} \frac{\Gamma(r+s-2)\Gamma(2-r)\Gamma(2-s)}{\Gamma(r)\Gamma(s)\Gamma(4-r-s)}. \tag{8.7}$$

We obtain

$$\int \frac{d^4k_E}{(2\pi)^4} \frac{(k_E^2)^{-\rho}}{k_E^2 (k_E - q_E)^2} = (q_E^2)^{-\rho} \frac{1}{(4\pi)^2} \frac{\Gamma(\rho)\Gamma(1-\rho)}{\Gamma(1+\rho)\Gamma(2-\rho)} = (q_E^2)^{-\rho} \frac{1}{(4\pi)^2} \frac{1}{\rho(1-\rho)}, \tag{8.8}$$

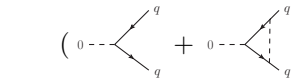
where we have employed the identity $x\Gamma(x) = \Gamma(1+x)$ twice. We call

$$F_1(\rho) := -(4\pi)^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{(k_E^2)^{-\rho}}{k_E^2 (k_E - \hat{q}_E)^2} = -\frac{1}{\rho(1-\rho)}, \tag{8.9}$$

the *Mellin transform* of first vertex primitive, where $\hat{q}_E = q_E/|q_E|$ is a unit vector. One can nicely see that the Mellin tranform it is invariant under the Möbius transformation $\rho \to 1-\rho$, often called *conformal transformation*. It is convenient to define $a := g^2/(4\pi)^2$ as the new coupling parameter. The renormalized value of the 1-loop vertex then is

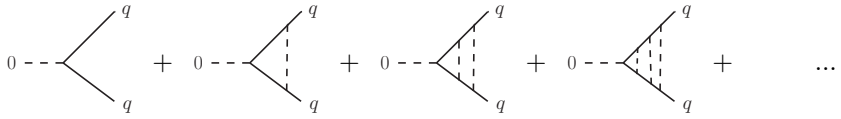
(8.10) $$(\ \text{}\ )_R = \lim_{\rho\to 0} a[(q_E^2)^{-\rho} - (\mu^2)^{-\rho}]F_1(\rho) = aL$$

where $L = \ln(q_E^2/\mu^2)$ is the Euclidean momentum parameter. Wick rotating back yields

(8.11) $$(\ \text{  }\ + ...)_R = -ig[1 + \frac{g^2}{(4\pi)^2}\log(-q^2/\mu^2) + ...]$$

Notice that the new coupling's power counts the number of loops whereas $g$ counts the vertices.

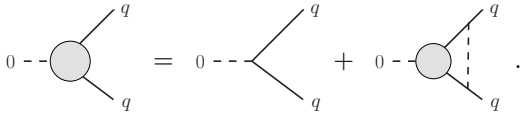
**Ladder approximation.** We can do much better than computing the first loop order contribution: it is in fact very easy to calculate the 1PI 'ladder' series

(8.12) 

which satisfies the linear Dyson-Schwinger equation

(8.13) $$X(a) = 1 + aB_+^u(X(a)),$$

where $u =$  is the first connected 1-loop primitive of the vertex. In terms of blob diagrams, this reads

(8.14) 

The linear DSE in (8.13) is combinatorially equivalent to (3.19) and we may again employ ladder trees to describe the terms in our ladder series (8.12). However, let us for a change consider another, and in fact very elegant way of writing the series.

**Shuffle algebra of words.** Given a set $\mathfrak{A}$ called *alphabet* with elements $a_1, a_2, ...$ named *letters*, one can generate a non-commutative polynomial algebra $\mathbb{Q}\langle\mathfrak{A}\rangle$, where an object of the form $w = a_{j_1}...a_{j_n}$ is called *word* and $e$ is the *empty word*, that is, the neutral element of the multiplication, i.e. $we = ew = w$ for any word $w \in \mathbb{Q}\langle\mathfrak{A}\rangle$. As a formal vector space, it is the words that provide a canonical basis. Note that $a_j a_k \neq a_k a_j$ if $k \neq j$. The elements in $\mathbb{Q}\langle\mathfrak{A}\rangle$ are generally linear combinations of words. In addition to this product called 'concatenation product', we introduce the *shuffle* product by

(8.15) $$w \sqcup\!\sqcup e = e \sqcup\!\sqcup w = w, \qquad a_j w_1 \sqcup\!\sqcup a_k w_2 = a_j(w_1 \sqcup\!\sqcup a_k w_2) + a_k(a_j w_1 \sqcup\!\sqcup w_2)$$

for any words $w, w_1, w_2 \in \mathbb{Q}\langle\mathfrak{A}\rangle$. The reader may prove that the shuffle powers of a single letter $a_j \in \mathfrak{A}$ fulfill

(8.16) $$a_j^{\sqcup\!\sqcup n} = n! a_j^n = n!\underbrace{a_j a_j ... a_j}_{n\text{-times}}.$$

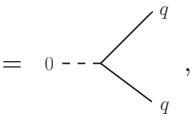
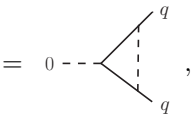
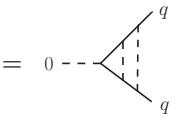
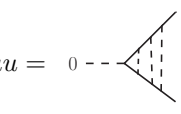
Using the shuffle product, one also defines a *shuffle exponential* of an element $x \in \mathbb{Q}\langle\mathfrak{A}\rangle$ by

(8.17) $$\exp_{\sqcup\!\sqcup}(x) := \sum_{n\geq 0} \frac{x^{\sqcup\!\sqcup n}}{n!}.$$

The pair $(\mathbb{Q}\langle\mathfrak{A}\rangle, \sqcup\!\sqcup)$ is what we call the *shuffle algebra* (of words). With all this, we may agree to write the ladder series in terms of the single letter alphabet $\mathfrak{A} = \{u\}$ such that it takes the form of a simple geometric series

(8.18) $$X(a) = 1 + au + a^2 uu + a^3 uuu + ... = 1 + (au)^2 + (au)^3 + ... = \frac{1}{1-au},$$

where

(8.19) $$1 = \ \text{  }\ , \quad u = \ \text{  }\ , \quad uu = \ \text{  }\ , \quad uuu = \ \text{  }\ ,$$

and so on and so forth. This means $B_+^u(u^k) = u^{k+1}$ for the insertion operator. In fact, one can also write the series as

$$X(a) = \exp_{\sqcup\!\sqcup}(au) \tag{8.20}$$

which relates the shuffle exponential to the geometric series in an interesting way, albeit only for a single letter. What this tells us is that all the information for the perturbation series of ladders is contained in the single letter $u$, corresponding to the single primitive 1-loop vertex graph plus some set of rules represented by the shuffle exponential. Therefore, the hope is, we may only be required to compute this single graph and then apply some easy rule to evaluate the Green's function given by

$$G_v(a, L) := \phi_R(X(a), L) = \phi_R(1 + au + a^2uu + a^3uuu + ..., L) = \phi_R(\exp_{\sqcup\!\sqcup}(au), L). \tag{8.21}$$

Before we see that this is indeed true, we first define both a deconcatenation coproduct $\Delta$ on the shuffle algebra $\mathbb{Q}\langle\mathfrak{A}\rangle$ by

$$\Delta(a_{j_1}...a_{j_n}) = a_{j_1}...a_{j_n} \otimes e + e \otimes a_{j_1}...a_{j_n} + \sum_{l=1}^{n-1} a_{j_1}...a_{j_l} \otimes a_{j_{l+1}}...a_{j_n} \tag{8.22}$$

and a counit $\overline{e}$ by $\overline{e}(e) = 1$ and vanishing on any non-empty word. The antipode $S$ is then recursively defined as usual. In fact, with these extra ingredients, the shuffle algebra is a Hopf algebra (see Appendix D), where the product is given by the shuffle product. This means in particular that the deconcatenation coproduct respects the shuffle product, i.e.

$$\Delta(x \sqcup\!\sqcup y) = \Delta(x) \sqcup\!\sqcup \Delta(y) \tag{8.23}$$

and therefore that $\Delta(\exp_{\sqcup\!\sqcup}(p)) = \exp_{\sqcup\!\sqcup}(p) \otimes \exp_{\sqcup\!\sqcup}(p)$ for a primitive element $p \in \mathbb{Q}\langle\mathfrak{A}\rangle$ by a brief calculation:

$$\begin{aligned} \Delta(\exp_{\sqcup\!\sqcup}(p)) &= \exp_{\sqcup\!\sqcup}(\Delta(p)) = \exp_{\sqcup\!\sqcup}(p \otimes e + e \otimes p) = \exp_{\sqcup\!\sqcup}(p \otimes e) \sqcup\!\sqcup \exp_{\sqcup\!\sqcup}(e \otimes p) \\ &= [\exp_{\sqcup\!\sqcup}(p) \otimes e] \sqcup\!\sqcup [e \otimes \exp_{\sqcup\!\sqcup}(p)] = [\exp_{\sqcup\!\sqcup}(p) \sqcup\!\sqcup e] \otimes [e \sqcup\!\sqcup \exp_{\sqcup\!\sqcup}(p)] \\ &= \exp_{\sqcup\!\sqcup}(p) \otimes \exp_{\sqcup\!\sqcup}(p). \end{aligned} \tag{8.24}$$

