1 Towards the Path Integral

Consider the Green’s function of a zero dimensional field theory.

\[ G_N = \frac{\int_{\mathbb{R}} x^N e^{-\frac{1}{2}ax^2+P(x)}dx}{\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2+P(x)}dx} \]

\( P(x) \) is a polynomial describing the interaction. Cut through the polynomials with a generating function:

\[ Z[J] = \sum_{N=0}^{\infty} \frac{J^N}{N!} \int_{\mathbb{R}} x^N e^{\frac{1}{2}ax^2+P(x)}dx = Z \sum_N J^N G_N \]

We can label \( G_N \) with 1PI graphs:

\[ G_N = \sum_{\Gamma} \frac{\#V(\Gamma) J^{E_I(\Gamma)}}{\#\text{Aut}(\Gamma) \#^{E_I(\Gamma)}} \]

where \( E_E(\Gamma) \) denotes the external edges of graph \( \Gamma \), \( E_I \) the internal edges, \( V \) the vertices, and \( \#(\cdot) \) generally denotes the number of (argument).

The whole idea of the path integral is to mimic this structure, but to appoint \( x \) to a field, as a function of spacetime points. The measure is transformed:

\[ \int_{\mathbb{R}} dx \rightarrow \int D\phi \quad \phi \in C^\infty (\mathbb{R}^D \rightarrow \mathbb{R}) \]

The Lagrangian density of a field theory, in general, is

\[ \mathcal{L}(\phi) = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - P(\phi) \]

where \( P(\phi) \) denotes the interacting polynomial, as above. The action is defined as the spacetime integral over the Lagrangian density,

\[ S_{\mathcal{L}}(\phi) = \int_{\mathbb{R}^D} \mathcal{L}(\phi) d^Dx \]

Quantum physics comes from a probability amplitude, which is somewhat associated with the action, which is a function of the fields. The amplitude is given by \( \exp \left( \frac{i}{\hbar} S_{\mathcal{L}}(\phi) \right) \). To compute a vacuum expectation value of an observable \( O \), we write

\[ \langle O(\phi) \rangle = \frac{\int O(\phi) e^{iS_{\mathcal{L}}(\phi)} D\phi}{\int e^{iS_{\mathcal{L}}(\phi)} D\phi} \]
The vacuum expectation value is normalized to one, by definition. Up until this point, we have not actually defined what the new measure $\mathcal{D}$ actually is.

The Green’s function depends on the Lagrangian:

$$G_N^L(x_1,\ldots,x_N) = \frac{\int \phi(x_1)\ldots\phi(x_N)e^{iS_L(\phi)}\mathcal{D}\phi}{\int e^{iS_L(\phi)}\mathcal{D}\phi}$$

This can be considered a formal definition of the Green’s function.

In principal, the (original) generating function is a Taylor expansion.

$$\int e^{iS_L(\phi)+\langle J,\phi \rangle}\mathcal{D}\phi$$

with the pairing

$$\langle J,\phi \rangle = \int_J \phi(x) dx$$

If we Wick rotate, the action acquires a relative sign change before the second and third term in the integrand,

$$S(\phi) = \int \left(\frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + P(\phi)\right) d^Dx$$

We can derive the Feynman rules from this expression. As a matter of fact, we could already have derived them from $G_N$: For each line, we get a propagator, which is the inverse of the coefficient of the quadratic term, and we get a $\lambda$ for each vertex).

Continue with the generating function:

$$Z[J] = \int \exp \left\{ -\int_{\mathbb{R}^D} \left(\frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + J(x)\phi(x)\right) d^Dx \right\} \mathcal{D}\phi$$

$$\frac{Z[J]}{Z[0]} = \sum_{N=0}^{\infty} J(x_1)\ldots J(x_N) \frac{G_N(x_1,\ldots,x_N)}{d^Dx_1\ldots d^Dx_N}$$

We derive the propagator from the Green’s function via Fourier transform:

$$(\partial \phi)^2 + m^2 \phi^2 \rightarrow (p^2 - m^2) \phi^2$$

### 1.1 An Abelian Gauge Theory: $SU(2)$

We know that the interaction Lagrangian in $SU(2)$ is given by

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu a}$$

Here, $\mu$ and $\nu$ are the Lorentz indices, which are contracted when they appear up and down. $a$, on the other hand, is a group index, it is contracted whenever it appears twice. Write the field tensor as

$$F^{a}_{\mu\nu} = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} + g e^{abc} A^b_{\mu} A^c_{\nu}$$
As far as the kinetic part is concerned, nothing has changed.

