Fock Space

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To explain the fundamental difference between classical and quantum mechanics, we consider the distinguishability of particles. In classical mechanics, any particle is assigned its path and therefore it is always possible to follow a particle's path back in order to discern it from other particles. In quantum mechanics, this is not the case. We use a Fock space as a notation of indistinguishability.

Consider a (bosonic) "two-particle state", that is, a state in a two-quantum Hibert space. Let $|\Psi(1,2;t)\rangle$ be a state of particles 1 and 2 which evolves in time and therefore satisfies some Schrödinger equation (SE). Then $|\Psi(2,1;t)\rangle$, where the two particles are swapped, satifies the same SE, and so do all superpositions of these two states. Then,

$$\left\{ \exp\left(i\left(\vec{k}_1\vec{x}_1 + \vec{k}_2\vec{x}_2\right)\right)\right\} \ , \ \exp\left(i\left(\vec{k}_2\vec{x}_1 + \vec{k}_1\vec{x}_2\right)\right)\Big\}$$

are used to describe Ψ.

1 Hilbert spaces

We now consider a two-dimensional one-particle Hilbert space $\mathcal{H}$ with an operator $A : \mathcal{H} \to \mathcal{H}$, which has two eigenstates $|a\rangle$ and $|b\rangle$. We write $|a(1)\rangle$ to describe that particle 1 is in state $|a\rangle$. For a two-particle system, we get the following possible states:

$$\Big\{ \ |a(1)\rangle\,|a(2)\rangle \ , \ |a(1)\rangle\,|b(2)\rangle \ , \ |b(1)\rangle\,|a(2)\rangle \ , \ |b(1)\rangle\,|b(2)\rangle \Big\}$$

Without further assumptions made, all states are equiprobable. In quantum mechanics, however, this is not what we observe. Instead, we get three bosonic and one fermionic state.

$$\left.\begin{array}{r} |a(1)\rangle\,|a(2)\rangle \\ |b(1)\rangle\,|b(2)\rangle \\ |a(1)\rangle\,|b(2)\rangle + |b(1)\rangle\,|a(2)\rangle \end{array}\right\} \quad \text{bosonic states}$$

$$|a(1)\rangle\,|b(2)\rangle - |b(1)\rangle\,|a(2)\rangle \qquad \text{fermionic state}$$

The bosonic states are symmetrical whereas the ferminic state is antisymmetrical ($\to$ Pauli principle).

Next, we focus our attention on creators (Erzeuger) and annihilators (Vernichter) acting on bosonic and fermionic states. In the space of quanta that can just be aggravated, so they can be counted, but not distinguished, we observe various types of quanta.

1.1 One-particle states

Starting with "one quantum" physics, states are considered as rays in a Hilbert space $\mathcal{H}_1$. We make the assumption that $\mathcal{H}_1$ has a discrete basis. Furthermore, we consider an observable $\hat{A}^{(1)}$. Let $\hat{A}^{(1)}$ have eigenvectors $|a_1\rangle, \ldots, |a_n\rangle$, $n \in \mathbb{N}$, with eigenvalues a_i. By writing $|a_i\rangle$ we mean that one particle occupies the i^{th} state.

$$\begin{aligned} |a_1\rangle &\simeq |1,0,\ldots,0\rangle_A \\ |a_2\rangle &\simeq |0,1,\ldots,0\rangle_A \\ &\ldots \end{aligned}$$

The subscript A indicates that the states are in the eigenbasis of $\hat{A}^{(1)}$.

1.2 Two-particle states

Let's move on to "two quantum" Hilbert space: In $\mathcal{H}_2$ we should have

$$
\begin{aligned}
&|2, 0, \ldots, 0\rangle_A \\
&|1, 1, 0, \ldots, 0\rangle_A \\
&|1, 0, 1, 0, \ldots, 0\rangle_A \\
&|0, 1, 1, 0, \ldots, 0\rangle_A \\
&\ldots
\end{aligned}
$$

Obviously, $\mathcal{H}_2 \neq \mathcal{H}_1 \otimes \mathcal{H}_1$! If this were the case, we would be able to distinguish quanta. Instead, $\mathcal{H}_2$ is isomorphic to $\mathcal{H}_1 \otimes_S \mathcal{H}_1$, where $\otimes_S$ is the symmtetric tensor product

$$h_1 \otimes_S h_2 = h_1 \otimes h_2 + h_2 \otimes h_1.$$

In general, there is some k^{th} state n_k for which all higher states are zero. In our notiation we omit all states beyond n_k.

$$
\begin{aligned}
&| \underbrace{n_1, \ldots, n_k}_{\text{Fock basis}}, 0, \ldots, 0\rangle_A \\
&|0, 0, \ldots, 0\rangle_A \equiv |0\rangle_A
\end{aligned}
$$

The vacuum state $|0\rangle_A$ is independent of the choice of $\hat{A}^{(1)}$.
For any two states $\left|\vec{n}\right\rangle$ and $\left|\vec{n}'\right\rangle$ for which $\vec{n} \neq \vec{n}'$, $\left|\vec{n}\right\rangle \perp \left|\vec{n}'\right\rangle$.
Experimentally, this means that we are able to measure the number of quanta in every state (that is, before other interactions), if they are free particles (outside the reach of an interaction).

2 Fock space

For bosonic Fock space, the i^{th} state can be occupied by any number of particles, including zero. For fermionic Fock space, each state can either be occupied by one or by no particle.

$$
\begin{aligned}
n_i &\in \mathbb{N}_{\geq 0} \qquad \text{(bosons)} \\
n_i &\in \{0, 1\} \quad \text{(fermions)}
\end{aligned}
$$

We define the number operator $\hat{N}_i^{(A)}$ which counts the number of quanta in the i^{th} eigenstate of $\hat{A}$. $\hat{N}_i^{(A)}$ is supposed to have real eigenvalues $n_i \in \mathbb{N} \subset \mathbb{R}$.

$$\hat{N}_i^{(A)} |n_1, \ldots, n_i, \ldots, n_k\rangle_A = n_i |n_1, \ldots, n_i, \ldots, n_k\rangle$$

n_i is the occupation number (Besetzungszahl) with respect to a fixed observable $\hat{A}$.
For fixed $\hat{A}$, the $\hat{N}_i^{(A)}$ form a maximal commuting set of operators on

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \ldots$$

Whereas $\mathcal{H}$ is a Fock space, $\mathcal{H}_1$ is a single quantum Hilbert space, $\mathcal{H}_2$ is a two-quantum Hilbert space isomorphic to $\mathcal{H}_1 \otimes_S \mathcal{H}_1$, and so on.
We can write an observable as $\hat{A} = \sum a_i \hat{N}_i^{(A)}$. Considering superpositions of $\left|\vec{n}\right\rangle$ and $\left|\vec{n}'\right\rangle$, we see that they are certainly no eigenstates of $\hat{N}_i^{(A)}$.

2.1 Creators and annihilators

Let us assume that $\hat{N}_i^{(A)}$ can be written using a creator $\hat{a}_i^\dagger$ and an annihilator $\hat{a}_i$. Then

$$\hat{N}_i^{(A)} = \hat{a}_i^\dagger \hat{a}_i \qquad \hat{a}_i^\dagger, \hat{a}_i \quad \text{not self-adjoint!}$$

2.1.1 Annihilator

Definition: The annihilator acting on the vacuum state will erase that state. The vacuum state is denoted by $|0\rangle$ and carries no particles whatsoever. If the annihilator acts on a non-vacuum state, the occupation number will be decreased by one.

$$\hat{a}_i |\ldots, n_i, \ldots\rangle = c(n_i) |\ldots, n_i - 1, \ldots\rangle \quad \text{if } n_1 > 0 \qquad \hat{a}_i |\ldots, n_i, \ldots\rangle = 0 \quad \text{if } n_1 = 0$$

To determine the action of $\hat{a}_i^\dagger$, we consider $\hat{a}_i^\dagger |\ldots, n_i, \ldots\rangle$ and keep in mind that in a scalar product with an operator acting on the ket vektor, we get the same result if we instead let the adjoint operator act on the bra vector.

$$\begin{aligned}\langle \ldots, n_i', \ldots | \hat{a}_i^\dagger |\ldots, n_i, \ldots\rangle &= \langle \ldots, n_i, \ldots | \hat{a}_i |\ldots, n_i', \ldots\rangle^* \\ &= \begin{cases} 0 & \text{if } n_i' = 0 \\ c^*(n_i') \underbrace{\langle \ldots, n_i, \ldots | \ldots, n_i' - 1, \ldots\rangle}_{\propto \delta_{n_i, n_i'-1}} & \text{if } n_i' \neq 0 \end{cases}\end{aligned}$$

Of course, the scalar product of two states is zero for different states and only non-zero for two identical states (if $n_i = n_i' - 1$). We therefore get

$$\hat{a}_i^\dagger |\ldots, n_i, \ldots\rangle = c^*(n_i + 1) |\ldots, n_i + 1, \ldots\rangle$$

and now take into account the previous definition of $\hat{N}_i^{(A)}$,

$$\begin{aligned} \hat{N}_i^{(A)} &\stackrel{!}{=} \hat{a}_i^\dagger \hat{a}_i \\ \Rightarrow c*(n_i) c(n_i) &= n_i \\ \Rightarrow c(n_i) &= \omega(n_i) \sqrt{n_i} \end{aligned}$$

$\omega(n_i)$ is a general complex phase. But since we work in a real Hilbert space, we can choose this phase once. We do that by setting $\omega(n_i) \equiv 1$ which gives us $c(n_i) = c^*(n_i) = \sqrt{n_i}$.

$$\left.\begin{aligned} \hat{a}_i |\ldots, n_i, \ldots\rangle &= \sqrt{n_i} |\ldots, n_i - 1, \ldots\rangle \\ \hat{a}_i^\dagger |\ldots, n_i, \ldots\rangle &= \sqrt{n_i + 1} |\ldots, n_i + 1, \ldots\rangle \end{aligned}\right\} \quad \hat{N}_i^{(A)} \equiv \hat{a}_i^\dagger \hat{a}_i$$

2.1.2 Commutator relations

The commutator relations for the creators and annihilators are as follows:

$$\begin{aligned} \left[\hat{a}_i, \hat{a}_j^\dagger\right] &= \delta_{ij} \hat{\mathbb{I}} \\ \left[\hat{a}_i, \hat{a}_j\right] &= 0 = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] \end{aligned}$$

$\hat{\mathbb{I}}$ denotes the unit operator in Hilbert space.

- Assume $i = j$.

$$\begin{aligned}
&\langle \ldots, n_i', \ldots | \left(\hat{a}_i \hat{a}_i^\dagger - \hat{a}_i^\dagger \hat{a}_i \right) | \ldots, n_i, \ldots \rangle \\
&\quad = \langle \ldots, n_i', \ldots | \, \hat{a}_i \hat{a}_i^\dagger \, | \ldots, n_i, \ldots \rangle - \langle \ldots, n_i', \ldots | \, \hat{a}_i^\dagger \hat{a}_i | \ldots, n_i, \ldots \rangle \\
&\quad = 1 \cdot \langle \ldots, n_i', \ldots | \ldots, n_i, \ldots \rangle \\
&\quad = \begin{cases} 1 & n_i' = n_i \\ 0 & n_i' \neq n_i \end{cases} \qquad \Rightarrow \left[\hat{a}_i, \hat{a}_i^\dagger \right] = \hat{\mathbb{I}}
\end{aligned}$$

- Assume $i \neq j$.

$$\begin{aligned}
&\left\langle \ldots, n_i', \ldots, n_j', \ldots \middle| \, \hat{a}_i \hat{a}_j^\dagger \, \middle| \ldots, n_i, \ldots, n_j, \ldots \right\rangle - \left\langle \ldots, n_i', \ldots, n_j', \ldots \middle| \, \hat{a}_j^\dagger \hat{a}_i \middle| \ldots, n_i, \ldots, n_j, \ldots \right\rangle \\
&\quad = \left(\sqrt{(n_i)(n_j + 1)} - \sqrt{(n_j + 1)(n_i)} \right) \cdot \left\langle \ldots, n_i', \ldots, n_j', \ldots | \ldots, n_i, \ldots, n_j, \ldots \right\rangle \\
&\quad = 0 \qquad \Rightarrow \left[\hat{a}_i, \hat{a}_j^\dagger \right] = 0 \quad \forall i \neq j
\end{aligned}$$

2.1.3 Fermionic case

For fermions, we still require that $\hat{N}_i^{(A)} \stackrel{!}{=} \hat{a}_i^\dagger \hat{a}_i$, but we also need $\left(\hat{a}_i^\dagger \right)^2 \stackrel{!}{=} 0$ to maintain the Pauli principle.

$$\begin{aligned}
\hat{a}_i | \ldots, 1_i, \ldots \rangle &= c(1) | \ldots, 0_i, \ldots \rangle \\
\hat{a}_i | \ldots, 0_i, \ldots \rangle &= 0
\end{aligned}$$

Since $c^*(1)c(1) = 1$ we know that c is a pure phase. The actual value of c depends on the content. The creator of the i^{th} state gives zero if that state is already occupied.

$$\left| \hat{a}_i^\dagger | 0, \ldots, 1_i, 0, \ldots \rangle \right|^2 > 0$$

2.1.4 Anticommutator relations

For fermions, we get anticommutator relations.

$$\begin{aligned}
\left[\hat{a}_i, \hat{a}_j^\dagger \right]_+ &= \delta_{ij} \hat{\mathbb{I}} \\
\left[\hat{a}_i, \hat{a}_j \right]_+ &= 0 = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger \right]_+
\end{aligned}$$

- Assume $i \neq j$ and consider $\hat{b}_S{}^\dagger := \frac{1}{\sqrt{2}} \left\{ \hat{a}_i^\dagger | 0 \rangle + \hat{a}_j^\dagger | 0 \rangle \right\}$. Note that since $\hat{b}_S{}^\dagger$ is a linear combination of one-quantum states it is therefore itself a one-quantum state. Of course, $\left(\hat{b}_S{}^\dagger \right)^2 = 0$.

$$\begin{aligned}
\left(\hat{b}_S{}^\dagger \right)^2 &= \frac{1}{2} \left(\hat{a}_i^\dagger + \hat{a}_j^\dagger \right) \left(\hat{a}_i^\dagger + \hat{a}_j^\dagger \right) = 0 \\
&= \frac{1}{2} \Bigg(\underbrace{\hat{a}_i^\dagger \hat{a}_i^\dagger}_{=0} + \underbrace{\hat{a}_j^\dagger \hat{a}_j^\dagger}_{=0} + \underbrace{\hat{a}_i^\dagger \hat{a}_j^\dagger + \hat{a}_j^\dagger \hat{a}_i^\dagger}_{= \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger \right]_+} \Bigg)
\end{aligned}$$

- Assume $i = j$. Considering all states $\left[\hat{a}_i, \hat{a}_i^\dagger \right]_+$ can act on, the result follows immediately.

2.2 Notation of states

To denote that the i^{th} (bosonic) state is occupied by n_i particles, the creator acts n_i-times on the vacuum state to creator the n_i particles in the i^{th} state.

$$|\ldots, n_i, \ldots\rangle = \frac{1}{\sqrt{n_i!}} \left(\hat{a}_i\right)^{n_i} |\ldots, 0_i, \ldots\rangle$$

$$\Rightarrow \quad |n_1, \ldots, n_i, \ldots n_k\rangle = \prod_{j=1}^{k} \frac{1}{\sqrt{n_j!}} \left(\hat{a}_j\right)^{n_j} |0_1, \ldots, 0_k\rangle$$

For fermions, each occupied state is denoted by the action of the creator on the vacuum state.

$$|1, \ldots, 1_k, \ldots\rangle = \hat{a}_1{}^{\dagger} \ldots \hat{a}_k{}^{\dagger} |0\rangle$$
$$\hat{a}_i^{\dagger} |\ldots, 0_i, \ldots\rangle = \pm |\ldots, 1_i, \ldots\rangle$$

with the plus sign if the number of occupied states before the i^{th} slot is even and the minus sign if it is odd.

From Fock Space to the Klein Gordon Equation

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Repitition: Fock space states are denoted by $|\dots, n_i, \dots\rangle$ which for bosons equals $\prod_i \frac{1}{\sqrt{n_i}}(\dots \hat{a}_i^\dagger \dots)|0\rangle$ where $|0\rangle$ denotes the vacuum state. For fermions, $\hat{a}_i^\dagger |\dots, 0_i, \dots\rangle = \pm|\dots, 1_i, \dots\rangle$ with plus sign is the number of occupied states before the i^{th} state $\left(\sum_{k=1}^{i-1} n_k\right)$ is even and minus sign if that number is odd.
We only need a few simple assumptions to derive the Klein Gordon equation (KGE) without any further knowledge of classial field theory.

3 Change of Basis

Change of basis in Fock space:

$$|\dots, n_i, \dots\rangle_A \xrightarrow{?} |\dots, n_i, \dots\rangle_B$$

To see that the change of basis in Fock space is given by a transition matrix, we expand an arbitrary state vector $|b_i\rangle$ from the B-basis in terms of vectors $\left|a_j\right\rangle$ from the A-basis (in $\mathcal{H}_1$).

$$|b_i\rangle = \sum_j \left|a_j\right\rangle \left\langle a_j \mid b_i\right\rangle = \sum_j c_{ji} \left|a_j\right\rangle$$

$$\hat{b}_i^\dagger |0\rangle = \sum_j c_{ji} \hat{a}_j^\dagger |0\rangle$$

$\hat{b}_i^\dagger$ and $\hat{b}_i$ have similar commutator relations to those of $\hat{a}_i^\dagger$ and $\hat{a}_i$. To show an example, let's look for

$$\left[\hat{b}_i, \hat{b}_{i'}^\dagger\right]_\pm = \sum_j \sum_{j'} c_{ji}{}^* c_{j'i'} \underbrace{\left[\hat{a}_i, \hat{a}_i^\dagger\right]_\pm}_{=\delta_{jj'}} = \sum_j \underbrace{c_{ji}{}^* c_{ji'}}_{c^*c=\mathbb{I}} = \delta_{ii'}$$

Up until this point, we have always assumed that the Hilbert space bases are discrete so they give us a discrete set of eigenvectors. This is not generally the case in physics, therefore we need to replace the $\sum$ with an $\int$ in case of free field theory.

3.1 Labels

Let's assume the continuous labels:

$\{\|k\rangle\}$	"momentum basis"	$k, p, q, \dots$
$\{\|x\rangle\}$	"position basis"	$x, y, z, \dots$

and assign a "field" to every point in space time:

$$\hat{\Psi}^\dagger(\vec{x}) = \underbrace{\int \mathrm{d}^3\tilde{k} \underbrace{\left\langle \vec{k} \mid \vec{x} \right\rangle}_{=e^{i\vec{k}\cdot\vec{x}}} \hat{a}_k(k)}_{\text{corresponds to Fourier expansion}}$$

The assumption we make when we write down a quantum field like this is that we have the ability to detect momenta of quanta and count the number of quanta in the different momentum states $n_{\vec{k}}$, $n_{\vec{k}'}$, and so on.

3.2 Measure

Let us now take a closer look at the measure. In the non-relativistic case, we set

$$\mathrm{d}^3\tilde{k} = \frac{\mathrm{d}^3\vec{k}}{(2\pi)^{\frac{3}{2}}}$$

but this is not a Lorentz covariant measure, therefore we include a dispersion funktion $\omega(\vec{k})$,

$$\begin{aligned}\mathrm{d}^3\tilde{k} &= \frac{\mathrm{d}^3\vec{k}}{(2\pi)^{\frac{3}{2}}\omega(\vec{k})}\\ \omega^2(\vec{k}) &= \vec{k}^2 + m^2\\ \omega(\vec{k}) &= \left|\sqrt{\vec{k}^2 + m^2}\right|\end{aligned}$$

This measure is indeed Lorentz covariant, even though it is still dependent on the three-vector $\vec{k}$.

4 Relativistic notion

We define the momentum operator $\hat{P}$ with eigenvectors $\left|\vec{k}\right\rangle$ and eigenvalues $\vec{k}$:

$$\hat{P}\left|\vec{k}\right\rangle = \vec{k}\left|\vec{k}\right\rangle$$

and also consider the resolution of identity:

$$\mathbb{I} = \int \mathrm{d}^3\vec{k} = \left|\vec{k}\right\rangle\left\langle\vec{k}\right|$$

To get a relativistic formulation, we combine energy and momentum into a four-vector. If energy and momentum should form a four-vector, the Hamiltonian squared minus $\left|\vec{p}\right|^2$ should transform as a Lorentz scalar and is therefore constant along the orbits of a Poincaré group.

$$\underbrace{H^2 - \left|\vec{p}\right|^2}_{\text{relativistic invariant notion?}} = \text{const.}\Big|_{\text{Poincaré}}$$

To determine the relation between H and $\left|\vec{p}\right|^2$, we regard the three vectors $\left|\vec{k}\right\rangle$ and $\left|\vec{k}'\right\rangle$ with the Lorentz transformation $\left|\vec{k}\right\rangle = \hat{\Lambda}\left|\vec{k}'\right\rangle$ between them. We also keep in mind that we know how the Hamiltonian should act on the labeled momenta because we know how $\left|\vec{p}\right|^2$ acts and we know that $H^2 - \left|\vec{p}\right|^2$ is a constant. there is one more assumption to be made.

Requirement: We demand that $H\left|\vec{k}\right\rangle \stackrel{!}{=} \left(\vec{k}^2 + \mu^2\right)^{\frac{1}{2}}\left|\vec{k}\right\rangle$. Thus the energy is not free, but determined by $\vec{k}$, so the three-momenta are essential for any calculations.

4.1 Unitary transformations

We are looking for a unitary operator in the form of

$$U(gh) = U(g)U(h)$$

and a translation that looks like

$$\Delta_a \Psi(x) = \Psi(x - a)$$

and therefore

$$U(\Delta) |\Psi\rangle \equiv |\Delta_a \Psi\rangle .$$

An example for a unitary transformation like this is the time evolution operator, which determines the time dependency of the Hamiltonian given by

$$e^{-iHt} |\Psi(x)\rangle = \left|\Psi(x_0 + t, \vec{x})\right\rangle .$$

Applying $U(\Delta_a)$ to $\left|\vec{k}\right\rangle$ will then give

$$U(\Delta_a) \left|\vec{k}\right\rangle = e^{ia\cdot\hat{p}} \left|\vec{k}\right\rangle = e^{ia\cdot k} \left|\vec{k}\right\rangle$$

with

$$a \cdot k = a_0 k_0 - \vec{a}\vec{k}$$
$$k_0 = \sqrt{\vec{k}^2 + \mu^2}$$

where μ is from the representation scheme of the Poincaré group.

- $\mu = 0$ propagates on the light cone.
- $\mu \neq 0$ propagates inside the light cone.

We call $k^2 = k_0^2 - \vec{k}^2 = \mu^2$ the "mass hyperboloid".

4.2 Lorentz-invariant measure

Definition: Heaviside step function

$$\Theta(t) := \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

We define the Lorentz-invariant measure $\mathrm{d}\lambda(k)$ by writing

$$\mathrm{d}\lambda(k) := \mathrm{d}^4 k \delta(k^2 - \mu^2)\Theta(k_0)$$

The term $\delta(k^2 - \mu^2)$ puts everyting on the mass shell whereas the term $\Theta(k_0)$ forbids negative energies. Since a Lorentz transformation never changes the sign of the 0^{th} component, this

measure is indeed Lorentz-invariant. Definition: Delta function

$$\begin{aligned}
\delta\Big(f(x)\Big) &= \sum_{\{\,k|f(k)=0\}} \\
\Rightarrow \delta(k^2-\mu^2)\Theta(k_0) &= \frac{1}{2\omega(\vec{k})}\Big\{\delta(k_0-\omega(\vec{k}))\Theta(k_0)+\delta(k_0+\omega(\vec{k}))\Theta(k_0)\Big\} \\
&= \frac{1}{2\omega(\vec{k})}\delta(k_0-\omega(\vec{k}))\Theta(k_0) \\
\Rightarrow \int \mathrm{d}\lambda(k) &= \int \mathrm{d}k_0 \frac{1}{}\delta(k_0-\omega(\vec{k}))\Theta(k_0) \\
&= \int \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})} f\Big(\omega(\vec{k}),\vec{k}\Big) \\
\Rightarrow \qquad \mathrm{d}\lambda(k) &= \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})} \quad ; \qquad k_0=\omega(\vec{k})
\end{aligned}$$

Apparently, the energies are given once the momenta are given. The expression also gives a scaling of the momentum states and implies the connection between three-momentum states and four-momentum states:

$$|k\rangle = \sqrt{2\omega(\vec{k})}(2\pi)^{\frac{3}{2}}\left|\vec{k}\right\rangle$$

where the squareroots are conventional. To see the normalization, we take the scalar produkt of two states k and k'.

$$\begin{aligned}
\langle k\,|\,k'\rangle &= 2\omega(\vec{k})(2\pi)^3\delta^3(\vec{k}-\vec{k}') \\
\Rightarrow \mathbb{I} &= \int \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3}\,|k\rangle\,\langle k|
\end{aligned}$$

which gives the resolution of $\mathbb{I}$ for four-momenta.

4.3 Unitary transformations

We take a unitary transformation :

$$U(\Lambda)\,|k\rangle = |\Lambda k\rangle$$

This is a unitary representation of the Lorentz group. Taking unitarity and the resolution of $\mathbb{I}$ gives

$$\begin{aligned}
U(\Lambda)U^\dagger(\Lambda) \stackrel{!}{=} \mathbb{I} &= \int \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3} U(\Lambda)\,|k\rangle\,\langle k|\,U^\dagger(\Lambda) \\
&= \int \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3}\,|\Lambda k\rangle\,\langle \Lambda k| \\
&= \mathbb{I}
\end{aligned}$$

Obviously, the measure is relativistically invariant.
$\Rightarrow$ It makes sense to construct Fock space states created out of the vacuum.

5 Free QFT

Taking a perturbative expansion of all possible free vacuum expectation values into account gives a quantum field theory.

$$\begin{aligned}
\hat{a}^\dagger(\vec{k})\,|0\rangle &= \left|\vec{k}\right\rangle \\
\hat{\alpha}^\dagger(k) &= (2\pi)^{\frac{3}{2}}\left(2\omega(\vec{k})\right)^{\frac{1}{2}}\hat{a}(k)\left|\vec{k}\right\rangle \\
\Rightarrow \hat{\alpha}^\dagger(k)\,|0\rangle &= |k\rangle \\
k_0 &= \omega(\vec{k}) \\
U(\Lambda)\hat{\alpha}^\dagger U^\dagger(\Lambda) &= \hat{\alpha}^\dagger(\Lambda k)
\end{aligned}$$

So far, there is no rule how to change between the momentum basis and the position basis. In the relativistic covariant fashion, there is no localization of states by position, but each position has a label, therefore we label states in a quantum field with their momenta.
The transition from one initial state $|i\rangle$ to some final state $|f\rangle$ is given by

$$\langle f|\,\phi(x)\,|i\rangle$$

where $\phi(x)$ is a function of space time. It can be suggested that this expectation value transforms like

$$\langle f|\,U^\dagger(\Lambda)\phi(x)U(\Lambda)\,|i\rangle = \langle f|\,\phi(\Lambda^{-1}x)\,|i\rangle$$

5.1 Axioms for a free quantum field theory

1. $\phi^\dagger(x) = \phi(x)$
   Monomials to compute real, measurable numbers

2. $U^\dagger(\Delta_a)\phi(x)U(\Delta_a) = \phi(x-a)$
   (translation)

3. $U^\dagger(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$
   (boost)

4. Causality: For space-like distances $|x-y|^2 < 0 \quad \Rightarrow \quad [\phi(x),\phi(y)] = 0$. The quantum fields commute.

Theorem: There exists a unique solution ϕ to axioms 1-3 which is linear in creation and annihilation operators $\hat{a}^\dagger, \hat{a}$, which then also funfills axiom 4.
The solution is:

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3}\left\{e^{ik\cdot x}\hat{\alpha}^\dagger(k) + e^{-ik\cdot x}\hat{\alpha}_(k)\right\}$$

or with a instead of α:

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{2\omega(\vec{k})}(2\pi)^3}\left\{e^{ik\cdot x}\hat{a}^\dagger(\vec{k}) + e^{-ik\cdot x}\hat{a}(\vec{k})\right\}$$

Proof: We take the ansatz

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3} \left\{ f_+(x,k)\hat{\alpha}^\dagger(k) + f_-(x,k)\hat{\alpha}(k) \right\}$$

Where $f_\pm(x,k)$ are general coefficients of x and k and the entire expression is linear in $\hat{\alpha}^\dagger(k)$ and $\hat{\alpha}(k)$. We now insert

$$\begin{aligned}\phi(x) &= e^{ix\cdot\hat{P}}\phi(0)e^{-ix\cdot\hat{P}} = \\ &= \int \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3} \left\{ f_+(0,k)e^{ixk}\hat{\alpha}^\dagger(k) + f_-(0,k)e^{-ixk}\hat{\alpha}(k) \right\}\end{aligned}$$

Next, we use axiom 3: $\phi(0) = U^\dagger(\Lambda)\phi(0)U(\Lambda)$

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3} \left\{ f_+(0,k)\hat{\alpha}^\dagger(\Lambda^{-1}k) + f_-(0,k)\hat{\alpha}(\Lambda^{-1}k) \right\}$$

But this is translation invariant, therefore we can write

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3} \left\{ f_+(0,\Lambda k)\hat{\alpha}^\dagger(k) + f_-(0,\Lambda k)\hat{\alpha}(k) \right\}$$

The result is a linear combination of plane waves. We compare coefficients and get

$$f_\pm(0,k) = f_\pm(0,\lambda k)$$

Determining $f_\pm$ in the rest frame, where $k = (\mu, \vec{0})$, we get

$$f_+^* = f_-$$

which is exactly the desired decomposition of free field theory. $f_\pm(0,k)$ fulfills the Klein-Gordon equation: $\left[\Box + \mu^2\right]\phi = 0$.

From Free Fields to Propagators

31.10.2011

Reminder: Four axioms of a free field theory

1. $\phi^\dagger(x) = \phi(x)$ (the field is a real field)
2. $U^\dagger(\Delta_a)\phi(x)U(\Delta_a) = \phi(x-a)$ (translation is a unitary transformation)
3. $U^\dagger(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$ (unitary field transform is given by Lorentz transform of the coordinate)
4. Causality: For space-like distances, $|x-y|^2 < 0 \;\Rightarrow\; [\phi(x), \phi(y)] = 0$. Fields of space-like distances commute.

To find a field that fulfills axioms (1) - (3) and that is linear in creators and annihilators, we take the most general ansatz

$$\begin{aligned}\phi(x) &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega(\vec{k})}\left\{e^{ixk}\hat{\alpha}^\dagger(k) + e^{-ixk}\hat{\alpha}(k)\right\} \\ &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega(\vec{k})}}\left\{e^{ixk}\hat{a}^\dagger(k) + e^{-ixk}\hat{a}(k)\right\}\end{aligned} \tag{1}$$

The free field is made up of creators and annihilators of vacuum.

Corollary The ansatz (1) fulfils causality.

6 Causality

Now let's have a look at the commutator of fields of space-like distances in order to show causality.

$$\begin{aligned}[\phi(x),\phi(y)] &= \left[\int \mathrm{d}\lambda(p)\left(e^{ixp}\hat{\alpha}^\dagger(p) + e^{-ixp}\hat{\alpha}(p)\right), \int \mathrm{d}\lambda(q)\left(e^{iyq}\hat{\alpha}^\dagger(q) + e^{-iyq}\hat{\alpha}(q)\right)\right] \\ &= \iint \mathrm{d}\lambda(p)\,\mathrm{d}\lambda(q)\Big(e^{ixp}e^{iyq}\left[\hat{\alpha}^\dagger(p),\hat{\alpha}^\dagger(q)\right] + e^{ixp}e^{-iyq}\left[\hat{\alpha}^\dagger(p),\hat{\alpha}(q)\right] + \\ &\qquad + e^{-ixp}e^{iyq}\left[\hat{\alpha}(p),\hat{\alpha}^\dagger(q)\right] + e^{-ixp}e^{-iyq}\left[\hat{\alpha}(p),\hat{\alpha}(q)\right]\Big) \\ &= \iint \mathrm{d}\lambda(p)\,\mathrm{d}\lambda(q)\left(e^{-i(x-y)p} - e^{-i(y-x)p}\right) \\ &= \Delta_+(x-y) - \Delta_+(y-x) \equiv \Delta(x-y)\end{aligned}$$

Of course, this is a Lorentz-invariant term because the measure and the scalar products in the integrand are Lorentz-invariant.
If $\Delta_+(z)$ is Lorentz-invariant, assume a space-like separation z, $|z|^2 < 0$. Then $\exists\Lambda$ with $\Lambda z = -z$.

$$\Rightarrow \qquad \Delta_+(z) = \Delta_+(-z) \qquad \Rightarrow \qquad [\phi(x),\phi(y)] = \Delta(x-y) \equiv 0$$

$\phi(x)$ fulfills the Klein Gordon equation: $\left(\Box + \mu^2\right)\phi = 0$ with $\Box := \partial x_0^2 - \sum_{i=1}^3 \partial x_i^2 = \partial_\mu\partial^\mu = \partial^\mu\partial_\mu$. The KGE is a relativistically covariant wave equation, but it is quadratic in time which is unusual for field theories.

7 Propagators

Let's reformulate the commutator $[\phi(x), \phi(y)] = i\Delta(x-y)$. In ordner to do that, we must be sure that the fields always commute for equal time.

$$[\phi(x), \phi(y)]\Big|_{x_0=y_0} = 0$$
$$[\phi(x), \partial_0\phi(y)]\Big|_{x_0=y_0} = i\delta^{(3)}(\vec{x}-\vec{y})$$

Before we continue, let's introduce the time ordering operator T.

$$T\Big(A(x)B(y)\Big) := \begin{cases} A(x)B(y) & x_0 > y_0 \qquad (y_0 \text{ happens before } x_0) \\ B(y)A(x) & x_0 < y_0 \qquad (x_0 \text{ happens before } y_0) \end{cases}$$

We use the time ordering operator in order to make sure that any product of fields is generated in the right order of time. We can then look at the matrix element fn two time ordered fields and the vacuum states, the vacuum expectation value or transition amplitude:

$$\langle 0|\, T\phi(x)\phi(y)\, |0\rangle = \lim_{\epsilon\to 0_+} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{-ie^{i(x-y)k}}{k^2-\mu^2+i\epsilon}$$

We call this expression the (Feynman) propagator. The physical interpretation of the Feynman propagator is that a quantum is created at spacetime point y, it then travels to spacetime point x and vanishes there. For antiparticles, the relation is vice versa: The antiparticle is created at x and annihilated at y.
$\rightarrow$ field: from $y \rightarrow x$
$\rightarrow$ antifield: from $x \rightarrow y$.
In quantum field theory, anything that can happen is a sum of subprocesses and can be written down using propagators and drawn using the respective Feynman graphs.

8 Lagrangian Density

The action $\mathcal{S}$ is defined using the Lagrangian density $\mathcal{L}$:

$$\mathcal{S} = \int \mathrm{d}^4x\, \mathcal{L}(x)$$

$\mathcal{S}$ is bosonic, hermitian, and Poincaré invariant. The Lagrangian density of a field and the partial derivatives of a field depends on the field points x, not on the actual fields themselves: We have a local Lagrangian density.

$$\mathcal{L}\Big[\phi_i(x), \partial_\mu\phi_i(x)\Big] \equiv \mathcal{L}(x)$$

$\mathcal{L}$ contains no second derivatives, and is at most quadratic in first derivatives. $\mathcal{L}$ has no constant term, and also no linear term. This can be shown, among other methods, using the path integral formalism. The terms quadratic in ϕ_i and $\partial_\mu\phi_i$ are called free Lagrangian. Higher orders of ϕ_i and $\partial_\mu\phi_i$ contribute to the interacting Lagrangian:

$\phi^n \qquad n > 2$	
$g_n\phi^n$	
$c_{2,L}\phi^2\partial_\mu\phi\partial_\mu\phi$	$c_{2,L}$: "coupling constant"
~~$\phi\partial_\mu\phi$~~	(would violate Poincaré invariance)

A short remark: The coupling "constant" $c_{2,L}$ is actually not a constant, but dependent on x. For now, we will however use it as a constant.

8.1 Variation principle

The variation of the action has to be zero, therefore we can derive the Euler-Lagrange equations from the definition of the action.

$$\begin{aligned} 0 = \delta\mathcal{S} &= \int \mathrm{d}^4 \delta\mathcal{L} \\ &= \int \mathrm{d}^4 \left[\delta\phi_i \frac{\partial\mathcal{L}}{\partial\phi_i} + \delta\Big(\partial_\mu\phi_i\Big) \frac{\partial\mathcal{L}}{\partial\Big(\partial_\mu\phi_i\Big)} \right] \\ &= \frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\Big(\partial_\mu\phi_i\Big)} \equiv 0 \end{aligned}$$

Euler-Lagrange equation

In the second line, we make the assumption that $\delta\partial_\mu\phi_i = \partial_\mu\delta\phi_i$.

8.2 Canonical coordinates and momenta

As we know from classical Lagrangian mechanics, a Hamiltonian density can be defined using canocinal coordinates and canonical momenta. The ϕ_i work as canocial coordinates, the canocial momenta are defined as usual.

$$\pi_i = \frac{\partial}{\partial\Big(\partial_0\phi_i\Big)}\mathcal{L}$$

π_i ist linear in in $\partial_0\phi_i$, therefore

$$\partial_0\phi_i = \Phi\Big(\phi_j, \partial_s\phi_j, \phi_j\Big) \qquad , s \in \{1,2,3\}$$

And once again the commutators for equal time are given by

$$\begin{aligned} \Big[\phi_i(x), \phi_j(y)\Big\}\Big|_{x_0=y_0} &= 0 \\ \Big[\pi_i(x), \pi_j(y)\Big\}\Big|_{x_0=y_0} &= 0 \\ \Big[\pi_i(x), \phi_j(y)\Big\}\Big|_{x_0=y_0} &= i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}) \end{aligned}$$

8.3 Hamiltonian density

The canonical coordinates and momenta given, we can easily define the Hamiltonian density $\mathcal{H}$ as

$$\begin{aligned} \mathcal{H} &:= \sum_i \partial_0\phi_i\pi_i - \mathcal{L}(x) \\ H &= \int \mathrm{d}^3\vec{x}\,\mathcal{H} \end{aligned}$$

From ETCR (equal time commutator relations) and the Euler-Lagrange equation, we derive the following commutator relations:

$$\left[i\mathcal{H}, \phi_j\right] = \partial_0 \phi_j$$
$$\left[i\mathcal{H}, \pi_j\right] = \partial_0 \pi_j$$

Here, i of course denotes the imaginary unit, $i = \sqrt{-1}$, and is not an index.
In general, the Hamiltonian and the Hamiltonian density are known as soon as $\mathcal{L}$ is known. We will now write H in terms of $\hat{\alpha}^\dagger$ and $\hat{\alpha}$:

$$\begin{aligned} H &= \int \mathrm{d}^3\vec{x} \frac{1}{2}\left[\pi^2(x) + \vec{\nabla}\phi \cdot \vec{\nabla}\phi + \mu^2\phi\right] \\ &= \dots \\ &= \frac{1}{2}\int \mathrm{d}^3\vec{k}\omega\vec{k}\left(\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)\right) \\ &= \underbrace{\frac{1}{2}\int \mathrm{d}^3\vec{k}\ \omega_{\vec{k}}\, \hat{a}^\dagger(k)\hat{a}(k)}_{\text{like harmonic oscillator}} + \underbrace{\frac{1}{2}\int \mathrm{d}^3\vec{k}\ \omega_{\vec{k}}\, \delta^{(3)}(\vec{0})}_{\text{ill-defined!}} \end{aligned}$$

To finish, define normal ordering, $:(\cdot):$, so that all annihilators in $(\cdot)$ are right of all creators.

$$\begin{aligned} H &= \int \mathrm{d}^3\vec{x} \frac{1}{2} :\left(\pi^2(x) + \vec{\nabla}\phi \cdot \vec{\nabla}\phi + \mu^2\phi\right): \\ &= \int \mathrm{d}^3\vec{x}\, \omega(\vec{k})\, \hat{a}^\dagger\hat{a} \end{aligned}$$

Propagators

07.11.2011

Reminder: In the last lecture, we defined the Hamiltonian using the creation and annihilation operators. One term showed up that made no sense because it included a δ-function of zero - this term will be disregarded by normal ordering. Normal ordering, once again, means that all creation operators will be left of all annihilation operators. Normal ordering is denoted by colons left and right of the expression it is supposed to act on.
The derived formula for the Hamiltonian was

$$H \coloneqq \int \mathrm{d}^3\vec{k}\omega(\vec{k})\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \frac{1}{2}\int \mathrm{d}^3\vec{k}\omega(\vec{k})\delta^{(3)}(0)$$

This lecture's project will be to collect all relevant formulas for propagators.