Since our graph $u$ is indeed primitive, we find

$$\Delta(X(a)) = \Delta(\exp_{\sqcup\!\sqcup}(au)) = \exp_{\sqcup\!\sqcup}(au) \otimes \exp_{\sqcup\!\sqcup}(au) = X(a) \otimes X(a) \tag{8.25}$$

and hence that the ladder series is *grouplike*. This implies for the renormalized Feynman rules

$$\phi_{R,\rho}(X(a), L) = (S^\phi_{R,\rho} * \phi_\rho)(X(a), L) = S^\phi_{R,\rho}(X(a))\phi_\rho(X(a), L) \tag{8.26}$$

with regulator $\rho \in \mathbb{C}$ and regularized Feynman rules $\phi_\rho$ given by

$$\phi_\rho(u, L) = \phi_\rho(B_+^u(1), L) = \int d^4k_E \; K_\rho^u(q_E, k_E)\phi_\rho(1, \ln(k_E^2/\mu^2)) = \int d^4k_E \; K_\rho^u(q_E, k_E), \tag{8.27}$$

with regularized integral kernel

$$K_\rho^u(q_E, k_E) = -\frac{1}{\pi^2}\frac{(k_E^2)^{-\rho}}{k_E^2(k_E - q_E)^2} \tag{8.28}$$

which the reader may compare with (8.3) and its Euclidean version in (8.5). The Mellin transform is can now be written as

$$F_1(\rho) = \int d^4k_E \; K_\rho^u(\hat{q}_E, k_E) = \int d^4k_E \; K_0^u(\hat{q}_E, k_E)(k_E^2)^{-\rho}. \tag{8.29}$$

If we apply these regularized Feynman rules $\phi_\rho$ to the cDSE (8.13), we find

$$\phi_\rho(X(a), L) = 1 + a\phi_\rho(B_+^u(X(a)), L) = 1 + a\int d^4k_E \; K_\rho^u(q_E, k_E)\phi_\rho(X(a), \ln(k_E^2/\mu^2)) \tag{8.30}$$

which is our analytical DSE. We can use this equation for the rhs of (8.26) and obtain

$$\phi_{R,\rho}(X(a), L) = Z_\rho(a) + a\int d^4k_E \; K_\rho^u(q_E, k_E)\phi_{R,\rho}(X(a), \ln(k_E^2/\mu^2)), \tag{8.31}$$

where $Z_\rho(a) := S^\phi_{R,\rho}(X(a))$ is the *renormalization Z-factor*. It serves as the counterterm to renormalize the skeletal integration and is thus given by

$$Z_\rho(a) = 1 - a\int d^4k_E \; K_\rho^u(\overline{q}_E, k_E)\phi_{R,\rho}(X(a), \ln(k_E^2/\mu^2)) \tag{8.32}$$

with Euclidean momentum $\overline{q}_E$ such that $\overline{q}_E^2 = \mu^2$. Eq.(8.31) is the analytical DSE for the Green's function $G_v(a, L)$ which may also be written in the form

$$G_v(a, L) = 1 + a \int d^4k_E \; K_R^u(q_E, k_E) G_v(a, \ln(k_E^2/\mu^2)), \tag{8.33}$$

where the limit $\rho \to 0$ has produced a perfectly convergent renormalized kernel

$$K_R^u(q_E, k_E) := K_0^u(q_E, k_E) - K_0^u(\overline{q}_E, k_E) \tag{8.34}$$

Let us next have a look at the exact solution of (8.33).

**Proposition 8.1.** *The vertex DSE in (8.33) is solved by the scaling solution*

$$G_v(a, L) = \exp(-\gamma_v(a)L) = \left(\frac{q_E^2}{\mu^2}\right)^{-\gamma_v(a)} = \exp\left\{-\gamma_v(a)\log(-q^2/\mu^2)\right\} \tag{8.35}$$

*with* $\gamma_v(a) = \frac{1}{2}(1 - \sqrt{1+4a})$.

*Proof.* First, the reader may check that the regularized integral measure scales according to

$$d^4(\lambda k_E) \; K_\rho^u(\lambda q_E, \lambda k_E) = \lambda^{-2\rho} d^4k_E \; K_\rho^u(q_E, k_E). \tag{8.36}$$

This means is particular

$$\int d^4k_E \; K_\rho^u(q_E, k_E) = (q_E^2)^{-\rho} \int d^4k_E \; K_\rho^u(|q_E|\hat{q}_E, k_E)(|q_E|^{-1})^{-2\rho} = (q_E^2)^{-\rho} F_1(\rho) \tag{8.37}$$

which we will use next when we plug the scaling ansatz into the integral on the rhs of the DSE (8.33):

(8.38)

$$\begin{aligned}
\int d^4k_E \; K_R^u(q_E, k_E) \left(\frac{q_E^2}{\mu^2}\right)^{-\gamma_v(a)} &= \int d^4k_E \; [K_0^u(q_E, k_E) \left(\frac{k_E^2}{\mu^2}\right)^{-\gamma_v(a)} - \; K_0^u(\overline{q}_E, k_E) \left(\frac{k_E^2}{\mu^2}\right)^{-\gamma_v(a)}] \\
&= \left(\frac{q_E^2}{\mu^2}\right)^{-\gamma_v(a)} F_1(\gamma_v(a)) - \left(\frac{\overline{q}_E^2}{\mu^2}\right)^{-\gamma_v(a)} F_1(\gamma_v(a)) \\
&= [\left(\frac{q_E^2}{\mu^2}\right)^{-\gamma_v(a)} - 1] F_1(\gamma_v(a))
\end{aligned}$$

where we recall that $\overline{q}_E^2 = \mu^2$ for the reference momentum $\overline{q}_E$. Finally, the DSE in (8.33) takes the form

$$\left(\frac{q_E^2}{\mu^2}\right)^{-\gamma_v(a)} = 1 + a[\left(\frac{q_E^2}{\mu^2}\right)^{-\gamma_v(a)} - 1] F_1(\gamma_v(a)) \tag{8.39}$$

which entails

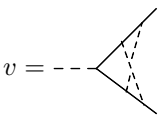
$$1 = aF_1(\gamma_v(a)) = \frac{-a}{\gamma_v(a)[1 - \gamma_v(a)]} \tag{8.40}$$

and thus $\gamma_v(a)^2 - \gamma_v(a) - a = 0$. We require $\gamma_v(0) = 0$ on the grounds that we want $G_v(0, L) = 1$ and therefore get the solution. □

This solution is similiar to the one of the propagator 'rainbow' DSE in (1.6) which we shall have a look at below (see also [Krei06]).

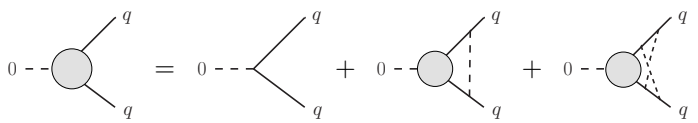
**Next-to-ladder approximation.** One can do even better than the ladder approximation: if we also take the primitive

(8.41)




into account, the DSE reads

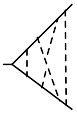
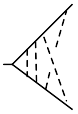
(8.42)




or, in 'words'

$$X(a) = 1 + aB_+^u(X(a)) + a^2 B_+^v(X(a)), \tag{8.43}$$

where $B_+^u(x) = xu$ and $B_+^v(x) = xv$ for any word $x \in \mathbb{Q}\langle u, v\rangle$. This means that as we read a word from left to right, we ascend from its deepest subdivergence all the way up to its skeleton graph, e.g.

(8.44) $B_+^u(B_+^v(u)) = uvu =$  , $B_+^v(B_+^u(u)) = uuv =$  .

This has been worked out in [BierKW] with the following results. Firstly, they find

(8.45) $$X(a) = \exp_{\sqcup\!\sqcup}(au + a^2 v) = 1 + au + a^2(v + uu) + a^3(uuu + uv + vu) + ...$$

and thus $\Delta(X(a)) = X(a) \otimes X(a)$ for the deconcatenation coproduct. The Mellin tranform for $v$ is

(8.46) $$F_2(\rho) = \int d^4k \int d^4l \; K_\rho^v(\hat{q}, k, l)$$

with Minkowksi space kernel

(8.47) $$K_\rho^v(\hat{q}, k, l) = -\frac{1}{\pi^4} \frac{\not{k}(\not{l} + \not{k})[(k+l)^2]^{-\rho}}{k^2(\hat{q}-k)^2(k+l)^2 l^2 (l+\hat{q})^2}.$$

Secondly, it turns out that, again, a scaling ansatz $G_R(a, L) = \exp(-\gamma_G(a)L)$ (in their notation) proves to be the right strategy in solving the renormalized DSE

(8.48) $$\begin{aligned} G_R(a, \log(-q^2/\mu^2)) = 1 &- a \int d^4k \; K_R^u(q,k) G_R(a, \log(-k^2/\mu^2) \\ &+ a^2 \int d^4k \int d^4l \; K_R^v(q,k,l) G_R(a, \log(-(k+l)^2/\mu^2) \end{aligned}$$

with $\gamma_G(a)$ such that

(8.49) $$1 = aF_1(\gamma_G(a)) + a^2 F_2(\gamma_G(a)).$$

Although $\gamma_G(a)$ remains to be given only implicitly, the Green's function can be represented as follows. Let $\mathcal{P}(u, v)$ be the set of all primitive elements in the Hopf algebra of words $\mathbb{Q}\langle u, v\rangle$. Then, the Green's function is

(8.50) $$G_R(a, L) = \phi_R(X(a), L) = \exp\left( \sum_{p \in \mathcal{P}(u,v)} a^{|p|} \phi_R(p, L) \right),$$

where $\phi_R(\cdot, L)$ are the renormalized Feynman rules and $|p|$ the number of letters of any term in $p \in \mathcal{P}(u, v)$ (primitives are homogeneous). For more see [BierKW] and references therein.

