Feynman Rules of Non-Abelian Gauge Theory

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0.1 The Lorenz gauge

In the Lorenz gauge, the constraint on the connection fields is

$$f_a(\vec{A}_\mu) = 0 = \partial^\mu A^a_\mu$$

For every group index a, there is one such equation, so there are three constraints overall. For a gauge variantion θ, the transformation is

$$U(\vec{\theta}) = 1 + i\vec{\theta}(x) \cdot \frac{\vec{\tau}}{2} + O(\theta^2)$$

The connection field components are given by

$$A^{a\theta}_\mu(x) = A^a_\mu(x) + \epsilon^{abc}\theta^b A^c_\mu - \frac{1}{g}\partial_\mu\theta^a(x)$$

The transformed gauge constraints are therefore

$$f_a(\vec{A}^\theta_\mu) = f_a(\vec{A}_\mu) + \partial^\mu \left\{\epsilon^{abc}\theta^b A^c_\mu - \frac{1}{g}\partial_\mu\theta^a(x)\right\}$$

which we will write as

$$f_a(\vec{A}^t_\mu heta) = f_a(\vec{A}_\mu) + \int \mathrm{d}^4y \left(M_f(x,y)_{ab}\theta^b(y)\right)\delta^{(4)}(x-y)$$

where a, b are the group indices.
In the Lorenz gauge, the gauge fixing and Faddeev-Popov-ghost terms in the action are

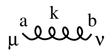
$$\mathcal{S}_{gf} = -\frac{1}{2\xi}\int \mathrm{d}^4x \left(\partial^\mu \vec{A}_\mu(x)\right)^2$$
$$\mathcal{S}_{FPGh} = \frac{1}{g}\int \mathrm{d}^4x \sum_{a,b} c^\dagger_a(x)\partial^\mu\left\{\delta_{ab}\partial_\mu - g\epsilon^{abc}A^c_\mu\right\}c_b(x)$$

The term in the Fadeev-Popov-ghost line which is in parentheses basically be viewed as a covariant derivative. In the Lagrangian, this produces , among others, the monomial $c^\dagger A_\mu c$, a ghost-ghost-connection field vertex. Similar as in QED, there is, once again, a spin one gauge boson.

1 Feynman Rules for QCD

The indices $a, b, \ldots$ denote the group indices. Let us call them ”color indices”. Moreover, call the gauge boson ”gluon”.

- The gluon propagator depends on Lorentz indices as well as color indices. For the massless gluon, the Feynman rule for the propagator is



$$\rightarrow \quad i(-\delta_{ab})\left\{g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right\}\frac{1}{k^2 + i\epsilon}$$

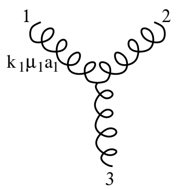
This is the term which comes from the quadratic terms in the A fields.

- The quadratic terms in the c fields gives the ghost propagator. The ghost is a very strange particle, because in the Lagrangian, it looks like a fermion, but its propagator is similar to that for bosons. Moreover, there are no external ghosts. It only exists for virtual lines.

$$\rightarrow \quad i(-\delta_{ab})\frac{1}{k^2 + i\epsilon}$$

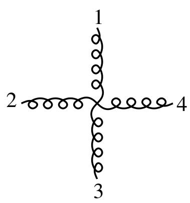
Even though there is no mass term in the propagator, one cannot for sure call the ghost massless, because it is neither boson nor fermion.

- We have already stated that the three-gluon vertex must depend on the momenta k, the Lorentz indices μ and the color indices a.