\[ \frac{1}{2} A^\mu_a (g^{\mu\nu} \Box - \partial^\mu \partial^\nu) A^\nu_a \]

but the operator \((g^{\mu\nu} \Box - \partial^\mu \partial^\nu)\) is actually not a good one to invert, because it is really a projector. For the transversal metric, we know

\[ g^\mu_\nu g^\nu_\rho = g^\mu_\rho \]

So we can write for the generating function:

\[ Z[J] = \int D A^1_\mu D A^2_\mu D A^3_\mu \exp \left\{ i \int d^Dx (L + J^\mu_a A^\mu_a) \right\} \]

The Lagrangian is gauge invariant under local phase transformation. But the measure might not be? We find a symmetry:

![SU(2) fibre](image)

Figure 1: The \(SU(2)\) fibre, with group action etc., over a manifold \(M\), with a connection field between the \(SU(2)\) fibre and a fibre in the neighborhood.

Let us try and isolate the volume of the gauge group.

\[ W = \int d x d y e^{i S(x,y)} = \int d \vec{r} e^{i S(\vec{r})} , \quad \vec{r} = (r, \theta) \]

\[ S(\vec{r}) \equiv S(\vec{r}_\varphi) , \quad \vec{r}_\varphi = (r, \theta + \varphi) \]

\[ \Rightarrow \quad W = 2\pi \int d r r e^{i S(r)} \]

\(2\pi\) is the volume of the gauge group. Introduce: \(1 = \int d \varphi \delta(\theta - \varphi)\). Rewrite \(W\):

\[ \Rightarrow \quad W = \int d \varphi \int d \vec{r} e^{i S(\vec{r})} \delta(\theta - \varphi) = \int d \varphi W_\varphi \]

\[ W_\varphi = W_{\varphi^\prime} \Rightarrow \quad W = 2\pi W_\varphi \]
Write down the identity:

\[
1 = \Delta g(\vec{r}) \int d\phi \delta(g(\vec{r}_\phi))
\]

\[
\Delta g(\vec{r}) = \left. \frac{\partial g(\vec{r})}{\partial \theta} \right|_{g=0}
\]

\[
\int d\phi \delta(g(\vec{r}_\phi + \phi')) = \int d\phi'' \delta(g(\vec{r}_{\phi''}))
\]

Obviously, the function \(g(\vec{r}_\phi)\) is invariant under rotation.

Let \(A_\mu^a\) be connection fields. They transform under gauge transformation:

\[
A_\mu^a \rightarrow A_\mu^{a\theta}
\]

\[
A_\mu^{a\theta} = U(\theta) \left[ A_\mu^a + \frac{1}{2} U^{-1}(\theta) \partial_\mu U(\theta) \right] U^{-1}(\theta)
\]

\[
U(\theta) = \exp \left\{ -i \left( \theta^a \tau^a \right) / 2 \right\}
\]

Here, \(\tau^a\) are the generators of the Lie group. For the connection fields, we choose a condition

\[
f_a(A_\mu^1, A_\mu^2, A_\mu^3) = 0
\]

to act as constraints. As a consequence, \(g\) can be any line form, not just straight lines, as long as there are no double values. Every radius can only be hit once.

Assume that we have a constraint \(f_a\) and that we can invert it. In the Lie algebra, Taylor expand the transformation to get a transformation law for infinitesimal angles \(\theta\):

\[
U(\theta) = 1 + i \theta^a \tau^a / 2 + O(\theta^2)
\]

with the measure

\[
[d\theta] = \prod_{a=1}^3 d\theta_a = D\theta
\]

\[
d\theta = d\theta'
\]
From \(1 = \Delta_g(r) \int d\varphi \delta(g(r_0^\varphi))\), it follows that

\[
1 = \Delta_f(A^a_\mu) \int \delta(f_a(A^1_\mu, A^2_\mu, A^3_\mu)) D\theta
\]

\[
\Delta_f(A^a_\mu) = \det M_f
\]

\[
(M_f)_{ab} = \frac{\delta f_a}{\delta \theta_b}
\]

\[
A^a_\mu = A^a_\mu + e^{abc} \theta^b A^c_\mu - \frac{1}{g} \partial_\mu \theta^a
\]

\[
\Rightarrow f_a(A^a_\mu(x)) = f_a(A_\mu(x)) + \int d^4y (M_f(x,y))_{ab} \theta_b(y) + \ldots
\]

\[
\Rightarrow Z_f[J] = \int D^3 \prod_{a=1}^3 A^a_\mu (\det M_f) \delta(f_a(A^a_\mu)) e^{i \int d^4x (L(x) + f_a^a \theta^a)}
\]

In the second line, we have introduced the Faddeev-Popov determinant. In the next step, we will exponentiate the determinant and the \(\delta\) function in order to write everything in just one exponential function, so to speak as one single Lagrangian density.

### 1.1.1 Exponentiating the Determinant \(\det M_f\)

At first, we rewrite the determinant in the following way:

\[
\det M_f = \exp \left\{ \text{Tr} \left( \ln (M_f) \right) \right\}
\]

And we expand \(M_f = 1 + L\). Then we can express the logarithm as a series in terms of \(L\):

\[
\exp \left\{ \text{Tr} \left( \ln (M_f) \right) \right\} \rightarrow \exp \left\{ \text{Tr} L + \frac{1}{2} \text{Tr} L^2 + \ldots + \frac{1}{n} \text{Tr} L^n \right\}
\]

\[
\rightarrow \exp \left\{ \int d^4x L_{aa}(x,x) + \frac{1}{2} \int d^4x d^4y L_{ab}(x,y) L_{ba}(y,x) + \ldots \right\}
\]

\[
\det M_f \sim \prod_a Dc^a \prod_b Dc_b^+ \exp \left\{ i \int d^4x d^4y \sum_{a,b} c^+_a M_f(x,y) c_b(y) \right\}
\]

The \(c\) and \(c^+\) are called Grassmann variables. They only show up in loops, not externally, because they were introduced for the sole reason to exponentiate \(\det M_f\).

### 1.1.2 Exponentiating the \(\delta\) Function \(\delta(f_a(A^a_\mu))\)

\(\delta(f_a(A^a_\mu))\) corresponds to the constraint \(f_a(A^a_\mu) = 0\). For the derivation, assume that \(f_a \neq 0\), but some arbitrary field \(\vec{B}\).

\[
f_a(A^a_\mu) = B_a(x)
\]

\[
\int D\theta \Delta_f(A^a_\mu) \delta(f_a(A^a_\mu) - B_a(x)) = 1
\]

\[
Z[J] = \int D\vec{A}_\mu D\vec{B} (\det M_f) \delta(f_a - B_a) \exp \left\{ i \int d^4x \left( L + J^\mu A^\mu - \frac{1}{2\xi} \vec{B} \cdot \vec{B} \right) \right\}
\]

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Formally, this is a Gaussian, but with the constraint $f_a = B_a$.
With these two functions exponentiated, the Lagrangian density acquires two more terms: one for gauge fixing, and one for the Faddeev Popov ghost particle.

\[
\mathcal{L} \rightarrow \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{FPGh}
\]

\[
\mathcal{L}_{gf} = -\frac{1}{2\xi} \int d^4x \ f_a (\vec{A}_\mu)^2
\]

\[
\mathcal{L}_{FPGh} = \int d^4x d^4y \sum_{a,b} \epsilon^x_a(x) M_f(x,y)_{ab} c_b(y)
\]

By choosing a gauge, we fix how the determinant has to be computed.

2 Remark on Feynman Rules

From foregoing discussions, we know that there must certainly be Feynman rules for the three-gluon vertex, $a_{1}\mu_1, a_{2}\mu_2, a_{3}\mu_3$, which should involve terms in $\mathcal{L}$ (the kinetic Lagrangian without subscript):

- $1,2,3$: indeces of $SU(2)$
- $p_i$: external momenta
- $\mu_i$: Lorentz indices

The three-gluon vertex must be completely symmetric under interchange of indices. Since the cubic interaction has one derivative, its Fourier transformation must be linear in the momenta $p_i$. Furthermore, it should transform covariantly. It must also carry the three group indices. The kinetic term would be

\[
-\frac{1}{4} F_{\mu \nu}^a F^{\mu \nu a} \quad \text{with} \quad F_{\mu \nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc} A_\mu^b A_\nu^c
\]

where the $SU(2)$ structure constant $f_{abc}$, is actually $\epsilon_{abc}$, the total antisymmetric tensor. In general, the $f_{a_1 a_2 a_3}$ is the group structure constant.

Moreover, the vertex should depend on

\[
(p_1 - p_2)_{\mu_3} g_{\mu_1 \mu_2} \quad \text{and cyclic.}
\]

To sum up,

\[
f_{a_1 a_2 a_3} \left\{ (p_1 - p_2)_{\mu_3} g_{\mu_1 \mu_2} + (p_2 - p_3)_{\mu_1} g_{\mu_2 \mu_3} + (p_3 - p_1)_{\mu_2} g_{\mu_3 \mu_1} \right\}
\]

is basically the only candidate one could write down for the three-gluon vertex.

We have written down the Feynman rule for the three-gluon vertex just by pure thought. A similar argument is possible for the four-gluon vertex, but then, one would eventually run into the problem of radiative corrections.