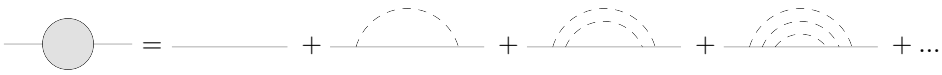
8.2. **Propagator series: rainbows and beyond.** We come back to the cDSE (2.4)

(8.51) $$X(a) = 1 + aB_+^{\overset{\frown}{\ }}(X(a)) \; .$$

or, in blobs

(8.52) 

whose solution

(8.53) 

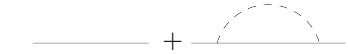
is the so-called *rainbow approximation* for the Yukawa fermion propagator. Just like the ladder approximation in (8.12), this equation has a *scaling solution*. We start by computing the skeleton graph in (8.52), by employing the Feynman rules (8.2) and obtain

(8.54)  $= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not{k} + i\epsilon} \frac{i}{(q-k)^2 + i\epsilon}.$

A suitably regularized version of this integral must depend on the external Minkowksi momentum $q$ and should be a Lorentz-invariant linear combination of Dirac $\gamma$-matrices (because the integrand is), i.e.

(8.55) $$(-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not{k} + i\epsilon} \frac{i}{(q-k)^2 + i\epsilon} = \not{q} A(-q^2).$$

is a first possible ansatz, where $A(\cdot)$ is what is called *form factor* or sometimes also *structure function*. The perturbation series starts like

(8.56)  $= \frac{i}{\not q} + \frac{i}{\not q} \not q A(-q^2) \frac{i}{\not q} + ... = \frac{i}{\not q}(1 + iA(-q^2) + ...),$

where the expression in brackets stands for the form factor of the whole series. When we apply the Feynman rules to the cDSE in (8.51), we get this very form factor. Let us quickly calculate the function $A(\cdot)$. To this end, we multiply both sides of (8.55) by the matrix $\not q$ and taking the trace we obtain

$$g^2 \int \frac{d^4k}{(2\pi)^4} \frac{kq}{k^2 + i\epsilon} \frac{1}{(q-k)^2 + i\epsilon} = q^2 A(-q^2). \tag{8.57}$$

A Wick rotation yields

$$ig^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{k_E q_E}{k_E^2 (q_E - k_E)^2} = q_E^2 A(q_E^2) \tag{8.58}$$

and finally using the rule

$$\int \frac{d^4k_E}{(k_E^2)^r} = \int \frac{d^4k_E}{((q_E - k_E)^2)^s} = 0 \tag{8.59}$$

for all $r, s \in \mathbb{R}$ and $2k_E q_E = -(q_E - k_E)^2 + k_E^2 + q_E^2$, we arrive at

$$A(q_E^2) = ia \int \frac{d^4k_E}{2\pi^2} \frac{1}{k_E^2 (q_E - k_E)^2}. \tag{8.60}$$

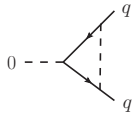
If we again introduce the convergence factor (8.6) and use the 'master integral formula' (8.7), we find

$$A_\rho(q_E^2) = ia \int \frac{d^4k_E}{2\pi^2} \frac{(k_E^2)^{-\rho}}{k_E^2 (q_E - k_E)^2} =: -ia \int d^4k_E \; K_\rho^p(q_E, k_E), \tag{8.61}$$

where $K_\rho^p(q_E, k_E)$ is the regularized kernel for the skeleton diagram

(8.62) $p =$  .

Notice that this is essentially the same function as the kernel for the vertex skeleton

(8.63) $u =$ 


in (8.28). The result for the regularized form factor can now be read off from (8.8) and gives

$$iA_\rho(q_E^2) = (q_E^2)^{-\rho} \frac{a}{2\rho(\rho - 1)} =: a \; (q_E^2)^{-\rho} F_p(\rho), \tag{8.64}$$

where $F_p(\rho)$ is the *Mellin transform* of the skeleton $p$. The renormalized value of this diagram is

(8.65) $\phi_R($  $, L) = \lim_{\rho \to 0} a[(q_E^2)^{-\rho} - (\mu^2)^{-\rho}] F_p(\rho) = \frac{a}{2} L.$

However, we are interested in the solution of the DSE

$$G_p(a, L) = 1 + a \int d^4k_E \; K_R^p(q_E, k_E) G_p(a, \ln(k_E^2/\mu^2)), \tag{8.66}$$

with renormalized kernel

$$K_R^p(q_E, k_E) := K_0^p(q_E, k_E) - K_0^p(\overline{q}_E, k_E) \tag{8.67}$$

for the renormalization point $\overline{q}_E^2 = \mu^2$, which is the renormalized analytical version of (8.51). As promised, the solution is again rather simple (see also [Krei06]) as shown in

**Proposition 8.2.** *The DSE in (8.66) is solved by the scaling solution*

$$G_p(a, L) = \exp(-\gamma_p(a) L) = \left( \frac{q_E^2}{\mu^2} \right)^{-\gamma_p(a)} = \exp\left\{ -\gamma_p(a) \log(-q^2/\mu^2) \right\} \tag{8.68}$$

*with* $\gamma_p(a) = \frac{1}{2}(1 - \sqrt{1 + 2a})$.

*Proof.* We apply the same procedure as in Prop.8.1 and find

$$1 = aF_p(\gamma_p(a)) = \frac{a}{2\gamma_p(a)[\gamma_p(a)-1]}, \tag{8.69}$$

which is solved by $\gamma_p^{\pm}(a) = \frac{1}{2}(1\pm\sqrt{1+2a})$. Choosing $\gamma_p^-(a)$ on account of $\gamma_p^-(0)=0$ yields the result. □

Question: What is the meaning of the peculiar construction

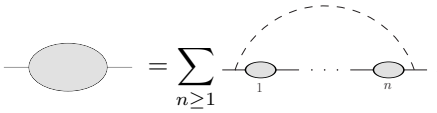
$$\frac{1}{2-G_p(a,L)} = \frac{1}{2-\exp\{-\gamma_p(a)\log(-q^2/\mu^2)\}} \tag{8.70}$$

and why does it make no sense at all for the vertex Green's function in (8.35)? Next, we will see that the DSE quickly gets highly nontrivial when we try to go beyond the rainbow approximation by considering the cDSE

$$X(\alpha) = 1 - \alpha B_+^{\frown}(1/X(\alpha)) \ . \tag{8.71}$$

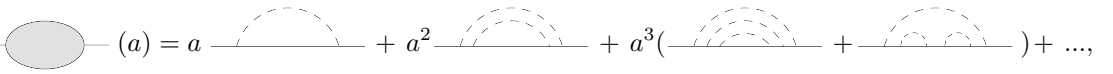
In this approximation, only fermion line corrections are accounted for, i.e. vertex and boson line corrections are omitted. The blob diagram version of this DSE is rather awkward to say the least:

(8.72)




where the blob does not start with the bare but the 1-loop propagator:

(8.73)




which corresponds to the series $X(a)-1$. The reader may check as an exercise that the analytic renormalized version of the DSE (8.71) is given by

$$G(a,L) = 1 - a\int d^4k_E\ \frac{K_R^p(q_E,k_E)}{G(a,\ln(k_E^2/\mu^2))} \tag{8.74}$$

and cannot be solved by a scaling ansatz as it leads to a contradiction. So far, no non-perturbative ansatz is known for this equation except for a method leading to an implicit equation involving the complementary error function which can be found in [BroK01]. However, one can do the following. For a start, the reader should check that

$$\lim_{\rho\downarrow 0}\frac{1}{G(a,-\partial_\rho)}\left(\frac{k_E^2}{\mu^2}\right)^{-\rho} = \frac{1}{G(a,\ln(k_E^2/\mu^2))} \tag{8.75}$$

for the ansatz $G(a,L) = 1-\sum_{j\geq 1}\gamma_j(a)L^j$. Once this has been verified, we see that (8.74) can be written in the form

$$G(a,L) = 1 - a\lim_{\rho\downarrow 0}\frac{1}{G(a,-\partial_\rho)}\int d^4k_E\ K_R^p(q_E,k_E)\left(\frac{k_E^2}{\mu^2}\right)^{-\rho}. \tag{8.76}$$

The integral on the rhs can be expressed in terms of the Mellin transform for the skeleton $p$ such that we finally get

$$G(a,L) = 1 - a\lim_{\rho\downarrow 0}\frac{1}{G(a,-\partial_\rho)}(e^{-\rho L}-1)F_p(\rho), \tag{8.77}$$

where we have made use of $\int d^4k_E\ K_\rho^p(q_E,k_E) = (q_E^2)^{-\rho}F_p(\rho)$ just like we did in (8.38). We can now take the DSE in (8.76) and find a DSE for each log-coefficient function by differentiating both sides with respect to $L$ and then setting $L=0$ giving

$$n!\gamma_n(a) = (-1)^n\frac{a}{2}\lim_{\rho\downarrow 0}\frac{1}{G(a,-\partial_\rho)}\frac{\rho^{n-1}}{\rho-1}, \tag{8.78}$$

for all $n\geq 1$. This is in fact a coupled system for these functions: for $\gamma_1(a)$ this yields

$$\gamma_1(a) = \frac{a}{2}[1+\sum_{1\leq n\leq j}(-1)^j j!\sum_{j_1+\dots j_n=j}\gamma_{j_1}(a)\dots\gamma_{j_n}(a)]. \tag{8.79}$$