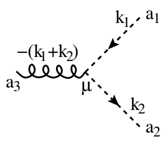
$$\rightarrow \quad ig\epsilon^{a_1a_2a_3}\left\{g_{\mu_1\mu_2}(k_1 - k_2)_{\mu_3} + g_{\mu_2\mu_3}(k_2 - k_3)_{\mu_1} + g_{\mu_3\mu_1}(k_3 - k_1)_{\mu_2}\right\}$$

- Next comes the Feynman rule for the four-gluon vertex. In the Lagrangian term $F_{\mu\nu}F^{\mu\nu}$, there appears a term with four connection fields, because of the non-vanishing commutator $\left[A_\mu, A_\nu\right]$ in a non-abelian gauge theory.



$$\begin{aligned} \rightarrow \quad g^2\epsilon^{a_1a_2c}\epsilon^{a_3a_4c} & \left\{g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3}\right\} \\ +\epsilon^{a_1a_3c}\epsilon^{a_2a_4c} & \left\{g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3}\right\} \\ +\epsilon^{a_1a_4c}\epsilon^{a_3a_2c} & \left\{g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_2}g_{\mu_3\mu_4}\right\} \end{aligned}$$

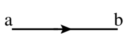
- The Feynman rule for the gluon-ghost vertex is a bit strange because it is not symmetric in the ghost momenta.



$$\rightarrow \quad ig\epsilon^{a_1a_2a_3}k_{1\mu}$$

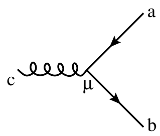
It is important to notice that only the monentum of the *incoming* ghost particle, that is the line directed towards the vertex, contributes, with the Lorentz index of the vertex. However, since the ghost only apears internally, the sum over all graphs still works out, even though this Feynman rule look counterintuitive.

- The Feynman rule for the fermion of the theory looks as usual, besides the fact that there is a contraction of color indices.



$$\rightarrow \quad i\delta_{ab}\frac{1}{\not{k} - m + i\epsilon}$$

- Finally, the Feynman rule for the gluon-fermion-vertex is



$$\rightarrow \quad ig(T^c)_{ab}\gamma^\mu$$

  where T^c can be any representation we choose, it just needs to be specified with which representation one is working.

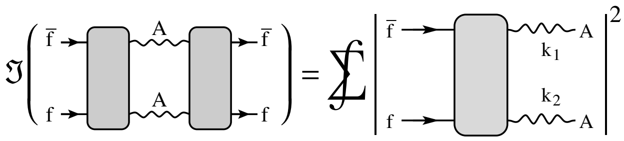
Of course, we have already thought about the ghost particle, what it is good for, why it is actually there at all, and whether or not the whole ghost formalism makes sense at all. To gain some more information, let us look into analyticity of the S-matrix and Cutkosky rules again.

1.1 Cutkosky in QCD

The S matrix is unitary: $SS^\dagger = \mathbb{I}$. If we write the S matrix as $S = \mathbb{I} + iT$, we get (in a rather symbolic notation)

$$\Im(T_{if}) = \sum_n T_{in} T_{nf}{}^\dagger \delta^{(4)}(p_{\text{in}} - p_{\text{out}})$$

Now, let us assume that a fermion antifermion pair interacts somehow, where a virtual connection field pair is produces, which again decays in some way into a fermion antifermion pair. According t Cutkosky,



1.1.1 Imaginary parts of propagators

Analyze the imaginary part of a gauge propagator with Lorentz indices $\mu\nu$ and color indices ab:

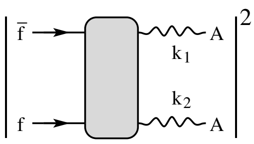
$$\begin{aligned}
\Delta^{ab}_{\mu\nu} &= \delta^{ab}(-g_{\mu\nu})\frac{1}{k^2 + i\epsilon} \\
&= \delta^{ab}(-g_{\mu\nu})\left(\mathcal{P}\left(\frac{1}{k^2}\right) + i\pi\delta(k^2)\right) \\
\Rightarrow \quad \Im(\Delta^{ab}_{\mu\nu}) &= \pi\delta^{ab}(g_{\mu\nu})\delta(k^2)\theta(\omega_k)
\end{aligned}$$

In the third line, $\mathcal{P}$ denotes the Cauchy principal value.
Consequently, the imaginary part of the ghost propagator is

$$\Im(\text{ - - - - - -}\!\!\succ\!\text{- - - - - - }) = \pi\delta^{ab}\delta(k^2)\theta(\omega_k).$$