9 The free field

A free field $\phi(t, \vec{x})$ is a function of space and time. It can be written using a Fourier transform, the Lorentz covariant measure, and the creation and annihilation operators.

$$\phi(t, \vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}(2\pi)^3}\left\{e^{ixp}\hat{a}^\dagger(\vec{p}) + e^{-ixp}\hat{a}(\vec{p})\right\}$$

With $\omega_{\vec{p}}$, we denote that which we formerly called $\omega(\vec{p})$, which is the same expression in two different notations.
Having a close look at the integrand, we realize that it can be read in a different way. Taking into account the creator, it seems to create a particle with momentum $\vec{p}$ at some point in space-time, whereas the annihilator apparently deletes a particle with momentum $\vec{p}$. A propagator describes the creation of a quantum in one space-time point and the annihilation in a generally different point in space-time. Usually, this is denoted by the product of two fields.
For $t_1 > t_2$, that means t_1 happens after t_2 (is later than t_2), we get

$$\phi(t_1, \vec{x}_1) \cdot \phi(t_2, \vec{x}_2) = \int \frac{\mathrm{d}^3\vec{p}}{2\omega_{\vec{p}}(2\pi)^6} e^{-ip(x_1 - x_2)}$$

Comparing this result with that for $t_2 > t_1$ and sorting it into odd and even pieces of $\vec{x}_2 - \vec{x}_1$, we find that

$$\phi(t_1, \vec{x}_1) \cdot \phi(t_2, \vec{x}_2) = -i\Delta_F(t_1 - t_2, \vec{x}_1 - \vec{x}_2)$$

Here, Δ_F is the **Feynman propagator** defined by

$$\Delta_F(t, \vec{x}) \coloneqq i\int \frac{e^{-i\omega_{\vec{p}}|t| + i\vec{p}\vec{x}}}{2\omega_{\vec{p}}}\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

Careful: Sometimes, Δ_F denotes the Feynman propagator, but in other cases the scalar product of Δ_F between bra and ket vacuum states, $\langle 0|\,\Delta_F\,|0\rangle$, is called Feynman propagator. It will be clear from the context "which Feynman propagator" is meant.

9.1 Properties of the Feynman Propagator

1. By construction, it fulfills $(\Box + m^2)\Delta_F = \delta_x$ where $\Box = \partial_t^2 - \vec{\nabla}_x \cdot \vec{\nabla}_x$. This yealds that Δ_F is the Green's funtion to the Klein-Gordon equation: It solves $Lx = \delta$ if L is an operator and δ is a δ-distribution.

2. In momentum space,

$$\hat{\Delta}_F(\omega_{\vec{p}}, \vec{p}) = \lim_{\epsilon \to 0_+} \frac{1}{-\omega^2 + \left|\vec{p}\right|^2 + i\epsilon} = \lim_{\epsilon \to 0_+} \frac{1}{-p^2 + m^2 - i\epsilon}$$

   In the second limit, we used $p^2 = p_0^2 - \vec{p}^2$. All limits are in the weak topology of tempered distributions.

3. Δ_F is invariant under the full Lorentz group.

4. Δ_F is a C^∞ function away from the light cone, supported in $\mathbb{R}^4$: It is nicely differentiable anywhere but on the light cone.

To show (1.), it is equivalent to show that $g(t) \coloneqq ie^{-i\omega_{\vec{p}}|t|}$ satisfies

$$\begin{aligned}
\left(\partial_t^2 + \left|\vec{p}\right|^2 + m^2\right) g(t) &= 2\omega_{\vec{p}}\delta(t) \\
\Rightarrow \quad \partial_t g(t) &= \omega_{\vec{p}} \operatorname{sgn}(t) e^{-i\omega_{\vec{p}}|t|} \\
\partial_t^2 g(t) &= 2\omega_{\vec{p}}\delta(t) e^{-i\omega_{\vec{p}}|t|} - i\omega_{\vec{p}}^2 e^{-i\omega_{\vec{p}}|t|} \\
&= \left(2\omega_{\vec{p}}\delta(t) - (\left|\vec{p}\right|^2 + m^2)\right) g(t) \qquad \square
\end{aligned}$$

We will also take into account the tempered distribution on $\mathbb{R}^4$ with Fourier transform:

$$\mathcal{F}_{\vec{x}}(\Delta_F)(t, \vec{p}) = \frac{ie^{-i\omega_{\vec{p}}}\,|t|}{2\omega_{\vec{p}}}$$

Let $\epsilon < 0$, $\vec{p} \in \mathbb{R}$, and let $\mathcal{U}(\epsilon, \vec{p})$ be the root of the denominator, $+i\epsilon - \left|\vec{p}\right|^2 - m^2$, with positive real part.

$$\begin{aligned}
\frac{1}{-\omega^2 + \left|\vec{p}\right|^2 + m^2 - i\epsilon} &= -\frac{1}{2\mathcal{U}}\left[\frac{1}{\mathcal{U} + i\omega} + \frac{1}{\mathcal{U} - i\omega}\right] \\
&= -\frac{1}{2\mathcal{U}}\left[\int_0^\infty e^{-(\mathcal{U}+i\omega)t}\,\mathrm{d}t + \int_{-\infty}^0 e^{-(\mathcal{U}-i\omega)t}\,\mathrm{d}t\right]
\end{aligned}$$

As a function of ω, $\left(-\omega^2 + \left|\vec{p}\right|^2 + m^2 - i\epsilon\right)^{-1}$ is $\mathcal{F}_t\left(-\frac{1}{2\mathcal{U}}e^{-\mathcal{U}|t|}\right)$. Computing

$$\begin{aligned}
\lim_{\epsilon \to 0} \mathcal{U} &= i\sqrt{\left|\vec{p}\right|^2 + m^2} = i\omega_{\vec{p}} \\
-\frac{1}{2\mathcal{U}}e^{-\mathcal{U}|t|} &\to \frac{i}{2\omega_{\vec{p}}}e^{-i\omega_{\vec{p}}|t|}
\end{aligned}$$

then (2.) and (3.) are obvious.
One representation of the Feynman propagator is

$$\Delta_F = \frac{im}{4\pi^2\sqrt{\left|\vec{x}\right|^2 - t^2}} K_1\left(m\sqrt{\left|\vec{x}\right|^2 - t^2}\right)$$

on $\mathbb{R}^4 \setminus \left\{(t,\vec{x}) : |t| = \left|\vec{x}\right|\right\}$. K_1 is the modified Bessel function of ordner one, Δ_F is defined everywhere in space-time besides on the light cone. Then, (4.) follows from documented properties of the Bessel function.

$$-\left(\frac{1}{p^2 - m^2 + i\epsilon}\right) \quad \hat{=} \quad \text{"Feynman propagator"}$$

10 More propagators

The Klein-Gordon equation $(\Box + m^2)\phi(x) = 0$ is not only solved by one propagator. In fact, there are quite a few propagators fulfilling this equation.

1.

$$\Delta(x, m^2) \equiv \frac{-i}{(2\pi)^3}\int \mathrm{d}^4 p \frac{p_0}{|p_0|}\delta(p^2 - m^2)e^{-ipx}$$

The δ-function puts everything on the mass shell.

2.

$$\Delta^{(1)}(x, m^2) \equiv \frac{1}{(2\pi)^3}\int \mathrm{d}^4 p \delta(p^2 - m^2)e^{-ipx}$$

Those two are called the "invariant δ-functions". Δ is odd under $x_\mu \to -x_\mu$, $\Delta^{(1)}$ is even under that same transformation. $\Delta = 0$ for spacelike arguments, that is if $x^2 < 0$.

10.1 Properties of invariant δ-functions

- $(\Box + m^2)\Delta(x, m^2) = 0$
- $\Delta(x, m^2)\Big|_{x_0=0} = 0$
- $\partial_0\Delta(x, m^2)\Big|_{x_0=0} = -\delta(\vec{x})$

Δ and $\Delta^{(1)}$ are Bessel functions of first order followed by terms including Θ-functions (Heaviside step functions) and δ-functions.
We will now examine how Δ and $\Delta^{(1)}$ behave near $x^2 = 0$.

$$\Delta(x, m^2) = -\frac{1}{2\pi}\frac{x_0}{|x_0|}\left\{\delta(x^2) + m^2\left(-\frac{1}{4} + O(x^2)\right)\Theta(x^2)\right\}$$

$$\Delta^{(1)} = -\frac{1}{2\pi^2}\left\{P\frac{1}{x^2} - \frac{m^2}{4}\log(m^2)\left|\vec{x}^2\right| + O(1)\right\}$$

Here, P denotes the Cauchy principal value. As a reminder, note that

$$\frac{1}{x + i\epsilon}\Big|_{\epsilon \ll 1} = P\frac{1}{x} + i\delta(x)$$

where the principal value gives the real part.

10.2 Important propagators in physics

Physicists usually define two more propagators, $\Delta^{(\pm)}$, as "positive frequency δ-functions". They are defined in the following way:

$$\Delta^{(\pm)}(x, m^2) := \mp(2\pi i)^3 \int \mathrm{d}^4 p \Theta(\pm p_0)\delta(p^2 - m^2)e^{-ipx}$$

And with that definition, we immediately see

$$\begin{aligned} i\Delta &= \Delta^{(+)} + \Delta^{(-)} \\ \Delta^{(1)} &= \Delta^{(+)} - \Delta^{(-)} \\ \Delta^{(+)}(-x, m^2) &= -\Delta^{(-)}(x, m^2) \end{aligned}$$

Again, these propagators are Green functions $\mathcal{G}$ to the Klein Gordon equation:

$$(\Box + m^2)\mathcal{G} = \delta^{(4)}(x)$$

and we define the propagators

$$\begin{aligned} \bar{\Delta}(x, m^2) &:= -\frac{1}{2}\frac{x_0}{|x_0|}\Delta(x, m^2) \\ \Delta_R(x, m^2) &:= -\Theta(x_0)\Delta(x, m^2) \\ \Delta_A(x, m^2) &:= \Theta(-x_0)\Delta(x, m^2) \end{aligned}$$

The R in the index of the second propagator stands for "retarded". Δ_R is the retarded Feynman propagator: It propagates a delay. Analogously, A stands for "advanced" since Δ_A refers to later events.
Finally, we can write the Feynman propagator as a linear combination of the above:

$$\begin{aligned} \Delta_F(x, m^2) &\equiv \bar{\Delta} + \frac{i}{2}\Delta^{(1)} \\ &= \Theta(x_0)\Delta^{(+)} - \Theta(-x_0)\Delta^{(-)} \end{aligned}$$

10.3 The role of the retarded and advanced propagator

Taking into account that

$$\mathcal{F}_t\left(-\frac{1}{2\mathcal{U}}e^{-\mathcal{U}|t|}\right) = \frac{1}{-\omega^2 + \left|\vec{p}\right|^2 + m^2 - i\epsilon},$$

we get for the retarded and advanced propagators

$$\begin{aligned} \Delta_R &= -(2\pi)^{-4} \int \mathrm{d}^4 p \frac{e^{-ipx}}{p^2 - m^2 + i\epsilon p_0} \\ \Delta_A &= -(2\pi)^{-4} \int \mathrm{d}^4 p \frac{e^{-ipx}}{p^2 - m^2 - i\epsilon p_0} \\ \Delta_F &= i(2\pi)^{-4} \int \mathrm{d}^4 p \frac{e^{-ipx}}{p^2 - m^2 + i\epsilon} \end{aligned}$$

To get from the advanced to the retarted operator, one has to change the sign before the imaginary term in the demoninator. This can be regarded as a sign flip in energy and can later on be associated with ghost and antighost particles.

10.3.1 Limit: $m \to 0$

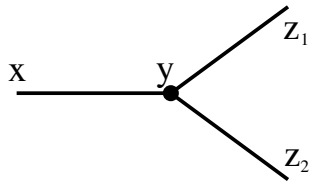
In the massless limit, that is the propagation of a massless particle such as the photon, the propagators become

$$\left.\begin{array}{l} \Delta \to D = (2\pi)^{-1} \frac{x_0}{|x_0|}\delta(x^2) \\ \Delta^{(1)} \to D^{(1)} = -(2\pi^2)^{-1} P\frac{1}{x^2} \end{array}\right\} \Rightarrow \begin{array}{l} D_R = (2\pi)^{-1}\Theta(x_0)\delta(x^2) \\ D_A = (2\pi)^{-1}\Theta(-x_0)\delta(x^2) \\ D_F = -(2\pi)^{-2}\frac{1}{x^2 - i\epsilon} \end{array}$$

D_F is the **massless Feynman propagator**.

10.4 Transistion amplitudes between states

We have regarded $\langle 0| \Delta_F |0\rangle$ as the creation of one particle somewhere in vacuum and its annihilation elsewhere. But how do we compute the probability for this process?
In order to understand that, let us focus on the creation of one particle, say, in the presence of one other particle. We then have an initial one-particle state and a final two-particle state. What is the probability for the creation of a second quantum at spacetime point y with free propagation afterwards?



If we assume that this process can happen with a probability $g < 1$ at any point in spacetime, g must be constant over $\mathbb{R}^4$.
But if we only know the initial and final states and have no means of measurement that can tell us exactly what happens at the interaction, we must not only consider the simplest transistion, but also all kinds of interactions that can leed to the desired final state. Quantum physics demands a sum over all possibilities in order to compute the correct transition amplitude and therefore the propability for the process of one particle decaying into two.

10.4.1 The simplest case

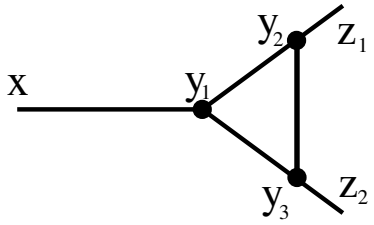
If we consider only one interaction, namely the creation of one particle in one point in spacetime, then the transistion amplitude must look something like

$$g \int \mathrm{d}^4 y \; \Delta_F(x-y)\Delta_F(y-z_1)\Delta_F(y-z_2)$$

and this will give us a nice, convergent answer. The factor g denotes the one vertex and quantifies the strength of the interaction. The Feynman propagators denote the motion of the respective particles in spacetime. The integration is carried out over all free variables, and since x, z_1 and z_2 are fixed, there is just y left to integrate over.

10.4.2 The next order

To go to the successing interaction process, we add a creation of a third particle and its annihilation.



Then the transition amplitude becomes

$$g^3 \int \mathrm{d}^4y_1\ \mathrm{d}^4y_2\ \mathrm{d}^4y_3\ \Delta_F(x-y_1)\Delta_F(y_1-y_2)\Delta_F(y_2-y_3)\Delta_F(y_3-y_1)\Delta_F(y_2-z_1)\Delta_F(y_3-z_2)$$

This graph also contributes to the transition amplitude of the entire process, but since $g < 1$, g^3 is even smaller so it won't play as big a role.
Taking into account that $\Delta_F(x-y_1) \sim \delta(x-y_1)$ and so forth, the subsequent integrations lead to an ill-defined expression:

$$\underbrace{\Delta_F(y_1-y_2)\Delta_F(y_2-y_3)}_{\Rightarrow\delta(y_1-y_3)}\underbrace{\Delta_F(y_3-y_1)}_{\Rightarrow\delta(y_3-y_1)} \sim \delta(y_1-y_3)\delta(y_3-y_1)$$

Two δ distributions which support the same argument: That's an ill-defined expression! Here's our first Feynman propagator calculation that diverges, even though the process itself seems highly possible. A physically sensible result would give a result whose square of the absolute value gives a number in between 0 and 1.

10.4.3 Possible solution

Mathematically, it is possible to define everything on configuration space and leave out the diagonals, then the problem does not exist and the expression becomes well-defined.

Noether's Theorem

14.11.2011

Reminder: In the last lecture, we found different Green's functions to the Klein-Gordon equation, each with a slightly different purpose. We also considered the massless limit.

$$\Delta_R = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4p \frac{e^{ipx}}{p^2 - m^2 + i\epsilon_+ p_0} \qquad m \to 0 \qquad D_R = \frac{1}{2\pi}\Theta(x_0)\delta(x^2)$$

$$\Delta_A = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4p \frac{e^{ipx}}{p^2 - m^2 - i\epsilon_+ p_0} \qquad m \to 0 \qquad D_A = \frac{1}{2\pi}\Theta(-x_0)\delta(x^2)$$

$$\Delta_F = \frac{i}{(2\pi)^4}\int \mathrm{d}^4p \frac{e^{ipx}}{p^2 - m^2 + i\epsilon_+} \qquad m \to 0 \qquad D_R = -\frac{1}{2\pi^2}\frac{1}{x^2 - i\epsilon_+}$$

The question, now, is: What constrains Feynman rules? How are diagrams calculated using propagators, what are the underlying standards? Generally, we can assume that vertices affect the calculation in some way, so do free propagators and free fields. Fields in general carry representations of groups. Those are:

- Transformations in spacetime, most importantly rotations, boosts, translations, but also dilatations and special conformal transformations.
- Local internal symmetries: These are the local symmetries that imply gauge freedom.
- Global internal symmetries: These have to do with aspects of the theory such as flavor, so they play an important role in particle physics.

11 Neother's Theorem (for classical fields)

Let $\{\phi_A\}$ be a set of scalar fields. We can define a Lagrangian density: $\mathcal{L} = \mathcal{L}\Big(\{\phi\}, \{\partial_\mu\phi\}\Big)$. Let us examine the variation of ϕ_A:

$$\begin{aligned} \phi_A &\to \phi'_A = \phi_A + \delta\phi_A \\ \mathcal{L} &\to \mathcal{L}' = \mathcal{L} + \delta\mathcal{L} \\ \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi_A}\delta\phi_A + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_A\right)}\partial_\mu\delta\phi_A \end{aligned} \tag{2}$$

In the last line, we have assumed that $\delta\partial_\mu\phi_A = \partial_\mu\delta\phi_A$, that the variation of the derivative equals the derivative of the variation. This comes from the simple fact that all fields are supposed to disappear in infinite space, therefore there are no boundary terms which would make a calculation difficult, like in classical mechanics.
Now we will take the action into account: The action $\mathcal{S}$ is defined as the integral of the Lagrangian density over spacetime, $\mathcal{S} = \int \mathrm{d}x\, \mathcal{L}$. Variation of $\mathcal{S}$ gives eventually

$$\frac{\delta\mathcal{S}}{\delta\phi_A} = \frac{\partial\mathcal{L}}{\partial\phi_A} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}$$

and this equation is equal to zero for equations of motion. Since we are considering a more general case here, we will leave it as it is for the moment, instead of setting it equal to zero right now. We will rather rewrite (2) using

$$\frac{\partial\mathcal{L}}{\partial\phi_A} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)} + \frac{\delta\mathcal{S}}{\delta\phi_A}$$

and this plugged into (2), the variation of $\mathcal{L}$, gives

$$\delta\mathcal{L} = \partial_\mu \underbrace{\left\{ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\delta\phi_A \right\}}_{=: j^\mu} + \frac{\delta\mathcal{S}}{\delta\phi_A}\delta\phi_A$$

Here we defined the Noether current $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\delta\phi_A$ which, if it is conserved, gives rise to equations of motion.

$$\partial^\mu j_\mu = \delta\mathcal{L} - \frac{\delta\mathcal{S}}{\delta\phi_A}\delta\phi_A \equiv 0 \qquad \text{for equations of motion}$$

Writing out the four-derivative of the Noether current, we find that we have a charge density and a current density:

$$\begin{aligned} \delta^\mu j_\mu = 0 \quad &\Leftrightarrow \quad \frac{\partial}{\partial t} j^0 + \vec{\nabla}\cdot\vec{j}(x) = 0 \\ j^0 : \quad &\text{charge density} \\ \vec{j}(x) : \quad &\text{current density} \end{aligned}$$

Noether's theorem now states that **if the Lagrangian is not invariant under a transformation, the four-current-density will not be conserved**. In other words, **a symmetry of the Lagrangian implies a conserved four-current.**

11.1 Invariance under rotation

As an example, let us work with the Lagrangian

$$\mathcal{L} = -\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi + \frac{1}{4}\lambda\left(\phi^\dagger\phi\right)^2$$

and check that it is invariant under rotation:

$$\begin{aligned} \phi \quad &\rightarrow \quad \phi' = e^{-i\alpha}\phi \\ \phi^\dagger \quad &\rightarrow \quad \phi'^\dagger = e^{i\alpha}\phi'^\dagger \end{aligned}$$

Here, we have assumed that the field ϕ and its hermitian conjugate $\phi^\dagger$ are independent. We could also use a different notation in which we regard the real and the imaginary parts of the field ϕ as our basis. The two notations are linear combinations of each other and therefore equivalent.

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi_1\partial_\mu\phi_1 - \frac{1}{2}\partial^\mu\phi_2\partial_\mu\phi_2 - \frac{1}{2}m^2\left(\phi_1^2 + \phi_2^2\right) - \frac{1}{16}\left(\phi_1^2 + \phi_2^2\right)^2$$

The transformation is consequently given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

The infinitesimal rotation, where $\alpha \ll 1$ and therefore $e^{i\alpha} \approx 1 + i\alpha$, is then given by

$$\begin{aligned} \phi \quad &\rightarrow \quad \phi' = \phi \underbrace{-i\alpha\phi}_{=\delta\phi} \\ \phi^\dagger \quad &\rightarrow \quad \phi'^\dagger = \phi^\dagger \underbrace{+i\alpha\phi^\dagger}_{=\delta\phi^\dagger} \end{aligned}$$

11.2 Noether's current

Let's go back to the Noether current and rewrite it for both the field and its hermitian conjugate. As a convention, we write αj_μ with α as the constant parameter for the gobal phase transformation.

$$\alpha j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\delta\phi^\dagger \quad \rightarrow \quad j^\mu = \alpha \Im\left(\phi^\dagger \overset{\leftrightarrow}{\partial^\mu} \phi\right)$$

where $\Im$ denotes the imaginary part and $\overset{\leftrightarrow}{\partial^\mu}$ means that the derivative acts to its left and to its right in a way such that

$$A \overset{\leftrightarrow}{\partial^\mu} B \coloneqq A\partial^\mu B - (\partial^\mu A)B$$

11.2.1 Charge

We define a charge by the space integral over the charge density:

$$Q \coloneqq \int \mathrm{d}^3\vec{x}\, j^0(x)$$

If we assume that the current density vanishes in "space-infinity" (that is, if the current density is finite), then charge is conserved.
Using the definitions for ϕ and $\phi^\dagger$, we can define charge using creation and annihilation operators:

$$\left.\begin{aligned}\phi(x) &= \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\left(\hat{a}(\vec{k})e^{ikx} + \hat{b}^\dagger(\vec{k})e^{-ikx}\right)\\ \phi^\dagger(x) &= \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\left(\hat{b}(\vec{k})e^{ikx} + \hat{a}^\dagger(\vec{k})e^{-ikx}\right)\end{aligned}\right\} \qquad Q = \int \frac{\mathrm{d}^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\left(\hat{a}^\dagger\hat{a}(\vec{k}) - \hat{b}^\dagger\hat{b}(\vec{k})\right)$$

11.3 From global to local symmetries

We have seen that the Lagrangian in invariant under global phase transformations, in which the phase α is not dependent on the spacetime point x and therefore the rotation is merely a multiplication with a constant. Now we are going to make α a function of x and see how this changes the Lagrangian. We will, once again, work with the specific Lagrangian

$$\mathcal{L} = -\partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi + \frac{1}{4}\lambda\left(\phi^\dagger\phi\right)^2$$

which, as we have seen, has a global symmetry. We will now make the transformation

$$\phi \quad \rightarrow \quad \phi' = e^{-i\alpha(x)}\phi$$

act on the fields, which is a <u>local</u> phase transformation. This produces an extra term in the Lagrangian, namely $e^{-i\alpha(x)}\left(\partial_\mu\alpha(x)\right)$. If we tranform ϕ, we must also transform transform $\phi^\dagger$:

$$\phi^\dagger \quad \rightarrow \quad \phi'^\dagger = \phi^\dagger e^{i\alpha(x)}$$

The Lagrangian is <u>not</u> invariant, but dependent on $\alpha(x)$. But we can use this dependence and add Maxwell's field tensor, $\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, and a coupling to a gauge field in the form of a covariant derivative, $\partial_\mu \rightarrow D_\mu = \partial_\mu - iA_\mu$. This will make the Lagrangian

invariant under local phase transformations if we transform the vector field A in accordance: $A_\mu \rightarrow A'_\mu = A_\mu + i\partial_\mu \alpha(x)$.
Experimentally, we know that all interactions ever observed are interactions between matter and gauge bosons. This is exactly what the transformed Lagrangian describes.
(This fact will eventually lead to requiring a Higgs boson, a spin 0 scalar field whose coupling to other fields is responsible for the masses of particles, since some gauge bosons are massive. The simple addition of a mass term would break the symmetry in the Lagrangian. Experience, however, tells us that all particles have a high amount of symmetry. Therefore, the theory cannot be complete without the Higgs boson.)

11.4 Noether and spacetime symmetries

One of the rather simple transformations mentioned above is a spatial translation. We denote this by

$$\begin{aligned} x^\nu &\rightarrow x^\nu - \epsilon^\nu \\ \phi_A(x) &\rightarrow \phi_A(x) + \epsilon^\nu \partial_\nu \phi_A(x) \\ \mathcal{L}(x) &\rightarrow \mathcal{L}(x) + \epsilon^\nu \partial_\nu \mathcal{L}(x) \\ (j^\mu)_\nu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \partial_\nu \phi_A - \delta^\mu_\nu \mathcal{L} \equiv T^\mu{}_\nu \end{aligned}$$

In the last line, the energy momentum tensor $T^\mu{}_\nu$ was introduced using the Noether current. The Noether current is conserved if and only if $\partial_\mu T^\mu{}_\nu = 0$.

12 Lorentz Transformation

Before the next lecture, we will finish with a short outlook on Lorentz transformations. A Lorentz transformation is the tranformation between two intertial systems in relativity. We denote an infinitesimal Lorentz transformation (LT) by

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

$\omega^\mu{}_\nu$ is antisymmetric.
Scalar fields transform like

$$\begin{aligned} \phi(x) \rightarrow \phi'(x) &= \phi\left(\Lambda^{-1} x\right) = \phi\left(x^\mu - \omega^\mu{}_\nu x^\nu\right) = \phi\left(x^\mu\right) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) \\ \delta\phi &= -\omega^\mu{}_\nu x^\nu \partial_\mu \phi \\ \delta\mathcal{L} &= -\partial_\mu \omega^\mu{}_\nu x^\nu \mathcal{L} \\ \Rightarrow \quad j^\mu &= -\omega^\rho{}_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} x^\nu \partial_\rho \phi - \delta^\mu_\rho x^\nu \mathcal{L} \right) \\ &= -\omega^\rho{}_\nu T^\mu{}_\rho x^\nu \end{aligned}$$

Since $\omega^{\rho}{}_{\nu}$ is antisymmetric, there are six independent values to be found. These are associated with charge.

$$
\begin{aligned}
(j^{\mu})^{\rho\sigma} &= x^{\rho}T^{\mu\sigma} - x^{\sigma}T^{\mu\rho} \\
Q^{ij} &= \int \mathrm{d}^3\vec{x}\left(x^i T^{0j} - x^j T^{0i}\right) \\
Q^{0i} &= \int \mathrm{d}^3\vec{x}\left(x^0 T^{0i} - x^i T^{00}\right) \\
0 = \frac{\mathrm{d}Q^{0i}}{\mathrm{d}t} &= \int \mathrm{d}^3\vec{x}\, T^{0i} + t\int \mathrm{d}^3\vec{x}\,\frac{\partial T^{0i}}{\partial t} - \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\vec{x}\, x^i T^{00} \\
&= P^i + \underbrace{\frac{\mathrm{d}P^i}{\mathrm{d}t}}_{=0} - \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\vec{x}\, x^i T^{00}
\end{aligned}
$$

The Lorentz Group

21.11.2011

13 Introduction to Lorentz groups

We saw that transformations in timespace can be described using Lorentz transformations. These Lorentz transformations have a group structure, namely that of the Lorentz group. We denote by $O(1,3)$ the group of linear transformations of $\mathbb{R}^4$ that preserve $A \in O(1,3)$. In other words, the scalar product of two four-vectors is invariant under Lorentz transformations:

$$(A_x)_\mu (A_y)^\mu = x_\mu y^\mu = x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \quad \text{is invariant.}$$

We denote this invariance by using the metric, g, which is usually $g = \text{diag}(1,-1,-1,-1)$, sometimes with reversed sign, which would not change the physics, but in order to be consistent in notation one has to choose one of the two metrics. We are only going to work with $g = \text{diag}(1,-1,-1,-1)$ because this is more common in most textbooks.
The invariance is given by

$$A^\dagger g A = g \equiv g_{\mu\nu} \ \ (= \eta_{\mu\nu}) \qquad \mu, \nu \in \{0,1,2,3\}$$

where A denotes any element in $O(1,3)$, $A \in O(1,3)$. The indices μ and ν start with a zero to point out that the first component (the "zeroth" component) is the time component, whereas the other three are the known spatial components, usually denoted with 1-3. If the metric is $g'_{ij} = \text{diag}(+1)$, $i, j \in \{1,2,3,4\}$, we have Euclidean space and the transformation laws are a bit different. We will focus on the given Minkowsky metric, $g = \text{diag}(1,-1,-1,-1)$.
$O(1,3)$ can be decomposed into four connected components, given by the determinant and the sign of the 00-component of an arbitrary transformation A. The Lorentz group then splits into four subspaces, which we can denote in the following way:

$$\begin{aligned} \det A = +1 \quad &\Rightarrow \quad SO(1,3) \\ \text{sgn}\left(A^0{}_0\right) = +1 \quad &\Rightarrow \quad O^\uparrow(1,3) \end{aligned}$$

$O^\uparrow(1,3)$ preserves time as there is no Lorentz transformation that reverses the sign of the time component, $t \to -t$, within $O^\uparrow(1,3)$. The subspace we are mainly going to deal with is called the proper orthochronous Lorentz group, it contains the identical transformation (and is therefore a group) and is given by

$$SO^\uparrow(1,3) = SO(1,3) \cap O^\uparrow(1,3)$$

This gives way to

$$SO(3) = O(3) \cap SO^\uparrow(1,3)$$

14 Lie groups and Lie algebras

14.1 Lie groups

We will now continue with Lie groups which lead to Lie algebras. Lie groups are denoted by $so(1,3)$ and give way to commutator relations: $X^\dagger g + gX = 0$ (antisymmetry of the generators).

We now define a matrix $e_{\mu\nu}$ as a 4x4-matrix, which has entry 1 at μ, ν, and 0 elsewhere, that means that its only non-vanishing entry is where the two indices point at, for example,

$$e_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that the columns and rows' enumerations start with a zero, not with a one.
We now define the generators of the Lie group by

$$\begin{array}{ll} X_{jk} = e_{kj} - e_{jk} & \text{for spatial components} \\ X_{k0} = -X_{0k} = e_{k0} + e_{0k} & \text{for mixed components} \\ X_{\mu\nu} = -X_{\nu\mu} & \text{for } \mu < \nu,\ X_{\mu\nu} \text{ is a basis for } so(1,3). \end{array}$$

In the second line, care must be taken: X_{k0} is defined by $e_{k0} + e_{0k}$, and X_{0k} is not derived by reversing indices, but is $-X_{k0}$. Even though $e_{k0} + e_{0k}$ is symmetric in 0 and k, X_{k0} is per definition antisymmetric in k and 0.

14.2 Lie algebras

Lie algebras are generated by a commutator. In $so(1,3)$,

$$\left[X_{\mu\nu}, X_{\rho\sigma}\right] = 0 \quad \begin{array}{l} \text{if} \{\mu,\nu\} = \{\rho,\sigma\} \\ \text{or} \{\mu,\nu\} \cap \{\rho,\sigma\} = \emptyset \end{array}$$

Obviously, the commutator is non-zero if and only if exactly one index of the two generators agree!

$$\left[X_{\mu\nu}, X_{\nu\rho}\right] = g_{\nu\nu} X_{\mu\rho}$$

14.2.1 Remarks

- $\exp(sX_{jk})$ is a rotation around the l-axis with rotation angle s; respectively (j,k,l)-axes cyclically.
- $\exp(sX_{0k})$ is a boost along the k-axis with boost parameter s.
- By group action, we mean the action of matrices on vectors on the respective vector space, that is $\Lambda^{\mu}{}_{\nu} x^{\nu} = x^{\mu}$. $O(1,3)$ acts by A, or $A^{\dagger^{-1}}$ (action in TM by $A \rightarrow$ action T*M by $A^{\dagger^{-1}}$)

14.2.2 Examples

$$X_{30} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

15 Physics and the Lorentz group

15.1 Lorentz transformations

As stated above, the Lorentz groups splits into subspaces. We can make a difference between objects on the mass shell (that is, if $p^2 = m^2$) and objects on the negative mass shell ($p^2 = -m^2$). The latter are mostly neglected by physicists, for obvious reason.

$$X_m^+ = \left\{p;\ \ p^2 = m^2,\ p_0 > 0\right\}$$
$$X_m^- = \left\{p;\ \ p^2 = m^2,\ p_0 < 0\right\}$$
$$Y_m = \left\{p;\ \ p^2 = -m^2\right\}$$

X_m^+ and X_m^- define the Lorentz group we work with: All four-vectors with the same mass, which is the invariant square, as we remember. The two groups differ in sign of the time component. In the special case that $m = 0$, the two mass shells become the light cone, X_0^+ forms the forward, X_0^- the backward light cone.
In general, a vector that is on the mass shell is denoted by

$$p = \begin{pmatrix} p_0 = \omega_{\vec{p}} \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

15.1.1 Examples

- Lorentz transformation with $\det A = -1,\ \ \mathrm{sgn}\left(A^0{}_0\right) = +1$: Space or Parity transformation

$$\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Lorentz transformation with $\det A = -1,\ \ \mathrm{sgn}\left(A^0{}_0\right) = -1$: Time transformation or time reversal

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

Both examples are not in connected components of $O(1,3)$ and neither is in $O^\uparrow(1,3)$.
Since the Lorentz transformations form a group, to every Lorentz transformation there is an inverse Lorentz transformation. If $\Lambda^\mu{}_\nu$ is such a Lorentz transformation and $\Lambda^\mu{}_\nu x^\nu = x^\mu$, then $(\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho$ is the inverse Lorentz transformation.

15.2 Unitary transformations

Now we assign to each Lorentz transformation Λ a unitary transformation $U(\Lambda)$: Each $\Lambda \in SO^\uparrow(1,3)$ gives rise to a unitary operator $U(\Lambda)$ such that $U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda)$. To consider

only infinitesimal transformations, one writes

$$U\Big(\mathbb{I}_{4x4} + \delta\omega_{\mu\nu}\Big) = \mathbb{I} + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}$$

where $M^{\mu\nu}$ are hermitian, antisymmetric operators:

$$M^{\mu\nu} = -M^{\nu\mu}$$

M are the infinitesimal generators of the Lorentz transformation, $\delta\omega$ are the coefficients of the respective M. For example, if M dictates a rotation around a set axis, $\delta\omega$ will determine the magnitude of the rotation.
For transformations of transformations we write

$$U(\Lambda^{-1})U(\tilde{\Lambda})U(\Lambda) = U\Big(\Lambda^{-1}\tilde{\Lambda}\Lambda\Big)$$

and we expand the infinitesimal transformation $\tilde{\Lambda} = \mathbb{I} + \delta\tilde{\omega}$:

$$\begin{aligned} U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) &= \Lambda^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \\ U^{-1}(\Lambda)p^{\mu}U(\Lambda) &= \Lambda^{\mu}{}_{\nu}p^{\nu} \\ [M^{\mu\nu}, M^{\rho\sigma}] &= i\hbar\Big(g^{\mu\rho}M^{\mu\sigma} - g^{\mu\nu}M^{\mu\sigma} - g^{\nu\rho}M^{\nu\sigma}\Big) \end{aligned}$$

similar to $X_{\mu\nu}$, $M^{\mu\nu}$ describe boosts and rotations.

- Rotations $\vec{J}$:

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$$

- Boosts $\vec{K}$:

$$K_i = M^{i0}$$

They fulfill the commutator relations:

$$\begin{aligned} [J_a, J_b] &= i\hbar\epsilon_{abc}J_c \\ [J_a, K_b] &= i\hbar\epsilon_{abc}K_c \\ [K_a, K_b] &= -i\hbar\epsilon_{abc}J_c \end{aligned}$$

These are the Lie algebra generators of the Poincaré group (Lorentz group together with translations). With the Lorentz group, these lead to unitary transformations. The commutators have the form

$$\begin{aligned} [P^{\mu}, M^{\rho\sigma}] &= i\hbar\Big(g^{\mu\sigma}P^{\rho} - g^{\mu\rho}P^{\sigma}\Big) \\ [P^{\mu}, P\nu] &= 0 \end{aligned}$$

Translations then have the explicit form:

$$\begin{aligned} T^{-1}(\alpha)\phi(x)T(\alpha) &= \phi(x-\alpha) \\ T(\alpha) &= e^{-\frac{i}{\hbar}p^{\mu}a_{\mu}} \\ U^{-1}(\Lambda)\phi(x)U(\Lambda) &= \phi(\Lambda^{-1}x) \\ U^{-1}(\Lambda)\partial_{\mu}\phi(x)U(\Lambda) &= \Lambda^{\mu}{}_{\rho}\bar{\partial}^{\rho}\phi(\Lambda^{-1}x) \end{aligned}$$

Where $\bar{\partial}$ denotes the derivative by the transformed parameter:

$$\bar{\partial}^{\rho} := \frac{\partial}{\partial(\Lambda^{-1}x)_{\rho}}$$

16 Connection between $\mathcal{SO}^{\uparrow}(1,3)$ and $\mathcal{SL}(2,\mathbb{C})$

Before we can get into $\mathcal{SO}^{\uparrow}(1,3)$ and $\mathcal{SL}(2,\mathbb{C})$ in particular, let us specify the names of groups.

- With $\mathcal{GL}(n,\mathbb{F})$ we denote a general linear group of degree n over a field $\mathbb{F}$.
- $\mathcal{SL}(n,\mathbb{F})$ denotes that the determinant equals one ("special linear group").
- In $\mathcal{SL}(2,\mathbb{F})$, there are 2x2-matrices with complex entries. As a basis, we can choose the Pauli matrices σ_j and the 2x2 unitary matrix.

  – Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  – Rotations and boosts:

$$J_i \equiv \frac{1}{2}\sigma_i \qquad K_i \equiv \frac{i}{2}\sigma_i$$

$$J_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}$$

  Rotations and boosts fulfill the commutator relations:

$$[J_a, J_b] = i\epsilon_{abc} J_c$$
$$[K_a, K_b] = i\epsilon_{abc} K_c$$

  And therefore we have the same Lie algebra as *so*.
  The general representation of the Lorentz group in $\mathcal{SL}(2,\mathbb{C})$ is

$$W = \exp\Big\{ - i \sum_{j=1}^{3} \big(\alpha_j J_j + \beta_j k_j\big)\Big\}$$

  with uni-modular W: $|W| = 1$.
  Since K_j is a non-hermitian generator, $\mathcal{SL}(2,\mathbb{C})$ is not a unitary group.

Set $K_j =: iS_j$. We use $J_i \equiv \frac{1}{2}\sigma_i$ and $K_i \equiv \frac{i}{2}\sigma_i$ and get

$$W = \exp\Big\{ - \frac{i}{2}\big(\zeta_1\sigma_1 + \zeta_2\sigma_2 + \zeta_3\sigma_3\big)\Big\}$$

with

$$\zeta_j = \alpha_j + i\beta_j \quad \in \mathbb{C}$$

Remark: Since the Pauli matrices are in $2 \times 2_{\mathbb{C}}$, we can express the same term in $\mathcal{SO}(1,3)$ which uses $4 \times 4_{\mathbb{R}}$ matrices, but we won't go into this here.
We define an X that will be a complex 2×2-matrix.

$$X := x^0 \mathbb{I}_{2x2} + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

The determinant of X gives $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$. This is the invariant scalar product of the Lorentz four-vectors in Minkowsky space. Lorentz transformations of the form

$$x \to \Lambda x \qquad \Rightarrow \qquad x^\mu \to \Lambda^\mu{}_\rho x^\rho$$

correspond to unitary transformations

$$X \to WXW^\dagger$$

which leave the determinant invariant, and therefore the scalar product. Each Lorentz transformation can be represented by $\pm W$.

Representation Theory of the Lorentz group

23.11.2011

Reminder: The Lorentz group splits into four parts, we are working in the subspace $SO^{\uparrow}(1,3)$ that contains identity. We have seen the Lorentz transformation of a scalar field:

$$\begin{aligned} \phi(x) \quad &\rightarrow \quad \phi'(x) = U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x) \\ \Lambda^{\mu}{}_{\rho}\bar{\partial}^{\rho}\phi(\Lambda^{-1}x) &= U^{-1}(\Lambda)\partial^{\mu}\phi(x)U(\Lambda) \end{aligned}$$

with

$$\bar{\partial}^{\rho} := \frac{\partial}{\partial(\Lambda^{-1}x)_{\rho}}$$

17 Lorentz transformation of vector and tensor fields

For vector and tensor fields, the Lorentz transformations are as follows:

$$\begin{aligned} U^{-1}(\Lambda)A^{\mu}U(\Lambda) &= \Lambda^{\mu}{}_{\rho}A^{\rho}(\Lambda^{-1}x) \\ U^{-1}(\Lambda)B^{\mu\nu}U(\Lambda) &= \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}B^{\rho\sigma}(\Lambda^{-1}x) \end{aligned}$$

To contract a field, we assign to every x a second order tensor $B^{\mu\nu}$ and contract with the metric tensor.

$$\begin{aligned} g_{\mu\nu}B^{\mu\nu} &= T(x) \\ g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} &= g_{\rho\sigma} \\ U^{-1}(\Lambda)T(x)U(\Lambda) &= T(\Lambda^{-1}x) \end{aligned}$$

B can be split into an antisymmetric part A, a traceless symmetric part S and a contracted tensor T.

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4}g_{\mu\nu}T(x)$$

18 Lorentz transformation of a field

We will now write the Lorentz transformation of a field ϕ_A in a more general way, namely that it tranforms into a different field ϕ_B using finite dimensional matrices L_A^B:

$$U^{-1}(\Lambda)\phi_A(x)U(\Lambda) = L_A^B(\Lambda)\phi_B(\Lambda^{-1}x)$$

We see that ϕ_B has transformed coordinates as above. Rules for L are

$$L_A^B(\Lambda')L_B^C(\Lambda) = L_A^C(\Lambda'\Lambda) \tag{3}$$

18.1 Infinitesimal Lorentz transformation

If we consider only infinitesimal Lorentz transformations, we look at transformations close to identity. We denote these infinitesimal Lorentz transformations with identity $\mathbb{I}$ and a small parameter $\delta\omega$.

$$U(\Lambda) \to U(\mathbb{I} + \delta\omega) = \mathbb{I} + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}$$

where M fulfills the commutator relation

$$\begin{aligned}\Big[M_{\mu\nu}, M_{\rho\sigma}\Big] &= i\Big(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\Big) \\ &= i\Big(\Big(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)\Big) - \Big(\rho \leftrightarrow \sigma\Big)\Big)\end{aligned} \tag{4}$$

Now we expand L^B_A, the matrices that describe the (infinitesimal) Lorentz transformation.

$$L_A{}^B(1 + \delta\omega) = \delta_A{}^B + \frac{i}{2}\delta\omega_{\mu\nu}\Big(S^{\mu\nu}\Big)_A{}^B$$

From (3) we derive that the commutator of the field ϕ_A with the tensor $M^{\mu\nu}$ is exactly

$$\begin{aligned}[\phi_A(x), M^{\mu\nu}] &= \mathcal{L}^{\mu\nu}\phi_A(x) + \Big(S^{\mu\nu}\Big)_A{}^B\phi_B(x) \\ \mathcal{L}^{\mu\nu} &= -i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\end{aligned}$$

We also note that $[\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}]$ and $[S^{\mu\nu}, S^{\rho\sigma}]$ both represent the same Lie algebra.

18.2 Representations of transformation matrices

Now we are trying to find representations for these matrices. As a reminder, we will consider non-relativistic quantum mechanics where the angular momentum J_i is $(2j + i) \times (2j + 1)$ dimensional, since $J_i = J_i^\dagger$ can be diagonalized along one carthesian axis. For convenience reasons, one almost always uses the third component axis (z-axis). J_i has eigenvalues $J_3 \in \{-j, -j + 1, \ldots, +j\}$. j is half integer.
Now, in relativistic quantum mechanics, or quantum field theory, we have sort of two copies of non-relativistic situations, one for rotations, one for boosts, and we denote them now by N.

$$\begin{aligned}N_j &= \frac{1}{2}\Big(J_j - iK_j\Big) \\ N_j^\dagger &= \frac{1}{2}\Big(J_j + iK_j\Big)\end{aligned}$$

with commutator relations derived from those of J_i and K_i:

$$\begin{aligned}[N_a, N_b] &= i\epsilon_{abc}N_c \\ \Big[N_a^\dagger, N_b^\dagger\Big] &= i\epsilon_{abc}N_c^\dagger \\ \Big[N_a, N_b^\dagger\Big] &= 0\end{aligned}$$

The first two commutator relations show that we have two copies of $\mathcal{SU}(2)$, connected by hermitian conjugation. From the third commutator we see that the two quantities are not interacting. The simplest representation is given by a set two numbers, (n, n'):

$(0,0)$	representative of a "singlett" or "scalar field"
$\left(\frac{1}{2},0\right)$	representation of a left-handed spinor
$\left(0,\frac{1}{2}\right)$	representation of a right-handed spinor
$\left(\frac{1}{2},\frac{1}{2}\right)$	representation of a vector

Remark: The number of components of n and n' is is $(2n+1)(2n'+1)$.
We can now write the angular momentum J_j as a linear combination of N_j and $N_j^\dagger$:

$$J_j = N_j + N_j^\dagger$$

J has eigenvalues from $|n-n'|,\ |n-n'|+1,\ \ldots, n+n'$. For $n = \frac{1}{2} = n'$ we get the two eigenvalues $j = \frac{1}{2} - \frac{1}{2} = 0$ and $j = \frac{1}{2} + \frac{1}{2} = 1$.

19 Local coordinates

19.1 Left-handed spinor field

Since in physics, it is often useful to use local coordinates, we will now take a closer look at *van der Werden's* transformation rules. We consider a representation of a two component left-handed spinor, $\psi \in \left(\frac{1}{2},0\right)$, and transform it as usual.

$$U^{-1}(\Lambda)\psi_a(x)U(\Lambda) = L_a^b(\Lambda)\psi_b(\Lambda^{-1}x) \qquad ,\psi \in \left(\frac{1}{2},0\right)$$

$$\text{and} \qquad L_a{}^b(\Lambda')L_b{}^c(\Lambda) = L_a{}^c(\Lambda'\Lambda)$$

We associate a with the spin direction which can be either up or down, for an arbitrary spatial axis. We denote the spin direction with arrows, $a \in \{\uparrow,\downarrow\}$.

19.2 Infinitesimal representation

We are looking for antisymmetric 2×2 matrices $\left(S_L^{\mu\nu}\right)_a{}^b$ which fulfill (4). Therefore the commutator relation must be

$$[\psi_a, M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\psi_a(x) + \left(S_L^{\mu\nu}\right)_a{}^b\psi_b(x)$$

Setting $x \equiv 0$ will make $\psi_a(x) \to \psi_a(0)$ a constant. Then the spatial part of the matrix M will be the total antisymmetric epsilon tensor times the angular momentum

$$M^{jk} = \epsilon^{jkl}J^l$$

with commutator relation

$$\epsilon^{jkl}[\psi_a(x), J_l] = \left(S_L^{jk}\right)_a{}^b\psi_b(0)$$

The matrix S^{jk} can be assiciated with the Pauli matrices since these matrices fulfill all the requirements.

19.2.1 Representation for S^{jk}: The Pauli matrices

The Pauli matrices, together with the unity matrix, are a basis of (2×2) complex matrices. They are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We can now express S^{jk} with the Pauli matrices in the following way:

$$S_L^{jk} = \frac{1}{2}\epsilon^{jkl}\sigma_l$$

With angular momentum J_j and momentum K_j given by

$$J_j = N_j + N_j^\dagger \qquad K_j = i\left(N_j - N_j^\dagger\right)$$

we denote S in general by

$$\left(S_L^{k_0}\right) = \frac{1}{2}\alpha\sigma_k \qquad \text{with } \alpha = \begin{cases} i & \text{for rotations} \\ 1 & \text{for boosts} \end{cases}$$

19.3 Right-handed spinor fields

In the right-handed notation, $\left(0, \frac{1}{2}\right)$, we add a dot to all indices. These dots are merely there to point out the difference between left-handed and right-handed representations and do not indicate something like time derivatives at all.

$$U^{-1}(\Lambda)\psi_{\dot a}^\dagger U(\Lambda) = R_{\dot a}{}^{\dot b}(\Lambda)\psi_{\dot b}^\dagger(\Lambda^{-1}x)$$
$$\left(S_R^{\mu\nu}\right) = -S_L^{\mu\nu *}$$

So what we want to do now is take fields ψ_a and $\psi_{\dot b}$ and try to find bilinears for the Lagrangian, so that we can finally find an equation of motion, such as the Dirac equation.

20 A Lagrangian for $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ fields

First of all, we define

$$\psi\psi := \epsilon^{ab}\psi_b\psi_a \qquad \epsilon^{ab} = -\epsilon^{ba}$$

Then we can try to find $i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi$ where we set

$$\sigma^\mu = \left(\mathbb{I}_{2\times2}, \sigma_1, \sigma_2, \sigma_3\right)^T$$
$$\bar\sigma^\mu = \left(\mathbb{I}_{2\times2}, -\sigma_1, -\sigma_2, -\sigma_3\right)^T = \begin{pmatrix} \mathbb{I}_{2\times2} \\ -\sigma_1 \\ -\sigma_2 \\ -\sigma_3 \end{pmatrix}$$

so we get

$$\begin{aligned}\left(i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi\right)^\dagger &= \left(i\psi_{\dot{a}}^\dagger\bar{\sigma}^{\mu\dot{a}c}\partial_\mu\psi_c\right)^\dagger \\ &= -i\partial_\mu\psi_{\dot{c}}^\dagger\left(\bar{\sigma}^{\mu a\dot{c}}\right)^*\psi_a \\ &= -i\partial_\mu\psi_{\dot{c}}^\dagger\bar{\sigma}^{\mu\dot{c}a}\psi_a \\ &= -i\psi_{\dot{c}}^\dagger\bar{\sigma}^{\mu\dot{c}a}\partial_\mu\psi_a - i\partial_\mu\left(\psi_{\dot{c}}^\dagger\bar{\sigma}^{\mu\dot{c}a}\psi_a\right) \\ &= -i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - i\partial_\mu\left(\psi'\bar{\sigma}^\mu\psi\right)\end{aligned}$$

Now we can write the Lagrangian density as

$$\Rightarrow \mathcal{L} = -i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}m\left(\psi\psi + \psi^\dagger\psi^\dagger\right)$$

and use the variation principle to derive

$$\begin{aligned}\frac{\delta S}{\delta\psi^\dagger} = 0 \qquad &\Rightarrow \qquad 0 = -i\bar{\sigma}^\mu\partial_\mu\psi + m\psi^\dagger \\ \frac{\delta S}{\delta\psi} = 0 \qquad &\Rightarrow \qquad 0 = -i\bar{\sigma}^\mu\partial_\mu\psi^\dagger + m\psi\end{aligned}$$

Now we define the γ matrices:

$$\gamma^\mu := \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$

They fulfill the Cifford algebra:

$$\left\{\gamma_\mu, \gamma_\nu\right\} = 2g_{\mu\nu}$$

Next, we set ψ to have four components whose first two components are those of the left-handed spinor representation, whereas the last two components are in right-handed spinor representation. This is still a spinor or Majorana field, not a vector field.

$$\psi = \begin{pmatrix} \psi_c \\ \psi_c^\dagger \end{pmatrix}$$

ψ describes spin $\frac{1}{2}$ particles and fulfills

$$\left(-i\gamma_\mu\partial^\mu + m\right)\psi = 0$$

which is the Dirac equation.

20.1 Derivation of the Dirac equation from representation theory

Let us take two Dirac fields ψ_1 and ψ_2 in the left-handed spinor representation, $\left(\frac{1}{2}, 0\right)$, and set

$$\begin{aligned}\chi &:= \frac{1}{\sqrt{2}}\left(\psi_1 + i\psi_2\right) \\ \zeta &:= \frac{1}{\sqrt{2}}\left(\psi_1 - i\psi_2\right)\end{aligned}$$

so we can set

$$\psi = \begin{pmatrix} \chi_c \\ \zeta^{\dagger \dot{c}} \end{pmatrix} \qquad \psi^\dagger = \begin{pmatrix} \chi^\dagger_{\dot{a}} \\ \zeta^a \end{pmatrix}$$

Now we define

$$\beta \coloneqq \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}$$

which is, in fact, equal to γ_0, the zeroth γ matrix.
With β, ζ and χ defined, we write $\bar{\psi}$ as

$$\begin{aligned} \bar{\psi} &\coloneqq \psi^\dagger \beta = \left(\zeta^a, \chi^\dagger_{\dot{a}}\right) \\ \Rightarrow \quad \bar{\psi}\psi &= \zeta^a \chi_a + \chi^\dagger_{\dot{a}} \zeta^{\dagger \dot{a}} \\ \bar{\psi} \not{\partial} \psi &= \zeta \sigma^\mu \partial_\mu \zeta^\dagger + \chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi \end{aligned}$$

with $\not{\partial} \coloneqq \gamma^\mu \partial_m u$.
The Lagrangian density for Dirac fields is then

$$\mathcal{L} = i\bar{\psi} \not{\partial} \psi - m\bar{\psi}\psi$$

20.2 Anticommutator relations

Since the Dirac equation describes fermions instead of bosons, we have to take anticommutator relations into account, instead of commutator relations. Among the most important are

$$\left\{\psi_\alpha(t, \vec{x}), \psi_\beta(t, \vec{y})\right\} = 0$$

(The equal time anticommutator vanishes.)

$$\left\{\psi_\alpha(t, \vec{x}), \bar{\psi}_\beta(t, \vec{y})\right\} = \left(\gamma^0\right)_{\alpha\beta} \delta^{(3)}(\vec{x} - \vec{y})$$

With

$$\frac{\partial \mathcal{L}}{\partial \partial_0 \psi} = i\bar{\psi}\gamma^0$$

and

$$\left(\gamma^0\right)^2 = \mathbb{I}$$

we get

$$(i\not{\partial} + m)\,\psi = 0$$

whose solutions give a plane wave expansion.

LSZ Formalism

28.11.2011

The discussion on the LSZ formalism is going to be an introduction to interacting field theory. Today's derivation will not be a proper, rigorous mathematical derivation, we will rather make some assumptions that may be sensible from a physical point of view, but have no mathematical hold. We need these assumptions in order to derive a quite useful formula for calculations.

21 Assumptions

- We assume that there is such a thing as a one-particle state, labeled with a 3-momentum vector $\vec{k}$.
- We assume that there is such a thing as a vacuum state.
- We assume that there is a Fock space with a creation operator acting on that vacuum in a way that it will produce a state with momentum $\vec{k}$

$$\begin{aligned} |k\rangle &= \hat{a}^\dagger(\vec{k})\,|0\rangle \\ \hat{a}^\dagger(\vec{k}) &\sim \int \mathrm{d}^3\vec{x}\;\; e^{ikx}\overset{\leftrightarrow}{\partial_0}\phi(x) \end{aligned} \tag{5}$$

  Further on, we demand that the annihilator will delete the vacuum and give a zero

$$\hat{a}(\vec{k})\,|0\rangle = 0$$

  Then, we want the vacuum and all other Fock space states to be normalized.

$$\begin{aligned} &\langle 0\,|\,0\rangle = 1 \\ &\langle k\,|\,k'\rangle = (2\pi)^3\omega_{\vec{k}}\delta^{(3)}(\vec{k}-\vec{k}') \qquad \left(\omega_{\vec{k}} = \sqrt{\vec{k}^2+m^2}\right) \end{aligned}$$

- The LSZ formalism will help find vacuum expectation values of operators, and to accomplish that it proves useful to smear the operators a little bit. To do this, we introduce a function f_1 that has smearing properties, for example a Gaussian wave packet.

$$\begin{aligned} \hat{a}^\dagger(\vec{k}) \;\to\; \hat{a}_1^\dagger &\coloneqq \int \mathrm{d}^3k\;\; f_1(\vec{k})\hat{a}^\dagger(\vec{k}) \\ f_1 &\sim \exp\left\{-\frac{(\vec{k}-\vec{k}_1)^2}{4\sigma^2}\right\} \end{aligned}$$

  Therefore, the momentum of the state being created by the creation operator will not be totally sharp, but still very localized.

- In the **Schrödinger picture**, the smeared state $\hat{a}_1^\dagger|0\rangle$ will propagate and spread out for $|t|\to\infty$. $\hat{a}_1^\dagger$ is localized only in the vicinity of $\vec{k}_1$, $\hat{a}_2^\dagger$ is localized only in the vicinity of $\vec{k}_2$, and so on. Therefore, $\hat{a}_1^\dagger\hat{a}_2^\dagger|0\rangle$ should look like two widely separated states in the limit $|t|\to\infty$, because the two maxima propagate and the distance in between them gets bigger. We call this effect "cluster decomposition".

Axiom States become multisingular, separated states in the limit $t \to \pm\infty$.

The axiom might hold long before or after an interaction. But in interacting field theory, $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ become time-dependent! (Note that this is still an assumption that makes sense for a physicist who works with interacting field theory, but not necessarily for a mathematician).

21.1 Initial and final state

We interpret the axiom, however, as two incoming, free particles, the so-called "initial state".

$$\lim_{t\to-\infty} \hat{a}_1^\dagger(t)\hat{a}_2^\dagger(t)\,|0\rangle =: |i\rangle$$

Here, $|i\rangle$ describes the initial state. We assume that we can normalize the initital state, since it is made up of free particle states:

$$\langle i \,|\, i\rangle = 1$$

For the same reason, we can interpret the limit on positively infinite time as the final state, also a set of free, separated particle states.

$$\lim_{t\to+\infty} \hat{a}_1^\dagger(t)\hat{a}_2^\dagger(t)\,|0\rangle =: |f\rangle$$
$$\langle f \,|\, f\rangle = 1$$

Example: Two particles approach each other, scatter, and the outcoming particles leave the interaction area and are separated.
Since we want the two intital (final) states to separate in the end, we must also assume that $\vec{k}_1 \neq \vec{k}_2$. Otherwise the two wave packets would superimpose and the two states would not be free, non-interacting particles outside each others ranges.

21.2 Scattering amplitude

Having defined the initial and final state as $|i\rangle$ and $|f\rangle$, the scattering amplitude must be the overlap of both states, the scalar product $\langle f \,|\, i\rangle$. This means that we compare a wave packet at $t = +\infty$ with a wave packet at $t = -\infty$. Following the fundamental theory of Perkins, we get

$$\hat{a}_1^\dagger(+\infty) - \hat{a}_1^\dagger(-\infty) = \int_{-\infty}^{+\infty} \mathrm{d}x_0\; \partial_0 \hat{a}_1^\dagger(t)$$

$$\begin{aligned}
(1) \Rightarrow \quad &= -i \int \mathrm{d}^3\vec{k}\; f_1(\vec{k}) \int \mathrm{d}^4x\; \partial_0 \left(e^{ikx} \overset{\leftrightarrow}{\partial_0}\, \phi(x) \right) \\
&= -i \int \mathrm{d}^3\vec{k}\; f_1(\vec{k}) \int \mathrm{d}^4x\; e^{ikx} \left(\partial_0^2 + \omega_{\vec{k}}^2 \right) \phi(x) \\
&= -i \int \mathrm{d}^3\vec{k}\; f_1(\vec{k}) \int \mathrm{d}^4x\; e^{ikx} \left(\partial_0^2 + \vec{k}^2 + m^2 \right) \phi(x) \\
&= -i \int \mathrm{d}^3\vec{k}\; f_1(\vec{k}) \int \mathrm{d}^4x\; e^{ikx} \left(\partial_0^2 - \left(\overleftarrow{\vec{\nabla}} \cdot \vec{\nabla} \right) + m^2 \right) \phi(x) \\
&= -i \int \mathrm{d}^3\vec{k}\; f_1(\vec{k}) \int \mathrm{d}^4x\; e^{ikx} \left(\partial_0^2 - \left(\vec{\nabla} \right)^2 + m^2 \right) \phi(x) \\
&= -i \int \mathrm{d}^3\vec{k}\; f_1(\vec{k}) \int \mathrm{d}^4x\; e^{ikx} \left(\Box + m^2 \right) \phi(x)
\end{aligned}$$

In the fourth line, we used the definition of $\omega_{\vec{k}}$. In the fifth line, we expressed $\vec{k}^2$ as a derivative acting on the left (on e^{ikx}). Finally, in the last line, we identified the Klein Gordon operator.
For free fields, $\left(\Box + m^2\right)\phi(x) = 0$. But in computing the scattering amplitude, obviously ϕ is not a free field, but an interacting field! Therefore, we cannot assume that the Klein Gordon operator acting on the field will generally give zero.
By the integral found above, wich equals $\hat{a}_1^\dagger(+\infty) - \hat{a}_1^\dagger(-\infty)$, we can express relations between the states at $t = \pm\infty$:

$$\left.\begin{aligned}\hat{a}_1^\dagger(-\infty) &= \hat{a}_1^\dagger(+\infty) + i\int \mathrm{d}^3\vec{k}\, f_1(\vec{k}) \int \mathrm{d}^4x\, e^{ikx}\left(\Box + m^2\right)\phi(x) \\ \hat{a}_1(+\infty) &= \hat{a}_1(-\infty) + i\int \mathrm{d}^3\vec{k}\, f_1(\vec{k}) \int \mathrm{d}^4x\, e^{-ikx}\left(\Box + m^2\right)\phi(x)\end{aligned}\right\} =: A$$

The second line was derived by Hermitian conjugation of the first line.
We use this result to express $\langle f \mid i\rangle$:

$$\begin{aligned}\langle f \mid i\rangle &= \left\langle 0 \left| \hat{a}_1(+\infty)\hat{a}_2(+\infty)\hat{a}_1^\dagger(-\infty)\hat{a}_2^\dagger(-\infty)\right| 0\right\rangle \\ &= \left\langle 0 \left| T\left[\hat{a}_1(+\infty)\hat{a}_2(+\infty)\hat{a}_1^\dagger(-\infty)\hat{a}_2^\dagger(-\infty)\right]\right| 0\right\rangle \\ &=: \langle 0\,|B|\,0\rangle\end{aligned}$$

In the first line, we see immediately that the operators are time-ordered. Therefore we can bring a time ordering operator into the expression without altering it.

21.3 The LSZ reduction formula

If we let $\sigma \to 0$, the formerly smeared wave packets become sharp again, they regain the form of three dimensional δ functions in momentum space, $\sim \delta^{(3)}(\vec{k} - \vec{k}')$. If we don't consider just two, but n scattering partners, and we use A in B, the formula for the scattering amplitude becomes

$$\langle f \mid i\rangle = i^{n+n'} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \int \mathrm{d}^4y_1 \ldots \mathrm{d}^4y_{n'} \left\{ e^{ik_1x_1}\left(\Box_{x_1} + m^2\right)\ldots e^{-ik'_1y_1}\left(\Box_{y_1} + m^2\right)\cdots \times \right.$$
$$\left. \times \langle 0\,|T\phi(x_1)\ldots\phi(x_n)\phi(y_1)\ldots\phi(y_{n'})|\,0\rangle + \delta \right\}$$

This is the **"LSZ reduction formula"**. The $x_1 \ldots x_n$ correspond to n incoming particles with momenta $k_1 \ldots k_n$. Accordingly, the $y_1 \ldots y_{n'}$ correspond to n' outgoing particles with momenta $k'_1 \ldots k'_n$. The term $+\delta$ is expressing the fact that there are extra δ functions carrying information about disconnected terms, that is if the two particles don't interact, but maintain their former paths. These terms are not interesting when talking about scattering processes, and will be neglected in further discussions.
The initial and final states are understood to be free, non-interacting states at $t \to \pm\infty$, so they can be expressed in terms of creation and annihilation operators. Feynman diagrams give a nice graphical interpretation of these terms.
Remark: A free particle interacts with itself and recombinates all the time.

22 Discussion

In the first part, we established the LSZ formalism. Now we will see how it can be used for purposes of calculation and interpretation.

22.1 The vacuum state

One axiom from part one was the assumption that the vacuum state $|0\rangle$ shall exist. Another axiom stated that there are well-defined one-particle states $\left|\vec{k}\right\rangle$ with energy $E = \omega_{\vec{k}} = \sqrt{\left|\vec{k}\right|^2 + m^2}$. We write $\left|\vec{k}_1\vec{k}_2\right\rangle$ as a two-particle state where the two involved particles carry momenta $\vec{k}_1$ and $\vec{k}_2$, and they are created out of the vacuum by the respective one-particle creation operators.

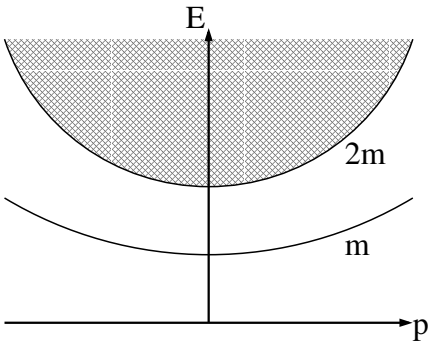


Figure 1: energy momentum diagram

The energy of such a two particle state would then be at least the sum of the masses of the two particles. If we assume that the two particles are identical, because they were created with the creation operators of the same kind, then the total energy is at least twice the mass of one particle. If the energy is exactly that, then the two particles will be at rest, with respect to each other. If there is relative movement, then the state also includes the kinetic energy connected to that movement.

$$E_{\left|\vec{k}_1\vec{k}_2\right\rangle} = 2m + \text{all kinetic energy} \geq 2m$$

22.2 Shifting the vacuum

If ϕ is not a free field, we can write it down expressing the interaction and use this for the vacuum expectation value.

$$\begin{aligned} \phi(x) &= e^{-iPx}\phi(0)e^{iPx} \\ \Rightarrow \quad \langle 0|\phi(x)|0\rangle &= \left\langle 0\left|e^{-iPx}\phi(0)e^{iPx}\right|0\right\rangle \end{aligned}$$

P is the momentum operator, whose eigenvalue to the vacuum state is 0.

$$\Rightarrow \quad \langle 0|\phi(x)|0\rangle = \langle 0|\phi(0)|0\rangle \quad \in \mathbb{C}$$

In general, the vacuum expectation value of the field ϕ at $x = 0$ gives some complex number. Since $\hat{a}_1^\dagger(\pm\infty)|0\rangle$ gives a pure one-particle state (in the limit of smearing without an overlap), then this complex number should actually be zero. If $\langle 0|\phi(0)|0\rangle$ were not zero, then $\hat{a}_1^\dagger(\pm\infty)|0\rangle$ would give a combination of states and no pure one-particle states. But we can assume that one-particle states exist, because detectors can identify single quanta.
Having established that $\langle 0|\phi(0)|0\rangle = 0$, shifting any field by its vacuum expectation value should not change anything.

$$\begin{aligned} \phi(x) \to \phi(x) - \langle 0|\phi(0)|0\rangle =: \tilde{\phi}(x) \\ \langle 0|0\rangle = 1 \\ \left\langle 0\left|\tilde{\phi}(x)\right|0\right\rangle = 0 \end{aligned}$$

The second line gives the normalization of the vacuum. We further on assume that all the calculations (which are still basically assumptions) are something that we can do.
As for notation, the tilde will be dropped from now on, because we expect that every field has had its vacuum expectation value subtracted.

22.3 Amplitude between vacuum and non-vacuum states

We have seen how to handle vacuum expectation values, but what happens if the final state is not the vacuum?

$$\begin{aligned}\langle p\,|\phi(x)|\,0\rangle &= \left\langle p\left|e^{-iPx}\phi(0)e^{iPx}\right|0\right\rangle \\ &= e^{-ipx}\,\langle p\,|\phi(0)|\,0\rangle \quad \in \mathbb{C}\end{aligned}$$

Here, we have used that $\langle p|\,e^{-iPx} = \langle p|\,e^{-ipx}$ with P being the momentum operator and p being the eigenvalue. The same goes for the right side: $e^{iPx}\,|0\rangle = 1\,|0\rangle$.
$\langle p\,|\phi(x)|\,0\rangle$ is a Lorentz scalar because ϕ is a scalar field. A Lorentz scalar as a function of p can not be proportional to p, only to $p^2 = m^2$.

22.4 Free field limit

In regarding $t \to \pm\infty$, we expect $\langle p\,|\phi(0)|\,0\rangle$ to give 1, since this is the case for free fields.

$$c = \lim_{t\to\pm\infty}\langle p\,|\phi(0)|\,0\rangle \stackrel{!}{=} 1 \quad \text{, to agree with free fields.}$$

If $c \neq 1$, we can rescale, so that it will give one. This is the second correction to the field ϕ that we are carrying out, the first one was a shift by $\langle 0\,|\phi(0)|\,0\rangle$, and now there is a rescaling.

$$\phi \quad\to\quad \phi Z = \tilde{\phi} \quad \text{such that} \quad \left\langle p\left|\tilde{\phi}(0)\right|0\right\rangle = 1.$$

22.5 Amplitude between vacuum and n-particle state

Now let us assume that ϕ is sandwiched between n-particle states, and later on we will check what happens for two fields $\phi(x)$ and $\phi(y)$.

22.5.1 n-particle states

We are choosing the notation $\langle P, n|$ for an outgoing state that is a collection of n particles moving with momenta P, where any relative movements are already included in the expression.
Then:

$$\begin{aligned}\langle P, n\,|\phi(x)|\,0\rangle &= \left\langle P, n\left|e^{i\hat{P}x}\phi(0)e^{i\hat{P}x}\right|0\right\rangle \\ &= e^{-iPx}\,\langle P, n\,|\phi(0)|\,0\rangle \\ &=: e^{-iPx}A_n(\vec{p}) \quad =: C\end{aligned}$$

The overlap between $\langle P, n|$ and the initial state should give zero:

$$\left\langle P, n\left|\hat{a}_1^\dagger(\pm\infty)\right|0\right\rangle \stackrel{!}{=} 0$$

If we, as stated above, assume that $\langle P, n|$ is a state with more than one particle, then we can define a wave packet for the multi-particle state:

$$\begin{aligned}|\psi\rangle &:= \sum\!\!\!\!\!\!\int \mathrm{d}^3p\;\psi_n(\vec{p})\,|P, n\rangle \\ \Rightarrow\quad \left\langle\psi\left|\hat{a}_1^\dagger\right|0\right\rangle &= -i\sum\!\!\!\!\!\!\int \mathrm{d}^3p\;\psi_n^*(\vec{p})\int \mathrm{d}^3k\;f_1(\vec{k})\int \mathrm{d}^3x\;e^{ikx}\overset{\leftrightarrow}{\partial_0}\langle P, n\,|\phi(x)|\,0\rangle \\ &= \sum\!\!\!\!\!\!\int \mathrm{d}^3p\;(2\pi)^3\psi_n^*(\vec{p})\int \mathrm{d}^3k\;f_1(\vec{k})\int \mathrm{d}^3x(p_0+k_0)e^{i(k-p)x}A_n(\vec{p}) \\ &= \sum\!\!\!\!\!\!\int \mathrm{d}^3p\;(2\pi)^3(p_0+k_0)\psi_n^*(\vec{p})f_1(\vec{p})A_n(\vec{p})e^{i(p_0-k_0)t}\end{aligned}$$

with

$$p^0 = \sqrt{\left|\vec{p}\right|^2 + M^2}$$
$$k^0 = \sqrt{\left|\vec{k}\right|^2 + m^2}$$
$$M \geq 2m > m \qquad \Rightarrow \quad p^0 < k^0$$

In the limit $t \to \pm\infty$, this entire expression will approach zero. So we can demand that

$$\langle 0 \left|\phi(x)\right| 0\rangle \stackrel{!}{=} 0$$
$$\langle k \left|\phi(x)\right| 0\rangle \stackrel{!}{=} e^{-ikx}$$

but we need to modify the Lagrangian a slight bit: We rescale every term and shift everything in order to get a well-defined Lagrangian.

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4 \\ &\to -\frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + Z_g\frac{g}{4!}\phi^4 + Z_s\phi\end{aligned}$$

Z_ϕ, Z_m, Z_g and Z_s are called the Z-factors.

22.5.2 Derivation of the Källén-Lehmann representation of the two-point Wightman function

We will examine the vacuum expectation value of two interacting fields before we get to the generalization of n fields. Since fields are described as quantum mechanical operators, we can expand $\langle 0 \left|\phi(x)\phi(y)\right| 0\rangle$.

$$\begin{aligned}\langle 0 \left|\phi(x)\phi(y)\right| 0\rangle = \langle 0 \left|\phi(x)\right| 0\rangle \langle 0 \left|\phi(y)\right| 0\rangle + \int \frac{\mathrm{d}^3\vec{k}}{2\omega_{\vec{k}}(2\pi)^3} \left\langle 0 \left|\phi(x)\right| \vec{k}\right\rangle \left\langle \vec{k} \left|\phi(y)\right| 0\right\rangle \\ + \sum\!\!\!\!\!\!\int \langle 0 \left|\phi(x)\right| K, n\rangle \langle K, n \left|\phi(y)\right| 0\rangle\end{aligned}$$

Once again, we write for $\phi(x) = e^{-ipx}\phi(0)e^{ipx}$ and that gives us

$$\langle 0 \left|\phi(x)\phi(y)\right| 0\rangle = \int \frac{\mathrm{d}^3\vec{k}}{2\omega_{\vec{k}}(2\pi)^3} e^{ik(x-y)} + \sum\!\!\!\!\!\!\int \frac{\mathrm{d}^3K}{2\omega_{\vec{K}}} e^{iK(x-y)} \left|\langle K, n \left|\phi(0)\right| 0\rangle\right|^2$$

We define

$$\begin{aligned}\rho(s) &:= \sum\!\!\!\!\!\!\int \left|\langle K, n \left|\phi(0)\right| 0\rangle\right|^2 \delta(s - M^2) \quad \text{if } s \geq 4m^2 \\ \text{and} \quad \rho(s) &\equiv 0 \qquad \text{if } s \leq 4m^2. \\ \text{Also,} \quad \rho(s) &\geq 0.\end{aligned}$$

Then we introduce time-ordering:

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon} = \Theta(x^0 - y^0) \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} e^{ik(x-y)} + \Theta(y^0 - x^0) \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} e^{-ik(x-y)}$$

All this preparation has lead to one really useful formula, the Källén-Lehmann representation of the two-point function in momentum space. This formula is built on the assumption that we can express $\hat{\Delta}(k^2)$, a Lorentz scalar, as the Fourier transform of $\langle 0 \,|T\phi(x)\phi(y)|\, 0\rangle = F(x-y)$.

$$\hat{\Delta}(k^2) = \frac{1}{k^2 - m^2 + i\epsilon} + \int_{4m^2}^{+\infty} \mathrm{d}s \, \rho(s) \frac{1}{k^2 - s + i\epsilon}$$

This formula, the Källén-Lehmann representation of the two-point Wightman function, holds in a full interacting free field theory. It gives us information about proper poles of interacting particles at fixed masses. Continuous parts (computations and measurements) agree well.

• Interacting QFT

5.12.2011

Lately we discussed the LSZ formalism, a reduction formula that involves many integrals and Klein Gordon operators, that we derived from physically sensible assumptions. It states that the transition amplitude between two free initial and final states is given by

$$\langle f \mid i\rangle = i^{n-n'} \int \mathrm{d}^4x_1 \ldots e^{ik_1x_1}(\Box_{x_1} + m^2) \ldots \; \mathrm{d}^4y_1 \ldots e^{-ik'_1y_1}(\Box_{y_1} + m^2) \ldots \langle 0 \,|T\phi(x_1) \ldots \phi(y_1) \ldots| \, 0\rangle +$$

$+$ "disconnected terms"

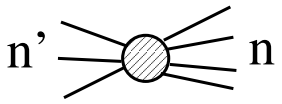


Figure 2: visualization of interaction with n' incoming and n outgoing edges

"Disconnected terms" means that there is no interaction at all, picturedly speaking the particles "don't hit, but miss each other". This case is not interesting if we want to examine interacting quantum field theory, therefore we name it here, but don't look at it further.
The Wightman function, the vacuum expectation value $\langle 0 \,|T \ldots| \, 0\rangle$, carries all information about the interaction.
It is time to talk about interacting quantum field theory, the perturbation expansion, and Feynman diagrams.

23 Pictures in Quantum Mechanics

As we know, and this will be but a short reminder, there are different quantum mechanical pictures to be taken into account when discribing quantum physics: the Schrödinger, the Heisenberg, and the interaction picture.

23.1 Schrödinger picture

In the Schrödinger picture, the states Ψ_S are time dependent, where this time dependence is given by the Schrödinger equation.

$$\partial_t \, |\Psi_S\rangle = \mathcal{H}(p_S, q_S, t) \, |\Psi_S\rangle$$

The time development of states is expressed using unitary transformations.

$$\begin{aligned} |\Psi_S(t)\rangle &= \mathcal{U}(t,t') \, |\Psi_S(t')\rangle \\ i\dot{\mathcal{U}} &= \mathcal{H}\mathcal{U} \\ \mathcal{U}(t',t') &= \mathbb{I} \end{aligned}$$

The time development operator is given using the exponentiation of the Hamiltonian:

$$\mathcal{U}(t,t') = e^{-i\mathcal{H}(t-t')}$$

Operators, and therefore fields as well, are time independent in the Schrödinger picture. The time development lies in the states alone.

