Quantization of the Photon Field
QED

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0.1 Reminder: Classical Electrodynamics

Before we start quantizing the photon field, let us reflect on classical electrodynamics.

- The Hamiltonian is given by

$$\mathcal{H} = \int d^3x \left(\vec{E}^2 + \vec{B}^2\right)$$

- For the electric field, we find

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \text{(for now)}$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- For the magnetic field, we find

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t}$$

- Define Maxwell's Field Tensor:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$
$$F^{0i} = \partial^0 A^i - \partial^i A^0$$
$$F^{ij} = \partial^i A^j - \partial^j A^i$$

so that Maxwell's equations are given by

$$\partial_\mu F^{\mu\nu} = 0$$
$$\partial_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0 \ (= j^\nu \text{ for sources})$$

1 Quantization of the Photon Field

The $\vec{A}$ field describes a spin one boson. The simplest equation that it must fulfill is the Klein Gordon equation so let us start with this. Call

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$$

the Lagrangian density for a connection field ($\mathcal{U}(1)$) with

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial_\mu \partial^\nu A^\mu = 0$$

If A_μ is a solution, then $A'_\mu = A_\mu + \partial_\mu \chi(x)$ is also a solution because of **gauge freedom**.
Call Π^μ the canonical conjugate momentum to A^μ. Π^0, the canonical conjugate momentum to A^0, vanishes.

1.1 Gauge constraints

In order to simplify future calculations, constaints can be imposed ("fixing the gauge").

Lorenz gauge The Lorenz gauge, after Ludvig Lorenz (and not Hendrik Antoon Lorentz after whom Lorentz transformations are named), states that $\partial\mu A^\mu$ vanishes. Fixing the Lorenz gauge is always possible. Imagine, $\partial_\mu A^\mu$ were not zero, but $\partial_\mu A^\mu = \Theta(x)$. Then we could use gauge freedom in A_μ to shift it to $A_\mu + \partial_\mu \chi$ with $\Box\chi = -\Theta(x)$. We will always be able to construct A_μ in a way that enables the Lorenz gauge.
What are the advantages of the Lorenz gauge? The most obvous advantage is that Maxwell's equations boil down to the Klein Gordon equation in each component,

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial_\mu \partial^\nu A^\mu = \Box A\nu = 0$$

One could think of quantizing every component independently like a scalar field. Then each component will fulfill the Klein Gordon equation, but we know that the photon has two polarizations, therefore only two degrees of freedom. But if we quantize the way we just proposed, we would have two "extra" degrees of freedom. This is a problem.
Let us try and quantize a bit differently. Focus on the wavelike structure:

$$A^\mu = N\epsilon^\mu e^{-ikx} \qquad ,k^2 = 0$$

Here, ϵ^μ denotes the polarization vector. e^{-ikx} emphasizes the wavelike structure. Even with the Lorenz condition,

$$k \cdot \epsilon = k\mu\epsilon^\mu = 0$$

there is still a remaining gauge freedom. $\chi(x)$ with $\Box\chi(x) = 0$ is free to choose. Shifting ϵ^μ to $\epsilon^\mu + \beta k^\mu$, we can make k^0 disappear. For that reason, the Lorenz condition is equivalent to $\vec{\epsilon} \cdot \vec{k} = 0$. Thus, the polarization can be chosen to be in a spacelike plane, orthogonal to the direction of propagation of the $\vec{A}$ field. For example:

$$\begin{aligned} \epsilon^1 &= (1,0,0) \\ \epsilon^2 &= (0,1,0) \\ \text{for} \quad \vec{k} &= (0,0,|\vec{k}|) \end{aligned}$$

Obviously, there are exactly two independent polarizations allowed, just as we wanted.
Because of the spacelike character of the polarization vector, the scalar product is negative-definite:

$$\epsilon^*(\lambda)\epsilon(\lambda') = -\delta\lambda\lambda' \qquad \lambda, \lambda' \in \{1,2\}$$

With this ansatz, let us quantize the photon field.

$$A^\mu = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega_k} \sum_{\lambda=1}^{2} \left\{ \epsilon^\mu(\vec{k},\lambda) a(\vec{k},\lambda) e^{-ikx} + \epsilon^{*\mu}(\vec{k},\lambda) a^\dagger(\vec{k},\lambda) e^{+ikx} \right\}$$

The Lagrangian density gives

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left(\partial_\mu A^\mu \right)^2$$

The canonical conjugate momentum is given by

$$\Pi^0 = -\partial\mu A^\mu$$
$$\Pi^i = \partial^i A^0 - \partial^0 A^i = E^i$$

Finally, we have arrived at the **equal time commutator relations**:

$$[A\mu(x), \Pi_\nu(y)]\big|_{x^0=y^0} = ig\mu\nu\delta^{(4)}(x-y)$$

The equal time commutator relation (ECTR) builds a commutator which transforms as a two-tensor under the Lorentz group. $g_{\mu\nu}$ is the metric tensor.
Even though it might look like a well-derived ECTR, it provokes a serious problem: Because of the Lorenz condition, $\partial_\mu [A\mu(x), \Pi_\nu(y)]$ should vanish, but it does not. Instead, it produces

$$[A\mu(x), \Pi_\nu(y)] = i\partial_\nu\delta^{(4)}(x-y)$$

which is some distribution, but certainly not generally zero.
Quantizing the photon field the way we did it is problematic! It does not work the usual way. Allowing a polarization vector with only two possible polarizations obviously leads to no good.

2 Quantization of the Photon Field 2.0

This time, let us try four instead of two polarizations. For $\vec{k}$ in the 3 direction, $k = (\omega_k, 0, 0, \pm\omega_k)$, we choose

- One timelike polarization: $(1,0,0,0)^T$
- Two transversal polarizations: $(0,1,0,0)^T$ and $(0,0,1,0)^T$
- One longitudinal polarization: $(0,0,0,1)^T$

The scalar product for polarizations is

$$\epsilon^*(k,\lambda)\cdot\epsilon(k,\lambda') = g_{\lambda\lambda'}$$
$$\sum_\lambda g_{\lambda\lambda}\epsilon^{*\mu}(k,\lambda)\epsilon^\nu(k,\lambda) = g^{\mu\nu}$$

Our new ansatz for the photon field is

$$A^\mu(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \left\{\epsilon^\mu(k,\lambda)a(k,\lambda)e^{-ikx} + \epsilon^{*\mu}(k,\lambda)a^\dagger(k,\lambda)e^{+ikx}\right\}$$

The new ETCR are

$$\left[a(k,\lambda), a^\dagger(k',\lambda')\right] = -g_{\lambda\lambda'}(2\pi)^3 2\omega_k\delta^{(3)}(\vec{k}-\vec{k}')$$

The right hand side is negative for timelike polarizations. This would mean a negative norm. This is a problem! Maybe we can fix this by keeping two out of the four degrees of freedom which we have, and arrange the other two in such a way that they erase each other so they will not play a role.

Let us now impose Lorenz's constraint, $\partial_\mu A^\mu$, in a "strong" sense. Start by keeping in mind that for all physical states $|\phi\rangle, |\psi\rangle$,

$$\partial^\mu A - \mu^{(+)} |\psi\rangle = 0 = \langle\phi| \partial^\mu A_\mu^{(-)}$$

where $\partial^\mu A_\mu^{(+)}$ denotes the positive frequency part, and $\partial^\mu A_\mu^{(-)}$ the negative frequency part. Therefore, the expectation value of $\partial_\mu A^\mu$ is also vanishing,

$$\left\langle \phi \middle| \partial_\mu A^\mu \middle| \psi \right\rangle = 0$$

Also, for longitudinal wave modes, which are in the 3-direction if $\vec{k}$ is in the 3-direction,

$$\left(a(\vec{k},0) - a(\vec{k},3)\right)|\psi\rangle = 0$$

Therefore, the term

$$\frac{1}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\left\{a^\dagger(k,0)a(k,0) - a^\dagger(k,3)a(k,3)\right\} = 0$$

vanishes and does not contribute to the Hamiltonian density, consequently. Only the 1- and 2-components appear in the Hamiltonian! In other words, only transverse modes generate dynamics.

2.1 The covariant ξ gauge

Instead of adding $(\partial_\mu A^\mu)^2$ to the Lagrangian as the kinetic term for the photon field, one could add the more generally scaled expression $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

The Klein-Gordon equation becomes

$$\left\{\Box g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right\}A_\nu \equiv 0$$

This is the equation of motion for a free connection field in a covariant ξ gauge.
For the **interactions**, replace ∂_μ with $D_\mu = \partial_\mu + ieA_\mu$ and A_μ with $A_\mu - i\partial\mu\alpha(x)$ to get

$$\bar{\psi}\partial\!\!\!/\psi \;\rightarrow\; \bar{\psi}D\!\!\!\!/\psi = \bar{\psi}\partial\!\!\!/\psi + ie\bar{\psi}A\!\!\!/\psi$$

The last term, $ie\bar{\psi}A\!\!\!/\psi$, gives the form of the only allowed vertex in quantum electrodynamics.

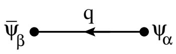
3 Feynman Rules of QED

Last, let us summarize all Feynman rules of QED. The Lagrangian is given by

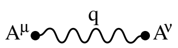
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(iD\!\!\!\!/ - m)\psi$$

3.1 Internal lines

There are two different kinds of propagators; on configuration space, the fermionic propagator $S^F_{\alpha\beta}(x-y)$ depends on the spin orientations α and β of the fermion fields created at y and annihilated at x. The photon propagator $G^{\mu\nu}$ is gauge dependent. If we choose $\xi = 0$, we get a transversal propagator. This is called the Landau gauge. If we choose $\xi = 1$, the Feynman gauge, the calculations become very easy.

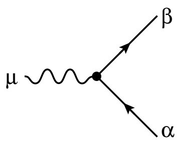


$$= \quad S^F_{\alpha\beta}(x-y) = \left\langle 0 \middle| T\psi_\alpha(x)\bar{\psi}_\beta(y) \middle| 0 \right\rangle = \left[\frac{i}{\not{q} - m + i\epsilon}\right]_{\alpha\beta}$$



$$= \quad G^{\mu\nu} = \left\langle 0 \middle| TA^\mu(x)A^\mu(y) \middle| 0 \right\rangle = i\frac{g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}}{q^2 + i\epsilon}$$

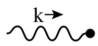
The vertex, or interaction diagram, with spinor indices α and β, corresponds to

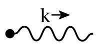


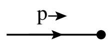
$$= \quad (ie\gamma^\mu)_{\beta\alpha}$$

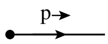
3.2 External lines

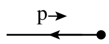
Here come the Feynman rules for incoming particles.

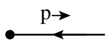
incoming photon:  $= \epsilon^\mu(\vec{k}, \lambda)$

outgoing photon:  $= \epsilon^{*\mu}(\vec{k}, \lambda)$

incoming electron:  $= u_\alpha(p, s)$

outgoing electron:  $= \bar{u}_\alpha(p, s)$

incoming positron:  $= \bar{v}_\alpha(p, s)$

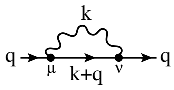
outgoing positron:  $= u_\alpha(p, s)$

3.3 Computing the Amplitude

In order to compute the amplitude, one must do more than just write down the expressions for each external particle, propagator, and vertex. Moreover, one must

- Multiply with (-1) for each closed fermion loop.
- Take into accound a relative sign between diagrams: There is a (-1) factor for each permutation or exchange of external lines.

Example Consider the graph  in a ξ gauge. Ignoring the external line contribution and any trivial overall factors, the loop integral gives



$$\rightarrow \int \gamma_\mu \frac{1}{\not{k}+\not{q}-m+i\epsilon}\gamma_\nu \frac{g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2}}{k^2+i\epsilon}\mathrm{d}^4k$$

$$= \underbrace{\int \gamma_\mu \frac{\not{k}+\not{q}+m}{(k+q)^2-m^2+i\epsilon}\gamma^\mu \frac{1}{k^2+i\epsilon}\mathrm{d}^4k}_{\text{Feynman gauge}} - (1-\xi)\int \frac{\not{k}(\not{k}+\not{q}-m\mathbb{I})\not{k}}{\big((k+q)^2-m+i\epsilon\big)(^2+i\epsilon)^2}\mathrm{d}^4k$$

In general, the photon propagators is gauge dependent. If we are not working in the Feynman gauge, every photon propagator contributes two terms. Thus, terms with several photons become nightmares.