1.1.2 Forward scattering

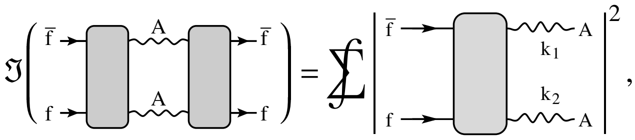
In the absolute square of the matrix element from above,



there is a phase space integrad in it which indicates forward scattering.

$$\int \mathrm{d}\rho_2 \left[\frac{1}{2} T^{ab}_{\mu\nu} T^{ab}_{\mu'\nu'}{}^{*} g^{\mu\mu'} g^{\nu\nu'} - S^{ab} S^{ab*} \right]$$

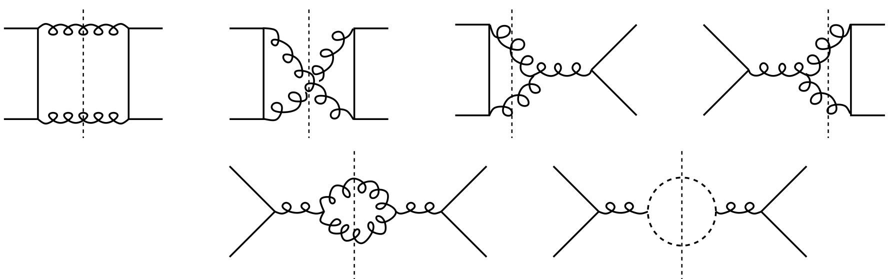
$T^{ab}_{\mu\nu}$ is the T matrix element for $f\bar{f} \to AA$ scattering. S^{ab} represents $f\bar{f} \to c\bar{c}$ scattering. The entire integral has an unphysical part, whose imaginary part is of interest.
If we compare the right-hand side and left-hand side of



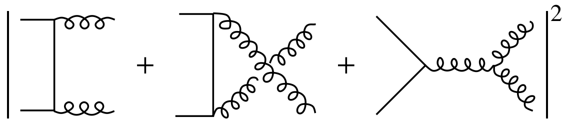
then this expression should be identical to

$$\frac{1}{2} \int \mathrm{d}\rho_2 \, T^{ab}_{\mu\nu} T^{ab}_{\mu'\nu'} P^{\mu\mu'}(k_1) P^{\nu\nu'}(k_2)$$

which is entirely physical. The two integrals should be the same.
These are all the contributions to Cutkosky cuts of the process:



The curly line denotes the gluon fields, the solid lide denotes the fermion fields, and the dashed line denotes the ghost field. Moreover, the dashed lines from top to bottom suggest the Cutkosky cuts. Since there is no external ghost particle in existence, the contribution to forward scattering consists of



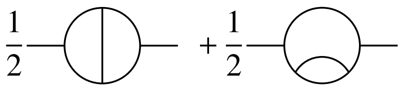
We are able to obtain a consistent result if we can show that

$$k_1^{\mu} T^{ab}_{\mu\nu} = -iS^{ab} k_{2\nu}$$
$$T^{ab}_{\mu\nu} k_2^{\nu} = -iS^{ab} k_{1\mu}$$

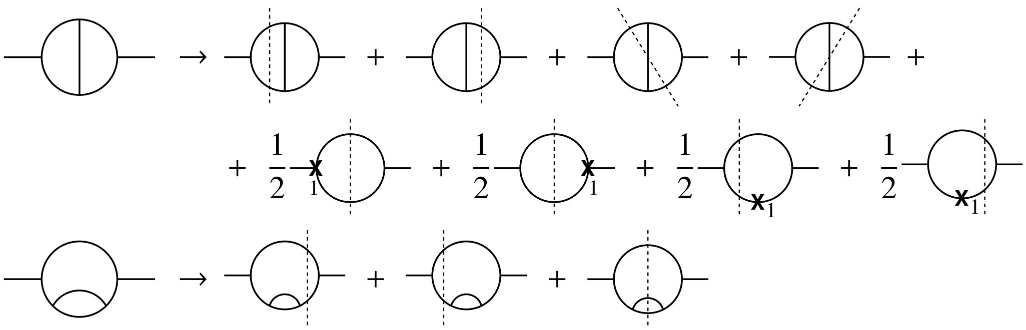
If we contract the entire expression with ${k_1}^{\mu} {k_2}^{\nu}$, we should get zero, because $k^2 = 0$ on the mass shell (gluons are massless). Therefore, $k_1^{\mu} T k_2^{\nu} = 0$.
As it turns out, once again, this whole integrals simply makes no sense without the ghost loop. This is the reason for its introduction.
There is a more systematic way to derive the introduction of the ghost loop, but it is too long for this class.