23.2 Heisenberg picture

In the Heisenberg picture, states are time independent, whereas fields are time dependent, as are all operators. Their time development is given by the conjugation with the time development operator, and the fields at time $t = 0$ are identical to the Schrödinger fields.

$$\begin{aligned}
\hat{q}_H(0) &= \hat{q}_S \\
\hat{p}_H(0) &= \hat{p}_S \\
\hat{q}_H(t) &= \mathcal{U}^\dagger(t,0)\, \hat{q}_H(0)\, \mathcal{U}(t,0) \\
\hat{p}_H(t) &= \mathcal{U}^\dagger(t,0)\, \hat{p}_H(0)\, \mathcal{U}(t,0) \\
|\Psi_H(t)\rangle &= |\Psi_H(0)\rangle
\end{aligned}$$

24 Interaction picture

We can express the Hamiltonian as a decomposition of a free and an interacting Hamiltonian.

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$$

The free Hamiltonian, $\mathcal{H}_0$, is quadratic in the fields, whereas the interaction Hamiltonian, $\mathcal{H}_I$, is a polynomials in the fields of degree greater than two.
We get the interaction picture, in which the fields and states at an arbitrary time $t = 0$ are exactly the Schrödinger fields and states (at that time)

$$\begin{aligned}
q_I(0) &= q_S \\
p_I(0) &= p_S \\
|\Psi_I(0)\rangle &= |\Psi_S(0)\rangle
\end{aligned}$$

Neither the fields not the states are time independent, but their time development is given only by the free Hamiltonian, $\mathcal{H}_0$:

$$\begin{aligned}
q_I(t) &= e^{i\mathcal{H}_0 t} q_I(0) e^{-i\mathcal{H}_0 t} \\
p_I(t) &= e^{i\mathcal{H}_0 t} p_I(0) e^{-i\mathcal{H}_0 t} \\
|\Psi_I(t)\rangle &= e^{i\mathcal{H}_0 t} |\Psi_I(0)\rangle
\end{aligned}$$

The matrix element M of an operator A is of course independent from the picture. To explain this, let us take a closer look at M in the Schrödinger picture.

$$\begin{aligned}
M(t) &= \langle \phi_S(t) | A_S | \Psi_S(t) \rangle \\
&= \left\langle \phi_I(t) \middle| e^{i\mathcal{H}_0 t} A_I(0) e^{-i\mathcal{H}_0 t} \middle| \Psi_I(t) \right\rangle \\
&= \langle \phi_I(t) | A_I(t) | \Psi_I(t) \rangle
\end{aligned}$$

Applying the time derivation operator on the interaction state gives the Schrödinger equation. We see immediately that only the interacting Hamiltonian plays a role here.

$$\begin{aligned}
i\partial_t |\Psi_I(t)\rangle &= e^{i\mathcal{H}_0 t} \Big(\mathcal{H}_0(p_S, q_S + \mathcal{H}_I(p_S, q_S, t) \Big) |\Psi_S(t)\rangle \\
&= e^{i\mathcal{H}_0 t} \mathcal{H}'(p_I(0), q_I(0)) e^{-i\mathcal{H}_0 t} |\Psi_I(t)\rangle \\
&= \mathcal{H}_I\Big(p_I(t), q_I(t), t\Big) |\Psi_I(t)\rangle
\end{aligned}$$

At first, we used $i\partial_t |\Psi\rangle = \mathcal{H}|\Psi\rangle$ and $|\Psi_I\rangle = e^{i\mathcal{H}_0 t}|\Psi_S\rangle$. Next, we commuted $\mathcal{H}$ and $e^{i\mathcal{H}_0 t}$ and wrote $|\Psi_S\rangle$ in the interaction picture again. After that, we identified the fields in the Hamiltonian to be at the time $t = 0$, and we used the relation $O_I(t) = e^{i\mathcal{H}_0 t} O_I(0) e^{-i\mathcal{H}_0 t}$, for any operator O, and $O_I(0) = O_S$.
This expression leads us to the time dependence of interaction states and a differential equation resulting from it:

$$\begin{aligned} \mathcal{U}_I(t,t')|\Psi_I(t')\rangle &= |\Psi_I(t)\rangle \\ i\partial_t \mathcal{U}_I(t',t) &= \mathcal{H}_I(t)\mathcal{U}_I(t,t') \\ \mathcal{U}_I(t',t') &= \mathbb{I} \end{aligned} \tag{6}$$

"Philosophical" remark: We assume that the interaction disappears at $t = \pm\infty$. We never justify this, and there is no evidence to suggest that this is the real case. (What is a "free" field? Aren't there interactions everywhere?) But we use it all the time, and for some reason, the right results come out.

25 Computing the transition

At first we write down the scattering matrix, $\mathcal{S}$, as a solution of (6).

$$\begin{aligned} \mathcal{S} &= \lim_{t\to+\infty} \lim_{t'\to-\infty} \mathcal{U}_I(t,t') \\ &= T\exp\left(-i\int_{-\infty}^{+\infty} \mathrm{d}t \mathcal{H}_I(t)\right) \end{aligned}$$

In order to compute $\mathcal{S}$ for any two times t_1, t_2, we have to expand:

$$\begin{aligned} \mathcal{S}(t_1,t_2) &= 1 - i\int_{t_1}^{t_2} \mathrm{d}t \mathcal{H}_I(t) - \frac{1}{2!}\int_{t_1}^{t_2}\int_{t_1}^{t_2} \mathrm{d}t\, \mathrm{d}t' T\Big[\mathcal{H}_I(t)\mathcal{H}_I(t')\Big] + \dots \\ &= 1 - i\int_{t_1}^{t_2} \mathrm{d}t \mathcal{H}_I(t) - \frac{1}{2!}\int_{t_1}^{t_2}\int_{t_1}^{t_2} \mathrm{d}t\, \mathrm{d}t' \mathcal{H}_I(t)\mathcal{H}_I(t') + \dots \end{aligned}$$

To solve this, we will use Wick's theorem, which we will learn in a minute. Before that, let us remind ourselves of field decompositions, and learn what contractions are.

25.1 Field decompositions (short reminder)

Free fields are linear in creation and annihilation operators.

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{k}}{2\sqrt{\omega_{\vec{k}}}} \left\{\hat{a}e^{-ikx} + \hat{a}^\dagger e^{ikx}\right\}$$

25.2 Contractions

Contractions are defined as the difference between time ordering and normal ordering.

$$\underbrace{A_i A_j} \equiv T(A_i A_j) - :A_i A_j:$$

If we assume $t_i < t_j$, then

$$T(A_iA_j) = \left(A_i^{(+)} + A_i^{(-)}\right)\left(A_j^{(+)} + A_j^{(-)}\right) =: A_iA_j : + \left[A_i^{(-)}, A_j^{(+)}\right]$$
$$\Rightarrow \quad \underbrace{A_iA_j} = \begin{cases} [A_i^{(-)}, A_j^{(+)}] & \text{if } t_i > t_j \\ [A_j^{(-)}, A_i^{(+)}] & \text{if } t_i < t_j \end{cases}$$

Since our computations involve the creation of states out of the vacuum, normal ordering makes no sense! It would annihilate the vacuum state, giving zero. For this reason, we discard $: X :$ for all X, as $\langle 0 |: X :| 0 \rangle = 0$.
Now we see that the contraction is exactly the Wightman function:

$$\underbrace{A_iA_j} = \left\langle 0 \left| TA_iA_j \right| 0 \right\rangle$$

It is important to realize that $\underbrace{A_iA_j}$ is not an operator, but a number. It is an operator sandwiched between two vacuum states, or a commutator which is either zero or proportional to some number and a δ distribution.
Last but not least, there is an important rule for calculating contractions:

$$: A_1\underbrace{A_2A_3A_4} : \quad \equiv \quad \underbrace{A_2A_4} : A_1A_3 :$$

We will see that it is a good idea to express everything in terms of contractions due to the graphical interpretation, which we will get to soon.

26 Wick's Theorem

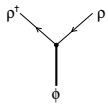
Theorem : *The time-ordered product of n free fields is equal to the sum of the normal-ordered product of all possible partial and complete contractions of all free fields in* $\left\{T \prod_j \phi_j\right\}$.
This does not only account for fields, but for anything linear in fields and their creators or annihilators. For now, we ignore the partial contractions.

26.1 Example

Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\nu\phi\partial^\nu\phi + \mu^2\phi^2\right) + \frac{1}{2}\left(\partial_\mu\rho^\dagger\partial^\mu\rho - m^2\rho^\dagger\rho\right) + \lambda\rho^\dagger\rho\phi$$

The first part in the sum gives a real, scalar field. The second part comes from a complex field, therefore there is a charge flow. Both fields correspond to massive particles. The third part describes the interaction, where λ is a small parameter, the coupling constant: $\lambda \ll 1$. The interaction Lagrangian is also the interaction Hamiltonian: $\mathcal{H}_I = \lambda\rho^\dagger(x)\rho(x)\phi(x)$.



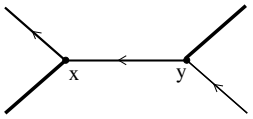
Interaction: Vertex with ρ corresponding to a half edge coming in, $\rho^\dagger$ corresponding to a half edge going out, and ϕ being a scalar field.

For every such vertex, we multiply with $(-i\lambda)$, where the $-i$ is from the time-ordered exponential, and λ is the coefficient in $\mathcal{H}_I$.
In this way, we can translate Wick diagrams to operations.

26.1.1 Wick diagrams with two vertices

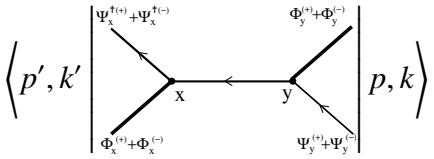
All possible Wick diagrams are made up from the basic vertex (see above), called **corolla**, just like puzzles. From the different ways to glue these corollas together, we get different overall interactions.
From the way the diagrams are drawn, we can extract the calculations: For every two half edges that are connected, the two corresponding fields are contracted.

- **Four-legged function** (four external edges)



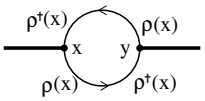
Contribution to the $\mathcal{S}$ matrix given by
$(-i\lambda)^2 \int \mathrm{d}^4x\, \mathrm{d}^4y\, \rho^\dagger(x)\rho(x)\phi(x)\rho^\dagger(y)\rho(y)\phi(y)$
$= (-i\lambda)^2 \int \mathrm{d}^4x\, \mathrm{d}^4y\, \rho^\dagger(x)\rho(y) : \rho(x)\rho^\dagger(y)\phi(x)\phi(y) :$

There are four fields outside the contraction (namely $\rho(x), \phi(x), \rho^\dagger(y)$, and $\phi(x)$), that means four external fields, which gives us 2^4 terms in the creation and annihilation operators $\alpha, \alpha^\dagger, \beta, \beta^\dagger, \gamma, \gamma^\dagger$. The contraction of $\rho^\dagger(x)$ and $\rho(y)$ will produce the Feynman operator, propagating the field from y to x:



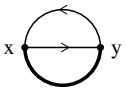
$\langle p', k' |$ $| p, k \rangle$

- **Two-legged function** (two external edges)



Two inner edges ⇒ two contractions
Contribution to the $\mathcal{S}$ matrix given by
$(-i\lambda)^2 \int \mathrm{d}^4x\, \mathrm{d}^4y\, \rho^\dagger(x)\rho(y)\rho(x)\rho^\dagger(y) : \phi(x)\phi(y) :$

- **Vacuum fluctuation** (no external edges)



No external edges ⇒ contract everything
Contribution to the $\mathcal{S}$ matrix given by
$(-i\lambda)^2 \int \mathrm{d}^4x\, \mathrm{d}^4y\, \rho^\dagger(x)\rho(y)\rho(x)\rho^\dagger(y)\phi(y)\phi(y)$

26.2 Expressing fields on states as plane waves

If we again consider the decomposition of fields into a creation and an annihilation part, we write

$$\left.\begin{array}{l} \phi = \phi_{(+)} + \phi_{(-)} \\ \mathrm{d}\lambda(\vec{k}) = \frac{\mathrm{d}\vec{k}}{(2\pi)^3 2\omega(\vec{k})} \end{array}\right\} \begin{array}{l} \phi_{(+)} = \int \mathrm{d}\lambda(\vec{k}) e^{ixk} \hat{\alpha}^\dagger(k) \\ \phi_{(-)} = \int \mathrm{d}\lambda(\vec{k}) e^{-ixk} \hat{\alpha}(k) \end{array}$$

$$\begin{array}{ll} \rho_{(+)} = \int \mathrm{d}\lambda(\vec{k}) e^{ixk} \hat{\gamma}^\dagger(k) & \rho_{(-)} = \int \mathrm{d}\lambda(\vec{k}) e^{-ixk} \hat{\beta}(k) \\ \rho^\dagger_{(+)} = \int \mathrm{d}\lambda(\vec{k}) e^{ixk} \hat{\beta}^\dagger(k) & \rho^\dagger_{(-)} = \int \mathrm{d}\lambda(\vec{k}) e^{-ixk} \hat{\gamma}(k) \end{array}$$

If we let a scalar field $\phi_{(-)}$ act on a state $|p\rangle$, we get

$$\begin{aligned} \phi_{(-)}(x)\,|p\rangle &= \int \mathrm{d}\lambda(\vec{k}) e^{-ikx} \hat{\alpha}(k) \alpha^\dagger(p)\,|0\rangle \\ &= \int \mathrm{d}^3\vec{k} e^{-ikx} \delta^{(3)}(\vec{k} - \vec{p})\,|0\rangle \\ &= e^{ixp}\,|0\rangle \qquad\qquad \text{with } p_0^2 = \vec{p}^2 + m^2 \equiv \vec{k}^2 + m^2 \end{aligned}$$

We see that a field $\phi_{(-)}$ acting on a state labeled with momentum p, will produce a plane wave with that momentum.

To the Feynman Rules

7.12.2011

We have learned about bicontractions of fields and that we can decompose scalar fields in creating and annihilating parts.

$$\phi = \phi^{(+)} + \phi^{(-)}$$
$$\phi^{(+)} = \int \mathrm{d}\lambda\ e^{ixk}\hat{\alpha}^\dagger(k)$$
$$\phi^{(-)} = \int \mathrm{d}\lambda\ e^{-ixk}\hat{\alpha}(k)$$

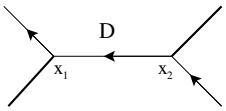
The annihilating part applied on a state simply gives a plane wave.

$$\phi^{(-)}(x)\,|p\rangle = e^{-ipx}\,|0\rangle$$

We saw that spin $\frac{1}{2}$ fields can be decomposed in a similar way:

$$\psi = \psi_{(+)} + \psi_{(-)} \qquad \psi^\dagger = \psi^\dagger_{(+)} + \psi^\dagger_{(-)}$$
$$\psi_{(+)} = \int \mathrm{d}\lambda\ e^{ixk}\hat{\beta}^\dagger(k) \qquad \psi_{(-)} = \int \mathrm{d}\lambda\ e^{-ixk}\hat{\gamma}(k)$$
$${\psi_{(+)}}^\dagger = \int \mathrm{d}\lambda\ e^{ixk}\hat{\beta}(k) \qquad {\psi_{(-)}}^\dagger = \int \mathrm{d}\lambda\ e^{ixk}\hat{\gamma}^\dagger(k)$$

And we drew a Wick diagram whose contributions to the scattering matrix was given by the following formula:



$$\mathcal{S}(D) = (-i\lambda)^2 \int \mathrm{d}^4x_1\mathrm{d}^4x_2 : \psi_1^\dagger \overbrace{\psi_1\phi_1\psi_2^\dagger}\psi_2\phi_2 :$$

Today we will focus on the symmetry of such expressions.

27 Contribution to the scattering matrix

There are four fields not contracted in

$$\mathcal{S}(D) = (-i\lambda)^2 \int \mathrm{d}^4x_1\mathrm{d}^4x_2 : \psi_1^\dagger \overbrace{\psi_1\phi_1\psi_2^\dagger}\psi_2\phi_2 :$$

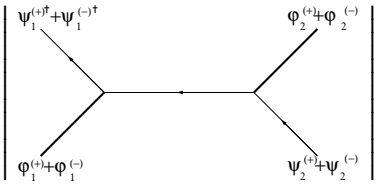
and all of these have a $(\pm)$ decomposition, therefore we have a lot of terms to consider. Since we don't want the contributions of disconnected pieces in a diagram, where there is no interaction, we compute

$$\langle p', k'\,|(\mathcal{S} - \mathbb{I})|\,p, k\rangle$$

instead of $\langle p', k'\,|\mathcal{S}|\,p, k\rangle$. Since different fields always commute, it suffices to compute the action of one field sandwiched between two states of which one is the vacuum.

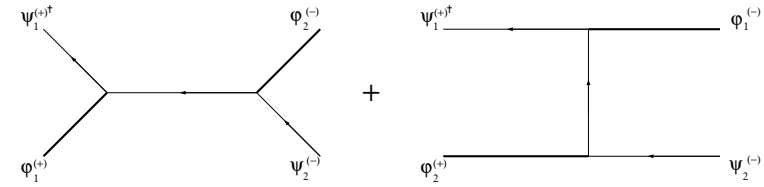
$$\langle 0\,|\phi|\,k\rangle = \left\langle 0\left|\phi^{(-)}\right|k\right\rangle = e^{-ixk}$$
$$\langle k'\,|\phi|\,0\rangle = \left\langle k'\left|\phi^{(+)}\right|0\right\rangle = e^{+ixk'}$$
$$\langle 0\,|\psi|\,p\rangle = \left\langle 0\left|\phi^{(-)}\right|p\right\rangle = e^{-ixp}$$
$$\left\langle p'\left|\psi^\dagger\right|0\right\rangle = \left\langle p'\left|{\psi^{(+)}}^\dagger\right|0\right\rangle = e^{+ixp'}$$

With the matrix element for D being

$$M(D) = \left\langle p', k' \left| \right. \right.$$


$$\left. \left. \right\rangle p, k \right\rangle$$

the only surviving diagrams give

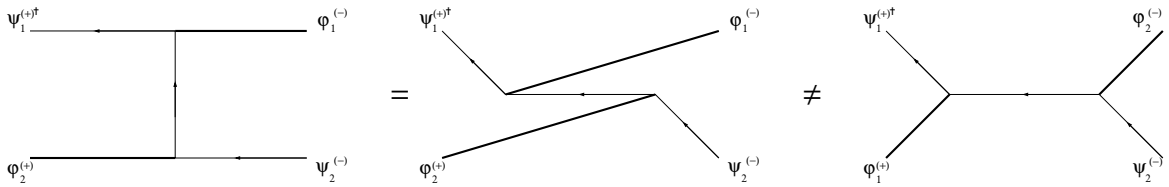
$$M(D) = \left\langle p', k' \left| \right. \right.$$


$$\left. \left. \right\rangle p, k \right\rangle$$

$$= (-i\lambda)^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left\{ \frac{i}{q^2 - m^2 + i\epsilon} \left(\underbrace{(2\pi)^4 \delta^{(4)}(k + p - q)(2\pi)^4 \delta^{(4)}(k' + p' - q)}_{\text{momentum conservation in the first diagram}} + \right. \right.$$

$$\left. \left. + \underbrace{(2\pi)^4 \delta^{(4)}(k' - p + q)(2\pi)^4 \delta^{(4)}(k - p' + q)}_{\text{momentum conservation in the second diagram}} \right) \right\}$$

There are no closed loops, hence these diagrams are called **first order tree level Feynman diagrams** contributing to the scattering of two fields. For every vertex, we have momentum conservation in the form of a δ function with respect to incoming and outgoing charge, and a multiplication with $i\lambda(2\pi)^4$.
Remark: the second diagram is drawn in a way that makes it easy to understand the interaction, but drawing it in a more distorted way makes it easier to realize the similarities and differences to the first diagram. Keep in mind:



27.1 General rules for drawing diagrams

- We draw diagrams in such a way that time flows from right to left. Therefore, "left comes later". This is only important for the initial and final states, since the interaction process is not observable and hence time order is unknown.
- At first, we write down the internal edges: Lines for incoming particles to the right, lines for outgoing particles to the left.
- For every vertex, we write down a $\delta^{(4)}$ function of the sum of all momenta with respect to the sign, and multiply by $i\lambda(2\pi)^4$.
- We sum over all possible diagrams with the same number of vertices.
- We integrate over all free momenta.
- We also have to take the symmetry factor into account, a rational factor before some terms which gives the rank of the automorphism group, $\mathrm{sym}\,(\Gamma) = |\mathcal{A}ut|$. (Especially important for self-interacting fields.)

28 Symmetry factors

Before we can compute symmetry factors, let us have a look at some basic integrals.

$$\int_{\mathbb{R}} \mathrm{d}x e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \quad , \quad a > 0 \tag{7}$$

$$\int_{\mathbb{R}} \mathrm{d}x e^{-\frac{1}{2}ax^2+Jx} = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{J^2}{2a}\right) \quad , \quad a > 0 \tag{8}$$

We can use (7) to define the mean of some even power of x:

$$\left\langle x^{2n} \right\rangle := \frac{\int_{\mathbb{R}} x^{2n} e^{-\frac{1}{2}ax^2} \, \mathrm{d}x}{\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} \, \mathrm{d}x} = \frac{(2n-1)!!}{a^n}$$

$$\text{where} \quad (2n-1)!! := (2n-1) \cdot (2n-3) \cdot_{\ldots} \cdot 3 \cdot 1$$

The numerator of $\left\langle x^{2n} \right\rangle$ gives the number of Wick contractions!

28.1 Action

Reminder: The action is defined by the four-integral over the Lagrangian density:

$$\mathcal{S} := \int \mathrm{d}^4 x \mathcal{L}(\phi) = \int \mathrm{d}^4 x \left(\mathcal{L}_0(\phi) + \mathcal{L}_I(\phi)\right)$$

Then, $\left\langle x^{2n} \right\rangle$ gives the number of pairings of $2n$ terms, where n is the number of propagators, or lines. The free Lagrangian, $\mathcal{L}_0$, is quadratic in the field ($\rightarrow x^2$), the interacting Lagrangian, $\mathcal{L}_I$, is a polynomial in the fields whose degree is greater than two.

29 Interacting field theory

Let us take (8) to a more general level by adding a polynomial $P(x)$ to the exponent. This polynomial is to play the role of interaction, and its degree should be greater than two.

$$\begin{aligned} Z(J) &= \int_{\mathbb{R}} \mathrm{d}x \, e^{-\frac{1}{2}ax^2+P(x)+Jx} \\ &= \int_{\mathbb{R}} \mathrm{d}x \underbrace{\left(\sum_{n=0}^{\infty} \frac{P^n(x)}{n!}\right)}_{=e^{P(x)}} e^{-\frac{1}{2}ax^2+Jx} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \mathrm{d}x \, P^n(x) e^{-\frac{1}{2}ax^2+Jx} \end{aligned}$$

We can express this integral using

$$\int \mathrm{d}x \, x^k e^{-\frac{1}{2}ax^2+Jx} = \int \mathrm{d}x \left(\frac{\mathrm{d}}{\mathrm{d}J}\right)^k e^{-\frac{1}{2}ax^2+Jx}$$

and with (8) this expression gives

$$\int \mathrm{d}x \left(\frac{\mathrm{d}}{\mathrm{d}J}\right)^k e^{-\frac{1}{2}ax^2+Jx} = \sqrt{\frac{2\pi}{a}} \left(\frac{\mathrm{d}}{\mathrm{d}J}\right)^k e^{\frac{J}{2a}}$$

Now we set $P(x) = \frac{\lambda}{k!}x^k$, $k \geq 3$:

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \mathrm{d}x \, \frac{P^n(x)}{n!} e^{-\frac{1}{2}ax^2+Jx} = \sum_{n=0}^{\infty} \int_{\mathbb{R}} \mathrm{d}x \, \frac{1}{n!} \left(\frac{\lambda}{k!}x^k\right)^n e^{-\frac{1}{2}ax^2+Jx}$$

$$\Rightarrow \quad Z(J) = \sqrt{\frac{2\pi}{a}} \exp\left\{\frac{\lambda}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}J}\right)^k + \frac{J^2}{2a}\right\}$$

29.1 Second approach

We expand $Z(J)$ in a different form in order to introduce a Green function.

$$Z(J) = \sum_{N=0}^{\infty} \frac{J^N}{N!} \int_{\mathbb{R}} \mathrm{d}x \, x^N e^{-\frac{1}{2}ax^2+P(x)} =: Z \sum_{N=0}^{\infty} J^N G_N$$

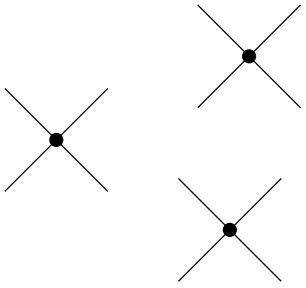
Here, G_N denotes the Green function (in the toy model). It counts how often there can be a vertex.

$$G_N := \frac{\int_{\mathbb{R}} \mathrm{d}x \, \frac{x^N}{N!} e^{-\frac{1}{2}ax^2+P(x)}}{\int_{\mathbb{R}} \mathrm{d}x \, e^{-\frac{1}{2}ax^2+P(x)}}$$

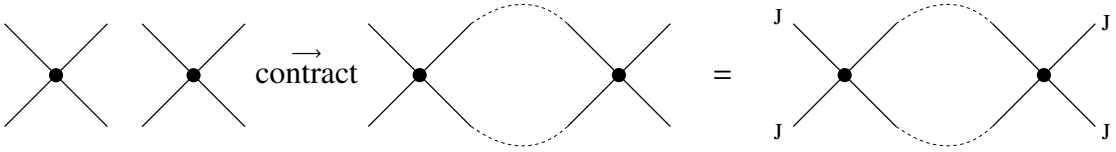
$$= \frac{\sum_{n=0}^{\infty} \int_{\mathbb{R}} \mathrm{d}x \, \frac{x^N}{N!} \left(\lambda \frac{x^k}{k!}\right)^n \frac{1}{n!} e^{-\frac{1}{2}ax^2}}{\sum_{n=0}^{\infty} \int_{\mathbb{R}} \mathrm{d}x \, \left(\lambda \frac{x^k}{k!}\right)^n \frac{1}{n!} e^{-\frac{1}{2}ax^2}}$$

29.1.1 Example

Set $P(x) = \lambda \frac{x^4}{4!}$. From the exponent being equal to four, we can already gather that the only possible vertices of this kind of interaction are the following:



Therefore, any diagrams are made up of this basic vertex, called **corolla**. One corolla consists of four half edges in one vertex. By putting two corollas next to each other and **contracting** two half edges at a time, we get interactions for external J fields.

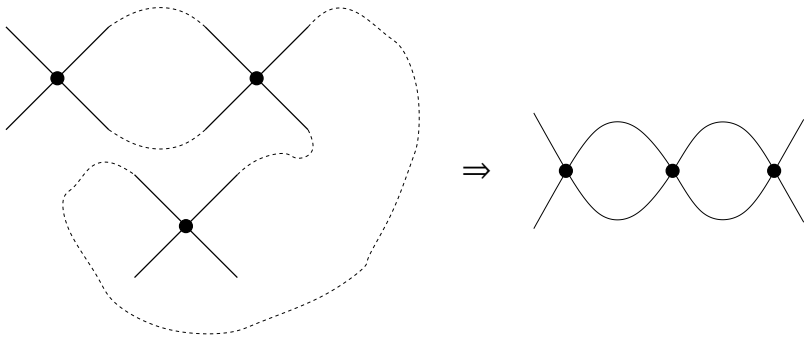


29.2 Symmetry factor

At the end of the day, we get for the number of Wick contractions the generic integrand

$$\int \mathrm{d}x x^N x^{kn} e^{-\frac{1}{2}ax^2} = \frac{(N+kn-1)!!}{N!n!(k!)^n} \tag{9}$$

Here, N denotes the number of external edges, n is the connection of extra edges (Wick contraction), and k is the power in the Lagrangian. For $k = 4$ and $N = 4$, we might get



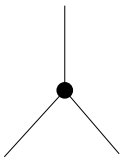
Obviously, we could have connected different edges in the three corollas and still would have gotten the same final graph. The number of different ways of gluing edges of corollas together is called the **symmetry factor**. It shows up in (9) when the summation in terms of graphs is carried out:

$$(9) \Rightarrow \sum_{\substack{\text{topologically} \\ \text{different graphs}}} \frac{\phi(\Gamma)}{\mathrm{sym}\,(\Gamma)}$$

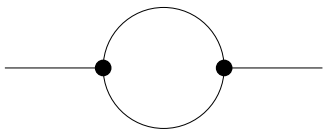
Here again, $\phi(\Gamma)$ denotes the Feynman rules of the graphs Γ.

29.2.1 Example

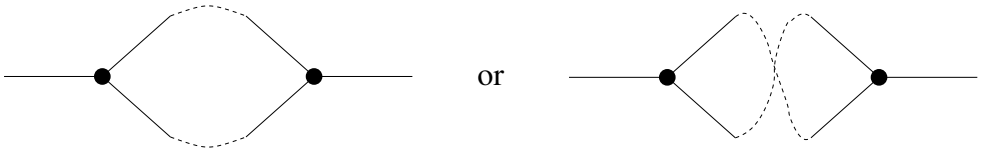
Let us consider ϕ^3 theory where the corolla is



If we wanted to build up the graph



we could either glue the two upper edges and the two lower edges together, or glue one of each.



Since there are two different ways to achieve the desired interaction, the symmetry factor of this graph is two.

30 Feynman rules of Feynman graphs Γ, in ϕ^3 theory

30.1 Vertex

For each vertex, we multiply with

$$(-i\lambda)(2\pi)^4\delta^{(4)}(k+p-q)$$

The signs in the delta function come from the orientation of the edges. Internal edges can be oriented arbitrarily, but the orientation must be kept alike for the entire expression.

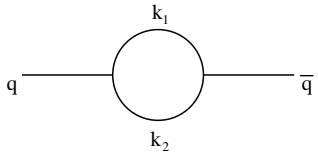
30.2 Internal edges

For each internal edge, we multiply with the scalar propagator and the volume element:

$$\text{———————} \quad \rightarrow \quad \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\epsilon}$$

30.3 Example

Let us look at the graph



For the contribution to the $\mathcal{S}$ matrix, we get

$$\mathcal{S}' = \lambda^2 \int \delta^{(4)}(k_1+k_2+q)\frac{1}{k_1^2-\mu^2+i\epsilon}\delta^{(4)}(-k_2-k_1+\bar{q})\frac{1}{k_2^2-\mu^2+i\epsilon}\,\mathrm{d}^4k_1\mathrm{d}^4k_2$$

where the first δ distribution from the left vertex, the second δ distribution comes from the right vertex, the first propagator (and integration over k_1) comes from the upper internal line, and the second propagator (and integration over k_2) comes from the lower internal edge.
Carrying out one integration in four dimension yields $k_1 = -k_2 - q$. With the other integration, we get $q = \bar{q}$.
Overall momentum is conserved, but there is still a free integration over k_2! This is non-trivial!

Topology of Wick Diagrams

12.12.2011

Last time we finished with the important formula for the number of Wick contractions (numerator)

$$\frac{(N+kn-1)!!}{N!n!(k!)^n}$$

for interactions of the type

$$\frac{\lambda}{k!}\phi^k$$

- k: number of fields interacting; exponent of the field in the Hamiltonian
- n: exchange of corollas
- N: exchange of external edges

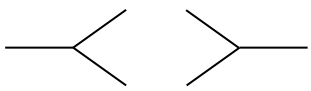
And if we take symmetry into account, we get

$$\sum_{\text{graphs }\Gamma} \frac{\lambda^{\#v(\Gamma)} J^{\#\text{external edges}(\Gamma)}}{|\mathcal{A}ut(\Gamma)|} \frac{1}{q^{\#\text{internal edges}(\Gamma)}}$$

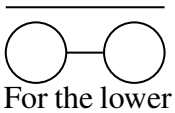
- $\#v(\Gamma)$ denotes the number of vertices in the graph Γ
- #external edges(Γ) denotes the number of external edges, that is, the number of incoming and outgoing lines
- $q^{\#\text{internal edges}(\Gamma)}$ denotes the number of internal edges, that is, the number of lines connecting two vertices
- The last part in the expression, $\frac{1}{q^{\#\text{internal edges}(\Gamma)}}$, is our "toy Fenyman rule".

31 Example

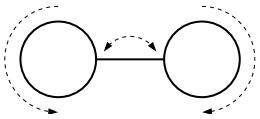
Order λ^2 (two vertices) and $k = 3$: two corollas ($\rightarrow$ two vertices), two-point function ($\rightarrow$ two external edges) $\rightarrow J^2$



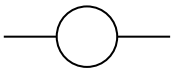
Number of Wick contractions and symmetry factor: $\frac{7 \cdot 5 \cdot 3}{(3!)^2 2!}$ Now the question is: What graphs can we glue together of these two corollas, and how often can we get the same graph?

1. 

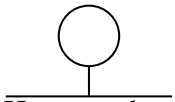
   For the lower graph, the only one involving corollas, we can connect each of the three legs of one corolla with each of the three legs of the other one, therefore are $3 \cdot 3$ possibilities to glue. With $N = 0$ (no external legs connected with a vertex), $n = 2$ (two corollas), and $k = 2$ (because of order $\lambda^k = \lambda^2$) we get for the symmetry factor $\frac{9}{0!2!(3!)^2} = \frac{9}{2 \cdot 36} = \frac{1}{8}$, and 8 is the rank of the automorphism group / the symmetry factor.

   This result can be interpreted as the flipping of two connected edges or rotating of the graph, where the result gives the same graph.

2. 

   We get for the inverse rank of the automorphism group: $\frac{1}{|\mathcal{A}ut|} = \frac{1}{2}$. The possibility to rotate the graph along the external lines and flip it without changing its appearance is the reason for the factor 2 in the denominator.

3. 

   Here, we have the same kind of symmetry as before. The bubble can be rotated, so that the two half-edges switch position, and the graph does not change. Once again, we get $\frac{1}{|\mathcal{A}ut|} = \frac{1}{2}$.

4. 

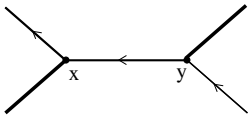
   Now, there are two loops, each of which can be rotate over without manipulating the appearance of the diagram. Both of these transformations give a factor of two to the denominator. Therefore, the inverse rank of the automorphism group is $\frac{1}{|\mathcal{A}ut|} = \frac{1}{2 \cdot 2} = \frac{1}{4}$

5. 

   In this last example, there are two axes along which there is a symmetry. First, there is a factor of $3! = 6$ because this is the number of cobinations for the three half-edges from the right vertex to the three half edges of the left vertex. Then there is an additional factor of two because this graph can be rotated again. $\frac{1}{|\mathcal{A}ut|} = \frac{1}{12}$

32 Tree level graphs

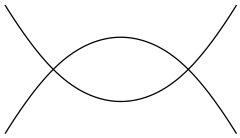
From now on, we will consider tree level graphs (graphs without closed loops). For example:



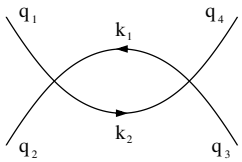
We are trying to set up an interacting quantum field theory using free quantum field theory (which we know) and math (which we can learn). We use propagators in order to describe free quantum field theory, and vertices to describe interacting quantum field theory. The actual form of the vertex can be derived from the Lagrangian.

32.1 ϕ^4 theory

Assume you look at the possibility of a scalar field interacting like $\lambda\frac{\phi^n}{n!}$ where λ is some coefficient classifying the strength of the interaction. If, in a four-dimensional spacetime, we assume $n = 4$, the interaction



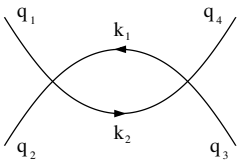
is constant under Lorentz transformation. This means that there exists a constant matrix, which is particularly not a function of the respective momenta. Therefore, the interaction cannot be proportional to a derivative.
The contribution to the $\mathcal{S}$ matrix is then given by



$$\propto \quad (-i\lambda)^2 \int \mathrm{d}^4k_1\, \mathrm{d}^4k_2\, \delta^{(4)}(q_1+q_2+k_1+k_2)\delta^{(4)}(q_3+q_4-k_1-k_2)\times \\ \times\frac{1}{k_1^2-m^2+i\epsilon}\frac{1}{k_2^2-m^2+i\epsilon}$$

$$\Rightarrow\ (-i\lambda)^2 \int \mathrm{d}^4k_1\delta^{(4)}(q_1+q_2+q_3+q_4)\frac{1}{k_1^2-m^2+i\epsilon}\frac{1}{(q_3+q_4-k_1)^2-m^2+i\epsilon}$$

The δ distribution is not a function of k_1, which is the variable to be integrated over, it "only" provides global momentum conservation. The propagators diverge logarithmically when integrated over. We solve this problem by using a cut-off because we assume that the physics we are describing is insensitive to high energies. We don't integrate from $-\infty$ to $+\infty$, but from $-\Lambda$ to $+\Lambda$ where Λ is the renormalization parameter. When regarding the limit $\Lambda \to \infty$, we regain the former expression.

$\to$ 

$$\propto \quad (-i\lambda)^2 \int_{-\Lambda}^{+\Lambda} \mathrm{d}^4k_1\delta^{(4)}(q_1+q_2+q_3+q_4)\frac{1}{k_1^2-m^2+i\epsilon}\times \\ \times\frac{1}{(q_3+q_4-k_1)^2-m^2+i\epsilon}$$

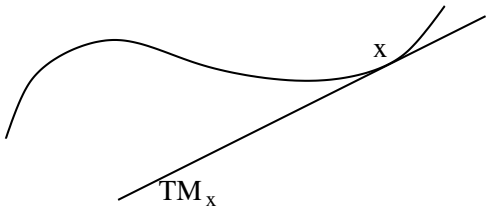
$$\propto \quad \log\Lambda + \text{finite terms} = \text{"const."}$$

$$\lim_{\Lambda\to\infty}\frac{1}{\log\Lambda} \quad \text{residue}$$

The residue of the interaction is the short term distance singularity and it gives the Feynman rule for the interaction.
There can be more complicated graphs, but they will always be built with the same vertices ($\to$ the same interaction). If the theory is self-consistent, then the residues can still be determined, and the behaviour of the system, no matter how complicated, is given by the simplest vertices.

33 Short distance behaviour

Now that we have local interaction rules, we need to come up with a manual for using them. Let us try and look at propagation and interaction from a mathematical point of view.
For any manifold M, we can define a tangent vector space in a point x, TM_x.



In TM, we have a Lorentz group, which yields a representation theory, which gives free propagators where we can find residues of convolutions of propagators which finally lead to vertices.

Lorentz group → representation theory → free propagators → residues of convolutions of propagators → vertices

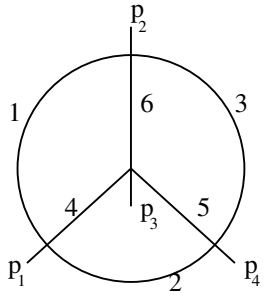
Feynman diagrams are the outcome of the respresentation theory on the tangent space.
Having Feynman rules, we will now try to write them in a (mathematically) nice way.

33.1 Kirchhoff polynomials

Example: For the Lagrangian

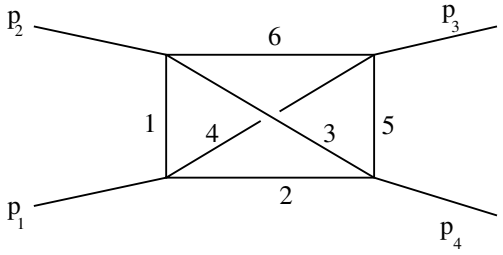
$$\mathcal{L} = \frac{1}{3}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - \frac{g}{4!}\phi^4$$

we have an interaction with vertices with four external fields, and we can, for example, get this graph with four external legs:



(10)

The six internal edges are labeled with the numbers $1 \ldots 6$. (Note that this is the same graph as



(11)

if we read the two lines in the middle in a way that they are solid diagonals, but do not interact with each other.)
In this graph, there are three independent loops, which (in a four-dimensional spacetime) leads

to a twelve-fold integration. This may not be not the easiest task, but at heart, it is doable. We use the well-known Γ function

$$\Gamma(x) = \int_0^\infty e^{-t} t^x \, \frac{\mathrm{d}x}{x}$$

and the relation ("Schwinger trick")

$$\frac{1}{U} = \int_0^\infty e^{-Ut} \, \mathrm{d}t$$

in order to write each of the three propagators in the following form:

$$\int_0^\infty \mathrm{d}A_1 \ldots \mathrm{d}A_6 \, \frac{\exp\left(- \dfrac{\phi(\Gamma)\{p_i\} + \left(\sum_{e=1}^6 A_e m_e^2\right)\Psi(\Gamma)}{\Psi(\Gamma)} \right)}{\Psi^2(\Gamma)}$$

This formula includes the **Kirchhoff Polynomials** which are defined by

$$\Psi(\Gamma) = \sum_{\text{spanning trees} T} \prod_{e \notin T} A_e$$

$$\phi(\Gamma)\{p_i\} = \sum_{\substack{\text{spanning two trees} \\ T_1 \cup T_2 \\ T_1 \cap T_2 = \emptyset}} \left\{ Q(T_1) \cdot Q(T_2) \prod_{e \notin T_1 \cup T_2} A_e \right\}$$

$$Q(T_i) = \sum_{\substack{\text{vertices} \\ v \text{ of } T_i}} q(v)$$

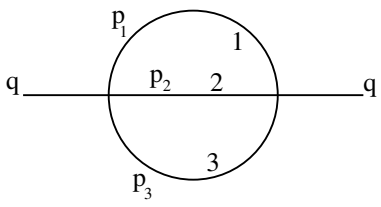
A **spanning tree** is a tree (graph with no closed loops) that is part of a bigger graph (that "spans" the bigger graph). It contains all vertices, but possibly fewer edges.
The product in $\phi(\Gamma)\{p_i\}$ means that the multiplication is carried out over all edges e that are not in the two-tree. In $Q(T_i)$, $q(v)$ denotes the external momentum incoming to v.
The contribution of this graph to the β function is $6\zeta(3)$, which connects the topology of graphs with algebraic geometry and number theory. This is rather counterintuitive!

33.2 Betti homology

So far, we have seen that a manifold M presents us with a tangent space TM on which respresentation theory $SO(1,3)$ gives free quantum field theory and locality leads to interacting field theory. With the Kirchhoff polynomials, we have made the connection to algebraic geometry - what comes next will be quite interesting.

33.2.1 Example

If we consider the graph



we see that there are three closed loops, made up of two labeled lines each: $(1,2)$, $(2,3)$, and $(1,3)$. We can get $(1,3)$ by subtracting $(2,3)$ from $(1,2)$, therefore there are only two independent loops in this graph. As a basis, we choose

$$l_1 = (1,2) = \text{“}1+2\text{”}$$
$$l_2 = (2,3) = \text{“}2+3\text{”}$$

They are the basis of a Betti homology.
For each momentum p, we define a 2×2 matrix:

$$p \quad \to \quad p_0\sigma_0 + p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3$$

and write the Kirchhoff parameters A_e in a matrix:

$$\begin{pmatrix} \overbrace{\begin{matrix} A_1+A_2 & A_2 \\ A_2 & A_2+A_3 \end{matrix}}^{=:M_0} & \begin{matrix} \mu_1A_1+\mu_2A_2 \\ \mu_2A_2+\mu_3A_3 \end{matrix} \\ \begin{matrix} \bar\mu_1A_1+\bar\mu_2A_2 & \bar\mu_2A_2+\bar\mu_3A_3 \end{matrix} & \bar\mu_1\mu_1A_1+\bar\mu_2\mu_2A_2+\bar\mu_3\mu_3A_3 \end{pmatrix} =: M$$

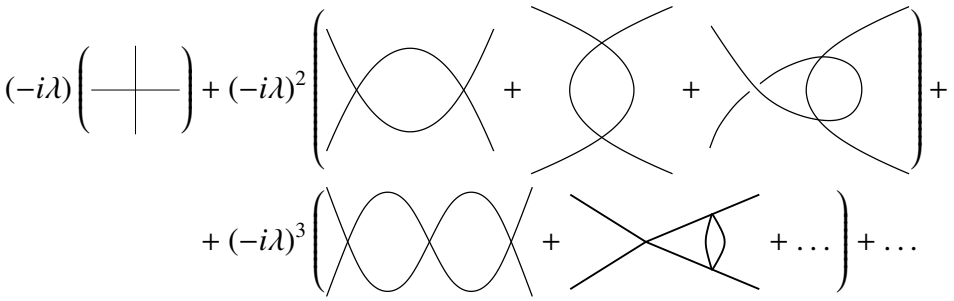
And the determinants of M_0 and M give the Kirchhoff polynomials:

$$\left|M_0^\Gamma\right| = \Psi(\Gamma)$$
$$\left|M^\Gamma\right| = \phi(\Gamma)$$

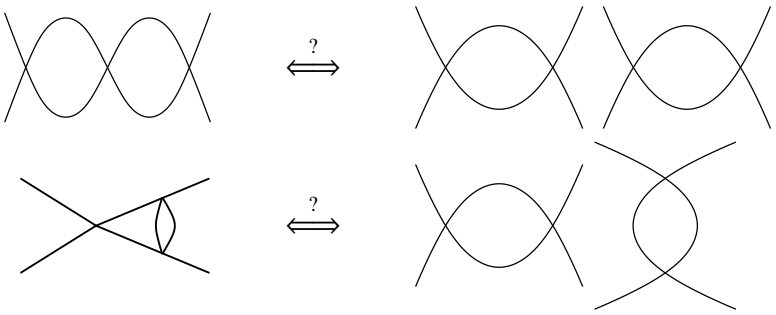
In graph theory, there is a sequence: $0 \to H_e^r \to \mathbb{Q}^\epsilon \xrightarrow{\partial} \mathbb{Q}^V \to 0$. Here, H_e^r denotes the Betti homology.

34 Hopf Algebra structure of graphs

If we expand ϕ^4 theory to order λ^3, we see some repetitions.



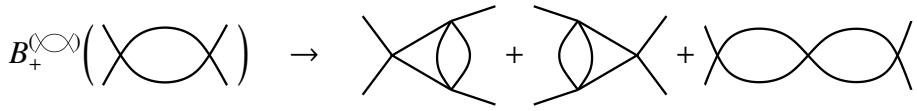
Of course, there is no evident reason to believe that this sum ever converges. But there is a lot of self-symmetry in the graphs of different orders:



For our further analysis, we will have to work out the Hopf algebra sturcture on graphs.

34.1 Hochschild 1-cocycle

We define an operator B_+^γ that will plug γ into another graph Γ (in places where it fits) to create a bigger graph $B_+^\gamma(\Gamma)$.



The four external lines of $\gamma =$  are connected with the vertices in $\Gamma =$ so γ pops up inside Γ. The sum states all the possible positions for γ in Γ.
There are fixpoint equations in the form of

$$X^n = 1 \pm \sum_{\substack{\text{primitive} \\ \text{graphs } \gamma}} \frac{g^{|\gamma|}}{|\mathcal{A}ut(\gamma)|} B_+^\gamma \left(X^n Q^{s|\gamma|} \right)$$

which are the equations of motion for this system, if $B_+^\gamma(\Gamma)$ is Hochschild closed. The + sign accounts for vertices, the − sign for propagators. $|\gamma|$ denotes the number of vertices in γ.
The Green's function

$$X^n = \sum_{\text{all graphs}} g^{|\Gamma|} \frac{\Gamma}{|\mathcal{A}ut(\Gamma)|}$$

is a solution to this fixpoint equation.
In a coalgebra, we have a coproduct $\Delta : \Gamma \to \sum \gamma \otimes \Gamma/\gamma$, which gives us all possible ways to decompose a graph Γ in smaller graphs γ and Γ/γ, following certain rules. γ and Γ/γ glued back together give Γ again.
A Hopf algebra has (among others) the properties of a coalgebra, so we still have the coproduct. We have the option to do a coradical filtration, which in the end gives information about how often subgraphs fit into graphs. We can then define a map that shifts from the coradical filtration n to the coradical filtration $n + 1$, and call this map "shift operator". Hochschild maps are those maps that glue the subgraphs γ into Γ.
The Green function for the propagator is then:

$$Q = \frac{X^4}{(X^2)^{\frac{4}{2}}}$$

This fixpoint equation is a Dyson Schwinger equation:

$$\longrightarrow \quad \sum_\gamma \frac{g^{|\gamma|}}{|\mathcal{A}ut|} B_+^\gamma$$

This points to number theory: Quantum field theory translated into mathematical questions in number theory. Quantum field theory is not mathematically well-defined, which means that it is not understood (yet). We are still facing a lot of "unknown math".

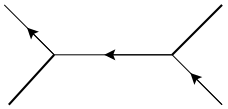
Review and Outlook

02.01.2012

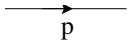
Today, our goal is to present an overview over what we have learned so far, and to grasp the aims for the upcoming weeks.
For the rest of this term, we will collect Feynman rules for different systems.

- Feynman rules for scalar fields, as in

$$\mathcal{L} = \partial_\nu \psi^\dagger \partial^\nu \psi - m^2 \psi^\dagger \psi + \lambda \psi^\dagger \psi \phi + \frac{1}{2} \partial_\nu \phi \partial^\nu \phi - \frac{1}{2} \mu^2 \phi^2$$

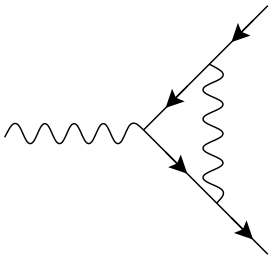


- Feynman rules for spin $\frac{1}{2}$ fermions



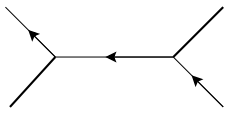
- Feynman rules for Yukawa theory: $\phi \bar{\psi} \psi$
- Feynman rules for QED (quantum electrodynamics)

Our set-up for accomplishing all that will be: to learn about tree level scattering first, which involves rather complicated, but straight-forward calculations which can basically be done by a computer. That which cannot be done by a computer is to evaluate closed internal loops, because they need to be understood rather than just written down.
After we had a deep look into closed internal loops, we will also speak about parametric representations.
In the summer semester course, we will get into gauge theories, renormalization, and path integrals.

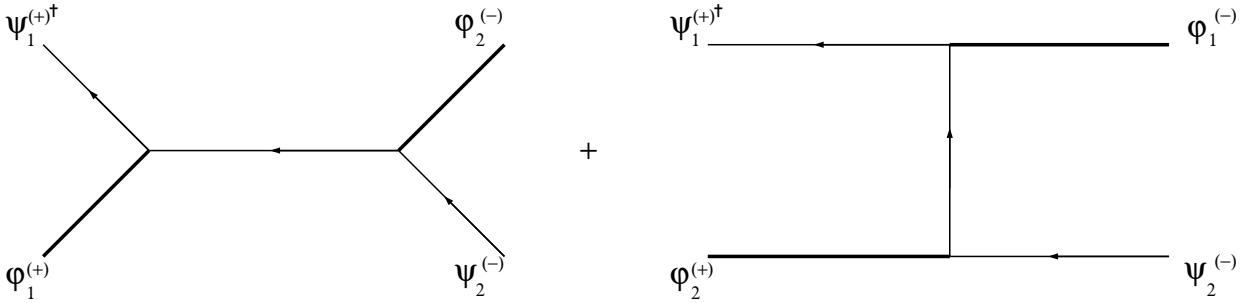


35 Review

We talked about Wick diagrams, such as



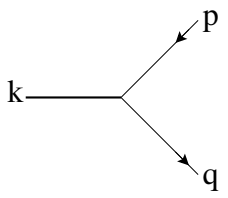
This is only one Wick diagram, but there are several Feynman diagrams corresponding to it, due to the decomposition of the external fields into creation and annihilation parts. In this case, there are 2^4 terms, since there are four external legs and two possibilities for each of them. Not all combinations contribute, though, and the only Feynman diagrams remaining are



They correspond to transition amplitudes which can easily be computed using Feynman rules.

35.1 Feynman rules of Yukawa theory

- **Vertex**: Real field coupling to a charged field.

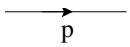


  For every vertex, we multiply with

$$(-i\lambda)(2\pi)^4\delta^{(4)}(k+p-q)$$

  In the argument of the δ distribution, which guarantees momentum conservation, we sum over the external momenta, with relative signs with respect to charge flow.

- **Charged field:** Propagator with momentum p.

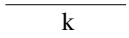


  For each internal line corresponding to a charged field, we multiply with

$$\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{\not{p}-m+i\epsilon}$$

  where m denotes the mass of the fermion.

- **Real field:** Propagator with momentum k.



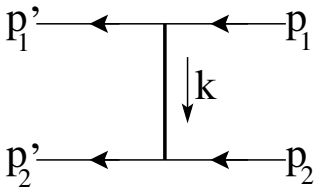
  For each internal line corresponding to a scalar field, we multiply with

$$\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\epsilon}$$

  where μ denotes the mass of the boson.

36 Computing scattering amplitudes

Let us compute the contribution of the following graph to the scattering amplitude of 2→2 scattering:



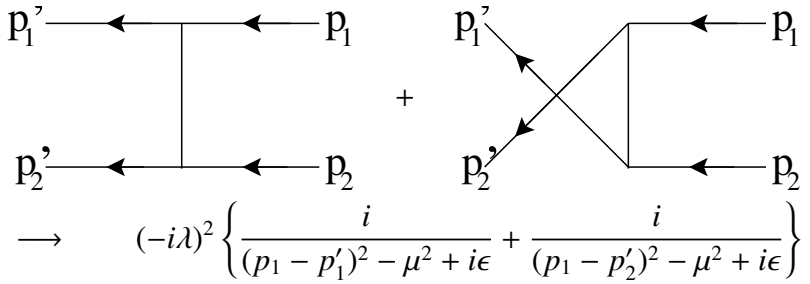
By applying the Feynman rules of Yukawa theory, we get:

$$\int \underbrace{\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\epsilon}}_{\text{internal scalar field}} \cdot \underbrace{(-i\lambda)(2\pi)^4\delta^{(4)}(p_1-k-p_1')}_{\text{upper vertex}} \cdot \underbrace{(-i\lambda)(2\pi)^4\delta^{(4)}(p_2+k-p_2')}_{\text{lower vertex}}$$

We chose for k to be directed downwards, but the result is insensitive to this choice if it is kept consistent.
The integration results in the δ distribution giving $k = p_2' - p_2$, and we get

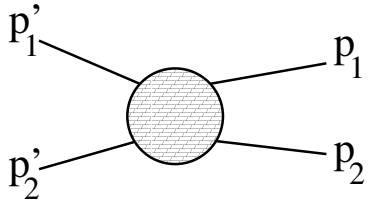
$$(2\pi)^4 \underbrace{\delta^{(4)}(p_1+p_2-p_1'-p_2')}_{\text{global momentum conserved}} (-i\lambda)^2 \frac{i}{(p_1-p_1')^2-\mu^2+i\epsilon}$$

But there is a second graph contributing to the scattering with these external legs and with the same order of λ. We can compute its contributions with a very similar calculation.



$$\longrightarrow \qquad (-i\lambda)^2 \left\{ \frac{i}{(p_1-p_1')^2-\mu^2+i\epsilon} + \frac{i}{(p_1-p_2')^2-\mu^2+i\epsilon} \right\}$$

This is the **invariant amplitude of order** λ^2 contributing to the process $\Psi^\dagger\Psi \to \Psi^\dagger\Psi$:

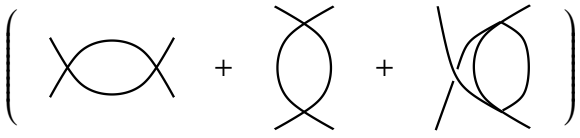


36.1 How to find Feynman rules

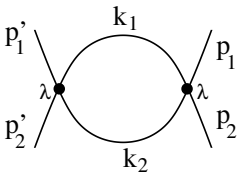
We can write down a theory with its transformation rules under the Poincaré group. For scalar fields, unfortunately, this does not really tgive away much about Feynman rules. For the interacting part of the Hamiltonian, we get something like

$$\mathcal{H}_I \propto \lambda \frac{\phi^n}{n!}$$

where n gives the number of legs running into a vertex. But which n is a good choice, which ones are bad choices? To find that out, we have to check the theory's consistency under renormalization. (We will talk a lot more about renormalization in the upcoming term.)
Eventuelly, it turns out that $n = 4$ is a good choice for spacetime dimensions $D = 4$. Then, we get for 2×2 scattering to order λ^2:



The contribution of the first of these three to $\mathcal{S}$ can be computed to give



$$\begin{aligned}
&\propto \int \mathrm{d}^4k_1\, \mathrm{d}^4k_2 \frac{1}{k_1^2 - m^2 + i\epsilon} \frac{1}{k_2^2 - m^2 + i\epsilon} \underbrace{\delta^{(4)}(p_1 + p_2 - k_1 - k_2)}_{\to\, k_1+k_2=p_1+p_2} \delta^{(4)}(k_1 + k_2 - p_1' - p_2') \\
&= \int \mathrm{d}^4k \frac{1}{(k - p_1 - p_2)^2 - m^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon} \underbrace{\delta^{(4)}(p_1 + p_2 - p_1' - p_2')}_{\text{no } k!} \\
&\propto \int \mathrm{d}^4k \frac{1}{(k - p_1 - p_2)^2 - m^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon}
\end{aligned}$$

This is clearly a divergent integral! To get around the ill-definedness for just now, we define a cut-off Λ for the integration boundaries. If we do that, the integral will be proportional to $\log \Lambda$.

36.2 Physicists' solution

We will try to redefine the coupling λ by introducing a second parameter λ_0, where the observed λ is a correction to the "real" coupling λ_0:

$$\lambda = \lambda_0(1 + c_1\lambda_0 \log \Lambda)$$

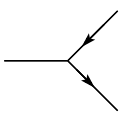
This is just a modification of the formula, no physical observables are changed by this. But it turns out that this modification absorbs the divergence! Even more, in $n = 4$, we can find suitable c_1, c_2, etc. for any graph to absorb the divergences! However, this will only work for $n = 4$, and for no other n (in ϕ^4 theory, at least). Consequently, $n = 4$ appears to be a very good choice for n.

37 Coupling of a spin $\frac{1}{2}$ field and a real field

Reminder: We associate with $\bar{\Psi}\Psi$ ——→—— a spin $\frac{1}{2}$ field
ϕ —————— a real field
and we keep in mind that Ψ is a column vector, $\bar{\Psi} = \Psi^\dagger\gamma_0$ is a row vector, ϕ is a scalar, and that $\bar{\Psi}(\cdot)_{4\times4}\Psi\phi$ eventually gives a scalar. $(\cdot)_{4\times4}$ can only be the unitary matrix, which becomes clear after consulting a little Clifford algebra and representation theory, because only $\mathbb{I}$ transforms like a scalar.

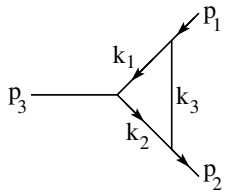
37.1 Vertex correction

Since vertices are fundamental for interactions, it is quite interesting to look at vertex corrections. By this, we denote further interactions within the primary vertex that shift the global transition amplitude. The single vertex,



,

contributes with a factor $(-i\lambda)$. The next highest order would result in the exchange of a boson between the two fermion lines and contributes with a factor $(-i\lambda)^3$:



$$\propto (-i\lambda)^3 \int \mathrm{d}^4k_1\, \mathrm{d}^4k_2\, \mathrm{d}^4k_3 \frac{i}{\not{k}_1 - m + i\epsilon} \frac{i}{\not{k}_2 - m + i\epsilon} \frac{i}{\not{k}_3 - \mu + i\epsilon} \times$$

$$\times \delta^{(4)}(p_1 - k_1 - k_3)\delta^{(4)}(k_1 - k_2)\delta^{(4)}(k_2 + k_3 - p_2)$$

Even though this is not generally the case, we have set $p_3 \equiv 0$ to make the calculation a little bit easier. With $\frac{i}{\not{k} - m + i\epsilon} = \frac{\not{k} + m}{k^2 - m^2 + i\epsilon}$, two of the integrals break down due to the δ distributions. We will be left with an overall δ function and an integral like

$$\begin{aligned}
&\int \mathrm{d}^4k \frac{1}{(\not{k} - m)(\not{k} - m)} \frac{1}{(k + p_1)^2 - \mu^2} \\
\to \quad &\int_{-\Lambda}^{+\Lambda} \mathrm{d}^4k \frac{\not{k}\not{k}}{(k^2 - m^2)^2(k + p_1)^2} \\
\to \quad &\int_{-\Lambda}^{+\Lambda} \mathrm{d}^4k \frac{1}{k^4} \\
\to \quad &c \cdot \log \Lambda
\end{aligned}$$

From the interaction, we get $c = \mathbb{I}_{4\times4}$.
In QED, ———————— is not a scalar field, but a vector field, and thus transforms like a vector, too. Since there are plenty of options for a matrix (in the vectorial case) instead of $\mathbb{I}$ (in the scalar case) to transform like a vector, things get a lot more interesting in QED!

Yukawa Theory, Spin Sums, and γ-matrices

04.01.2012

Before we start, let us accept that today, we will gain but a small insight in the technology of Yukawa theory, spin sums, and γ-matrices. For the latter, we will focus especially on the multiplication and calculation on machines.

38 Yukawa theory

The interaction Lagrangian of Yukawa theory is

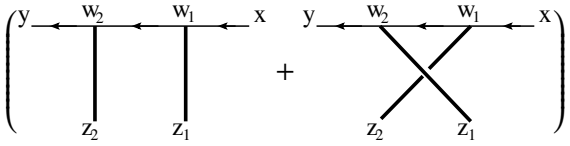
$$\mathcal{L}_{int} = g\phi\bar{\psi}\psi$$

and the propagators for the ψ and ϕ fields are (in momentum space):

$$\begin{aligned} \psi: \qquad & S(p) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \\ \phi: \qquad & \Delta(p) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\epsilon} \end{aligned}$$

where m and M are the masses associated with the respective fields.
In Yukawa theory, the Feynman rules are quite easy. To demonstrate that, and to get used to the theory, let's try to calculate the vacuum expectation value of $T\psi_\alpha(x)\bar{\psi}_\beta(y)\phi(z_1)\phi(z_2)$, for connected graphs to the lowest order (which is g^2).
In coordinate space:

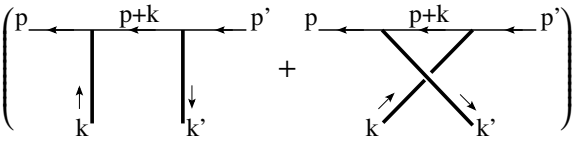
$$\left\langle 0 \left| T\psi_\alpha(x)\bar{\psi}_\beta(y)\phi(z_1)\phi(z_2) \right| 0 \right\rangle_C \propto$$



$$\propto g^2 \int \mathrm{d}^4 w_1 \mathrm{d}^4 w_2 \left\{ \left(S(x - w_1) S(w_1 - w_2) S(w_2 - y) \right)_{\alpha\beta} \times \right.$$
$$\left. \times \left(\underbrace{\Delta(z_2 - w_2)\Delta(z_1 - w_1)}_{\text{first graph}} + \underbrace{\Delta(z_1 - w_2)\Delta(z_2 - w_1)}_{\text{second graph}} \right) \right\}$$

In momentum space:

$$\langle 0 | \ldots | 0 \rangle_C \propto g^2$$

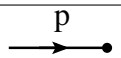
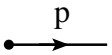
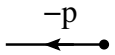
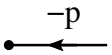
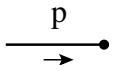
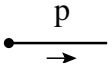
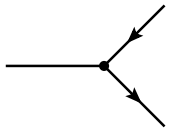
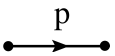
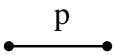


$$\propto g^2 \bar{u}_{s'}(p') \left[\frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2 + i\epsilon} + \frac{\not{p} - \not{k}' + m}{(p+k')^2 - m^2 + i\epsilon} \right] u_s(p)$$

This is already a bit lengthy if we try to calculate it on paper, but it is still doable, in the sense that nothing unexpected shows up and the approach is straight-forward. However, for more complicated graphs, these expressions become rather hard, if not hopeless.

39 Feynman rules of Yukawa theory

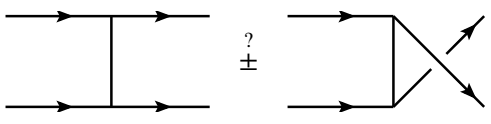
For this discussion, we assume that the spin $\frac{1}{2}$ fermion is an electron, the respective antifermion is a positron, and that the boson has spin 0 and could therefore be a Higgs particle. Let us also remind ourselves that the arrow in the fermion lines indicates charge flow.

	field			multiply with
•		incoming e^- with momentum p	$\rightarrow$	$u_s(p)$
•		outgoing e^- with momentum p	$\rightarrow$	$\bar{u}_s(p) \quad \left(= u_s^\dagger(p)\gamma^0\right)$
•		incoming e^+ with momentum p	$\rightarrow$	$\bar{v}_s(p)$
•		outgoing e^+ with momentum p	$\rightarrow$	$v_s(p)$
•		incoming Higgs with momentum p	$\rightarrow$	1
•		outgoing Higgs with momentum p	$\rightarrow$	1
•		e^--e^+-Higgs vertex	$\rightarrow$	$ig\mathbb{I}_{4\times4}$ ($\mathbb{I}$ usually omitted)
•		internal e^- line with momentum p	$\rightarrow$	propagator: $\dfrac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$
•		internal Higgs line with momentum p	$\rightarrow$	propagator: $\dfrac{i}{p^2-M^2+i\epsilon}$

For all spinors with spin index, s gives the spin (+ or −) in the direction of the propagation.
So far, so good, but the more difficult part is to find the right spin indices.
We can only allow for fermion lines with consistent orientation. To write down the transition amplitude in the correct order, start at the end of a fermion line and then read the fermion line backwards. then we can compute the probability amplitude with only the use of traces.

39.1 Relative sign in fermionic expressions

Consider the exchange of one boson between two fermions,



,

where we have identical incoming and outcoming particles. What is the relative sign between the two transitions?
We can obtain the relative sign by arbitrarily choosing one diagram, giving this diagram a + sign, and determine the number of permutations needed in order to transform this graph into the other graph. The overall sign cannot be obtained, but since the observable quantity is the

absolute square, it is not important to know the global phase, only the relative phase is of importance. In this case, we find a − sign.

40 Spin sums

Let us we write the above transition amplitude in momentum space with the abbreviations

$$\begin{aligned}
\bar{u}' &:= \bar{u}_{s'}(p') \\
u &:= u_s(p) \\
A &:= \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2 + i\epsilon} + \frac{\not{p} - \not{k}' + m}{(p+k')^2 - m^2 + i\epsilon} \\
\Rightarrow \quad & \left\langle 0 \left| T\psi_\alpha(x)\bar{\psi}_\beta(y)\phi(z_1)\phi(z_2) \right| 0 \right\rangle_C \propto g^2 \bar{u}' A u
\end{aligned}$$

Since $\bar{u}'$ is a row vector, u is a column vector, and A is a 4×4 matrix, the overall transition amplitude is a scalar, as it is supposed to be.
The calculations can be simplified for just a little bit, if we keep in mind that in the end all we need is the absolute square of the transition amplitude, $|T|^2$.
If we sum over all possible (two) spins, we get the identity

$$\sum_{s=\pm} u_s(\vec{p})\bar{u}_s(\bar{p}) = \not{p} + m$$

then we can express $|T|^2$ as

$$|T|^2 = \left(\bar{u}' A u\right)\left(\bar{u} A u'\right) \equiv \mathrm{Tr}\left[\bar{u}\bar{u}' A u u'\right]$$

We know that in A there are two slashed spinors and therefore two γ matrices, and with $\bar{u}\bar{u}' = \not{p}' + m$ and $uu' = \not{p} + m$, the trace must be a product of (at most) four γ matrices.

41 γ matrix technology

We will use the Clifford algebra in order to find out how to multiply γ matrices in an efficient way. The so-called γ-matric technology is about the calculus of Clifford algebras: traces of products of γ matrices.

41.1 Basic rule

There is one elementary rule for γ matrices, their anticommutator:

$$\left\{\gamma_\mu, \gamma_\nu\right\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\mathbb{I}_{n'\times n'} \tag{12}$$

$$\Rightarrow \qquad \gamma_0^2 = \mathbb{I}, \quad \gamma_i^2 = -\mathbb{I} \tag{13}$$

Here, $n' = 2^{\frac{n}{2}}$ is the dimension of the matrices, where n is the dimension of spacetime. In our case, where $n = 4$, we get $n' = 4$. $g_{\mu\nu}$ denotes the metric tensor. The μ and ν are spacetime indeces, $\mu, \nu \in \{0, 1, 2, 3\}$.

41.2 γ_5

Besides the γ_μ, there is another quite interesting and useful matrix:

$$\gamma_0\gamma_1\gamma_2\gamma_3 =: c \cdot \gamma_5$$

where c is either 1, i, or $-i$. The choice of c is not important, but it needs to be consistent throughout the calculations. γ_5 has the properties that

$$\begin{aligned} &\gamma_5^2 \sim \mathbb{I} \\ \text{and} \quad &\{\gamma_5, \gamma_\mu\} = 0. \end{aligned}$$

41.3 Computing traces

For an interaction with external fermions, we write down the spinors and the propagators, bracket the propagator matrix, and compute traces of the resulting products of γ matrices. But how do we calculate these traces?
We find two different types of traces $\mathrm{Tr}\left(\gamma_{\mu_1} \dots \gamma_{\mu_k}\right)$:

- Traces with no γ_5 involved
- Traces with one γ_5 involved.

Since γ_5 and γ_μ anticommute, we can always move all γ_5's within the expression to one spot, and with $\gamma_5^2 = \mathbb{I}$, we are left with either one or no γ_5.
In the end, we find four different cases of traces:

1. $\mathrm{Tr}\left[\gamma_{\mu_1}, \dots, \gamma_{\mu_{2k}}\right]$: even numbers of γ's, no γ_5
2. $\mathrm{Tr}\left[\gamma_{\mu_1}, \dots, \gamma_{\mu_{2k+1}}\right]$: odd numbers of γ's, no γ_5
3. $\mathrm{Tr}\left[\gamma_5\gamma_{\mu_1}, \dots, \gamma_{\mu_{2k}}\right]$: even numbers of γ's, one γ_5
4. $\mathrm{Tr}\left[\gamma_5\gamma_{\mu_1}, \dots, \gamma_{\mu_{2k+1}}\right]$: odd numbers of γ's, one γ_5

It turns out that all traces with an odd number of γ's is identical zero. We will see about that in a minute.

41.4 even number of γ's, no γ_5

41.4.1 two γ matrices

To compute $\mathrm{Tr}\left[\gamma_{\mu_1}, \dots, \gamma_{\mu_{2k}}\right]$, we start with the simplest case of two γ matrices.

$$\begin{aligned} \mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right] &= \mathrm{Tr}\left[\frac{1}{2}\Big(\underbrace{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu}_{=2g_{\mu\nu}\mathbb{I}}\Big) + \frac{1}{2}\Big(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\Big)\right] \\ &= g_{\mu\nu}\underbrace{\mathrm{Tr}\left(\mathbb{I}\right)}_{=4} + \frac{1}{2}\Big(\mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right] - \underbrace{\mathrm{Tr}\left[\gamma_\nu\gamma_\mu\right]}_{=\mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right]}\Big) \\ \rightarrow \quad \mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right] &= g_{\mu\nu}\,\mathrm{Tr}\left[\mathbb{I}\right] = 4g_{\mu\nu} \end{aligned}$$

In the second line, we used the linearity of the trace, and in the third line the cyclicity ($\mathrm{Tr}\left[ABC\right] = \mathrm{Tr}\left[CAB\right] = \mathrm{Tr}\left[BCA\right]$).

41.4.2 Clifford algebra representations

Looking for a basis of 4×4 matrices, we can use Clifford algebra representations.

- $\mathbb{I}$: one matrix
- γ_μ with $\mu \in \{0, 1, 2, 3\}$: four matrices
- $\gamma_\mu \wedge \gamma_\nu$ (external product): six matrices $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$
- $\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho$: four matrices
  with the total antisymmetric tensor (Levi-Cevita tensor):

$$\epsilon_{0123} := +1$$

$$\text{and} \quad \epsilon_{\mu\nu\rho\sigma} := \begin{cases} +1 & \text{for even permutations of } \{0,1,2,3\} \\ -1 & \text{for odd permutations of } \{0,1,2,3\} \\ 0 & \text{else} \end{cases}$$

- $\gamma_5 = \frac{1}{24}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$: 1 matrix

This set of 16 independent matrices is a basis of 4×4 matrices.

41.4.3 four γ matrices

For the trace

$$\mathrm{Tr}\left[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right] = \mathrm{Tr}\left[\gamma_\mu\gamma_\nu(\gamma_\sigma\gamma_\rho + 2g_{\rho\sigma}\mathbb{I})\right] = \mathrm{Tr}\left[\gamma_\sigma\gamma_\mu\gamma_\nu\gamma_\rho\right]$$

When we use the Clifford algebra relation from the beginning, and also use the cyclicity of the trace, then we will realize that anticommuting two γ matrices will eventually give either the same trace again, or terms with fewer γ matrices. In that way, we can reduce the number of γ matrices in the trace step by step. We have found an inductive way to determine the trace. This does not only hold for four, but for any even number of γ matrices with no γ_5 in the expression.

41.5 odd number of γ's, no γ_5

To convince ourselves that $\mathrm{Tr}\left[\gamma_{\mu_1}, \ldots, \gamma_{\mu_{2k+1}}\right] = 0$, we will start by computing $\mathrm{Tr}\,[\gamma_\mu]$. To do that, we use

$$\begin{aligned} & (\gamma_5)^2 = \mathbb{I} \\ \text{and} \quad & \left[\gamma_5, \gamma_\mu\right] = 0 \\ \Rightarrow \quad & \mathrm{Tr}\,[\gamma_\mu] = \mathrm{Tr}\,[(\gamma_5)^2\,\gamma_\mu] = -\,\mathrm{Tr}\,[\gamma_5\gamma_\mu\gamma_5] = \mathrm{Tr}\,[\gamma_\mu\,(\gamma_5)^2] = -\,\mathrm{Tr}\,[\gamma_\mu] \\ \Rightarrow \quad & \mathrm{Tr}\,[\gamma_\mu] = 0 \end{aligned}$$

In the second step, we used the anticommutator, and in the third step the cyclicity of the trace. Similarly, we can always plug in $(\gamma_5)^2$ anywhere and use the anticommutator and cyclicity to make the expression zero.

41.6 even number of γ's, one γ_5

Before we compute $\mathrm{Tr}\left[\gamma_5\gamma_\mu\gamma_\nu\right]$ we consider the longer, yet simpler expression

$$\mathrm{Tr}\left[\gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_4}\right] = \mathrm{Tr}\left[\gamma_{\nu_1}\dots\gamma_{\nu_4}\gamma_{\mu_1}\dots\gamma_{\mu_4}\right]\cdot\frac{1}{2}\epsilon^{\nu_1\dots\nu_4}$$

To solve this, we use the fact that this expression is antisymmetric in ν_i, and we use the cyclicity again. Then we will see that this trace is a product of linear combinations of four metric tensors, where all combinations are allowed:

$$\begin{aligned}\mathrm{Tr}\left[\gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_4}\right] &= g_{\nu_1\nu_2}g_{\nu_3\nu_4}g_{\mu_1\mu_2}g_{\mu_3\mu_4} + \dots + g_{\nu_1\mu_1}g_{\dots}\dots \\ &\propto \epsilon^{\mu_1\mu_2\mu_3\mu_4}\end{aligned}$$

The proportionality to the Levi Cevita tensor is given because of the complete antisymmetry in ν_i.

42 Physical computations

We have see that the trace of an odd number of γ matrices vanishes, and that the trace of an even number of γ matrices and one γ_5 is totally antisymmetric. In particle physics, one often comes across computations like

$$\mathrm{Tr}\left[u'\bar{u}'Au\bar{u}A\right]$$

with

$$\begin{aligned}u'\bar{u}' &= \not{p}' + m \\ u\bar{u} &= \not{p} + m \\ A &\propto \not{k}\end{aligned}$$

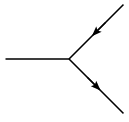
So, in principal we get

$$\begin{aligned}\mathrm{Tr}\left[\not{p}'\not{k}\not{p}\not{k}\right] &= p'_\alpha k_\beta p_\gamma k_\delta \,\mathrm{Tr}\left[\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\right] \\ &= -\,\mathrm{Tr}\left[\not{p}'\not{k}\not{k}\not{p}\right] + 2(p\cdot k)\,\mathrm{Tr}\left[\not{p}'\not{k}\right] \\ &= -k^2\,\mathrm{Tr}\left[\not{p}'\not{p}\right] + (p\cdot k)\,(p'\cdot k)\,\mathrm{Tr}\left[\mathbb{I}\right] \\ &= -k^2\,(p'\cdot p)\,\mathrm{Tr}\left[\mathbb{I}\right] + (p\cdot k)\,(p'\cdot k)\,\mathrm{Tr}\left[\mathbb{I}\right] \\ &= -4k^2\,(p'\cdot p) + 4(p\cdot k)\,(p'\cdot k)\end{aligned}$$

In the second line, we used the Clifford algebra relations to swap $\not{p}$ and $\not{k}$. In the third line, we used $\not{k}\not{k} = k^2$ and $\not{k}\not{p} = k\cdot p$, where $(\cdot)$ denotes the scalar product.

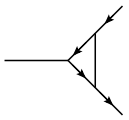
42.1 Vertex correction: one loop

For the plain vertex,



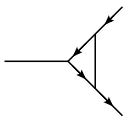
$\propto g$,

we can compute a correction,



$$\propto g^3 \,,$$

for now in the limit of massless fermions and bosons, $m = 0 = M$. We will also omit all constant factors like $(2\pi)^4$ in the calculation for simplicity.



$$\begin{aligned} &\propto g^3 \int \mathrm{d}^4k \frac{\not{k}\not{k}}{k^2k^2(k+q)^2} \\ &= g^3 \int \mathrm{d}^4k \frac{1}{k^2(k+q)^2} \end{aligned}$$

This integral is logarithmically divergent and thus not very well-defined. In order to try to navigate around the divergence, we use

$$\frac{1}{U} = \int_0^\infty \mathrm{d}A \; e^{-AU}$$

and we set k^2 or $(k+q)^2$ for U.

$$\Rightarrow \quad g^2 \int \frac{\mathrm{d}^4k}{k^2(k+q)^2} = g^3 \int \mathrm{d}^4k \int_0^\infty \mathrm{d}A \int_0^\infty \mathrm{d}B \; e^{-Ak^2 - B(k+q)^2}$$

Next time, we will explicitely evaluate this integral. For now, let me illustrate how it is to be solved.
At first, we aim at carrying out the k integration first, so we complete the square, and we get

$$\begin{aligned} -Ak^2 - B(k+q)^2 &= -(A+B)\left\{k^2 + \frac{1k\cdot qB}{A+B} + \frac{Bq^2}{A+B}\right\} \\ &= -(A+B)\left\{\left(k + q\frac{B}{A+B}\right)^2 - \frac{q^2B^2}{(A+B)^2} + \frac{(A+B)Bq^2}{(A+B)^2}\right\} \end{aligned}$$

and since the k integral is translation invariant, we can shift k to $k + q\frac{B}{A+B}$ so that the integral looks

$$\int_0^\infty \mathrm{d}A \int_0^\infty \mathrm{d}B \; e^{-\frac{q^2AB}{A+B}} \int \mathrm{d}^4k \; e^{-(A+B)k^2} = 4\pi^2 \int_0^\infty \mathrm{d}A \int_0^\infty \mathrm{d}B \; \frac{e^{-\frac{q^2AB}{A+B}}}{A+B}$$

This integral is still problematic for $q^2 = 0$: It is highly divergent at lower boundaries.

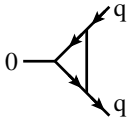
An integral towards QED

09.01.2012

At the end of the last lecture, we saw a draft of the computation of the one-loop correction to the vertex function. We will now finish this computation in more detail.

43 Vertex correction in Yukawa theory

Applying the Feynman rules of (massless) Yukawa theory, the one loop vertex function with no incoming momentum translates to



$$\begin{aligned} &\to \int \mathbb{I}\frac{\not k}{k^2+i\epsilon}\mathbb{I}\frac{\not k}{k^2+i\epsilon}\mathbb{I}\frac{1}{(k+q)^2+i\epsilon} \\ &\to \mathbb{I}\int \mathrm{d}^4k \frac{k^2}{(k^2+i\epsilon)^2}\frac{1}{(k+q)^2+i\epsilon} \\ &\to \int \mathrm{d}^4k \frac{1}{k^2(k+q)^2} \end{aligned}$$

where we set $\not k_1 = \not k_2 = \not k$ because there is no momentum exchange. We are reminded that

$$\not k = \sum_{\mu=0}^{3} k_\mu \gamma^\mu \equiv k_\mu \gamma^\mu \qquad \text{Einstein's summation convention}$$

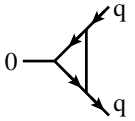
$$\not k \not k = k_\alpha \gamma^\alpha k_\beta \gamma^\beta = k_\alpha k_\beta \gamma^\alpha \gamma^\beta = k_\alpha k_\beta \left\{ \frac{1}{2}\Big(\underbrace{\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha}_{=2g^{\alpha\beta}\mathbb{I}} \Big) + \frac{1}{2}\Big(\underbrace{\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha}_{=0} \Big) \right\}$$

$$= k^2$$

The first term in just the Clifford algebra anticommutator applied, the second term must be zero because it is antisymmetric in α and β, but $k_\alpha k_\beta$ is symmetric, so the antisymmetric part has to vanish.
To try to avoid the divergence of the vertex correction, we used

$$\frac{1}{U} = \int_0^\infty \mathrm{d}A \; e^{-AU}$$

in order to rewrite the integral to give



$$\to \int \mathrm{d}^4k \iint_0^\infty \mathrm{d}^4A \; \mathrm{d}^4B \; e^{-Ak^2 - B(k+q)^2}$$

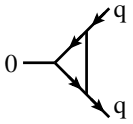
We will now do the k integration first and leave the parametric integration be for just now, so that maybe we understand the matter a little bit better. We complete the square:

$$\begin{aligned} -Ak^2 - B(k+q)^2 &= -k^2(A+B) + 2k\cdot qB + B^2q^2 \\ &= -(A+B)\left\{ \left(k + \frac{qB}{A+B} \right)^2 - \frac{q^2B^2}{(A+B)^2} + \frac{q^2B(A+B)}{(A+B)^2} \right\} \\ &= -(A+B)\left\{ \left(k + \frac{qB}{A+B} \right)^2 - \frac{q^2AB}{(A+B)^2} \right\} \end{aligned}$$

We get for the k integration:

$$\int_{\mathbb{R}^4} \mathrm{d}^4k \; e^{-(A+B)k^2} = \frac{4\pi^2}{(A+B)^2}$$

where the exponent of the denominator is half of the spacetime dimension.
Altogether, the integral gives



$$\rightarrow \quad \int_{\mathbb{R}^2_+} \mathrm{d}A\mathrm{d}B \; \frac{e^{-\frac{q^2AB}{(A+B)}}}{(A+B)^2}$$

We set $B = A \cdot b$. The integral will then give

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}A\mathrm{d}b}{A(1+b)^2} e^{-q^2A\frac{b}{1+b}} =: I(q^2)$$

And this expression would also be equal to

$$\int_0^{\infty} \frac{\mathrm{d}t}{t} \frac{e^{-tq^2\frac{ab}{(a+b)}}}{(a+b)^2} \Big(a\mathrm{d}b - b\mathrm{d}a\Big)$$

No matter how we alter the expression, we always find a divergence for (A, B) close to $(0, 0)$. Therefore, in order to find out the functional structure of the integral, we introduce a cut-off at the lower boundary. We use the exponential integral:

$$Ei(t) = \int_c^{\infty} \frac{e^{-}t}{t}\mathrm{d}t \quad \sim \quad \log c + \gamma_E + O(c) \quad , \qquad 0 < c \ll 1$$

where γ_E is the Euler Mascheroni constant. In that way, we navigate around the singularity at the origin by leaving out a quarter disc.

Our integral above would give

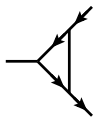
$$I = \log c + \log q^2 + O(c) + \gamma_E$$

and the divergence is "only" logarithmic and not a function of q^2, the physical observable. We can erase the divergence by simply comparing the integral with the same integral at a different scale $\bar{q}^2$.

$$I(q^2) - I(\bar{q}^2) = \log\left(\frac{q^2}{\bar{q}^2}\right)$$

With this knowledge, we can find that the one-loop vertex correction depends on $\log c$, where c is a parameter $0 < c \ll 1$, which is a constant independent of any physical parameter, so

we can modify or shift the expression by a term which may be divergent, but is independent of all parameters and does not alter any physically important quantity, because it is unobservable. Therefore, after shifting, we get

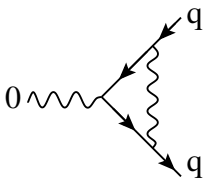


$$\sim \quad \log c + \ldots \quad \Rightarrow \quad \sim \mathbb{I} + \text{ finite terms}$$

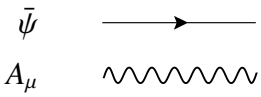
For any logarithmic divergences, we can avoid singularities by simply comparing two integrals ("measuring twice / at two different scales"). For divergences of a higher order, say, linear or quadratic divergences, this is not so easy. However, since $k^2 = k_o^2 - \vec{k}^2$ has a zero, we can derive that all problems are independent of the short-distance scale. Still, for large k, there is a problem, which does not go away by simply transfering it from one spot to another. In our case, we went from a momentum integral to a parameter integral, which made it easier to understand the divergence, which is still there after all.

44 QED corrections

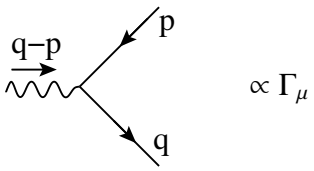
So far, we found a vertex correction in massless Yukawa theory. In order to find out what the photon propagator and the vertex function in QED should look like, we will use the Clifford algebra to derive possible Feynman rules for QED.
What we actually do know about quantum electrodynamics if that there are three fields $\bar{\psi}, \psi, A_\mu$, where $\bar{\psi}$ and ψ correspond to charged fermion fields and A_μ is the photon field, denoted in a diagram by $\rightarrow\!\!-$ and $\sim\!\!\sim$, respectively.
We can find Feynman rules of



just by looking at



There should exist a simple vertex with non-vanishing incoming momentum, proportional to some matrix Γ_μ:



So that we get an expression for incoming and outgoing fermions like

$$\bar{\psi}\Gamma_\mu\psi \qquad \Leftrightarrow \qquad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

For a general ansatz to finding Γ_μ, we write

$$\Gamma_\mu \sim c_1\gamma_\mu + c_2\frac{\not{q}q_\mu}{q^2} + \ldots$$

44.1 Vertex correction in QED

We know what the scalar field propagator, $\langle 0\,|\phi\phi|\,0\rangle$, graphically given by ---------- , does. For photons, however, the fields are not scalar, but vector fields. We try to find the bosonic field propagator

$$\left\langle 0\left|A_\mu A_\nu\right|0\right\rangle \qquad \Leftrightarrow \qquad \mu \overset{q}{\sim\!\sim} \nu$$

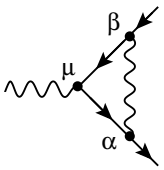
Since photons are bosons, $A_\mu A_\nu$ is symmetric in μ and ν, so we need a symmetric two-tensor. This tensor could only depend on the metric or external momenta, since there are no other quantities involved. Possible terms of the photon propagator are

- $g_{\mu\nu}$
- $q_\mu q_\nu$

So the most general ansatz for the photon propagator is

$$\mu \overset{q}{\sim\!\sim} \nu \quad \propto \quad \frac{\bar{c}_1 g_{\mu\nu} + \bar{c}_2 \frac{q_\mu q_\nu}{q^2}}{p^2 + i\epsilon}$$

With this ansatz, we calculate the one-loop correction to the photon-fermion-antifermion-vertex.



$$\rightarrow \quad \int \gamma_\alpha \frac{1}{\not{k} + i\epsilon} \gamma_\mu \frac{1}{\not{k} + i\epsilon} \gamma_\beta \left\{ \bar{c}_1 g_{\alpha\beta} + \bar{c}_2 \frac{(k+q)_\alpha (k+q)_\beta}{(k+q)^2 + i\epsilon} \right\}$$

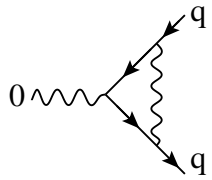
$$\rightarrow \quad \bar{c}_1 \underbrace{\int \frac{\gamma_\alpha \not{k} \gamma_\mu \not{k} \gamma^\alpha}{(k^2)^2 (k+q)^2}}_{(A)} + \bar{c}_2 \underbrace{\int \frac{(\not{k} + \not{q}) \not{k} (\not{k} + \not{q})}{(k^2)^2 (k+q)^2}}_{(B)}$$

$$(A) \quad \Rightarrow \quad \not{k}\gamma_\mu \not{k} = \not{k}\gamma_\mu \gamma_\beta k^\beta = -\not{k}\gamma_\beta \gamma_\mu k^\beta + 2 g_{\mu\beta} k^\beta \not{k} = \underline{\underline{-k^2 \gamma_\mu + 2 k_\mu \not{k}}}$$

$$\gamma_\alpha \gamma_\mu \gamma^\alpha = -\gamma_\mu \gamma_\alpha \gamma^\alpha + g_{\mu\alpha}\gamma^\alpha = -4\gamma_\mu + 2\gamma_\nu = \underline{\underline{-2\gamma_\mu}}$$

$$\gamma_\alpha \not{k} \gamma^\alpha = -\not{k}\gamma_\alpha \gamma^\alpha + 2 k_\alpha \gamma^\alpha = -4\not{k} + 2 k_\alpha \gamma^\alpha = \underline{\underline{-2\not{k}}}$$

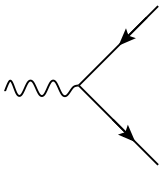
From the calculation of (B), it follows that  must transform as a vector under the Lorentz group. So we can write it as



$$\rightarrow \quad F_1\left(q^2, \bar{c}_1, \bar{c}_2, \{c_i\}\right) \cdot \gamma_\mu + F_2\left(q^2, \bar{c}_1, \bar{c}_2, \{c_i\}\right)$$

The functions F_1 and F_2 are called form factors. F_1 is proportional to $\log c$ and finite terms, where c is the same parameter we used before when we cut out a quarter disc around the origin with radius c. F_2 gives only finite terms.

$\Rightarrow$ All short-distance singularities in QED sit in the γ_μ form factor because $\log c$ is the only term in the expression that actually "sees" the singularity.

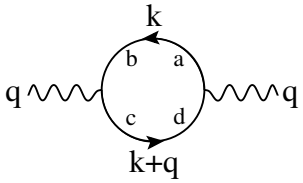


$$\overset{!}{\sim} \quad \gamma_\mu \qquad \Rightarrow \quad \bar{\psi}\gamma_\mu\psi A_\mu \overset{!}{\sim} \mathcal{H}_I$$

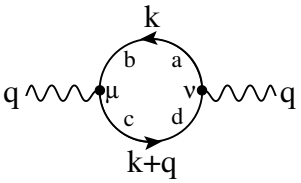
Since  is proportional to γ_μ, the matrix in between the two fermionic spinors has to be proportional to γ_μ as well. The interaction Hamiltonian of QED follows.

44.2 Photon propagator correction

Now that we know the vertex, we can compute a one-loop correction to the photon propagator, called **self-energy**.



We will give a short draft today and focus on the detailed computation next time. For the closed fermion loop, we know that we have to compute the trace of the respective matrices, since we get an expression like $M_{ab}M_{bc}M_{cd}M_{da}$, where the M are matrices and their indices indicate the trace. From the above results, we suspect that the one-loop correction should be proportional to $q_\mu q_\nu$ and / or $q^2 g_{\mu\nu}$, since there are no other quantities available that have an impact.



$$\rightarrow \quad \int \mathrm{d}^4k \ \mathrm{Tr}\left(\gamma_\mu \frac{1}{\not{k}} \gamma_\nu \frac{1}{\not{k}+\not{q}}\right)$$

$$= \underbrace{\int \mathrm{Tr}\left(\frac{\gamma_\mu \not{k} \gamma_\nu (\not{k}+\not{q})}{k^2(k+q)^2}\right)}_{=:(+)}$$

$$= q^2 g_{\mu\nu} F_1(\dots) + q_\mu q_\nu F_2(\dots)$$

F_1 and F_2 are dimensionless variables. This expression gives the general structure of the propagator, but for reasons of gauge invariance, we claim that there shall be only one F:

$$(+) \ \overset{!}{\sim} \ \left(q^2 g_{\mu\nu} - q_\mu q_\nu\right) F$$

We multiply (+) with q_μ, write $\not{q} = (\not{k}+\not{q}) - \not{k}$, and use cyclicity of the trace:

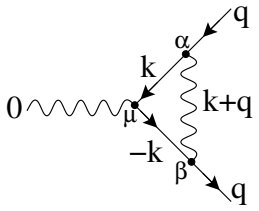
$$\mathrm{Tr}\left(\not{q}\not{k}\gamma_\mu(\not{k}+\not{q})\right)$$

Through cyclicity, the propagator goes away, then the tensor vanished, and therefore the integral also vanishes.

The Structure of the QED Vertex and Propagators

16.01.2012

This lecture will give some general remarks to the structure of QED, its vertices and propagators. As an example, we will examine the one-loop vertex $\gamma \to e^+e^-$. Our considerations will cover the case of zero momentum transfer.



We suspect that  $\sim \gamma_\mu$, which is only dependent on the Lorentz index of the vertex and not on external momenta, therefore  should just be proportional to an intergral over a single loop.

$$\sim \int \mathrm{d}^4k\, \gamma_\alpha \frac{1}{\not{k} - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} - m + i\epsilon} \gamma_\beta \underbrace{D_{\alpha\beta}(k+q)}_{\substack{\text{photon} \\ \text{propagator}}} \tag{14}$$

45 Photon propagator

The propagator for a spin one boson, which is a solution to the Maxwell equations, must be the inverse of the Fourier transform, $\sim \frac{1}{(k+q)^2 + i\epsilon}$, and its numerator should have two Lorentz indices: A_μ, A_ν are the two fields, and vacuum expectation values of bosonic fields are symmetric in μ and ν. Therefore the general form of the numerator is

$$D_{\alpha\beta}(k+q) \sim g_{\alpha\beta} + \xi (k+q)_\alpha (k+q)_\beta$$

This is the general form that the photon propagator must have. To simplify things, we will choose $\xi = 0 = m$. Then (**??**) becomes

$$\int \mathrm{d}^4k\, \gamma_\alpha \frac{1}{\not{k} + i\epsilon} \gamma_\mu \frac{1}{\not{k} + i\epsilon} \gamma_\beta \frac{g_{\alpha\beta}}{(k+q)^2 + i\epsilon} =: \Lambda_{1\mu}(q)$$

The index 1 of $\Lambda_{1\mu}(q)$ covers the one-loop correction, whereas μ is the Lorentz index stating that Λ transforms as a vector under the Lorentz group. We have omitted the fermion masses for simplicity.

45.1 Form factor decomposition

In order to compute $\Lambda_{1\mu}$, we make the structural ansatz

$$\Lambda_{1\mu}(q) = F_1(q^2)\gamma_\mu + F_2(q^2)\frac{\not{q} q_\mu}{q^2}$$

This ansatz is called form factor decomposition. Now we have to determine F_1 and F_2. To do that, we multiply Λ with γ_μ or $\frac{\not{q}q^\mu}{q^2}$ from the right.

$$\begin{aligned}
\Lambda_{1\mu}(q)\gamma^\mu &= F_1(q^2)\underbrace{\gamma_\mu\gamma^\mu}_{=4\mathbb{I}} + F_2(q^2)\underbrace{\frac{\not{q}q_\mu\gamma^\mu}{q^2}}_{=\mathbb{I}} \\
&= \Big(4F_1(q^2) + F_2(q^2)\Big)\mathbb{I} \\
\Lambda_{1\mu}(q)\frac{\not{q}q^\mu}{q^2} &= F_1\mathbb{I} + F_2\mathbb{I} \\
\Rightarrow \quad F_1(q^2) &= \frac{1}{3}\Big(\Lambda_{1\mu}\gamma^\mu - \Lambda_{1\mu}\frac{\not{q}q^\mu}{q^2}\Big) \\
F_2(q^2) &= \frac{4}{3}\Lambda_{1\mu}\frac{\not{q}q^\mu}{q^2} - \frac{1}{3}\Lambda_{1\mu}\gamma^\mu
\end{aligned}$$

So we have two integrals to compute:

$$\begin{aligned}
\Lambda_{1\mu}\gamma^\mu \quad &\to \quad \int \mathrm{d}^4k\ \gamma_\alpha\frac{1}{\not{k}+i\epsilon}\gamma_\mu\frac{1}{\not{k}+i\epsilon}\gamma_\beta\gamma^\mu\frac{g_{\alpha\beta}}{(k+q)^2+i\epsilon} \\
&= \int \mathrm{d}^4k\frac{\gamma_\alpha\not{k}\gamma_\mu\not{k}\gamma^\alpha\gamma^\mu}{(k^2+i\epsilon)^2\Big((k+q)^2+i\epsilon\Big)} \\
&\text{with} \quad \gamma_\mu\not{k} = -\not{k}\gamma_\mu + 2k_\mu \\
&\text{and} \quad \gamma_\alpha\not{q}\gamma^\alpha = -2\not{q} \\
\Lambda_{1\mu}\frac{\not{q}q_\mu}{q^2} \quad &\to \quad \int \mathrm{d}^4k\ \gamma_\alpha\frac{1}{\not{k}+i\epsilon}\gamma_\mu\frac{1}{\not{k}+i\epsilon}\gamma_\beta\frac{\not{q}q_\mu}{q^2}\frac{g_{\alpha\beta}}{(k+q)^2+i\epsilon}
\end{aligned}$$

These integrals are doable, see ensuing discussions. Eventually, this reconfirms  $\sim \gamma_\mu$, so we assume this further on.

45.2 One-loop correction

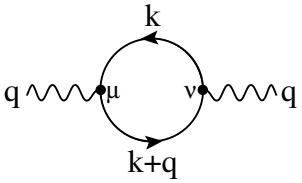
Since we don't really know much about the photon propagator yet, but there is also no internal photon line in the one-loop correction, luckily we know enough to compute fermion loops. Because of the anticommutation of fermion fields, we get a relative minus sign from the closed fermion loop. The interaction Hamiltonian is

$$\mathcal{H}_I = \bar{\psi}(x)\not{A}(x)\psi(x)\bar{\psi}(y)\not{A}(y)\psi(y) \quad ,$$

since we consider the case of two vertices at spacetime points x and y, with the associated fields. The only possible contraction that leaves two external photon fields is the following.

$$\mathcal{H}_I \to \bar{\psi}(x)\not{A}(x)\psi(x)\bar{\psi}(y)\not{A}(y)\psi(y)$$

This corresponds to the expected graph and the assocated integral:



$$\begin{aligned} \propto \quad & -\int \mathrm{d}^4k\, \gamma_\mu \frac{1}{\not k}\gamma_\nu \frac{1}{\not k + \not q} \\ & = -\operatorname{Tr}\int \mathrm{d}^4k \frac{\gamma_\mu \not k \gamma_\nu (\not k + \not q)}{k^2(k+q)^2} \\ & =: \Pi_{1\mu\nu}(q) \\ & = A(q^2)q^2 g_{\mu\nu} + B(q^2) q_\mu q_\nu \end{aligned}$$

The index 1 states that this is the first correction to the bosonic propagator. If we demand transversality, that is, $A + B \stackrel{!}{=} 0$, we get

$$\begin{aligned} \Pi_{1\mu\nu}(q) &= A(q^2)\Big(q^2 g_{\mu\nu} - q_\mu q_\nu\Big) \\ &= -\operatorname{Tr}\int \mathrm{d}^4k \frac{\gamma_\mu \not k \gamma_\nu (\not k + \not q}{k^2(k+q)^2} \end{aligned}$$

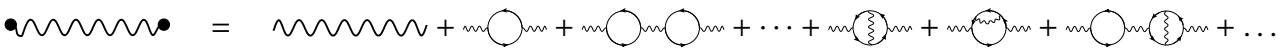
The term after the $A(q^2)$ illustrates transversality of the photon. Because of the closed loop, we have to compute the trace. If we multiply $\Pi_{1\mu\nu}(q)$ from the right with q_μ, we get

$$\begin{aligned} \Pi_{1\mu\nu}(q)q_\mu &\to -\operatorname{Tr}\Big(\int \mathrm{d}^4k \frac{\not q \not k \gamma_\nu (\not k + \not q)}{k^2(k+q)^2}\Big) \\ &= -\operatorname{Tr}\Big(\int \mathrm{d}^4k \Big(\frac{\not k \gamma_\nu}{k^2} - \frac{\gamma_\nu(\not k + \not q)}{k^2(k+q)^2}\Big)\Big) \end{aligned}$$

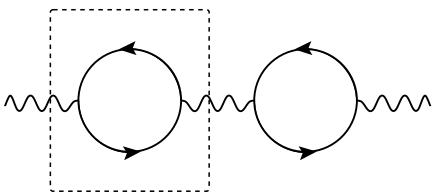
In the second line, we used $\not q = (\not k + \not q) - \not k$ and the cyclicity of the trace. This entire expression approaches zero if we maintain a translation invariant loop measure.
Before, we chose $\xi = 0$, but in general,  depends on ξ because there is an internal photon line. Now, for  , the result is always independent of ξ!

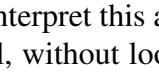
45.3 Full photon propagator

To consider not just one-loop corrections, but any possible corrections to the photon propagator, we would have to draw an infinite number of graphs:



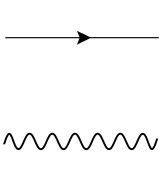
This can be interpreted as a geometric series in the self-energy $\Pi^1_{\mu\nu}$ we just computed. But not many graphs are really different from other graphs. For example, in



we can interpret this as twice the same type of one-loop correction,  $\sim \Big(g_{\mu\nu}q^2 - q_\mu q_\nu\Big)$. At tree level, without loop integrals, everything can easily be calculated, but even with loops, we must only consider those graphs which are one-particle irreducible.

46 The fermion propagator

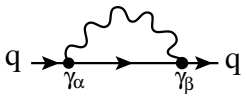
We know the bare fermion propagator, and we lately derived the photon propagator.



$$\frac{1}{\not{p} - m + i\epsilon}$$

$$\frac{g_{\alpha\beta} + \xi\frac{k_\alpha k_\beta}{k^2}}{k^2 + i\epsilon}$$

Then the one-loop correction to the fermion propagator looks like



$$\to \int \mathrm{d}^4k\, \gamma_\alpha \frac{1}{\not{p} + \not{k} - m + i\epsilon}\gamma_\beta D_{\alpha\beta} =: \Sigma_1(q)$$

From one of the exercises, we remember that

$$\frac{\partial}{\partial q_\mu} S_F(q) = -S_F(q)\gamma_\mu S_F(q)$$

and we know

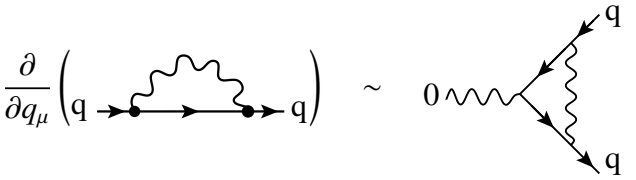
$$\frac{\partial}{\partial q_\mu}\frac{1}{\not{q}} = -\frac{1}{\not{q}}\gamma_\mu\frac{1}{\not{q}}$$

and $$\frac{\partial}{\partial q_\mu}\not{q} = \gamma_\mu$$

Which gives us a **Ward Identity**:

$$\frac{\partial}{\partial q_\mu}\Sigma_1(q) \sim \Lambda_{1\mu}(q)$$

$\Rightarrow$



We only need to consider graphs which are one-particle irreducible (1PI)! 1PI means that the removal of any one single line will not result in the graph falling apart into two disjoint diagrams. 1PI graphs give any contributing graphs (up to a given loop order) because all other graphs can be constructed from 1PI graphs.

47 Techniques and Applications

It can be said that no matter how we twist it, eventually we will encounter an integral like

$$\int \mathrm{d}^D k \frac{1}{\left(k^2\right)^\alpha\left((k+q)^2\right)^\beta} = F(q^2; D, \alpha, \beta) \qquad D, \alpha, \beta \in \mathbb{C}$$

At first it seems insane that D, α and β should be complex! This has no physical interpretation, we cannot identify $\frac{1}{k^2}$ with a propagator anymore if its power is not an integer. Also should the spacetime dimension be four, or at least any integer. As it turns out, however, it may make sense to handle D, α and β as complex numbers for regularization purposes.

47.1 Regularization Techniques

47.1.1 Cut-off Regularization

If we have an integral that diverges at the limits being infinity, we put it into a four dimensional box by integrating not up to infinity, but to some cut-off parameter Λ, and so we make the result a function of Λ

$$\int_{-\infty}^{+\infty} \mathrm{d}^4k\ F(k,\{q\},m) \ \rightarrow \int_{-\Lambda}^{+\Lambda} \mathrm{d}^4k\ F(k,\{q\},m) = F_{\text{cut-off}}(k,\{q\},m,\Lambda)$$

where $\{q\}$ is the set of external momenta. This approach is quite ancient, and it has the disadvantage of not being translation invariant, which stands in contrast to gauge theories for which translation invariance is crucial.

47.1.2 No regulator

A second, also ancient method is to subtract on the level of the integrand. This works well for QED and Yang-Mills theories, like $(g-2)$ in QED.
It is a lot more complicated than cut-off.

47.1.3 Dimensional Regularization

Dimensional regularization is basically the method of choice nowadays. It works with a gauge invariant regulator outside chiral group theory. We will see about dimensional regularization in the next lecture.

47.1.4 Analytic regularization

For analytic regularization, the quantum equations of motion produce fields that self-regulate by anomalous dimensions.

47.2 Wick Rotations

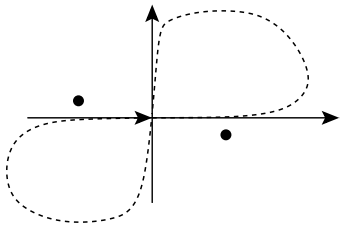
Wick rotations are used when the Minkowsky square need to be transformed into a euklidean square. This proves useful when calculating the well-known integral

$$\begin{aligned}\int \mathrm{d}^4k \frac{1}{k^2(k+q)^2} \quad &\rightarrow \quad \int \mathrm{d}^4k \iint_0^\infty \mathrm{d}A\mathrm{d}B\ e^{-k^2A-(k+q)^2B} \\ &= \int \mathrm{d}^4\bar{k} \iint_0^\infty \mathrm{d}A\mathrm{d}B\ e^{-\bar{k}^2 f(A+B)} e^{\cdots}\end{aligned}$$

where we complete the square, use translation invariance and so on, as we have done plenty of times. Now we have to remember that we always considered only the leading order in k, but the integral looks actually more like

$$\int_{-\infty}^{+\infty} \mathrm{d}k_0\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\ \frac{1}{(k^2+i\epsilon)\left((k+q)^2+i\epsilon\right)}$$

and the two terms $+i\epsilon$ from the propagators correspond to the shifting of the poles in the complex plane off the real axis. When we integrate, we close the contour either above or below the real axis. But we could also close it in a different way, namely half of it above and half of it below the real axis, by integration along the imaginary axis as well.



This change of contour will result in a shift

$$\begin{aligned} k_0 &\rightarrow ik_0 \\ k_0^2 &\rightarrow -k_0^2 \end{aligned}$$

and therefore

$$k_0^2 - k_1^2 - k_2^2 - k_3^2 \quad \rightarrow \quad -k_0^2 - k_1^2 - k_2^2 - k_3^2$$

and so the Miskowsky square becomes a euklidean square.
Of course, this method only works like this if all the poles lie outside the new "inside" of the contour. If we have more than just two propagators, the poles might lie somewhere else and the Wick rotation gets more complicated.
The method of Wick rotations is explained in great detail in "QTF" by Itzykson & Zuber.

47.3 Remark to the number of scalar products

Consider an argument of a discussion with the structure

$$\int \mathrm{d}^4k_1 \ldots \mathrm{d}^4k_n \; F\Big(\{k\},\{q\},\{m\}\Big)$$

for N external momenta $\{q\}$. Of course the result of the integration will be a scalar, so it can only be proportional to scalar products of the external momenta. For the number of invariant scalar products, we find:

N	scalar products	number of invariant scalar products
2	$q_1 \cdot q_1$	1
3	$q_1^2, q_2^2, \{q_3^2$ or $q_1 \cdot q_2\}$	3
4	...	6
N	...	$4N-10$

If

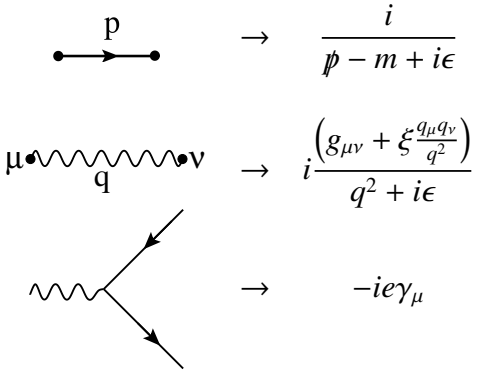
$$\left(\sum_{k=1}^{N} u_k q_k\right)^2 \leq 0 \qquad \forall \text{ real } u_k,$$

then there exists a timelike vector $\hat{n}$ such that $\hat{n} \cdot q_k = 0 \;\; \forall k$.

Epilogue and Prologue for QED

18.01.2012

Today we are going to focus on the detailed computation of a Feynman diagram. From our previous studies, we can assume that we know the Feynman rules for internal lines and the vertex. In quantum electrodynamics, these are internal photon and fermion lines, and the photon-fermion-antifermion vertex.

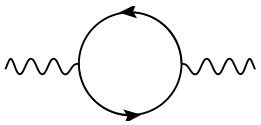


$$\rightarrow \quad \frac{i}{\not{p} - m + i\epsilon}$$

$$\rightarrow \quad i\frac{\left(g_{\mu\nu} + \xi\frac{q_\mu q_\nu}{q^2}\right)}{q^2 + i\epsilon}$$

$$\rightarrow \quad -ie\gamma_\mu$$

We need to know the Feynman rules for external lines, for example incoming or outgoing photons. For example, when we talked about the one-loop corrections to the photon propagator, we mainly focused on the fermion loop, but we never really gave much thought to the transversal photon.



$$\rightarrow \quad g_{\mu\nu}q^2 - q_\mu q_\nu$$

$$\rightarrow$$ =?

48 Integrals

Here is a compilation of some important integrals we are going to use throughout the calculations.

48.1 The Γ function

The hopefully well-known Γ function has the properties

$$z\,\Gamma(z) = \Gamma(z+1)$$
$$\Gamma(1) = 1$$
$$\Gamma(0) \text{ has a pole}$$

This is an integral representation of the Γ function:

$$\Gamma(\rho) = \int_0^\infty e^{-A} A^\rho \frac{\mathrm{d}A}{A}$$

The measure $\frac{\mathrm{d}A}{A}$ is invariant under rescaling. We set $A = ua$:

$$\Gamma(\rho) = \int_0^\infty e^{-ua}(ua)^\rho \frac{\mathrm{d}a}{a} \tag{15}$$

$$= u^\rho \int_0^\infty e^{-ua} a^\rho \frac{\mathrm{d}a}{a} \tag{16}$$

$$\Rightarrow \qquad u^{-\rho} = \frac{1}{\Gamma(\rho)} \int_0^\infty e^{-ua} a^\rho \frac{\mathrm{d}a}{a} \tag{17}$$

48.2 A second identity

We know that in some D dimensional spacetime,

$$\int \mathrm{d}^D k \, e^{-Xk^2} = \frac{1}{(2\pi X)^{\frac{D}{2}}} \qquad \forall D \in \mathbb{Z}$$

for all integers D, this is well-known. We take this as a **definition for a measure with complex D**:

$$\frac{1}{(2\pi X)^{\frac{D}{2}}} \coloneqq \text{“} \int \mathrm{d}^D k \text{ ”} \; e^{-Xk^2} \tag{18}$$

We integrate in a complex-dimensional spacetime. This makes no real-life sense, but at some point, we want to do dimensional regularization, and this is exactly what we would have to do there. In fact, most integrals we will want to compute look like

$$I \coloneqq \int \mathrm{d}^D k \, \frac{1}{\left(k^2\right)^\alpha \left((k+q)^2\right)^\beta} \qquad \forall \; \alpha, \beta, D \in \mathbb{C}$$

This integral is crucial for quantum field theory. There is no way around this giant.

49 Computation

By using (**??**), we set

$$I = \int \mathrm{d}^D k \, \frac{1}{\left(k^2\right)^\alpha \left((k+q)^2\right)^\beta} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int \mathrm{d}^D k \iint_0^\infty e^{-k^2 A - (k+q)^2 B} A^\alpha B^\beta \frac{\mathrm{d}A \mathrm{d}B}{AB}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int \mathrm{d}^D k \iint_0^\infty \mathrm{d}A \mathrm{d}B \, A^{\alpha-1} B^{\beta-1} e^{-(A+B)\left\{k^2 + 2k\cdot q \frac{B}{A+B} + q^2 \frac{B}{A+B}\right\}}$$

Now we complete the square and go from k to $\bar{k}$ by using translation invariance.

$$I = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int \mathrm{d}^D \bar{k} \iint_0^\infty \mathrm{d}A \mathrm{d}B \, A^{\alpha-1} B^{\beta-1} e^{-(A+B)\left\{\bar{k}^2 - q^2 \frac{B^2}{(A+B)^2} + q^2 \frac{B}{A+B}\right\}}$$

$$= \frac{(2\pi)^{-\frac{D}{2}}}{\Gamma(\alpha)\Gamma(\beta)} \iint_0^\infty \mathrm{d}A \mathrm{d}B \, A^{\alpha-1} B^{\beta-1} \frac{1}{(A+B)^{\frac{D}{2}}} e^{-q^2 \frac{AB}{A+B}}$$

In the last line, we used (**??**) in order to compute the $\bar{k}$-integral. Next, we set $B =: Ab$.

$$I = \frac{(2\pi)^{-\frac{D}{2}}}{\Gamma(\alpha)\Gamma(\beta)} \iint_0^\infty \mathrm{d}A\mathrm{d}b\, A^{\alpha+\beta-1} b^{\beta-1} \frac{1}{A^{\frac{D}{2}}(1+b)^{\frac{D}{2}}} e^{-q^2 \frac{Ab}{1+b}}$$

Next, we make the transformation $q^2 A \frac{b}{1+b} =: a \quad \rightarrow \quad A = a\frac{1+b}{bq^2}$

$$\begin{aligned}
\Rightarrow \quad I &= (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{(2\pi)^{-\frac{D}{2}}}{\Gamma(\alpha)\Gamma(\beta)} \iint \mathrm{d}a\mathrm{d}b\, \frac{1+b}{b} \left(a\frac{1+b}{b}\right)^{\alpha+\beta-\frac{D}{2}-1} \frac{b^{\beta-1}}{(1+b)^{\frac{D}{2}}} e^{-a} \\
&= (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{(2\pi)^{-\frac{D}{2}}}{\Gamma(\alpha)\Gamma(\beta)} \int \mathrm{d}a\, a^{\alpha+\beta-\frac{D}{2}-1} e^{-a} \int \mathrm{d}b \left(\frac{1+b}{b}\right)^{\alpha+\beta-\frac{D}{2}} \frac{b^{\beta-1}}{(1+b)^{\frac{D}{2}}}
\end{aligned}$$

The integral over a is exactly the integral representation of the Γ function!

$$\Rightarrow \quad I = (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{(2\pi)^{-\frac{D}{2}} \Gamma\left(\alpha+\beta-\frac{D}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int \mathrm{d}b \left(\frac{1+b}{b}\right)^{\alpha+\beta-\frac{D}{2}} \frac{b^{\beta-1}}{(1+b)^{\frac{D}{2}}}}_{=:I'}$$

In order to calculate I', we change variables again and set $b' := \frac{b}{1+b}$, $\mathrm{d}b = \frac{1}{(1-b')^2}\mathrm{d}b'$:

$$\begin{aligned}
I' &= \int_0^1 \mathrm{d}b'\, (1-b')^{\frac{D}{2}-\beta-1} (b')^{\frac{D}{2}-\alpha-1} \\
&\equiv B\left(\frac{D}{2}-\alpha, \frac{D}{2}-\beta\right) \\
&\equiv \frac{\Gamma\left(\frac{D}{2}-\alpha\right)\Gamma\left(\frac{D}{2}-\beta\right)}{\Gamma\left(D-\alpha-\beta\right)}
\end{aligned}$$

This is the integral representation of the **B (Beta) function**!
We have computed the entire integral, called the **Master Integral**. The result is

$$\int \mathrm{d}^D k\, \frac{1}{\left(k^2\right)^\alpha \left((k+q)^2\right)^\beta} = \frac{(q^2)^{\frac{D}{2}-\alpha-\beta}(2\pi)^{-\frac{D}{2}} \Gamma\left(\alpha+\beta-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\alpha\right)\Gamma\left(\frac{D}{2}-\beta\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma\left(D-\alpha-\beta\right)} \tag{19}$$

As expected and required, it is translation invariant in k and symmetric in α and β. We still need to discuss an analytic continuation of the master formula.

49.1 Consequences

If we assume α or β to be equal to zero, the denominator will diverge and therefore the entire integral will vanish. This will always happen if one propagator does not appear (its power is zero), so we know without further computation that

$$\int \mathrm{d}^D k\, (k^2)^{-\alpha} \equiv 0 \quad \forall \alpha$$

As a consequence, all tadpole contributions vanish.
Let us consider the most natural Feynman integral, where $D = 4$ and $\alpha = \beta = 1$. Here, $\Gamma\left(\alpha+\beta-\frac{D}{2}\right)$ has a pole! This might seem like a problem, but it is only natural and makes sense because the integral it came from, namely $\int \mathrm{d}^4k \frac{1}{k^4}$ approaches infinity for $|k| \to \infty$ or $|k| \gg 1$. Therefore, $\Gamma\left(\alpha+\beta-\frac{D}{2}\right)$ should have a pole!!
Of course, the integral is much more complicated for more propagators. On the other hand,

$$\int \frac{\mathrm{d}^4k}{(k^2)^4}$$

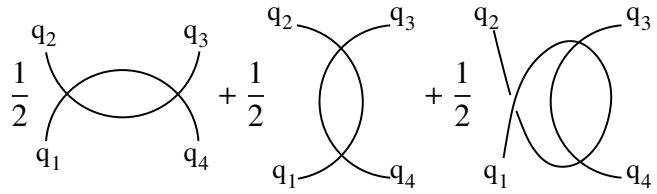
has a pole for small $|k|$. A regulator z is introduced, $D = 4 - 2z$, where z is a small complex quantity with positive real part. Then $\Gamma\left(\frac{D}{2}-\alpha\right)$ becomes $\Gamma(-z)$.
The question remains, what to do with mixed poles?

50 ϕ^4 theory in four dimensions

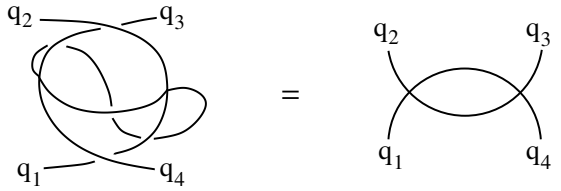
In ϕ^4 theory, the Lagrangian is

$$\mathcal{L} = \frac{1}{4}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$$

The vertex $+ \sim ig$ corresponds to 2×2 scattering and its one-loop corrections divided by their symmetry factors are

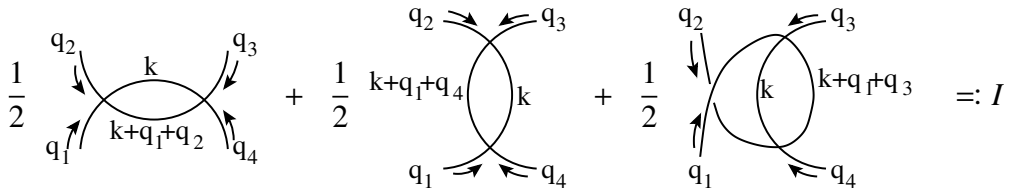


Note that the incoming momenta q_1 and q_2 and the outgoing momenta q_3 and q_4 are fixed. These three diagrams are the only one-loop contributions to 2×2 scattering, any other one-loop graph with four externa edges would be a distortion of one of the above, for example:



50.1 Computation of the one-loop integral

For the two internal momenta k_1 and k_2, we already know from momentum conservation that we can eliminate one of the two and express it via the q_i and k. This gives us



When integrating out, we get three times almost the same integral. This time, we leave the spacetime dimension D as a free parameter. When talking about the **one-loop correction to the vertex**, the order is g, or g^2, depending on whether or not one takes into account that the vertex already has order g. One could say that the one-loop integral has order g^2 and the correction to the vertex is of order g.

$$\begin{aligned} I \quad \Rightarrow \quad & \frac{1}{2}g^2\left\{\int \mathrm{d}^D k \frac{1}{k^2(k+q_1+q_2)^2} + \int \mathrm{d}^D k \frac{1}{k^2(k+q_1+q_4)^2} + \int \mathrm{d}^D k \frac{1}{k^2(k+q_1+q_3)^2}\right\} \\ & = \frac{1}{2}g^2(2\pi)^{\frac{D}{2}} \frac{\Gamma(2-\frac{D}{2})\Gamma^2(\frac{D}{2}-1)}{\Gamma^2(1)\Gamma(D-2)}\left(\left((q_1+q_2)^2\right)^{\frac{D}{2}-2} + \left((q_1+q_4)^2\right)^{\frac{D}{2}-2} + \left((q_1+q_3)^2\right)^{\frac{D}{2}-2}\right) \end{aligned}$$

Now we set $D := 4 - 2z$ by introducing the regulator z, $|z| \ll 1$, $z \neq 0$, $\Re(z) > 0$.

$$\Rightarrow I = \frac{1}{2}g^2 \frac{(2\pi)^z}{(2\pi)^2} \frac{\Gamma(z)\Gamma^2(1-z)}{\Gamma(2-2z)}\left(\underbrace{\left((q_1+q_2)^2\right)^{-z}}_{\text{s-channel}} + \underbrace{\left((q_1+q_4)^2\right)^{-z}}_{\text{t-channel}} + \underbrace{\left((q_1+q_3)^2\right)^{-z}}_{\text{u-channel}}\right)$$

For $z \to 0$, the $\frac{\Gamma^2(1-z)}{\Gamma(2-2z)}$ part stays regular, but the $\Gamma(z)$ in the numerator will be singular. This is a problem, yet not unexpected. To get a better impression of the brackets with the channels, we expand them in z using

$$\begin{aligned} U^{-z} &= e^{-z\ln U} \\ \Rightarrow \quad \left((q_1+q_2)^2\right)^{-z} &= 1 - z\ln(q_1+q_2)^2 + O(z^2) \end{aligned}$$

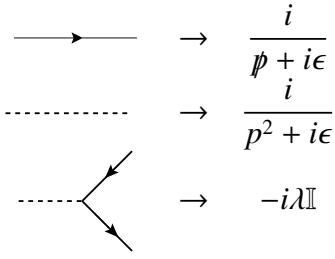
Basically what this is telling us is that our understanding of quantum field theory almost makes sense but not entirely (yet). By using the expansion in I, we get

$$I \to \frac{1}{2}g^2 \frac{1}{(2\pi)^2}\left(1 + z\ln 2\pi + O(z^2)\right)\frac{1}{z}(1+\ldots)\left\{1+1+1 - z\cdot\left(\ln(q_1+q_2)^2 + \ln(q_1+q_4)^2 + \ln(q_1+q_3)^2\right)\right\}$$

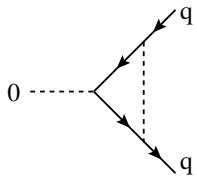
Which gives us a pole part of $\frac{3g^2}{2(2\pi)^2}\frac{1}{z}$ and finite parts. Hopefully, we can find a good argument why this pole should vanish. This might indeed be true, since the pole is independent of the q_i.

51 Massless Yukawa Theory

As a reminder, here are the Feynman rules for massless Yukawa theory:

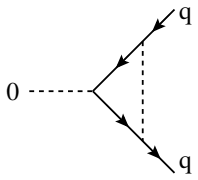


To get the one-loop correction to the vertex, we have



$$\sim \quad (-i\lambda)^3 \int \mathrm{d}^D k \, \frac{1}{\not{k}}\frac{1}{\not{k}}\frac{1}{\left((k+q)^2\right)^{1+\rho}}$$

where we omit the $\mathbb{I}$ for the vertices. For dimensional regularization, we regulate in D, and assuming that the bosonic propagator has a scaling behavious with ρ, we also regulate in ρ.



$$\rightarrow \quad +\lambda^3 \frac{(2\pi)^{\frac{D}{2}}}{\Gamma(1)\Gamma(1+\rho)} \frac{\Gamma(2+\rho-2+z)\Gamma(1-z)\Gamma(1+\rho-z)}{\Gamma(2-2z-\rho)} \left(q^2\right)^{-z-\rho}$$

As we can read off from $\Gamma(2+\rho-2+z) = \Gamma(\rho+z)$, there is a pole part for small ρ and z. Thereby,

$$\frac{1}{z+\rho}\left(q^2\right)^{-(z+\rho)} = \frac{1}{z+\rho}\left(1-(z+\rho)\ln q^2\right)$$

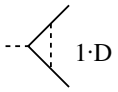
so we can regulate by z, or ρ, or even both of them. Anyway, we get a pole with a coefficient $\ln q^2$. How do we get rid of this pole? These computations are a lot more complicated for graphs like, say,  . We cannot compute this graph, but we can at least compute that it has a pole, and we may even be able to physically interpret this pole.

52 Divergence

In the one-loop correction to the vertex, we have found a logarithmic divergence. If we want to find out for other graphs what their order of divergence is, without having to compute all of them, there is an easy way to do so.

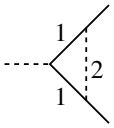
- We weight each internal line with its power in the Fourier transform:

- We count the number of loops and multiply it with the dimension of spacetime, D:

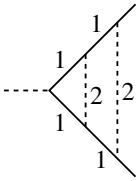


- We take the number of loops multiplied with D and subtract the sum over all weights.

For example, look at



The sum over all weights is 4. The spacetime dimension is $D = 4 - 2z$, we get $-2z \rightarrow 0$ if $z \rightarrow 0$. This graph is logarithmically divergent.
Not let's look at



This graph is also logarithmically divergent, since the sum over all weights is 8 and there are two closed loops, therefore we get $2 \cdot D - 8 = -4z \rightarrow 0$ if $z \rightarrow 0$.
In this way, we can determine whether or not a graph is logarithmically divergent.

53 Outlook: Renormalization

The techniques of removing poles is called renormalization. We need to find out what tricks are used, how they are justified and why the results make sense. Also, the Lagrangian formulation of renormalization is worth a look.

Introduction to Renormalization

23.01.2012

54 Review

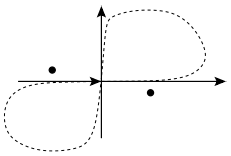
In the first part of today's lecture, we will give an overview about our previous calculations.

54.1 Vertex corrective integral

We have considered the integral

$$I_E(D,\alpha,\beta) = \int \mathrm{d}^D k \frac{1}{\left(k^2\right)^\alpha \left((k\ +\ q)^2\right)^\beta}$$

for which we introduced the Wick rotation which shifts k_0 to ik_0 by changing the path of integration from along the real axis and closing either above or below, to



As a consequence, $k_0^2 - k_1^2 - k_2^2 - k_3^2$
becomes $-k_0^2 - k_1^2 - k_2^2 - k_3^2$
and thus, with the $k^2 = k_0^2 + k_1^2 + k_2^2 + k_3^2$
minus sign pulled out, $q^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$

$$\Rightarrow \quad \ldots \quad \rightarrow \quad \ldots \frac{(-1)}{\text{positive definite propagators}}$$

Since k^2 and $(k+q)^2$ are now positive quadratics, the intergral is a well-defined expression $F(D,\alpha,\beta,q^2)$. It should be possible to continue the variable q^2 analytically to negative real values to undo the Wick rotation. This is merely a matter of complex analysis.
Remark: The same result would have been obtained if the integration would have been carried out without the Wick rotation and the undoing.
The integral is known:

$$I_E(D,\alpha,\beta) = \int \mathrm{d}^D k \frac{1}{\left(k^2\right)^\alpha \left((k\ +\ q)^2\right)^\beta} = \pi^{-\frac{D}{2}} (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{\Gamma\left(\frac{D}{2}-\alpha-\beta\right)\Gamma\left(\frac{D}{2}-\alpha\right)\Gamma\left(\frac{D}{2}-\beta\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma\left(D-\alpha-\beta\right)}$$

54.2 Vertex corrections in ϕ^4 theory

We looked at one-loop vertex correction in ϕ^4 theory and the symmetry factors of the graphs and we calculated them using dimensional regulation.

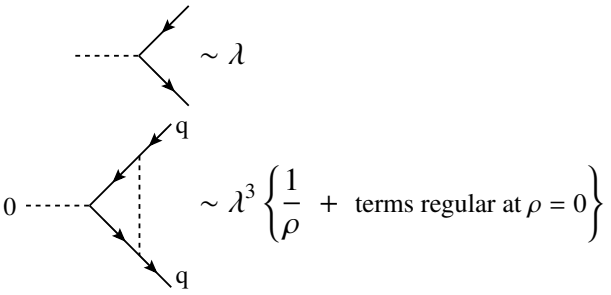
$$\frac{1}{2}\left\{ \quad + \quad + \quad \right\} \rightarrow \left(\frac{1}{2}\frac{3}{z} \ +\ \text{terms regular at } z=0\right) g^2$$

There is a pole in $z = 0$, that's true, but this pole is independent of any momenta associated with the graphs, and therefore our hope is that it might not be a physical observable, since it does not rely on any, and could therefore be removed. Our goal is to reveal how to handle ϕ^4 in four dimensions of spacetime, but for $D = 4 - 2z$ (dimensional regularization) we get a pole in z. The regular terms are indeed dependent on the momenta: They are given by

$$\frac{1}{2}\left\{\sum_{i=2}^{4} \ln\left(q_1 + q_i\right)^2\right\}$$

54.3 Massless Yukawa theory

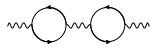
We also looked at the one-loop vertex correction in massless Yukawa theory, graphically given by



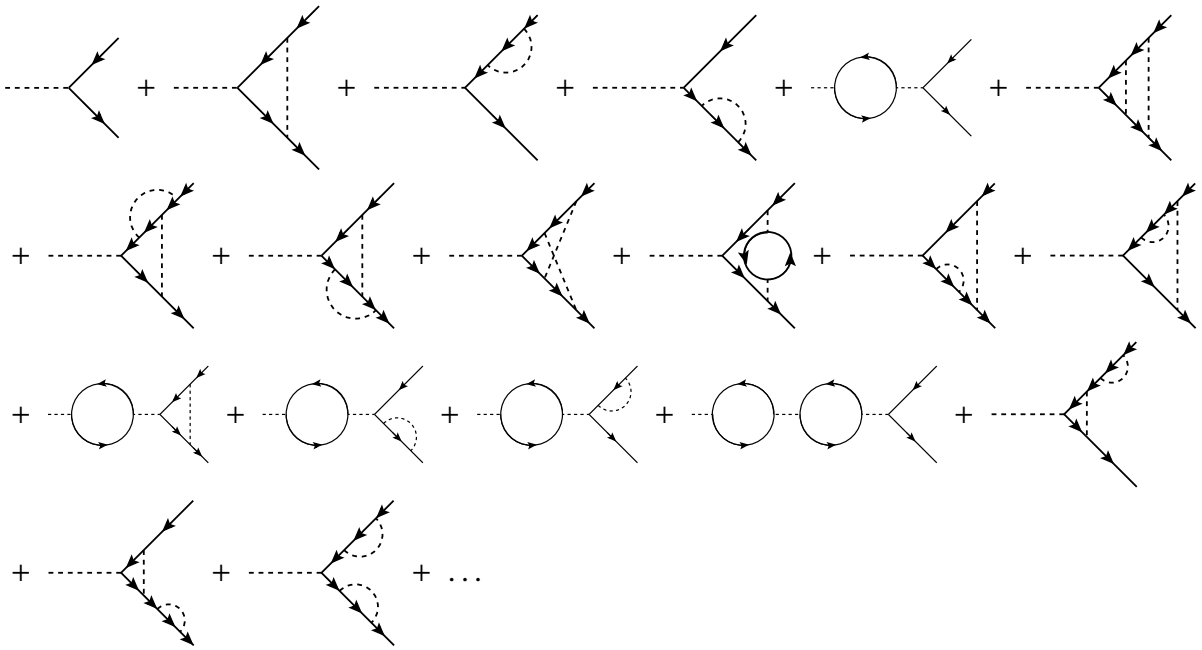
It seems we always come accross those "annoying" poles. Do they have to be there? Is there an approach to quantum field theory without the poles? The answer is yes, they have to be there, they are fundamental and not a result of a formulation. They are physically not oservable, but experiments only show how effects change when the energy scale changes, so it is only natural that the energy scale must be taken into account. The poles are necessary for identifying short-distance behaviour, and they dictate coefficients in the renormalization group equations and the dynamics of variables.
Either way, we have to find out how the limits $D \to 4$ or $z \to 0$ can be taken without the physically measurable expressions to diverge. For that, we need to understand the analytical structure of the integrals.

55 Four ways of thinking in renormalization

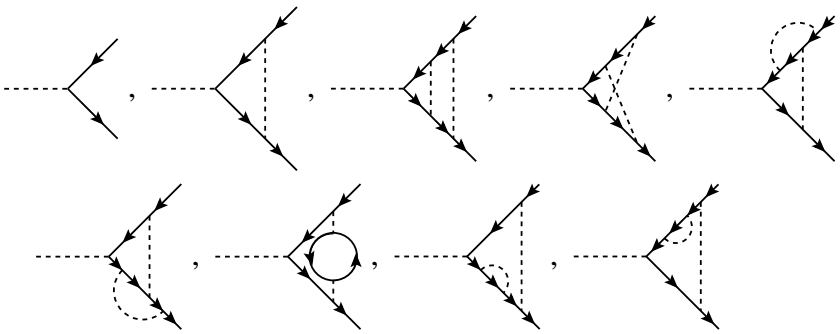
We will get started on renormalization today and study some of the approaches in detail. We start with a definition:

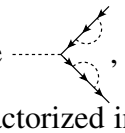
Definition We call a Feynman graph ***n*-particle irreducible (*n*-PI)**, if it is connected when we remove n internal edges. A **connected graph** is 0-PI.
Example:  is 0-PI, but not 1-PI.

The vertex in Yukawa theory, with corrections up to two loops, is given by

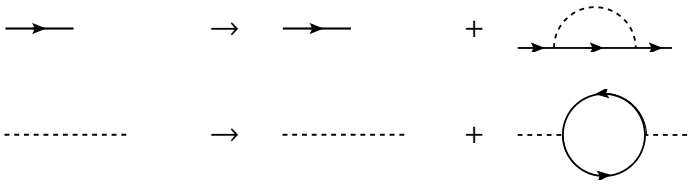


But almost all of these graphs are superfluous, the only 1-PI graphs are



Any of the other graphs, for example  , can be derived from the knowledge of .

In fact, all non-1-PI graphs can be factorized into 1-PI pieces, thus it suffices to compute only 1-PI graphs. Once we know the one-loop propagator corrections,



we basically know all 0-PI two-loop graphs. It needs no mention that it is helpful to work with just the 1-PI graphs.

56 To find out if a 1-PI graph is divergent

Since we are eventually facing an infinite number of graphs, it makes sense to sort them in some way to classify which set of graphs is ”problematic” and which is not.
Further on, we work with 1-PI graphs only. Not only do we neglect 0-PI graphs, but also

2-PI and any higher order of connectedness. It turns out that only 1-PI graphs are actually advantageous.
Remark: Mathematicians call 1-PI graphs two-connected because there are two lines between final and initial states / two lines need to be removed in order to disconnect final and initial state. In order to classify 1-PI graphs, we need another definition.

Definition In $D \in \mathbb{N}_+$ dimensions, let

$$\omega_D(\Gamma) := |\Gamma|\, D - \sum_{e \in \Gamma^{[1]}_{\text{int}}} w(e) - \sum_{v \in \Gamma^{[0]}} w(v) \quad ,$$

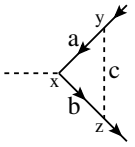
where

- $|\Gamma|$ is the first Betti number of the graph Γ (the number of independent cycles, or the loop number)
- $\Gamma^{[1]}_{\text{int}}$ is the set of internal edges in Γ
- $\Gamma^{[0]}$ is the set of vertices in Γ
- $w(e)$, $w(v)$ are weights, where

$$w(e) = \begin{cases} 1 & \text{if } e \text{ is a fermionic edge} \\ 2 & \text{if } e \text{ is a bonosic edge} \end{cases}$$

and $w(v)$ is the mass dimension of the coupling at the vertex v. In our examples so far, $w(v) \equiv 0 \ \forall v$. (This is about to change in the summer term)

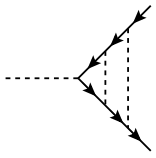
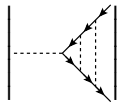
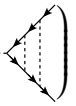
56.1 Yukawa theory



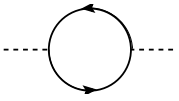
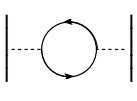
is called the **vertex-function at one loop**, and it is 1-PI.

$$|\Gamma| = 1 \quad \Rightarrow \quad \omega_4(\Gamma) = 1 \cdot 4 - \underbrace{w(a)}_{=1} - \underbrace{w(b)}_{=1} - \underbrace{w(c)}_{=2} - \underbrace{w(x)}_{=0} - \underbrace{w(y)}_{=0} - \underbrace{w(z)}_{=0}$$
$$= 4 - 4 = 0$$

Since $\omega_4\left($  $\right) = 0$,  is logarithmically divergent. Now if we consider

 $\Rightarrow$  $= 2 \quad \Rightarrow \quad \omega_4\Big($  $\Big) = 2 \cdot 4 - 4 \cdot 1 - 2 \cdot 2 = 0$

 $\Rightarrow$  $= 1 \quad \Rightarrow \quad \omega_4\,($  $) = 1 \cdot 4 - 1 \cdot 1 - 1 \cdot 2 = +1$

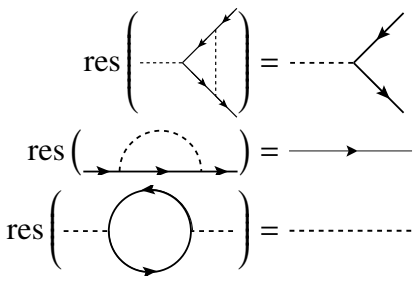
 $\Rightarrow$  $= 1 \quad \Rightarrow \quad \omega_4\Big($  $\Big) = 1 \cdot 4 - 2 \cdot 1 - 0 \cdot 2 = +2$

56.2 Grading of graphs

Let Γ^{Yuk} be the set of all 1-PI graphs in Yukawa theory. In Γ^{Yuk}, the graphs can be graded by their Betti numbers (numbers of loops):

$$\Gamma^{\text{Yuk}} = \Gamma^0 \cup \Gamma^1 \cup \Gamma^2 \cup \dots$$

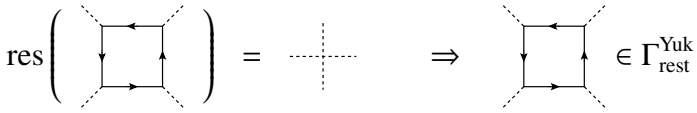
But there is another way to decompose Γ^{Yuk}: Let $\text{res}(\Gamma)$ be the graph obtained by shrinking all internal edges $e \in \Gamma^{[1]}_{\text{int}}$ to zero length.
Example:



Then we classify the graphs in Γ^{Yuk} by their residues:

$$\Rightarrow \qquad \Gamma^{\text{Yuk}} = \Gamma^{\text{Yuk}}_{\diagup\!\diagdown} \cup \Gamma^{\text{Yuk}}_{\text{---}} \cup \Gamma^{\text{Yuk}}_{\cdots} \cup \Gamma^{\text{Yuk}}_{\text{rest}} \quad ,$$

where $\Gamma^{\text{Yuk}}_{\diagup\!\diagdown}$ is the set of all 1-PI graphs Γ with $\text{res}(\Gamma) = \diagup\!\diagdown$, and so on. All graphs whose residues are neither $\diagup\!\diagdown$, nor —, nor $\cdots$, are in $\Gamma^{\text{Yuk}}_{\text{rest}}$.
Example:



Theorem

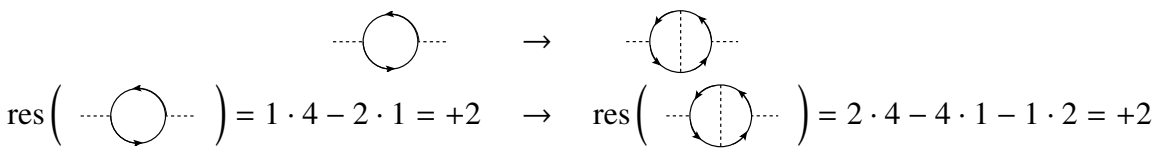
$$\omega_4(\Gamma) = 0 \qquad \forall \Gamma \in \Gamma^{\text{Yuk}}_{\diagup\!\diagdown}$$

$$\omega_4(\Gamma) = +1 \quad \forall \Gamma \in \Gamma^{\text{Yuk}}_{\text{---}}$$

$$\omega_4(\Gamma) = +2 \quad \forall \Gamma \in \Gamma^{\text{Yuk}}_{\cdots}$$

$$\omega_4(\Gamma) =< 0 \quad \forall \Gamma \in \Gamma^{\text{Yuk}}_{\text{rest}}$$

Proof Obvious for one-loops graphs (see above). Euler-characteristics tells you the rest, so actually the proof is over at this point. To visualize this, we look at the changes made to get from a one-loop graph to a two-loop graph. In fact, we get one additional loop, so the expression adds four, and we also get one additional bosonic edge and two additional fermionic edges.



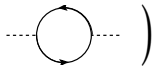
Remark: ω_4  $= 0$, not < 0, but there is an argument by Caveat why this process does not appear in Yukawa theory. (Confer Caveat: Four boson interactions, Stokes theorem)

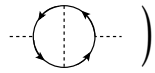
56.3 Divergence in graphs

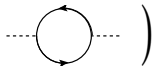
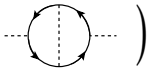
For $\omega_4 = 0$, we get a logarithmic divergence, which has the nice property that the divergence drops out in the sum over all graphs, therefore we can treat logarithmically divergent graphs as "basically convergent"! This is a major advantage of power counting.
Still, for $\omega_4 = +1$ we get linear divergence, for $\omega_4 = +2$ even quadratic divergence, which is not so easy to get rid of. Analysis of ω_D helps find out how bad the problem is by at least identifying all divergent graphs.

Definition A theory for which a $D > 0$ exists such that $\omega_D(\Gamma) = \omega_D(\tilde{\Gamma})\ \forall\ \Gamma, \tilde{\Gamma}$ with $\text{res}\,(\Gamma) = \text{res}\,(\tilde{\Gamma})$, or in other words, where Γ and $\tilde{\Gamma}$ have the same external leg structure, is called **renormalizable in D dimensions**. ($\Rightarrow$ Yukawa theory is renormalizable in four dimensions of spacetime)
We have considered ω_4 mainly because it is most interesting, since we live in a world with four dimensions. However, it is also worth looking at other values of D. For example, what happens to Yukawa theory in two dimensions?

56.3.1 Yukawa theory in two dimensions

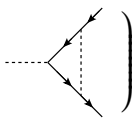
ω_2() $= 2 - 2 = 0 \quad \Rightarrow$ logarithmically divergent

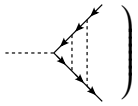
ω_2() $= 2 \cdot 2 - 4 - 2 = -2 \quad \Rightarrow$ convergent at two loops

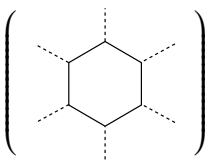
but : res() = res() =

In $D < 4$ dimensions, $\omega_D(\Gamma) < 0$ for $|\Gamma|$ large enough / with $|\Gamma| < n_0$. A theory for which $\omega_D(\Gamma) < 0\ \forall\ \Gamma$ is called **superrenormalizable in D dimensions**.

56.3.2 Yukawa theory in six dimensions

ω_6() $= 6 - 4 = +2 \quad \Rightarrow$ quadratically divergent

ω_6() $= 2 \cdot 12 - 8 = +4 \quad \Rightarrow$ quartically divergent

ω_6() $= 2 \cdot 6 - 6 = 0 \quad \Rightarrow$ logarithmically divergent

( is certainly convergent in $D = 4$!)

A theory for which $\omega_D(\Gamma) \geq 0$ for all $\text{res}\,(\Gamma)$ for $|\Gamma|$ large enough is called **non-renormalizable**.
(Our) QFTs are either renormalizable, or superrenormalizable, or non-renormalizable.
For some reason which we do not know, nature organizes in renormalizable QFTs, as far as we know (exception: quantum gravity might or might not be renormalizable).

57 Theories and their renormalizability

We finish today with a list of theories and their degrees of interaction, and the dimensions of spacetime in which they are renormalizable.

Theory	*Interaction*	*Renormalizability*	*Decomposition*
ϕ^3 scalar theory	cubic	$D = 6$	$\Gamma^{\phi^3_6} = \Gamma^{\phi^3_6}_{\text{}} \cup \Gamma^{\phi^3_6}_{\text{}} \cup \Gamma^{\phi^3_6}_{\text{rest}}$
ϕ^4 scalar theory	quartic	$D = 4$	$\Gamma^{\phi^4_4} = \Gamma^{\phi^4_4}_{\text{}} \cup \Gamma^{\phi^4_4}_{\text{}} \cup \Gamma^{\phi^4_4}_{\text{rest}}$
QED	cubic	$D = 4$	$\Gamma^{QED_4} = \Gamma^{QED_4}_{\text{  }} \cup \Gamma^{QED_4}_{\text{  }} \cup \Gamma^{QED_4}_{\text{  }} \cup \Gamma^{QED_4}_{\text{rest}}$
Yukawa theory	cubic	$D = 4$	$\Gamma^{\text{Yuk}_4} = \Gamma^{\text{Yuk}_4}_{\text{}} \cup \Gamma^{\text{Yuk}_4}_{\text{}} \cup \Gamma^{\text{Yuk}_4}_{\text{}} \cup \Gamma^{\text{Yuk}_4}_{\text{rest}}$

Renormalization (1)

30.01.2012

Review So far, we got for our dimensionally regularized integral (omitting arbitrary factors of 2π because sometimes, $\mathrm{d}^D k := \frac{\mathrm{d}^D k}{(2\pi)^D}$):

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{\left(k^2\right)^\alpha \left((k+q)^2\right)^\beta} = (4\pi)^{-\frac{D}{2}} (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{\Gamma\left(\alpha+\beta-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\alpha\right)\Gamma\left(\frac{D}{2}-\beta\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D-\alpha-\beta)} \tag{20}$$

This expression is divergent for $D = 4, \alpha = \beta = 1$.

58 Renormalization

To renormalize, we rescale all quantities in the Lagrangian density and write the scalings as series in the coupling constant. In that way, we hope to get rid of all divergencies without changing any physics.

58.1 Scalings in ϕ^4 theory

We will do one-loop renormalization of the vertex function in massless ϕ^4 theory. The interaction Lagrangian is given by

$$\mathcal{L}_I = \frac{g}{4!}\phi^4$$

where ϕ is a real scalar field, and the Lagrangian is a function of the field, the derivative of the field, the coupling g, and in general the mass m (not here).

$$\mathcal{L} = \mathcal{L}\left(\phi, \partial_\mu \phi, g, m\right)$$

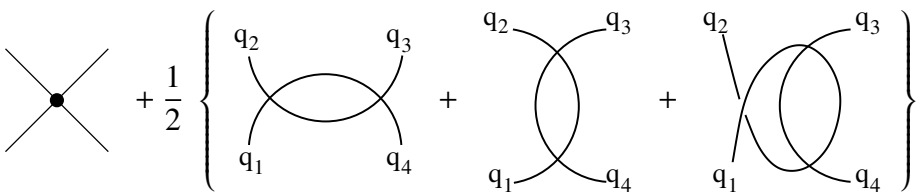
We rewrite the Lagrangian with the following scalings:

$$\mathcal{L} \to \mathcal{L}\left(Z_\phi^{\frac{1}{2}}\phi, Z_\phi^{\frac{1}{2}}\partial_\mu\phi, Z_g g, Z_m m\right)$$
$$Z_i = 1 + g c_{1i} + g^2 c_{2i} + \dots$$

We use these scalings Z_i to make sense of initially ill-defined expressions. The square root of Z_ϕ is there for historical reasons. Temporarily, yet not generally, we set $Z_\phi = 1 = Z_m$, so only Z_g is left.

58.1.1 1-PI vertex function

Expanding the vertex in graphs up to one loop, we get



where the first term is of order g, the last three terms are of order g^2. We compute the 1-PI vertex function of ϕ^4 theory to order g^2 using dimensional regularization (**??**).
We call the vertex function Green function $\mathcal{G}^{\times}\Big(g, \{p_i\}\Big)$

$$\begin{aligned}\mathcal{G}^{\times}\Big(g, \{p_i\}\Big) &= g + \frac{g^2}{2}\left\{\int \mathrm{d}^D k \left(\frac{1}{k^2(k+p_1+p_2)^2} + \frac{1}{k^2(k+p_1+p_4)^2} + \frac{1}{k^2(k+p_1+p_3)^2}\right)\right\} \\ &= g + \frac{g^2}{2}\left\{\Big(p_1+p_2)^2\Big)^{-z} + \Big(p_1+p_4)^2\Big)^{-z} + \Big(p_1+p_3)^2\Big)^{-z}\right\} \frac{\Gamma(z)\Gamma^2(1-z)}{\Gamma^2(1)\Gamma(2-2z)} + O(g^3)\end{aligned}$$

In the second line, we have set $D = 4 - 2z$. Because of the term $\Gamma(z)$ in the numerator, this expression diverges for $z = 0$, of course.

58.1.2 Rewriting the Γ function

Using a property of the Γ function,

$$z\Gamma(z) = \Gamma(1+z) \qquad ,$$

and that $\Gamma(1+z)$ is regular at $z = 0$, we can derive that $\Gamma(z)$ has a pole of first order at $z = 0$. We then use the Taylor expansion of $\Gamma(1+z)$:

$$\Gamma(1+z) = e^{\gamma_E z} \exp\left(-\sum_{k=2}^{\infty} \frac{\zeta(k)}{k}(-z)^k\right)$$

where $\zeta(k)$ gives the Zeta value of z. Plugging this expression into $\mathcal{G}^{\times}$, we get

$$\begin{aligned}\mathcal{G}^{\times}\Big(g, \{p_i\}\Big) = g + \frac{g^2}{2}\Big\{ &\Big(1 - z\ln(p_1+p_2)^2 + O(z^2)\Big) + \Big(1 - z\ln(p_1+p_4)^2 + O(z^2)\Big) + \\ &+ \Big(1 - z\ln(p_1+p_3)^2 + O(z^2)\Big)\Big\}\end{aligned}$$

Now we introduce the scaling of the coupling, namely $g \to Z_g g$ in $\mathcal{L}$:

$$\begin{aligned}\mathcal{G}^{\times}\Big(g, \{p_i\}\Big) = Z_g g + \frac{Z_g^2 g^2}{2}\Big\{ &\Big(1 - z\ln(p_1+p_2)^2 + O(z^2)\Big) + \Big(1 - z\ln(p_1+p_4)^2 + O(z^2)\Big) + \\ &+ \Big(1 - z\ln(p_1+p_3)^2 + O(z^2)\Big)\Big\}\end{aligned}$$

58.1.3 Determining Z_g

Our goal now is to establish Z_g in such a way that the limit $z \to 0$ can be taken for the Green function as a function of g and Z_g.
Remark: The Lagrangian is well-defined for any z other that $z = 0$, even though this is necessary for the natural dimension $N = 4$. But nature does not realize Lagrangians, but Green functions and Wightman functions, so the Lagrangian might as well diverge.
Z_g is a series in g, $Z_g = 1 + gc_{1g} + g^2 c_{2g} + \ldots$, but we only wanted to go up to order g^2. Consequently, the Green function gives

$$\mathcal{G}^{\times}\Big(g, \{p_i\}\Big) = g + g^2 c_{1g} + \frac{g^2}{2}\left\{\frac{3}{2}\frac{1}{z} + \left(\sum_{j=2}^{4} \ln(p_1+p_j)^2\right)\cdot\Big(1 + O(z)\Big) - \frac{3}{2}\gamma_E + O(z)\right\}$$

where c_{1g} is a Laurent-Taylor series in z.

58.1.4 Making $\mathcal{G}$ finite

To make $\mathcal{G}^\times$ finite, we certainly need

$$c_{1g} = -\frac{3}{2}\frac{1}{z} - \frac{1}{2} \cdot (\text{finite term})$$

The finite term is a term that we are free to choose! It is not fixed by mathematics and it is independent of the kinematics of the scattering process. This choice determines (up to one loop) my **renormalization scheme**.

$$\mathcal{G}^\times \quad \to \quad \mathcal{G}^\times_{\text{ren}}\Big(\text{scheme}, g, \{p_i\}\Big)$$

58.2 Renormalization schemes

The mathematician's job is done at the point where they find out that they can choose a renormalization scheme. However, physicists try to make some sense out of different schemes. Some choices might be "better" than others - what are the "good" choices?

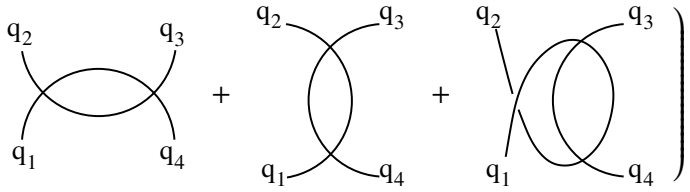
58.2.1 Kinematic renormalization scheme

At Euklidean $p_1^2 = p_2^2 = p_3^2 = p_4^2 \equiv \mu^2$ (uniform angles), the renormalized Green function is

$$\mathcal{G}^\times_{\text{ren}} \stackrel{!}{=} g$$

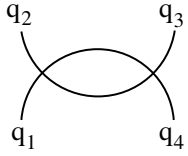
The one-loop terms in the ϕ^4 vertex are

$$\frac{1}{2}\left(\quad + \quad + \quad\right)$$



and they give three integrals of the same kind. We will only work with one of them, for example

$$\frac{1}{2} \quad \to \quad \frac{1}{2}\int \mathrm{d}^4k \frac{1}{k^2(k+p_1+p_2)^2}$$



With dimensional regularization being omitted here, this expression is ill-defined! But we know that it is "only" logarithmically divergent, so if we subtract at the integrand at some other value of k in the kinematic renormalization scheme, we get the expression

$$\frac{1}{2}\int \mathrm{d}^4k \left(\frac{1}{k^2(k+p_1+p_2)^2} - \frac{1}{k^2(k+\bar{p}_1+\bar{p}_2)^2}\right)$$

which exists!
If we now sum over all three graphs, we get

$$\frac{1}{2}\sum_{j=2}^{4}\left(\frac{(p_1+p_j)^2}{(p_1+\bar{p}_j)^2}\right)$$

The two limits (subtraction and integration) interchange, but they do so only in the kinematic renormalization scheme! (It is generally not possible to subtract at the integrand in other schemes.)
This choice of scheme determines c_{1g} completely.

$$\mathcal{G}^{\times}_{\text{ren}} = g + \frac{g^2}{2}\underbrace{\left(\sum_{j=2}^{4}\left(\frac{(p_1+p_j)^2}{(p_1+\bar{p}_j)^2}\right)\right)}_{=0} \quad , \text{ where } \bar{p}_i^2 = \mu^2$$

The sum vanishes because $\ln(1) = 0$.
This scheme can be set up at any order of the coupling constant.

58.2.2 Minimal subtraction scheme / MS scheme

At first, we set

$$c_{1g} = -\frac{3}{2}\frac{1}{z}$$

This takes care of the pole, but since a logarithm cannot be taken of a non-dimensionless quantity, we need something additional for the logarithm.
In an earlier exercise, we determined the mass dimensions of fields. Let us do this again because it gives a good argument where to get the scale from:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{4!}\phi^4$$
$$\mathcal{S} = \int_{\mathbb{M}} \mathcal{L}\,\mathrm{d}^4x$$

The action $\mathcal{S}$ is a dimensionless quantity we get by integrating the Lagrangian density over the entire Minkowsky space $\mathbb{M}$. This helps us determine the mass dimensions of fields, since the mass dimensions of d^4x is 4, $[\mathrm{d}x]_m = 1$. Now we know that $[\partial]_m = 1$, so $[\phi]_m$ has to be 1 as well. Hence, the mass dimension of the coupling g is 0.
In dimensional regularization, though, the mass dimensions are regulated as well.

$$\mathcal{S} = \int_{\mathbb{M}} \mathcal{L}\,\mathrm{d}^4x \quad , \quad D = 4 - 2z$$
$$\Rightarrow \quad [\phi]_m = 1 - z \qquad \Rightarrow \quad [\phi^4]_m = 4 - 4z$$
$$g \to g_0(z) = \frac{g}{(\mu^2)^{-z}}$$

These dimensions do note make any physical sense, they are just there for regularization purposes. In the minimal subtraction scheme, the logarithm becomes

$$\ln\left(\frac{(p_1+p_i)^2}{\mu^2}\right)$$

And the Green function will be

$$\mathcal{G}^{\times}_{\text{ren}} = g + g^2\left(\sum_{j=2}^{4}\ln\left(\frac{(p_1+p_i)^2}{\mu^2}\right) - \frac{3}{2}\gamma_E\right)$$

We basically threw the pole term out. The advantage of this method is that it is really quick, but the disadvantage is the strange parts that occur in the Green function.
Remark "$-\frac{1}{k^2(k+\bar{p}_1+\bar{p}_2)^2}$" is called a **counterterm** at the level of an integrand. "$\left(-\frac{1}{2}\frac{1}{z} + \text{finite}\right)$ is a counterterm as a function of the regulator.

58.3 The renormalized Lagrangian

In the minimal subtraction scheme, we saw that we threw away pole terms. Since pole terms contain useful information, maybe this is not exactly what we want to do from the start. Let us have a look at the Lagrangian:

$$\mathcal{L} = \mathcal{L}\Big(Z_\phi, \phi, Z_g, g, Z_m, m\Big)(z) \ .$$

We want to fix the Z_i in such a way that

$$\lim_{z\to 0} \mathcal{G}^\times \Big(Z_\phi, \phi, Z_g, g, Z_m, m, \{p_i\}\Big)$$

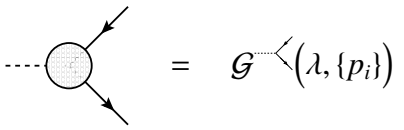
exists. We need that Z_i cancel all the poles that occur!
The limit exists and is called **renormalized Green function in the XYZ scheme**.

$$\lim_{z\to 0} \mathcal{G}^\times(\dots)(z) = \mathcal{G}^\times_{\text{ren, scheme}}(g, m, \{p_i\}, \mu)$$

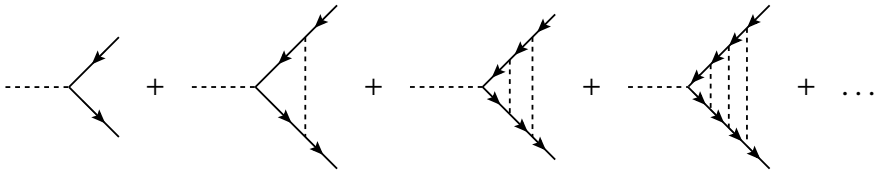
The renormalization parameter μ is dictated by the scheme.
Remark: It is not a trivial statement that the limit exists, neither is it a trivial statement that the renormalization factors can be worked out.

58.4 Quantum equations of motion

For the vertex correction in Yukawa theory, we can imagine a Green function giving a simplified version of quantum equations of motion:



As a simplification, we will only sum over the graphs with interactions like



(This simplification is called **Ladder Approximation**). As an ansatz, we take

$$\mathcal{G} \quad \to \quad \mathcal{G}_{\text{ren}}(\lambda, q) = \left(\frac{q^2}{\mu^2}\right)^{-\gamma(\lambda)}$$
$$= 1 - \gamma(\lambda)\ln\left(\frac{q^2}{\mu^2}\right) + \frac{\gamma^2}{2!}\left(\ln\left(\frac{q^2}{\mu^2}\right)\right)^2 + \dots$$

We claim that with this ansatz, $\mathcal{G}_{\text{ren}}$ gives a solution of the integral equation

$$\mathcal{G}_{\text{ren}}(\lambda, q) = 1 + \lambda^2 \left\{ \left(\int \mathrm{d}^4k \frac{1}{\not k} \mathcal{G}(\lambda, k) \frac{1}{\not k} \frac{1}{(k+q)^2} \right) - \left(\int \mathrm{d}^4k \frac{1}{\not k} \mathcal{G}(\lambda, k) \frac{1}{\not k} \frac{1}{(k+q)^2} \right)\Bigg|_{q^2=\mu^2} \right\}$$

and that $\mathcal{G} \cdot \lambda$ gives the transition amplitude. We will see about that in the next lecture.

Renormalization (2)

01.02.2012

Review Lately, we focused on vertex renormalization to one loop in ϕ^4 theory, in dimensional regularization, computed in the kinematic renormalization scheme and the minimal subtraction scheme. To do that, we rescaled the coupling g with some power series in g, to get

$$\mathcal{L}_I = \frac{g}{4!}\phi^4 \qquad , \quad g \to Z_g g \quad , \quad Z_g = 1 + c_{1g}g + c_{2g}g^2 + \dots$$

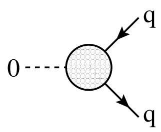
Followingly $\mathcal{L}$ will include a number of monomials with infinite coefficients, called counterterms.

59 Vertex Renormalization

We have already seen vertex renormalization in massless ϕ^4 theory. Today, we will focus on massless Yukawa theory and iterate a subclass of graphs.

59.1 The vertex function of Yukawa theory

By writing down



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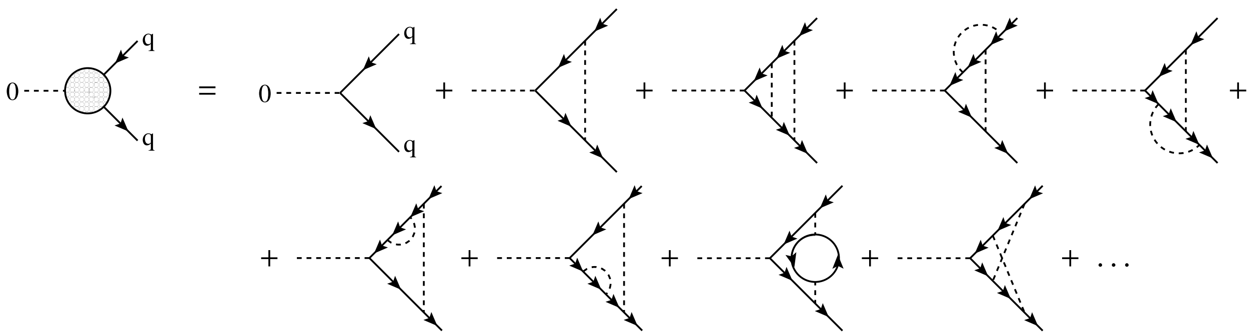
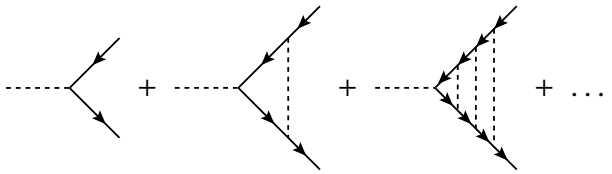
we mean the vertex function of Yukawa theory, ergo all possible interactions at the vertex. The vertex function should be a series, lead by the tree level graph.



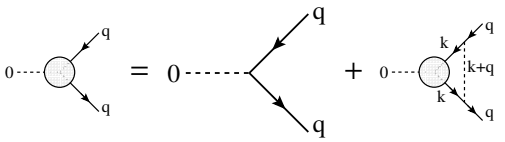
The tree level graph is of order λ, the one-loop graphs are of order λ^3, the two-loop graphs are of order λ^5, and all following graphs must be at least of order λ^7.
A problem about renormalization is that all internal propagators are to be renormalized as well because they carry divergences on their own. What seems to be a neverending story is supposedly very similar to what we have down up until here.

59.2 A subclass of graphs

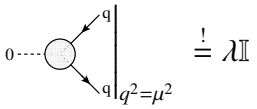
Let us try to classify graphs and consider all graphs with zero-momentum at the bosonic line, which have the form of



This simplification is called **Ladder Approximation**, for obvious reasons.
We claim that this simplified series of graphs is a solution to an integral equation.



We solve this integral equation with the boundary condition that at some fixed external momentum $q^2 = \mu^2$, the vertex function should be proportional just to λ.



$$\stackrel{!}{=} \lambda\mathbb{I}$$

We write for the integral equation of the vertex function in terms of the coupling λ and the Green function:

$$\lambda\mathcal{G}(\lambda, q^2, \mu^2) \;=\; \lambda\mathbb{I} \;+\; \lambda^3 \int \frac{1}{\not{k}}\mathcal{G}(\lambda, k^2, \mu^2)\frac{1}{\not{k}}\frac{1}{(k+q)^2}\,\mathrm{d}^4k$$

Since the second term should vanish at $q^2 = \mu^2$ (this is our boundary condition), we can subtract it. Or in other words, by subtracting it, we make sure that the whole integral epression vanishes at $q^2 = \mu^2$.

$$\lambda\mathcal{G}(\lambda, q^2, \mu^2) \;=\; \lambda\mathbb{I} \;+\; \lambda^3 \left\{ \int \frac{1}{\not{k}}\mathcal{G}(\lambda, k^2, \mu^2)\frac{1}{\not{k}}\frac{1}{(k+q)^2}\,\mathrm{d}^4k \;-\; \int \frac{1}{\not{k}}\mathcal{G}(\lambda, k^2, \mu^2)\frac{1}{\not{k}}\frac{1}{(k+q)^2}\,\mathrm{d}^4k\Big|_{q^2=\mu^2} \right\}$$

Dividing by λ gives the fixpoint equation

$$\mathcal{G}(\lambda, q^2, \mu^2) \;=\; \mathbb{I} \;+\; \lambda^2 \left\{ \int \frac{1}{\not{k}}\mathcal{G}(\lambda, k^2, \mu^2)\frac{1}{\not{k}}\frac{1}{(k+q)^2}\,\mathrm{d}^4k \;-\; \int \frac{1}{\not{k}}\mathcal{G}(\lambda, k^2, \mu^2)\frac{1}{\not{k}}\frac{1}{(k+q)^2}\,\mathrm{d}^4k\Big|_{q^2=\mu^2} \right\}$$

If we expand the expression in parentheses, we get the overall quadratic term of λ from the constant term in $\mathcal{G}$.

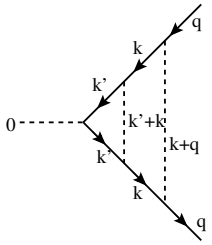
$$\mathcal{G}(\lambda, q^2, \mu^2) \;=\; \mathbb{I} \;+\; \lambda^2 \left\{ \int \frac{1}{\not{k}}\mathbb{I}\frac{1}{\not{k}}\frac{1}{(k+q)^2}\,\mathrm{d}^4k \;-\; \int \frac{1}{\not{k}}\mathbb{I}\frac{1}{\not{k}}\frac{1}{(k+q)^2}\,\mathrm{d}^4k\Big|_{q^2=\mu^2} \right\} \;+\; \mathcal{O}(\lambda^4)$$

In subtracting at some fixed value of q^2, $q^2 = \mu^2$, we got a well-defined integral because the infinite constants simply drop out. The Green function then goes like $\propto \ln\left(\frac{q^2}{\mu^2}\right)$.
Remark: This only works because the graph is logarithmically divergent and not linearly or

even worse.
In the integral, if we go up to order λ^4, we get:

$$\mathcal{G}(\lambda, q^2, \mu^2) = \mathbb{I} + \lambda^2 \left\{ \int \frac{1}{\not{k}} \mathbb{I} \frac{1}{\not{k}} \frac{1}{(k+q)^2} \, \mathrm{d}^4 k - \int \frac{1}{\not{k}} \mathbb{I} \frac{1}{\not{k}} \frac{1}{(k+q)^2} \, \mathrm{d}^4 k \Big|_{q^2=\mu^2} \right\} + \\ + \lambda^4 \left\{ \int \mathrm{d}^4 k \frac{1}{\not{k}} (\dots) \frac{1}{\not{k}} \frac{1}{(k+q)^2} - \int \mathrm{d}^4 k \frac{1}{\not{k}} (\dots) \frac{1}{\not{k}} \frac{1}{(k+q)^2} \Big|_{q^2=\mu^2} \right\}$$

where $(\dots)$ represents the integral associated with λ^4, but with external momentum k, and internal momentum k' integrated out. This corresponds to the graph



If we solve the integral equation for higher and higher orders of λ, it will reproduce all the Feynman rules needed for the next vertex each time.

59.3 Bootstrap equations

Let's try to write the integral equation in a more sensible way. We make an ansatz for the Green function:

$$\mathcal{G}(\lambda, q^2, \mu^2) = \left(\frac{q^2}{\mu^2} \right)^{-\gamma(\lambda)} \mathbb{I}$$

We need to derive $\gamma(\lambda)$ in order to have a working Green function. The integral equation (up to order λ^2) becomes

$$\left(\frac{q^2}{\mu^2} \right)^{-\gamma(\lambda)} \mathbb{I} = \mathbb{I} + \lambda^2 \left\{ \underbrace{\int \frac{1}{\not{k}} \left(\frac{q^2}{\mu^2} \right)^{-\gamma(\lambda)} \frac{1}{\not{k}} \frac{1}{(k+q)^2} \, \mathrm{d}^4 k}_{=:A} - \int \frac{1}{\not{k}} \left(\frac{q^2}{\mu^2} \right)^{-\gamma(\lambda)} \frac{1}{\not{k}} \frac{1}{(k+q)^2} \, \mathrm{d}^4 k \Big|_{q^2=\mu^2} \right\}$$

To solve this (part of the) integral equation, let us have a closer look at A.

$$\begin{aligned}
A &= \int \mathrm{d}^4k \, \frac{1}{\not{k}} \left(\frac{q^2}{\mu^2}\right)^{-\gamma(\lambda)} \frac{1}{\not{k}} \frac{1}{(k+q)^2} \\
&= \left(\frac{1}{\mu^2}\right)^{-\gamma(\lambda)} \mathbb{I} \int \mathrm{d}^4k \frac{1}{(k^2)^{q+\gamma(\lambda)}(k+q)^2} \\
&= \left(\frac{1}{\mu^2}\right)^{-\gamma(\lambda)} \mathbb{I}\pi^2 q^{-\gamma(\lambda)} \frac{\Gamma\Big(\gamma(\lambda)\Big)\Gamma\Big(1-\gamma(\lambda)\Big)\Gamma(1)}{\Gamma(1)\Gamma\Big(1+\gamma(\lambda)\Big)\Gamma\Big(2-\gamma(\lambda)\Big)} \\
&= \left(\frac{q^2}{\mu^2}\right)^{-\gamma(\lambda)} \mathbb{I} \frac{1}{\gamma(\lambda)} \frac{1}{1-\gamma(\lambda)} \\
\Leftrightarrow \quad & \left(\left(\frac{q^2}{\mu^2}\right)^{-\gamma(\lambda)} - 1\right)\mathbb{I} = \lambda^4 \left(\left(\frac{q^2}{\mu^2}\right)^{-\gamma(\lambda)} - 1\right) \mathbb{I} \frac{1}{\gamma(\lambda)} \frac{1}{1-\gamma(\lambda)} \\
\Rightarrow \quad & \mathbb{I} = \frac{\lambda^2}{\gamma(\lambda)\Big(1-\gamma(\lambda)\Big)} \\
\Rightarrow \quad & \gamma(\lambda) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda^2}
\end{aligned}$$

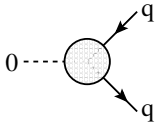
We got solving for $\gamma(\lambda)$ down to just one $\pm$ sign. Since λ is the coupling, $\lambda = 0$ would result in no interaction at all, if the coupling strength is zero, then there would be no vertices. Consequently, we would expect that the Green function would just become the identity in the case $\lambda = 0$. Therefore, the $\pm$ sign has to be a $-$.

$$\mathcal{G}(\lambda, q^2, \mu^2) = \left(\frac{q^2}{\mu^2}\right)^{-\frac{1}{2}+\sqrt{\frac{1}{4}+\lambda^2}} \mathbb{I}$$

This Green function resums all graphs from the ladder approximation with boundary conditions $\mathcal{G}(q^2 = \mu^2) = \mathbb{I}$.
Remark (1): I admit that it is quite possible that a factor π^2 is missing somewhere.
Remark (2): Is a set of graph self-regulated, or is self-renormalizable, then the vertex function will develop an anomalous dimension $\gamma(\lambda)$ which shows up in the integrand.

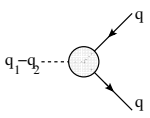
59.4 Propagator corrections in the vertex

If we make the ansatz from above,

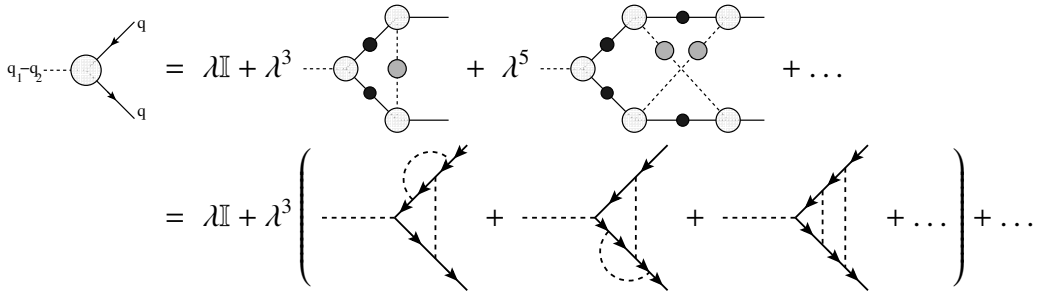


$$= \lambda \left(\frac{q^2}{\mu^2}\right)^{-\gamma(\lambda)} \mathbb{I} = \lambda \mathcal{G} \quad ,$$

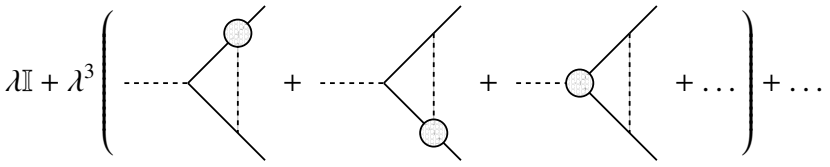
then we can write down the same graph, but with a non-vanishing incoming momentum on the bosonic line.



But there is no ansatz for it, if we want to take all propagators and their corrections into account. We would get something like



which basically corresponds to



where the vertex functions go up to one loop.
With this graphic explanation, any imaginable graph is produced. Problematically, there is no useful ansatz for it yet, even though there is proof that this ansatz exists.
But even though we don't have an explicit form of the Green function, maybe we can get some answers to the dominant kinematical behaviour. We may not know how to express a solution, but we can find some relations or dependencies about it.

59.5 Back to perturbation theory

Still today, there are no solutions known that are non-perturbative. So we go back to perturbation theory and see what else we can find out. We remember the master integral,

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(k^2)^\alpha \left((k+q)^2\right)^\beta} \;\to\; (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{\Gamma\left(\alpha+\beta-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\alpha\right)\Gamma\left(\frac{D}{2}-\beta\right)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(D-\alpha-\beta)} \quad ,$$

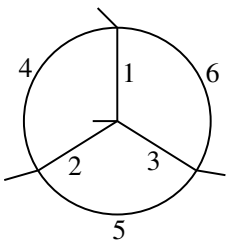
and we also remember that we got this expression from the integral representation of the Γ function,

$$\Gamma(z) = \int_0^\infty e^{-x} x^z \frac{\mathrm{d}x}{x}$$

60 Symanzik Polynomials

If we look at some arbitrary graph, for example

Γ :=



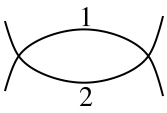
,

from ϕ^4 theory, where the six internal edges are labeled, we can assign a variable A_i to each edge i in order to distinguish the edges.

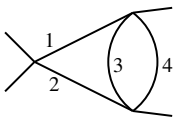
Definition A spanning tree of Γ is a simply-connected subset of Γ which contains all vertices of Γ.
Simply-connected means that there are no internal loops. For example, this Γ given above has sixteen spanning trees. We denote by Ψ_Γ the **first Symanzik polynomial** or dual Kirchhoff polynomial,

$$\Psi_\Gamma := \sum_{\substack{\text{spanning trees} \\ T \text{ of } \Gamma}} \prod_{e \notin T} A_e$$

Even though we have already defined the Kirchhoff polynomials, we talk about the Symanzik polynomials as well because sometimes they are more convenient for Feynman graphs.
Examples:

$\Gamma =$ 

$\Psi_\Gamma = A_1 + A_1$

$\Gamma =$ 

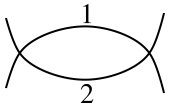
$\Psi_\Gamma = (A_1 + A_2)(A_3 + A_4) + A_3A_4$

This should immediatedly remind us of the calculation of the graph  , where we came across some factor of $\sim (A_1 + A_2)^{-D}$, which comes exactly from this.

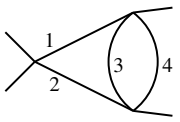
Definition A two-forest is a set of two disjoint trees $T_1, T_2 \subsetneq T$ such that $T_1 \cup T_2$ contains all vertices.
In other words, a two-forest consists of two proper subsets of the spanning tree in such a way that they is merely one edge removed.
We define the **second Symanzik polynomial** as

$$\varphi_\Gamma := \sum_{T_1 \cup T_2} Q(T_1) \cdot Q(T_2) \prod_{e \notin T_1 \cup T_2} A_e$$

where $Q(T_i)$ denotes the sum of all external momenta of the vertices of T_i.
Examples:

$\Gamma =$ 

$$\begin{aligned} \varphi_\Gamma &= (q_1 + q_2)(q_3 + q_4)A_1A_1 \\ &= -(q_1 + q_2)^2 A_1A_2 \\ &= -(q_3 + q_4)^2 A_1A_2 \end{aligned}$$

$\Gamma =$ 

$$\begin{aligned} \varphi_\Gamma &= (q_1 + q_2)(q_3 + q_4)A_1A_2(A_3 + A_4) + \\ &\quad + q_3(q_1 + q_2 + q_4)A_1A_2A_3 + q_4(q_1 + q_2 + q_3)A_2A_3A_4 \\ &= (q_1 + q_2)(q_3 + q_4)A_1A_2(A_3 + A_4) - q_3^2 A_1A_2A_3 - q_4^2 A_2A_3A_4 \end{aligned}$$

Here, we used overall momentum conservation, that the sum over all initial momenta equals the sum over all momenta in the final state. For massless propagators, the Feynman rules for a graph Γ are available through the Symanzik polynomials and given by

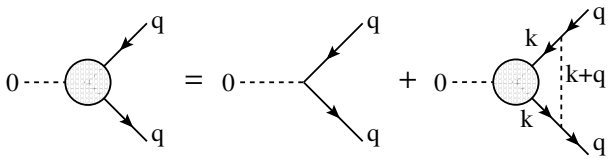
$$\Phi_\Gamma = \int \frac{\exp\left(-\frac{\varphi_\Gamma}{\Psi_\Gamma}\right)}{(\Psi_\Gamma)^{\frac{D}{2}}} \, \mathrm{d}A_1 \ldots \mathrm{d}A_E$$

where E denotes the number of internal edges in Γ.
Thus, all we need to do to compute a graph is to write down some polynomials which are defined combinatorically, and integrate over all edges.

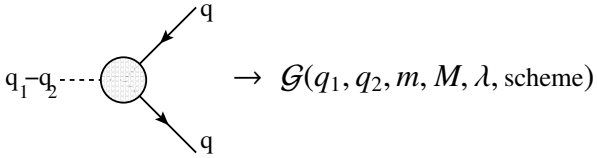
Renormalization (3)

06.02.2012

Review Last time, we focused on vertex renormalization in massless Yukawa theory. The ladder approximation gave us the identity



which led us to an integral equation for the massless vertex function in Yukawa theory at zero momentum transfer in the ladder approximation. To make a long story short, we made our lives easier. Typically, though, there are masses involved which we cannot neglect, and there will be a momentum transfer and so the incoming boson will carry a momentum other than zero.



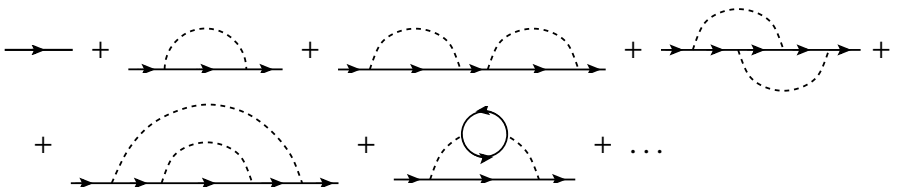
Usually, the ladder approximation will not be sufficient either because we cannot generally neglect diagrams like ⊣⟨ or ⊣⟨, for which we would need more integral kernels.

- The ladder approximation is an extreme simplification. This is why we could do it so easily.
- But: Once we've understood the ladder approximation and all computations related to it, we will be able to break down everything that is more complicated into the same structure.

In any case, we will always have to deal with a fix point equation for the Green function. In last week's case, there was just one fix point equation for the vertex function.

61 Propagators

Since we have dealt with the vertex function, it is time to move on to propagators. For the propagator function, we can (again) start with the tree level graph and sum it with graphs of loop order 1, 2, and so forth.

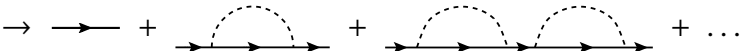


We know that applying the Feynman rules to any one of these diagrams will give rise to ill-defined integrals. At least we know that some graphs, for example ⌒⌒, are not one-particle irreducible. The question is: How do I get all connected diagrams contributing to a propagator, just from 1-PI graphs?

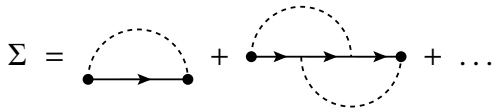
61.1 Propagator as a geometric series

Let us call -●-, the sum over all 1-PI graphs contributing to the fermion propagator, $\Sigma(p)$, where p is the external momentum of the fermion line. Let us also call the tree level propagaor, ⟶, $S_F(p)$. Multiplying $S_F(p)$ by $\frac{1}{1-S_F(p)\cdot\Sigma}$ will give

$$\Rightarrow \qquad S_F(p)\left(\frac{1}{1-S_F(p)\cdot\Sigma}\right) \rightarrow S_F(p) + S_F(p)\,\Sigma\,S_F(p) + S_F(p)\,\Sigma\,S_F(p)\,\Sigma\,S_F(p) + \ldots$$



Now we can write out Σ, starting with one loop. We also amputate the external fermion lines ("amputated propagators").



61.2 Computing the self-energy function

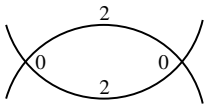
Today we are going to see one example of propagator renormalization, and we will see why it is a bit more complicated than the vertex function (it needs wave function renormalization and mass renormalization). To make life simpler, as usual, we will start with scalar field theory, where there is only one propagator. We can choose from ϕ_4^4 theory and ϕ_6^3 theory.

61.2.1 ϕ^4 theory

In ϕ^4 theory, the Lagrangian looks like this.

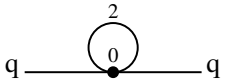
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\phi^2 - \frac{g}{4!}\phi^4$$

There are two monomials in the Lagrangian that appear quadratically, and we need both of them in order to renormalize properly. Let's review power counting. The lowest-order vertex correction is of the form

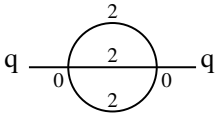


, with weight $w = 4 - 4 = 0$,

so the vertex function is logarithmically divergent and a single subtraction at a different value for the external momenta will make it finite.
As for the propagator, let us consider one-loop and two-loop corrections.



$w = 4 - 2 = 2$



$w = 8 - 6 = 2$

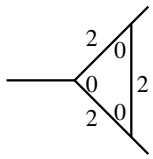
The first order correction does not depend on q or q^2, since there is only one single integral, $\int \mathrm{d}^4k\ (k^2 - m^2 + i\epsilon)^{-1}$, but this is merely an accident of the theory than anything fundamental. By looking at the power counting, we see that the propagator is quadratically divergent. This is by far more difficult to handle than a logarithmic divergence. A single subtraction will not suffice for quadratically divergent graphs. On the contrary, we will also need to subtract on the level of the first term in the Taylor expansion at $q^2 = \mu^2$, and even more.

61.2.2 ϕ^3 theory

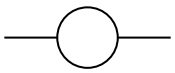
In ϕ^3 theory, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\phi^2 - \frac{g}{3!}\phi^3$$

ϕ^3 theory is renormalizable in six dimensions, which is the reason why it is sometimes denoted by ϕ^3_6. Let us again review the vertex and the propagator correction at lowest order, and do some power counting.

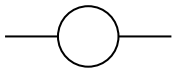


$w = 6 - 6 = 0$ logarithmically divergent



$w = 6 - 4 = 2$ quadratically divergent

As a nice little combinatorical exercise, it is left to the reader to show that all graphs with the same external leg structure have the same order of divergence.
Let us try to compute the propagator correction.



$$\begin{aligned}
&\to \quad \frac{1}{2}g^2\int\frac{\mathrm{d}^6k}{(2\pi)^6}\frac{1}{\Big(\underbrace{k^2-m^2+i\epsilon}_{:=P_1}\Big)\Big(\underbrace{(k+q)^2-m^2+i\epsilon}_{:=P_2}\Big)}\\
&\propto \quad \int\frac{1}{P_1}\frac{1}{P_2}\\
&= \quad \int\frac{P_2-P_1}{P_1P_2^2}+\int\frac{1}{P_2^2}\\
&= \quad \int\left\{\frac{(P_2-P_1)^2}{P_1P_2^3}+\frac{P_2-P_1}{P_2^3}+\frac{1}{P_2^2}\right\}
\end{aligned}$$

We have two totally different problems.

- $\int\frac{4k\cdot q^2}{P_1P_2^3}$ $\Rightarrow$ power counting: $\frac{8\text{dim}}{8\text{dim}}$ $\Rightarrow$ logarithmically divergent $\Rightarrow$ simple subtraction suffices
- $\int\frac{q^2}{P_2^3}$ $\Rightarrow$ $\frac{1}{P_2^3}$ may be independent of q^2, so a simple subtraction would suffice, but $\frac{q^2}{P_1^2}$ is not so easy.
- $\int\frac{1}{P_2^2}$ can be shifted to $\int\frac{k^2-m^2}{P_2^2}$

Apparenty, we need two subtractions. Since this is very technical, we will work out the details next term. So far, we have dealt with kinematic renormalization and minimal subtraction. Now, we have two structure functions of q^2 and m^2 to renormalize, and it is not always clear which scheme to choose.

- Propagators might have different masses. In the Standard Model, for example, there are loads of particles with all kinds of different masses. We also have to distinguish between external and internal lines because they don't generally have the same mass scale: external lines are always on-shell, while internal lines are off-shell.

- If $\Sigma(p^2)$ is the self-energy of a massive scalar boson with mass μ^2, then this mass parameter in the Lagrangian is not necessarily the physical mass. This effect is comparable to the coupling: When we set $g \to Z_g g$, we distinguished between the physical coupling and the bare coupling, an unphysical quantity. The same accounts for the mass parameter in the Lagrangian, which need not even be finite. To distinguish more easily, we give it a different name and denote it by μ_0^2. Then the propagator becomes

$$\begin{aligned} i\Delta &= \frac{i}{p^2-\mu_0^2+i\epsilon} + \frac{i}{p^2-\mu_0^2+i\epsilon}\left(-i\Sigma(p^2)\right)\frac{i}{p^2-\mu_0^2+i\epsilon} + \dots \\ &= \frac{i}{p^2-\mu_0^2+i\epsilon}\left[\frac{1}{1+\frac{i\Sigma(p^2)}{p^2-\mu_0^2+i\epsilon}}\right] \end{aligned}$$

So we need to renormalize. The Lagrangian gives a propagator, which is a geometric series in the self-energy. Sensible renormalization conditions for the propagator are:

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2-\mu^2)\Sigma'(\mu^2) + \tilde{\Sigma}(p^2)$$

The propagator should have a pole at the physical mass, or in other words, the inverse propagator should vanish at the physical mass. This is exactly how experimentists detect particles. The derivative should also vanish at $p^2 = \mu^2$, so here, $\tilde{\Sigma}$ denotes the remaining parts of the Taylor expansion in $p^2 = \mu^2$, with $\tilde{\Sigma}(\mu^2) = 0$ and $\tilde{\Sigma}'(\mu^2) = \left.\frac{\partial\tilde{\Sigma}(p^2)}{\partial(p^2)}\right|_{p^2=\mu^2} = 0$. Then, the propagator $i\Delta$ becomes

$$\begin{aligned} i\Delta &\to \frac{i}{p^2-\mu_0^2-\Sigma(\mu^2)-(p^2-\mu^2)\Sigma'(p^2)-\tilde{\Sigma}(p^2)+i\epsilon} \\ &\to \frac{i}{(p^2-\mu^2)\left(1-\Sigma(\mu^2)-\tilde{\Sigma}(p^2)\right)+i\epsilon} \\ \Rightarrow \quad i\Delta(p^2) &= \frac{Z_\phi}{p^2-\mu^2-\tilde{\Sigma}(p^2)+i\epsilon} \end{aligned}$$

Here, we introduced

$$\begin{aligned} Z_\phi &= \left(1-\Sigma'(\mu^2)\right)^{-1} \\ \Rightarrow \quad \phi_{\text{ren}} &= Z_\phi^{-\frac{1}{2}}\phi \\ p^2-\mu_0^2-\Sigma(\mu^2) &=: \mu^2 Z_{\mu^2} \\ \Rightarrow \quad i\Delta_{\text{ren}} &= \frac{i}{p^2-\mu^2-\tilde{\Sigma}(p^2)} \end{aligned}$$

We will have to carry out two subtractions:

- One at $p^2 = \mu^2$.
- One at the level of the derivative, which should also vanish at $p^2 = \mu^2$.

The Lagrangian then becomes

$$\mathcal{L} = Z_\phi \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - Z_m m^2\phi^2 - Z_g\frac{g}{3!}\phi^3$$

Since the Z_i are power series in g, at two loops, they are already dependent on g.
This was just a taste of things to come. In the next terms, we will learn how to do renormalization explicitly, learn about the renormalization group,...

62 Combinatorics

We have already discussed some graph polynomials, the Kirchhoff and Symanzik polynomials. Let Γ be a 1-PI graph, then the Symanzik polynomials are given by

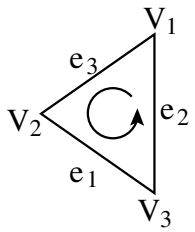
$$\Psi_\Gamma = \sum_{\substack{\text{spanning} \\ \text{trees } T}} \prod_{e \notin T} A_e$$

$$\phi_\Gamma = \sum_{\substack{\text{spanning} \\ \text{two-trees} \\ T_1 \cup T_2}} Q(T_1) \cdot Q(T_2) \prod_{e \notin T_1 \cup T_2} A_e$$

From elementary observation, we can set up a sequence of maps, where the image of each map lies in the kernel of the next map.

$$0 \;\to\; H^1_\Gamma \;\to\; \mathbb{Q}^E \;\xrightarrow{\partial}\; \mathbb{Q}^V \;\to\; 0$$

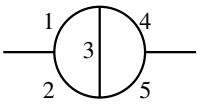
H^1_Γ denotes the first Betti homology number (the loop numbers). E stands for edges, V for vertices. For example, consider the graph



where the arrow represents the orientation of the edges. Then the derivatives of the edges map onto the set of vertices in the following way:

$$\partial(e_1) = v_3 - v_2$$
$$\partial(e_2) = v_1 - v_3$$
$$\partial(e_3) = v_2 - v_1$$

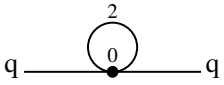
As a second example,



has three loops, namely $\{$  $\}$, but only two independent loops.
We can compute the polynomials Ψ_Γ and ϕ_Γ by looking at the choice of loop basis.

62.1 Ψ_Γ

Let us work with the graph



and define the basis

$$h_1 := \text{“}1 + 2 + 3\text{”}$$
$$h_2 := \text{“}1 + 2 + 4\text{”}$$

Let us set up a matrix in the homology of h_1 and h_2, by writing in each cell the common elements (edges) from colomn and row, where h_i is in the i^{th} row or column.

$$
\begin{aligned}
& & h_1 & & h_2 \\
M \to & \begin{matrix} h_1 \\ h_2 \end{matrix} & \begin{pmatrix} A_1+A_2+A_3 & A_1+A_2 \\ A_1+A_2 & A_1+A_2+A_3 \end{pmatrix} \\
\Rightarrow \quad & M := & \begin{pmatrix} A_1+A_2+A_3 & A_1+A_2 \\ A_1+A_2 & A_1+A_2+A_3 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \det M = |M| &= (A_1+A_2+A_3)(A_1+A_2+A_4) - (A_1+A_2)^2 = \\
&= (A_1+A_2)^2 + A_1A_3 + A_2A_3 + A_1A_4 + A_2A_4 + A_3A_4 - (A_1+A_2)^2 = \\
&= A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4 = \\
&= \sum_{\substack{\text{spanning} \\ \text{trees } T}} \prod_{e \notin T} A_e
\end{aligned}
$$

If we had chosen a different basis, for example

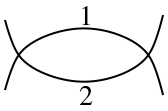
$$
h_1 := \text{“}1+2+3\text{”} \quad h_2' \qquad\qquad := \text{“}3+4\text{”}
$$

the result would have been the same:

$$
\begin{aligned}
M' &:= \begin{pmatrix} A_1+A_2+A_3 & A_3 \\ A_3 & A_3+A_4 \end{pmatrix} \\
\Rightarrow \quad \det M' = |M'| &= (A_1+A_2+A_3)(A_3+A_4) - A_3^2 = \\
&= A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_3 + A_3A_4 - A_3^2 = \\
&= A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4 = \\
&= \sum_{\substack{\text{spanning} \\ \text{trees } T}} \prod_{e \notin T} A_e \equiv \det M
\end{aligned}
$$

62.2 ϕ_Γ

For ϕ_Γ, we define the matrix a bit differently. To avoid confusion, let us work with a simpler graph,



The we define

$$
\begin{aligned}
\begin{vmatrix} A_1+A_2 & \mu_1A_1+\mu_2A_2 \\ \mu_1^TA_1+\mu_2A_2^T & \mu_1\mu_1^TA_1+\mu_2\mu_2^TA_2 \end{vmatrix} &= (A_1+A_2)(\mu_1\mu_1^TA_1+\mu_2\mu_2^TA_2) - (\mu_1A_1+\mu_2A_2)(\mu_1^TA_1+\mu_2A_2^T) \\
&= \dots \\
&= (\mu_1 \pm \mu_2)(\mu_1^T \mp \mu_2^T)A_1A_2
\end{aligned}
$$

We identify $\mu_1 - \mu_2$ as the momentum incoming into a vertex. If we remember momentum and spin representations of the Lorentz group, we will see that

$$
\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \to p_0\mathbb{I}_{2\times2} - p_i \cdot \sigma_i =: \mu
$$

where σ_i denotes the Pauli matrices.
In general, we can set up the matrix with

$$\begin{array}{ccc} (A_1+A_2+A_3) & (A_3+A_4) & \sum_{e\in h_i}\mu_i A_i \\ (A_3+A_4) & (A_1+A_2+A_4) & (\quad) \\ (\sum_{e\in h_i}\mu_i A_i)^T & (\quad)^T & \sum_{\text{edges } e}\mu_e\mu_e^T A_e \end{array}$$

62.3 Kirchhoff polynomials

From the Kirchhoff polynimials, we get the Feynman rules by

$$\int \frac{e^{-\frac{\phi}{4}}}{\Psi^{\frac{D}{2}}} \quad \rightarrow \quad \text{Feynman rules}$$

where properties of the integral and the polynomials can be understood just by looking at the determinants of matrices of the kind like we just set up. While Ψ is completely understood, ϕ gives unknown terms already at two loops.