Replacing all log-coefficient functions by their perturbation series $\gamma_k(a) = \sum_{j\geq k} \gamma_{j,k} a^j$ yields a set of recursion formulae which can then be solved by a computer. The first few terms are

$$\gamma_1(a) = \frac{a}{2}[1 + (-1)\gamma_1(a) + (-1)^2 2!\{\gamma_1(a)^2 + \gamma_2(a)\} + ...] \tag{8.80}$$

and yield for $\gamma_1(a)$:

$$\gamma_{1,1} = \frac{1}{2}, \qquad \gamma_{1,2} = \frac{1}{2}\gamma_{1,1} = \left(\frac{1}{2}\right)^2 \qquad \gamma_{1,3} = \frac{1}{2}\gamma_{1,2} - (\gamma_{1,1}^2 + \gamma_{2,2}) = -\left(\frac{1}{2}\right)^3 - \gamma_{2,2}. \tag{8.81}$$

To carry on, we need the DSEs for the higher log-order functions. It is noteworthy that the first two are related by

$$\gamma_1(a) + 2\gamma_2(a) = \frac{a}{2} \tag{8.82}$$

which implies $\gamma_{1,l} + 2\gamma_{2,l} = 0$ for $l \geq 2$. Then $\gamma_{1,3} = 0$ follows. The rest remains to be done by a computer. This example is discussed in [KrY06] and the coefficients of $\gamma_1(a)$ have been calculated by a computer up to a staggering 500 loops using a different method in [BroK01]. For the sake of completeness, we mention that the log-coefficient functions are also related by the recursion

$$(n+1)\gamma_{n+1}(a) = \gamma_1(a)(2a\partial_a - 1)\gamma_n(a), \tag{8.83}$$

see [KrY06] for more. Together with (8.82), this formula implies a non-linear ODE

$$\gamma_1(a) + \gamma_1(a)(2a\partial_a - 1)\gamma_1(a) = \frac{a}{2}. \tag{8.84}$$

We can employ a computer algebra software to find that the solution satisfies

$$\sqrt{\frac{a}{\pi}} e^{-Z(a)^2} = 1 + \mathrm{erf}(Z(a)) \tag{8.85}$$

with $Z(a) = (\gamma(a) - 1)/\sqrt{a}$. We remind the reader of the both error functions:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\ e^{-t^2}, \qquad \mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt\ e^{-t^2}. \tag{8.86}$$

Question: Does the result $\gamma_{1,1} = 1/2$ contradict the rainbow solution in Prop. 8.2 where

$$\gamma_p(a) = -a/2 + \mathcal{O}(a^2) \qquad ? \tag{8.87}$$

Hint: Ponder over eq.(8.70) again!

## 9. Field Diffeomorphisms and Perturbation Theory

Let $F(x) \in \mathbb{R}[[x]]$ be a formal power series with $F(0) = 0$ and $F'(0) = 1$. Objects of this kind are also referred to as *formal diffeomorphisms*. We use $F(x)$ to transform a free scalar quantum field $\varphi$ of a massless Klein-Gordon particle into the field $\phi$ according to

$$\varphi = F(\phi) = \sum_{k\geq 0} a_k \phi^{k+1} \tag{9.1}$$

and thereby change the Lagrangian $\mathscr{L}(\varphi, \partial\varphi) = \frac{1}{2}(\partial\varphi)^2$ into

$$\mathscr{L}_F(\phi, \partial\phi) := \mathscr{L}(F(\phi), \partial F(\phi)) = \mathscr{L}(F(\phi), F'(\phi)\partial\phi) = \frac{1}{2}F'(\phi)^2(\partial\phi)^2. \tag{9.2}$$

This yields explicitly

$$\mathscr{L}_F(\phi, \partial\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\phi)^2 \sum_{n\geq 1} \frac{d_n}{n!}\phi^n, \tag{9.3}$$

where $d_n = n! \sum_{j=0}^n (j+1)(n-j+1)a_j a_{n-j}$ are the coefficients of what now seem to be interaction terms. A transformation of 'field variables' which does not involve time derivatives like our example is in physics called *point transformation*. These transformations do not change the equations of motion and hence leave the physics untouched. We shall quickly check this. The functional derivative of $\mathscr{L}$ with respect to $\varphi = F(\phi)$ reads

$$\frac{\delta\mathscr{L}}{\delta\varphi} = \frac{\delta\mathscr{L}}{\delta F(\phi)} = -\partial_\mu\partial^\mu F(\phi) = -F''(\phi)(\partial\phi)^2 - F'(\phi)\partial_\mu\partial^\mu\phi, \tag{9.4}$$

where

$$\frac{\delta\mathscr{L}}{\delta\varphi} := \frac{\partial\mathscr{L}}{\partial\varphi} - \partial_\mu \frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi)}. \tag{9.5}$$

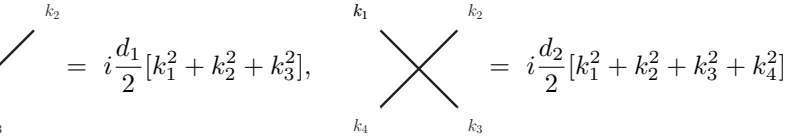
The functional derivative of $\mathscr{L}_F$ is

$$\frac{\delta\mathscr{L}_F}{\delta\phi} = F'(\phi)F''(\phi)(\partial\phi)^2 - \partial_\mu(F'(\phi)^2\partial^\mu\phi) = ... = -F'(\phi)[F''(\phi)(\partial\phi)^2 + F'(\phi)\partial_\mu\partial^\mu\phi] \tag{9.6}$$

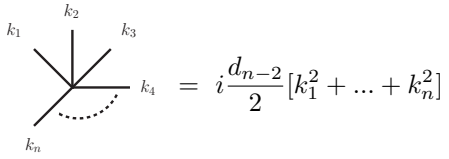
and thus we have

$$\frac{\delta\mathscr{L}_F}{\delta\phi} = F'(\phi)\frac{\delta\mathscr{L}}{\delta\varphi}. \tag{9.7}$$

Therefore, the equation of motion for the field $\phi$ is satisfied if that for the original field $\varphi$ is. Next, we read off the Feynman rules from the Lagrangian:

(9.8) 


and

(9.9) 

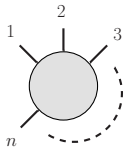

for a general vertex of valence $n$. With these Feynman rules, the power counting for a graph $\Gamma$ yields

$$\omega_4(\Gamma) = -2(|\Gamma| + 1), \tag{9.10}$$

which says that *all amplitudes are superficially divergent* and *the degree of divergence gets worse with increasing number of loops no matter how many external legs are involved.* This is similiar to the situation in quantum gravity [Krei07]. Let us now see what the Dyson-Schwinger equations look like when we restrict ourselves to 1-loop primitives. The 1-loop primitives for the two-leg amplitude of the theory are

(9.11) 


To dress these skeletons, we need 3 series: the propagator, the 3-point and the 4-point amplitude. For the sake of convenience, we introduce the notation

(9.12) $X_n(\alpha) :=$ 


for the 1PI $n$-point amplitude. The DSE for the propagator series $X_2(\alpha)$ is

$$X_2 = \mathbb{I} - \alpha B_+^{\circ}(X_4X_2^{-1}) - \alpha B_+^{\circ\!-\!\circ}(X_3^2X_2^{-2}). \tag{9.13}$$

This requires us to write down those DSEs of $X_3(\alpha)$ and $X_4(\alpha)$. That of $X_3(\alpha)$ is

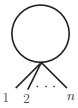
$$X_3 = \mathbb{I} + \alpha B_+^{\curlywedge}(X_3^3X_2^{-3}) + \alpha B_+^{\curlywedge}(X_4X_3X_2^{-2}) + \alpha B_+^{\curlywedge}(X_5X_2^{-1}), \tag{9.14}$$

where the operator $B_+^{\curlywedge}$ consists of 3 terms:

$$B_+^{\curlywedge} = \sum_{j=1}^{3} B_+^{\curlywedge_j}, \tag{9.15}$$

one for each possibility of a leg to be the singled out one. The last term tells us that we additionally need the 5-point amplitude. In general, if we aim to formulate the DSE for the $n$-point amplitude, it is the animal

(9.16)




which necessitates the $(n+2)$-point amplitude to be taken into account. This procedure goes on and on and there is no finite number of independent DSEs unless we find a symmetry like

$$X_{n+1}X_{n-1} = X_n^2 \tag{9.17}$$

such that the corresponding replacements lead to a decoupled system of DSEs. Once this system is solved, (9.17) will then deliver all other amplitudes. This is in fact what we expect for our seemingly interacting field theory with Lagrangian $\mathscr{L}_F$ in (9.3): we expect all $n$-point amplitudes to vanish for $n \geq 3$ as our field theory is a free theory disguised as an interacting theory. Before we come to this very important point, let us reformulate our DSEs in (9.13) and (9.14) using an a particularly useful definition of *invariant charges*. For every vertex amplitude $X_n$, $n \geq 3$, we define a charge

$$Q_n := X_n X_2^{-\frac{n}{2}}. \tag{9.18}$$

This is a series with everything needed to dress an $n$-valent vertex with all possible vertex corrections and its attached half-edges with half a full propagator. Dressing every vertex with its charge will then provide for all radiative corrections needed for vertices and internal edges. The only problem is that external propagators will also be dressed. To account for that, we have to multiply the charges by the corresponding half-propagators such that

$$X_2^{\frac{\ell(\gamma)}{2}} \prod_{v \in V(\gamma)} Q_{|v|} \tag{9.19}$$

is the series that contains everything needed to dress the skeleton graph $\gamma$ with $\ell(\gamma)$ external legs and vertex set $V(\gamma)$. $|v|$ denotes the valence of the vertex $v \in V(\gamma)$. The above DSEs then take the form

$$X_2 = \mathbb{I} - \alpha B_+^{\circ}(Q_4 X_2) - \alpha B_+^{\circ\!\!-\!\!\circ}(Q_3^2 X_2) \tag{9.20}$$

and

$$X_3 = \mathbb{I} + \alpha B_+^{\curlyvee}(Q_3^3 X_2^{3/2}) + \alpha B_+^{\curlyvee}(Q_4 Q_3 X_2^{3/2}) + \alpha B_+^{\curlyvee}(Q_5 X_2^{3/2}). \tag{9.21}$$

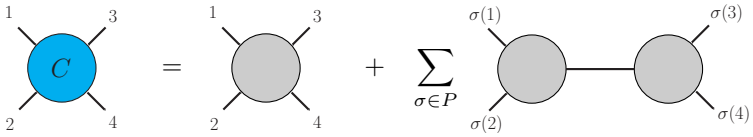
If $\mathcal{M}_n$ is the set of all 1PI 1-loop primitives with $n$ external legs, then the system of all amplitudes in this theory can now be written in the form

$$X_n = \mathbb{I} \pm \alpha \sum_{\gamma \in \mathcal{M}_n} B_+^{\gamma}(X_2^{\ell(\gamma)/2} \prod_{v \in V(\gamma)} Q_{|v|}), \tag{9.22}$$

where there is a plus sign for all series except for the propagator series with $n = 2$.

However, let us see what the tree-level amplitudes are and whether they vanish on-shell which they should for the very reason that our theory is a free theory in disguise. It is easy to see that the 3-point amplitude vanishes on-shell, just set $k_j^2 = 0$. More interesting is the 4-point amplitude. First note that the full 4-point amplitude consists of two parts:

(9.23)

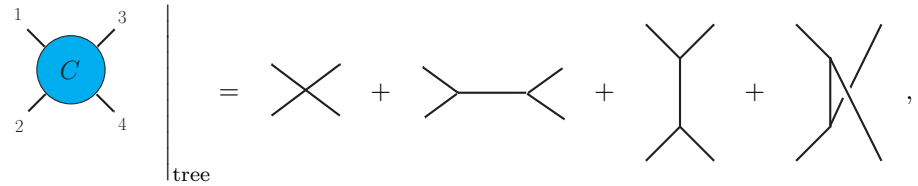



where

$$P = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \right\} \tag{9.24}$$

is the set of admissible external leg permutations. The blobs on the rhs are all 1PI and the blob labelled 'C' includes all connected parts, that is, also *one-particle reducible* diagrams. The tree-level contributions

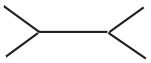
are

(9.25) 

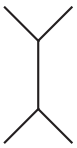

where we have now expressed the leg permutations in a graphical way, without labels. The first term is vanishes on-shell since

(9.26)  $= i\frac{d_{n-2}}{2}[p_1^2 + ... + p_4^2]\Big|_{\text{on-shell}} = 0$

when $p_j^2 = 0$ for $j = 1, 2, 3, 4$. The second term yields

(9.27) 
$$= i\frac{d_1}{2}[p_1^2 + p_2^2 + (p_1 + p_2)^2]\frac{i}{(p_1 + p_2)^2}i\frac{d_1}{2}[p_3^2 + p_4^2 + (p_3 + p_4)^2]\Big|_{\text{on-shell}}$$
$$= -i\frac{d_1^2}{2}p_1 \cdot p_2$$

which does not vanish on-shell. The third term and fourth terms are

(9.28) 
$$= i\frac{d_1}{2}[p_1^2 + p_3^2 + (p_1 + p_3)^2]\frac{i}{(p_1 + p_3)^2}i\frac{d_1}{2}[p_2^2 + p_4^2 + (p_2 + p_4)^2]\Big|_{\text{on-shell}}$$
$$= -i\frac{d_1^2}{2}p_2 \cdot p_4$$

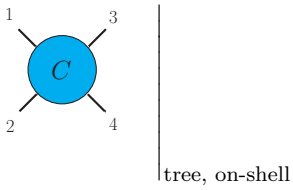
and

(9.29) 
$$= i\frac{d_1}{2}[p_1^2 + p_4^2 + (p_1 + p_4)^2]\frac{i}{(p_1 + p_3)^2}i\frac{d_1}{2}[p_2^2 + p_3^2 + (p_2 + p_3)^2]\Big|_{\text{on-shell}}$$
$$= -i\frac{d_1^2}{2}p_2 \cdot p_3.$$

Neither of them vanishes on-shell individually but their sum does:

$$-i\frac{d_1^2}{2}p_2 \cdot [p_1 + p_3 + p_4] = i\frac{d_1^2}{2}p_2 \cdot p_2 = 0, \tag{9.30}$$

and hence we have the result

(9.31) 


These two results can be used to prove (9.31) for all higher number of external legs.

**Proposition 9.1.** *All tree-level amplitudes of the above theory with more than 2 external legs vanish on-shell.*

*Proof.* See [VelKrei13] . □

Moreover, by arguments akin to those in the proof for the optical theorem (see [PesSchr]), one can prove this assertion for all 1-loop contributions. However, as already mentioned, by acknowledging that our theory is a disguised free theory, all vertex functions should vanish and hence all loop orders, even though it is not clear how to prove it.

Regarding loop diagrams, there is a subtle issue. Because *individual* diagrams evaluate by the Feynman rules to divergent integrals, one has to choose a renormalization scheme. It turns out that the minimal

subtraction scheme (MS) is a bad choice for which the assertion about the vanishing of amplitudes at all loop orders is not true. Considering the 'unphysicalness' of this scheme, this need not worry us: a (physically reasonable) *renormalized theory is not sufficiently characterized by its Lagrangian but rather by its renormalized Lagrangian*. Therefore, a choice about the renormalization scheme is to be made. A formal diffeomorphism transforming one theory into another must also make sure the renormalization scheme is 'co-transformed'. [VelKrei13] discuss a few requirements a reasonable transformation of a renormalized QFT needs to meet. Among them: 'cut-reconstructability'.

This property refers to the fact that one can reconstruct a non-analytic function with a branch cut in the complex plane from its behaviour along this cut by means of its dispersion relation. Renormalized values of loop diagrams in momentum scheme do for example fulfill this requirement with respect to their momentum dependence whereas those obtained through the MS scheme do not. This scheme notoriously yields constant terms which are trivially entire functions and hence void of branch cuts.

However, the reason why we have discussed this model is that it bears some resemblance to *quantum gravity*: the power counting for pure gravity diagrams (without matter fields) is similiar and entails non-renormalizability. Yet the free theory it stems from is renormalizable. It owes its apparent non-renormalizability to the 'awkward formulation' brought about by the field diffeomorphism (the point transformation). Who knows whether *quantum gravity is only seemingly non-renormalizable due to the awkward form of its currently known Lagrangian*?

## 10. The Core Hopf algebra

Given a graph $\Gamma$, we recall that its (superficial) degree of divergence is given by the general formula

$$\omega_D(\Gamma) = \sum_{e\in E(\Gamma)} \omega(e) + \sum_{v\in V(\Gamma)} \omega(v) - D|\Gamma|, \tag{10.1}$$

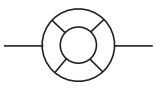
where $\omega : E(\Gamma) \cup V(\Gamma) \to \mathbb{Z}$ is the weight function. From this definition, it is clear that any graph is divergent if only the dimension $D$ is high enough. Because the coproduct cuts out a proper subgraph only if it is divergent, we find that there is a Hopf algebra of Feynman graphs $\mathcal{H}_D$ for every dimension. We denote its coproduct by $\Delta_D$. As we increase the dimension of spacetime, we have to include more and more graphs and get a *tower of Hopf algebras*

$$\mathcal{H}_0 \subset \mathcal{H}_2 \subset \mathcal{H}_4 \subset \mathcal{H}_6 \subset ... \tag{10.2}$$

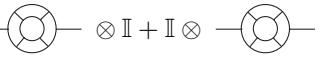
and finally the *core Hopf algebra* $\mathcal{H}_\infty$ defined by

$$\mathcal{H}_\infty := \bigcup_{D=0}^{\infty} \mathcal{H}_D. \tag{10.3}$$

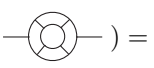
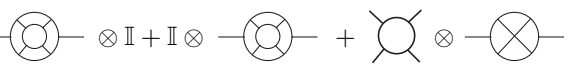
Containing graphs with any number of external legs, this Hopf algebra is the combinatorial Hopf algebra for theories which feature a power counting like that of quantum gravity. Each Hopf algebra in the tower has a clearly defined coproduct, but what, if existent, is the coproduct $\Delta_\infty$ of the core Hopf algebra? This is easy to understand: it operates exactly like all the others with the exception that it treats every proper subgraph with at least one loop as divergent. For example, the scalar graph

(10.4) 

has the proper subgraph  with degree of divergence $\omega_D($  $) = 8 - D$. Therefore, the coproduct yields

(10.5) $\Delta_D($  $) =$ 

for $D \leq 6$ but

(10.6) $\Delta_D($  $) =$ 

for $D \geq 8$. What is a primitive graph in $\mathcal{H}_0, ..., \mathcal{H}_6$ is not primitive in $\mathcal{H}_D$ for $D \geq 8$. This implies for the two extremes $\mathcal{H}_0$ and $\mathcal{H}_\infty$ that in the former all graphs are primitive whereas in the latter, a graph must have precisely one loop to be primitive.

**Grading.** The loop number defines a natural grading on $\mathcal{H}_\infty$. Another possible grading is what is known as the *multi-grading*: let $V_n(\Gamma)$ be the number of vertices of valence $n$ of a graph $\Gamma$, then we define

$$k_n(\Gamma) := V_n(\Gamma) - \delta_{n,\ell(\Gamma)}, \tag{10.7}$$

where the Kronecker delta vanishes unless $\Gamma$ has $n$ external legs, i.e. when $\ell(\Gamma) = n$. The multivertex-degree of $\Gamma$ is then given by the sequence

$$k(\Gamma) = (k_3(\Gamma), k_4(\Gamma), ...). \tag{10.8}$$

This does indeed provide for a grading: if $\gamma \otimes \Gamma/\gamma$ is a term produced by the 'core' coproduct, one finds

$$k(\Gamma) = (k_3(\Gamma), k_4(\Gamma), ...) = (k_3(\gamma) + k_3(\Gamma/\gamma), k_4(\gamma) + k_3(\Gamma/\gamma), ...) = k(\gamma) + k(\Gamma/\gamma), \tag{10.9}$$

which the reader is encouraged to check for the coproduct in (10.6). The Hopf algebra tower can be understood as a sequence of subalgebras, where $\mathcal{H}_D$ is a subalgebra of $\mathcal{H}_{D+n}$. As their coproducts differ, it is not a Hopf subalgebra of $\mathcal{H}_{D+n}$. However, it can be viewed as a quotient Hopf algebra as follows. Let $\mathcal{G}$ be the set of all Feynman graphs. In $\mathcal{H}_{D+n}$ we take the ideal

$$\langle\, \Gamma \in \mathcal{G} : \omega_D(\Gamma) > 0, \omega_{D+n}(\Gamma) \leq 0 \,\rangle_{\mathbb{Q}} \tag{10.10}$$

generated by all graphs found in $\mathcal{H}_{D+n}$ but not in $\mathcal{H}_D$. The conditions $\omega_D(\Gamma) > 0$ and $\omega_{D+n}(\Gamma) \leq 0$ imply that $\Gamma \notin \mathcal{H}_D$ and $\Gamma \in \mathcal{H}_{D+n}$, respectively. Then, for the quotient of $\mathcal{H}_{D+n}$ with respect to this ideal, we have

$$\mathcal{H}_D \simeq \mathcal{H}_{D+n} / \langle\, \Gamma \in \mathcal{G} : \omega_D(\Gamma) > 0, \omega_D(\Gamma) \leq 0 \,\rangle_{\mathbb{Q}}\,, \tag{10.11}$$

in the sense of Hopf algebras.

## Appendix A. Hopf algebra: a concise introduction

In the following, we assume all vector spaces to be vector spaces over the field $\mathbb{K}$, either $\mathbb{R}$ or $\mathbb{C}$. In some cases $\mathbb{Q}$ or even simpler fields are sufficient. A Hopf algebra is a set equipped with unusually many algebraic structures. The reader is asked for patience during the course of the following passages.

A.1. **Tensor space.** Let $A, B$ be vector spaces and $\{a_i : i \in I\}$ a basis in $A$, $\{b_j : j \in J\}$ a basis in $B$ with index sets $I, J \subset \mathbb{N}$, not necessarily finite. The *tensor space* $A \otimes B$ is the vector space over $\mathbb{Q}$ spanned by pairs of the form $e_{jk} = a_j \otimes b_k$ with the following properties:

$$\begin{aligned} \lambda a_j \otimes b_k &= a_j \otimes \lambda b_k \qquad \forall \lambda \in \mathbb{K}, \\ a_j \otimes b_k + a_i \otimes b_k &= (a_j + a_i) \otimes b_k, \qquad a_j \otimes b_i + a_j \otimes b_k = a_j \otimes (b_i + b_k). \end{aligned} \tag{A.1}$$

This implies $A \cong \mathbb{K} \otimes A \cong A \otimes \mathbb{K}$ since for example $\lambda \otimes a = 1 \otimes \lambda a$ for any $a \in A$ and any $\lambda \in \mathbb{K}$, i.e. the basis in $\mathbb{K}$ is simply given by 1. We will always write $\lambda \otimes a = \lambda a$ and identify such objects if they arise. For two linear maps $f : A \to A$ and $g : B \to B$ we can define a linear map $f \otimes g$ on $A \otimes B$ by setting

$$(f \otimes g)(a \otimes b) := f(a) \otimes g(b). \tag{A.2}$$

If $A = \mathbb{K}$, then $(f \otimes g)(\lambda \otimes b) = f(\lambda)g(b) = \lambda f(1)g(b) = f(1)g(\lambda b) = g(\lambda f(1)b)$ by linearity of $f$ and $g$.

A.2. **Algebra.** We define an *algebra* $A$ as a vector space with an associative product, distributive with respect to the addition and containing a neutral element $1_A$ called unit. We view the product as a linear map $m : A \otimes A \to A$ and write the product of two elements $x, y \in A$ as

$$m(x \otimes y) = xy, \tag{A.3}$$

i.e. as a simple juxtaposition. Then we have $1_A a = a 1_A = a$ for the unit. Associativity can then be expressed in the form $m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)$ because of

$$m(m(x \otimes y) \otimes z) = m(xy \otimes z) = (xy)z = x(yz) = m(x \otimes yz) = m(x \otimes m(y \otimes z)). \tag{A.4}$$

The *tensor algebra* of two algebras $A$ and $B$ is the tensor space $A \otimes B$ with associative product

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'. \tag{A.5}$$

The reader may check that the associativity is inherited. We often need the so-called unit map $u : \mathbb{K} \to A$ which simply takes a scalar $\lambda$ to $\lambda 1_A$. Then, more formally, an algebra is the triple $(A, m, u)$. Examples are polynomials $\mathbb{K}[X]$ in a variable $X$, continuous functions $\mathcal{C}^0(\mathbb{K})$ and matrices $\mathbb{K}^{n \times n}$. This should be familiar to the reader. The only perhaps new aspect is the linearity of the product $m : A \otimes A \to A$, which is easily illustrated by

$$m(a \otimes b + c \otimes d) = m(a \otimes b) + m(c \otimes d) = ab + cd \tag{A.6}$$

for $a, b, c, d \in A$.

A.3. **Coalgebra.** Given an algebra $A$, we may be interested in the dual vector space $A^*$ of linear functionals $A \to \mathbb{K}$, also known as covectors. Let $f \in A^*$. We write $f(a) = \langle f, a \rangle$ for its action on a vector $a \in A$. What is the map dual to the product $m$? If we denote it by $\Delta$, it has to satify

$$\langle f, m(a \otimes b) \rangle = \langle \Delta(f), a \otimes b \rangle, \tag{A.7}$$

and surely map $A^*$ to $A^* \otimes A^* \simeq (A \otimes A)^*$, where $\langle f \otimes g, a \otimes b \rangle := \langle f, a \rangle \langle g, b \rangle$. A quick calculation shows that associativity of the product requires

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta \tag{A.8}$$

to hold on $A^*$. The reader is recommended to prove this property which is known as *coassociativity*. This linear map $\Delta$ is called *coproduct*. The unit map $u : A \to \mathbb{K}$ does also have a dual which we denote by $\overline{e}$ and refer to as *counit*. Because of

$$\langle f, 1_A \rangle = \langle f, u(1) \rangle = \langle \overline{e}(f), 1 \rangle = \overline{e}(f) 1 = \overline{e}(f) \tag{A.9}$$

it must map $A^*$ to $\mathbb{K}$. Additionally, by

$$\langle f, a \rangle = \langle f, 1_A a \rangle = \langle f, u(1)a \rangle = \langle f, m(u(1) \otimes a) \rangle = \langle \Delta(f), u(1) \otimes a \rangle = \langle (\overline{e} \otimes \mathrm{id})\Delta(f), 1 \otimes a \rangle \tag{A.10}$$

and the same for $a = a 1_A = a u(1)$ it is required to fulfil

$$(\mathrm{id} \otimes \overline{e})\Delta = (\overline{e} \otimes \mathrm{id})\Delta = \mathrm{id} \tag{A.11}$$

In general, without having to be characterized as a dual, a vector space $C$ equipped with a coproduct $\Delta$ and counit $\overline{e}$ such that (A.11) is called *coalgebra*. An example is the vector space $\mathbb{K}[X]$ of polynomials:

$$\Delta(X^n) := \sum_{j=0}^{n} \binom{n}{j} X^j \otimes X^{n-j} \qquad (n \in \mathbb{N}) \tag{A.12}$$

which defines $\Delta$ uniquely. The counit is given by $\overline{e}(X^n) = 0$ for $n \geq 1$ and $\overline{e}(1) = 1$. It is a nice exercise to prove that these so defined linear maps really do establish a coalgebra structure on $\mathbb{K}[X]$ and also to find that the binomial coefficient in (A.12) can be dropped with no harm. Another example is the vector space $\mathbb{R}[\partial_x]$ of polynomials, where $\partial_x$ is the usual differential operator acting on smooth functions $\mathbb{R} \to \mathbb{R}$. The structures $\Delta$ and $\overline{e}$ are defined in the same way as for the variable $X$. The reader may try to prove the identity

$$\partial_x^n(f(x)g(x)) = \Delta(\partial_x^n)(f(x) \otimes g(x)). \tag{A.13}$$

for any smooth $f, g \in \mathcal{C}^\infty(\mathbb{R})$.

A.4. **Bialgebra.** We are now very close to a Hopf algebra. Consider again the space of polynomials $\mathbb{K}[X]$. We can clearly multiply polynomials and have a neutral element with respect to this operation. Therefore, $\mathbb{K}[X]$ is an algebra. We have seen that on the other hand, it can be equipped with a coalgebra structure. All structures together, i.e. the product $m$, unit $u$, coproduct $\Delta$ and counit $\overline{e}$ are the ingredients of what is known as a *bialgebra* $B$ if two conditions are fulfilled:

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad \overline{e}(ab) = \overline{e}(a)\overline{e}(b), \tag{A.14}$$

in words: the both coalgebra structures $\Delta$ and $\overline{e}$ must respect the algebra structures, i.e. they both are required to be multiplicative and linear. It is revealing to see that in the case $B = \mathbb{K}[X]$, the coproduct, as defined in (A.12), cannot be multiplicative if the binomial factor is omitted. In summary, a bialgebra is a quintuple $(B, m, u, \Delta, \overline{e})$ with the corresponding properties specified above.

A.5. **Hopf algebra.** Suppose $H$ is a bialgebra, i.e. given by the quadruple $(H, m, u, \Delta, \overline{e})$. With these structures, we can now establish an associative bilinear operation on the space $\mathcal{L}(H)$ of linear maps from $H$ to itself by setting

$$f * g := m(f \otimes g)\Delta, \tag{A.15}$$

which means $(f * g)(x) = m(f \otimes g)\Delta(x) = m(f \otimes g)(\sum_{(x)} x' \otimes x'') = \sum_{(x)} f(x')g(x'') \in H$ if we use a variant of Sweedler's notation for the coproduct given by $\Delta(x) = \sum_{(x)} x' \otimes x''$. This operation is called *convolution product*. Note that $f * g : H \to H$ is again linear and the composition $e = u \circ \overline{e}$ turns out to be the neutral element of the convolution:

$$(f * e)(x) = \sum_{(x)} f(x')e(x'') = \sum_{(x)} f(x')\overline{e}(x'')1_H = \sum_{(x)} f(x'\overline{e}(x'')) = f(\sum_{(x)} x'\overline{e}(x'')) = f(x) \tag{A.16}$$

where $(\mathrm{id} \otimes \overline{e})\Delta = \mathrm{id}$ is a property of the coalgebra. $(e * f)(x) = f(x)$ goes along the same lines. We now ask whether there is an inverse of a map $f \in \mathcal{L}(H)$ with respect to the convolution product. In particular, for $f = \mathrm{id}$. If it exists, we call it the *antipode*(or coinverse) $S$ and write its defining property as

$$S * \mathrm{id} = \mathrm{id} * S = e. \tag{A.17}$$

Now, there we are. A bialgebra $H$ that has the luxury of an antipode is called *Hopf algebra*. In the example $H = \mathbb{K}[X]$, the antipode is recursively given by

$$S(X^n) = -X^n - \sum_{j=1}^{n-1} \binom{n}{j} S(X^j)X^{n-j} = -X^n - \sum_{j=1}^{n-1} \binom{n}{j} X^j S(X^{n-j}) \tag{A.18}$$

for a monomial $X^n$ with $n \geq 1$ which follows from $(\mathrm{id} * S)(X^n) = (S * \mathrm{id})(X^n) = e(X^n) = 0$. Because of $\overline{e}(1) = 1$, the antipode preserves the unit $S(1) = 1$ by $\Delta(1) = 1 \otimes 1$ which follows from (A.12) and its defining property in (A.17).

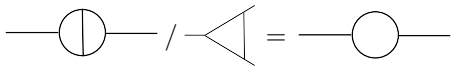
## Appendix B. Hopf algebra of Feynman graphs

One can endow the set of Feynman graphs with a Hopf algebra structure as follows. Consider the set of all connected and superficially divergent 1PI Feynman graphs in some theory(1PI means the graph stays connected upon removal of any chosen edge). Now we take the polynomial algebra in which these graphs are the variables, where the unit $\mathbb{I}$, i.e. the neutral element of the product, is given by the empty graph. Tree diagrams do not participate or, if you will, are considered as empty graphs, hence indentified with the unit $\mathbb{I}$. Let $\Gamma \neq \mathbb{I}$ be one such Feynman graph. The coproduct is defined as a linear and multiplicative map given by $\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$ and

$$\Delta(\Gamma) = \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I} + \sum_{\gamma} \gamma \otimes \Gamma/\gamma, \tag{B.1}$$

where the sum is over all subdivergences $\gamma \subsetneq \Gamma$ of the form $\gamma = \prod_j \gamma_j$ such that $\gamma_j$ is a proper divergent 1PI subgraph. The graph $\Gamma/\gamma$ is called *cograph*, obtained by shrinking the subgraph $\gamma$ to a single point in $\Gamma$. An example is

(B.2)



In some cases, depending on whether we have self-loop and tadpol graphs in our Hopf algebra, we use an altered definition of the coproduct. The counit $\overline{e}$ is given by $\overline{e}(\mathbb{I}) = 1$ and vanishing on all non-empty graphs. The antipode is then again given recursively by its defining property in (A.17).

## Appendix C. Ideals

**Ideals.** Let $A$ be an algebra over a field $\mathbb{K}$. A subspace $I \subset A$ is called *left ideal* if $AI \subset I$, and *right ideal* $IA \subset I$, i.e. if $ax \in I$ for a left and $xa \in I$ for a right ideal whenever $x \in I$ and $a \in A$. If both conditions are satisfied, then $I$ is called (two-sided) *ideal*. Note that, trivially, (left/right) ideals are subalgebras by definition and, of course, if the product is commutative, both right and left ideals coincide. Here is a simple example. Take the set of polynomials $A = \mathbb{K}[X]$ in one variable. Let $c \in \mathbb{K}$ be any number. The set of polynomials defined by

$$I_c := \{p \in \mathbb{K}[X] \mid p(c) = 0\} \tag{C.1}$$

clearly form a subspace and, surely, a subalgebra. Multiplication and linear combination are the only operations we allow for and we find they do not carry us out of $I_c$. It is also an ideal, since $q(c)p(c) = 0$ even if $q(c) \neq 0$ for $q \notin I_c$. We can in fact choose any polynomial $q \in A$ and generate an ideal

$$(q) := \{aq \mid a \in A\}, \tag{C.2}$$

known as *principle ideal*(German Hauptideal). This really is an ideal since any $r \in (q)$ is of the form $r = aq$ and we can multiply it with anything $w \in A$ and find $wr = waq \in (q)$ since $wa \in A$.

**Hopf ideals.** A less trivial question is whether an ideal $I \subset H$ is also a so-called *coideal* of a Hopf algebra $H$, i.e. if

$$\Delta(I) \subset I \otimes H + H \otimes I. \tag{C.3}$$

Furthermore, we may ask whether the antipode respects it: $S(I) \subset I$. If these two conditions are satisfied, $I$ is referred to as *Hopf ideal*. Let us see what the answer is in the case of the Hopf algebra $H = \mathbb{K}[X]$ of polynomials with $I$ being the ideal of polynomials vanishing at some fixed number $c \in \mathbb{K}$. Then clearly $p(X) = X - c \in I_c$. The coproduct gives

$$\Delta(p(X)) = X \otimes 1 + 1 \otimes X - c1 \otimes 1 = (X - c) \otimes 1 + 1 \otimes X. \tag{C.4}$$

Only for $c = 0$ is this an element in $H \otimes I + I \otimes H$. We choose $c = 0$. The coproduct of a monomial $X^n \in I_0$ for $n \neq 0$ is

$$\Delta(X^n) = \sum_{j=0}^{n} \binom{n}{k} X^j \otimes X^{n-j} = 1 \otimes X + X \otimes 1 + \sum_{j=1}^{n-1} \binom{n}{k} X^j \otimes X^{n-j} \tag{C.5}$$

where $1 \otimes X + X \otimes 1 \in H \otimes I + I \otimes H$ and the reduced part is actually in $I \otimes I \subset I \otimes H + H \otimes I$. Since this holds for all monomials, we have for any polynomial $p \in I_0$

$$\Delta(p(X)) \subset H \otimes I + I \otimes H \tag{C.6}$$

since $p(X)$ must be a linear combination of monomials $X^n$, $n \neq 0$. Therefore $I_0 \subset H = \mathbb{K}[X]$ is indeed a coideal. One can show that $S(I_0) \subset I_0$ by the antipode's multiplicativity:

$$S(X^n) = S(X)^n = (-X)^n = (-1)^n X^n \in I_0, \tag{C.7}$$

where $S(X) = -X$ follows from $0 = (S * \mathrm{id})(X) = m(S \otimes \mathrm{id})\Delta(X) = S(X)1 + S(1)X$ and $S(1) = 1$. We conclude: the ideal $I_c$ is a Hopf ideal iff $c = 0$ (the double ff being no typo).

## Appendix D. The Hopf Algebra of Words

Let $\mathfrak{A}$ be a countable set called *alphabet* with elements $a \in \mathfrak{A}$ named *letters*. We let this set be the generator set of the non-commutative algebra $W := \mathbb{Q}\langle\mathfrak{A}\rangle$. A product of letters is called *word*. By $|w|$ we denote the number of letters in $w$. This endows $W$ with a vector space grading $W = \bigoplus_{n\geq 0} W_n$. Let $e \in W$ denote the empty word, i.e. $xe = ex = x$ for any $x \in W$. We introduce a bilinear operation on $W$ by

$$w \sqcup\!\sqcup e = e \sqcup\!\sqcup w = w, \qquad aw \sqcup\!\sqcup bv = a(w \sqcup\!\sqcup bv) + b(aw \sqcup\!\sqcup v) \tag{D.1}$$

for any words $w, v \in W$ and and call it the *shuffle product*.

**Proposition D.1.** *The pair $(W, \sqcup\!\sqcup)$ is a commutative algebra, i.e. the shuffle product is both associative and commutative.*

*Proof.* We proceed by induction. First commutativity. Let $a, b \in \mathfrak{A}$ be two letters, then

$$a \sqcup\!\sqcup b = a(e \sqcup\!\sqcup b) + b(a \sqcup\!\sqcup e) = ab + ba. \tag{D.2}$$

Assume we have proved it for a shuffle product giving an $n$-letter word. Let $w, v \in W$ be words such that $|w| + |v| = n - 1$. Then,

$$aw \sqcup\!\sqcup bv = a(w \sqcup\!\sqcup bv) + b(aw \sqcup\!\sqcup v) = a(bv \sqcup\!\sqcup w) + b(v \sqcup\!\sqcup aw) = bv \sqcup\!\sqcup aw. \tag{D.3}$$

Now associativity. It is trivial for shuffle products resulting in words in $W_n$, $n \leq 4$. Let now $a, b, c \in \mathfrak{A}$ be letters and $w, v, t \in W$ be words such that $|w| + |v| + |t| = n - 2$. Then,

$$\begin{aligned} aw \sqcup\!\sqcup (bv \sqcup\!\sqcup ct) &= aw \sqcup\!\sqcup [b(v \sqcup\!\sqcup ct) + c(bv \sqcup\!\sqcup t)] \\ &= a\{w \sqcup\!\sqcup b(v \sqcup\!\sqcup ct) + c(bv \sqcup\!\sqcup t)\} + b\{aw \sqcup\!\sqcup (v \sqcup\!\sqcup ct)\} + c\{aw \sqcup\!\sqcup (bv \sqcup\!\sqcup t)\} \\ &= a\{w \sqcup\!\sqcup (bv \sqcup\!\sqcup ct)\} + b\{aw \sqcup\!\sqcup (v \sqcup\!\sqcup ct)\} + c\{aw \sqcup\!\sqcup (bv \sqcup\!\sqcup t)\} \end{aligned} \tag{D.4}$$

and

$$\begin{aligned} (aw \sqcup\!\sqcup bv) \sqcup\!\sqcup ct &= [a(w \sqcup\!\sqcup bv) + b(aw \sqcup\!\sqcup v)] \sqcup\!\sqcup ct \\ &= a\{(w \sqcup\!\sqcup bv) \sqcup\!\sqcup ct\} + b\{(aw \sqcup\!\sqcup v) \sqcup\!\sqcup ct\} + c\{[a(w \sqcup\!\sqcup bv) + b(aw \sqcup\!\sqcup v)] \sqcup\!\sqcup t\} \\ &= a\{(w \sqcup\!\sqcup bv) \sqcup\!\sqcup ct\} + b\{(aw \sqcup\!\sqcup v) \sqcup\!\sqcup ct\} + c\{(aw \sqcup\!\sqcup bv) \sqcup\!\sqcup t\} \end{aligned} \tag{D.5}$$

are equal by associativity of the shuffles in the curly brackets $\{...\}$. □

Next, we introduce the coproduct. It is given by

$$\Delta(a_{j_1}...a_{j_n}) = a_{j_1}...a_{j_n} \otimes e + e \otimes a_{j_1}...a_{j_n} + \sum_{l=1}^{n-1} a_{j_1}...a_{j_l} \otimes a_{j_{l+1}}...a_{j_n}. \tag{D.6}$$

We write $\Delta(x) = x' \otimes x''$ as a convenient shorthand notation. The counit $\bar{e}$ is given by $\bar{e}(e) = 1$ and $\bar{e}(a) = 0$ for any $x \in W \setminus \{e\}$. Coassociativity can be most easily proved inductively. The induction start for $e$ and a single letter word are trivial. Assume it holds for $n$-letter words. We take a word $w \in W_n$ and a letter $a \in \mathfrak{A}$ and compute

$$(\Delta \otimes \mathrm{id})\Delta(aw) = \Delta(e) \otimes aw + \Delta(aw') \otimes w'' = e \otimes e \otimes aw + e \otimes aw' \otimes w'' + a(w')' \otimes (w')'' \otimes w'' \tag{D.7}$$

as well as

$$(\mathrm{id} \otimes \Delta)\Delta(aw) = e \otimes \Delta(aw) + aw' \otimes \Delta(w'') = e \otimes e \otimes aw + e \otimes aw' \otimes w'' + aw' \otimes (w'')' \otimes (w'')''. \tag{D.8}$$

On the grounds that the coproduct is coassociative for $w$, the last two terms are equal and hence $\Delta$ is coassociative on $W$. Due to the next assertion, $W$ is a bialgebra. Since the antipode is uniquely determined recursively, it is moreover a Hopf algebra, i.e. the *Hopf algebra of words*.

**Proposition D.2.** *The coproduct respects the shuffle product, i.e. $\Delta(x \sqcup\!\sqcup y) = \Delta(x) \sqcup\!\sqcup \Delta(y)$ for $x, y \in W$.*

*Proof.* By induction. First let $a, b \in X$. Then,

$$\begin{aligned}\Delta(a \sqcup b) &= \Delta(ab + ba) = ab \otimes e + e \otimes ab + a \otimes b + ba \otimes e + e \otimes ba + b \otimes a \\ &= (a \sqcup b) \otimes e + e \otimes (a \sqcup b) + a \otimes b + b \otimes a \\ &= (a \sqcup b) \otimes e + e \otimes (a \sqcup b) + (a \otimes e) \sqcup (e \otimes b) + (b \otimes e) \sqcup (e \otimes a) \\ &= (a \otimes e) \sqcup (b \otimes e) + (e \otimes a) \sqcup (e \otimes b) + (a \otimes e) \sqcup (e \otimes b) + (b \otimes e) \sqcup (e \otimes a) \\ &= (a \otimes e) \sqcup [b \otimes e + e \otimes b] + (e \otimes a) \sqcup [e \otimes b + b \otimes e] \\ &= [a \otimes e + e \otimes a] \sqcup [b \otimes e + e \otimes b] = \Delta(a) \sqcup \Delta(b).\end{aligned} \tag{D.9}$$

Now let it be true for a shuffle of words in $W_n$. Let now $w, v \in W$ be words such that $|w| + |v| = n - 1$. First compute

$$\begin{aligned}\Delta(aw \sqcup bv) &= \Delta(a(w \sqcup bv)) + \Delta(b(aw \sqcup v)) \\ &= e \otimes (aw \sqcup bv) + a(w \sqcup bv)' \otimes (w \sqcup bv)'' + b(aw \sqcup v)' \otimes (aw \sqcup v)'' \\ &= e \otimes (aw \sqcup bv) + a(w' \sqcup (bv)') \otimes (w'' \sqcup (bv)'') + b((aw)' \sqcup v') \otimes ((aw)'' \sqcup v'')\end{aligned} \tag{D.10}$$

The last two terms are

$$\begin{aligned}a(w' \sqcup (bv)') \otimes (w'' \sqcup (bv)'') &= a(w' \sqcup e) \otimes (w'' \sqcup bv) + a(w' \sqcup bv') \otimes (w'' \sqcup v'') \\ &= aw' \otimes (w'' \sqcup bv) + a(w' \sqcup bv') \otimes (w'' \sqcup v'')\end{aligned} \tag{D.11}$$

and

$$\begin{aligned}b((aw)' \sqcup v') \otimes ((aw)'' \sqcup v'') &= b(e \sqcup v') \otimes (aw \sqcup v'') + b(aw' \sqcup v') \otimes (w'' \sqcup v'') \\ &= bv' \otimes (aw \sqcup v'') + b(aw' \sqcup v') \otimes (w'' \sqcup v'').\end{aligned} \tag{D.12}$$

Their sum is

$$(aw' \otimes w'') \sqcup (e \otimes bv) + (e \otimes aw) \sqcup (bv' \otimes v'') + (aw' \sqcup bv') \otimes (w'' \sqcup v''), \tag{D.13}$$

and, together with '$e \otimes (aw \sqcup bv)$', they equal

$$\begin{aligned}\Delta(aw) \sqcup \Delta(bv) = (e \otimes aw + aw' \otimes w'') \sqcup (e \otimes bv + bv' \otimes v'') &= e \otimes (aw \sqcup bv) \\ + (e \otimes aw) \sqcup (bv' \otimes v'') + (aw' \otimes w'') \sqcup (e \otimes bv) &+ (aw' \otimes w'') \sqcup (bv' \otimes v'').\end{aligned} \tag{D.14}$$

□

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