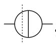
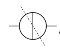
2 Optical Theorem and Renormalizability

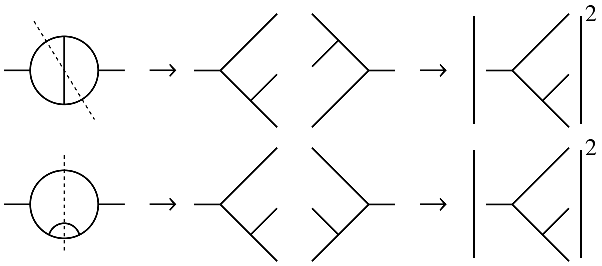
The optical theorem and renormalizability to not trivially or automatically both hold. Still, there is quite a systematic approach to it.
Let us work in ϕ_6^3 theory here. The two-loop 1PI graphs for the propagator are



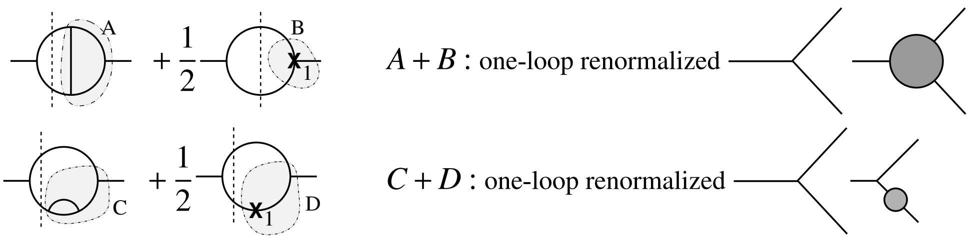
To get the imaginary part, we look at Cutkosky cuts:



Of course, counterterms can be cut, too. For each of the cuts, we get branch cut ambiguities. For the graphs with two cut lines, like , it starts at $2m^2$, for three cut lines, line , it starts at $3m^2$, and so on.
The forward scattering is given by, for example,



There are also short-distance singularities:



⇒ Only if we allow counterterms, there can be renormalization and Cutkosky at the same time.
⇒ The optical theorem is meaningful even for short-distance singularities.
Showing that the optical theorem also holds for unrenormalized amplitudes is not so easy. One would need to regulate, and for that, the gauge bosons have to be massless, which is not generally the case. If the gauge bosons were massive, then there would be no gauge-invariant regulator.

3 Slavnov Taylor Identities

Back to gauge theories: Let us derive Slavnov Taylor identities. Write down the Lagrangian for a fermion and boson field:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}{}^{a} F^{\mu\nu a} + \bar{\psi} i \not{D} \psi - m \bar{\psi} \psi$$

where

$$\begin{aligned} \not{D} &= \gamma_\mu D^\mu \\ D_\mu \psi &= \left(\partial_\mu - i g A^a \mu T^a \right) \psi \\ F_{\mu\nu}{}^{a} &= \partial_\mu A_\nu{}^{a} - \partial_\nu A_\mu{}^{a} + g \epsilon^{abc} A_\mu{}^{b} A_\nu{}^{c} \end{aligned}$$

If we transform the fermionic field, it depends on the representation we use:

$$\delta\psi = -i T^a \theta^a \psi$$

The transformation of the connection field is

$$\delta A_\mu^a = \epsilon^{abc} \theta^b A_\mu^c - \frac{1}{g} \partial_\mu \theta^a$$

The Lagrangian for the gauge fixing is

$$\mathcal{L}_{gf} = -\frac{1}{2\xi} \left(\partial^\mu A_\mu^a \right)^2$$

and for the Faddeev Popov ghost

$$\mathcal{L}_{FPGh} = i c_a^\dagger \partial^\mu \left(\delta_{ab} \partial_\mu - g \epsilon^{abc} A_\mu^c \right) c_b$$

Now, we write the ghost field in terms of complex fermion fields.

$$\begin{aligned} c_a &= \frac{1}{\sqrt{2}} (\rho_a + i \sigma_a) \\ c_a^\dagger &= \frac{1}{\sqrt{2}} (\rho_a - i \sigma_a) \end{aligned}$$

That allows us to decompose the ghost fields c and $c^\dagger$ in real and imaginary fermionic fields ρ and σ. The Faddeev Popov ghost Lagrangian becomes

$$\Rightarrow \quad \mathcal{L}_{FPGh} = -i \partial^\mu \rho_a D_\mu \sigma_a$$

with

$$D_\mu \sigma_a = \partial_\mu \sigma_a - g \epsilon_{abc} \sigma_b A_\mu^c$$

This looks nice! Our gauge condition is the real part times the covariant derivative of the imaginary part.

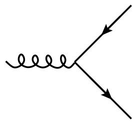
We introduce a variable ω as infinitesimal Grassmann variable. It anticommutes with any other Grassmann variable and commutes with any scalar field. Infinitesimal transformations are

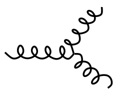
$$\begin{aligned}
\delta A\mu^a &= \omega D_\mu \sigma^a \\
\delta\psi &= ig\omega(T^a\sigma^a)\psi \\
\delta\rho^a &= -\frac{i}{\xi}\omega\partial^\mu A_\mu^a \\
\delta\sigma^a &= -g\omega\epsilon^{abc}\frac{\sigma^b\sigma^c}{2} \\
U(\vec{\theta}) &= \exp\left\{ i\sum_a \theta^a(x)\frac{\tau^a}{2} \right\} \\
\theta^a(x) &= -g\omega\sigma^a(x)
\end{aligned}$$

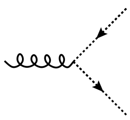
The last two lines are the formal transformation laws of the fixed Lagrangian $\mathcal{L}$.
Let us consider the full Lagrangian $\mathcal{L}+\mathcal{L}_{FPGh}+\mathcal{L}_{gf}$. Is it invariant? $\mathcal{L}$ is certainly invariant, $\mathcal{L}_{FPGh}$ and $\mathcal{L}_{gf}$ are certainly not invariant. We claim, without proof (it should be in any textbook), that the sum $\mathcal{L}_{FPGh}+\mathcal{L}_{gf}$ is invariant.

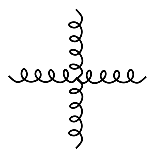
4 Ward Identities

If we take a look at the couplings,

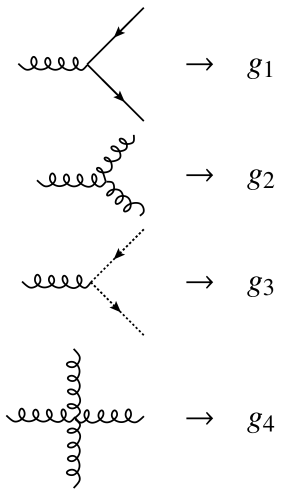
 $\rightarrow \quad g$

 $\rightarrow \quad g$

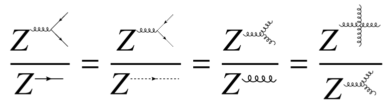
 $\rightarrow \quad g$

 $\rightarrow \quad g^2$

there has got to be a Ward identity! Imagine the couplings were different for each vertex.



Then there would be different g's in different monomials in the Lagrangian, like $\bar{\psi}\not{A}\psi$ and $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. This would result in a total loss of gauge invariance, which would destroy all formal proofs.
The Ward identity, which holds not only for the Z factors, but also for the complete Green's functions, is



The last two terms can be interpreted graphically: