Renormalization
&
Renormalization Group

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Lecture Notes

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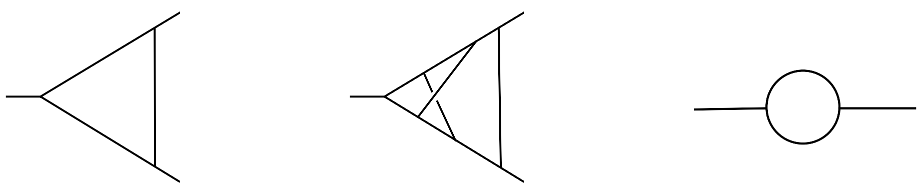
Contents

CHAPTER 1

Graph, Weights and Forests

1.1. Introduction: Feynman graphs

Basic definitions. Modern physics describes elementary particles and their interactions by the heavy machinery of perturbative quantum field theory(pQFT). Within this framework, graphical objects known as *Feynman graphs* play a prominent role. Pictorial representations of such graphs are



for example. Accommodated to the needs of physical theory, these graphs are non-standard as will become apparent in the following. Rather than giving a formal definition loaded with technicalities, we shall adopt a more informal and narrative style of describing them.

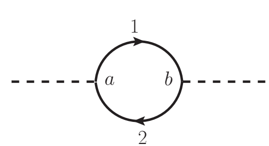
For a start, depending on what one focusses on, Feynman graphs are generally *labelled*. That is, equipped with maps that assign information of physical interest(momentum, position, etc.) to its edges and vertices. We will further elaborate on these maps as we go along.

Let Γ be one such Feynman graph. We shall use the terms 'Feynman graph' and 'graph' interchangably henceforth. Γ consists of its vertex set $\Gamma^{[0]}$ and a set of edges $\Gamma^{[1]}$. We distinguish between *external* and *internal edges*: if an edge $e \in \Gamma^{[1]}$ connects to only one vertex, i.e.

$$|e \cap \Gamma^{[0]}| = 1 \tag{1.1}$$

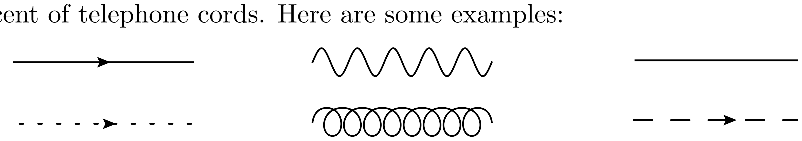
we speak of it as external, and if $|e \cap \Gamma^{[0]}| = 2$, the edge is called internal(it connects two vertices). Their sets are denoted $\Gamma^{[1]}_{ext}$ and $\Gamma^{[1]}_{int}$, respectively. This may seem a bit strange at first, but external edges are 'open' towards one end and are not, as in standard graph theory, a pair of vertices. The edges of a Feynman graph should rather be thought of as extra elements with data on which vertex they connect to and, moreover, are subdivided into *half-edges*: internal edges are two joint half-edges, whereas an external edge is made up of a single half-edge.

Though this may sometimes not be of interest, edges are *oriented*. For example, in Yukawa theory one encounters the graph



with oriented internal edges 1,2 and vertices a,b. The *source* $s(e)$ of an edge $e \in \Gamma^{[1]}$ is the vertex it is oriented away from, while its *target* $t(e)$ is the one it is oriented towards. In our example, if we choose the orientation suggested by the little arrow, we have $s(\text{edge } 1) = a$ and $t(\text{edge } 1) = b$.

Feynman graphs are built from various edge and vertex types, each corresponding to a type of quantum particle and type of interaction, respectively[1]. Edges come in the form of straight, wiggly and dashed lines, amongst other somewhat fancy line styles. For example, the lines used for gluons are strongly reminiscent of telephone cords. Here are some examples:



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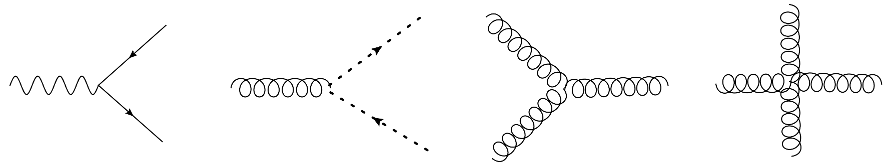
[1]The corresponding notion in standard graph theory is that of a coloured graph.

Vertices together with their adjacent half-edges are called *corollas*, such as



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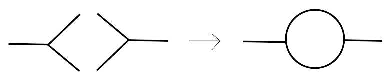
A Feynman graph can therefore be thought of as constructed by glueing together corollas, as in



.

If an edge is made of two half-edges that connect to the same vertex, we get self-loops like

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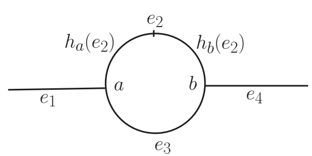
where the latter belongs to what has been dubbed 'tadpoles'(inspired by their shape). Though they are genuine Feynman graphs which a physicist might play around with(prior to what is known as renormalization), we will forbid self-loops. Given a vertex $v \in \Gamma^{[0]}$ and an adjacent edge $e \in \Gamma^{[1]}$, then $h_v(e)$ is the half-edge of e which is attached to v. By $n(v)$ we denote the set of all adjacent edges of the vertex v. Consider the graph

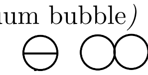


,

where we have marked the two half-edges $h_a(e_2), h_b(e_2)$ of edge e_2. The adjacent edges of the two vertices a and b are $n(a) = \{e_1, e_2, e_3\}$ and $n(b) = \{e_2, e_3, e_4\}$ with $|n(a)| = |n(b)| = 3$(cardinality).

Definition 1.1.1. *Let* Γ *be a connected Feynman graph.* Γ *is said to be* one-particle irreducible(1PI) *or* 2-connected *if it stays connected after removal of any internal edge. Furthermore, it is called*

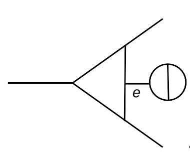
(1) vacuum graph*(or* vacuum bubble*) if* $|\Gamma^{[1]}_{ext}| = 0$*, i.e. if* Γ *has no external edges, like*



;

(2) tadpole graph *if* $|\Gamma^{[1]}_{ext}| = 1$*;*
(3) propagator *or* self-energy graph *if* $|\Gamma^{[1]}_{ext}| = 2$
(4) *and* vertex graph *if* $|\Gamma^{[1]}_{ext}| \geq 3$.

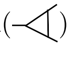
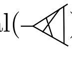
Throughout this lecture we will only consider 1PI propagator and vertex graphs, discarding all the rest. Take the vertex graph



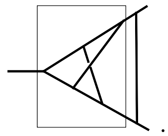
.

It is not 1PI on account of the tadpole being attached to it: upon removal of edge e, we are left with two components, namely a vertex graph and a vacuum bubble.

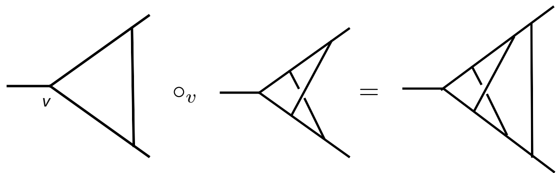
Evaluating graphs. Physicists assign numbers or certain functions to graphs. For example, say the assignment is a map called eval, mapping Feynman graphs to the algebra $\mathbb{C}[L]$ of polynomials in one variable L, then we might write

(1.2) eval() $= d_1 L + d_0$, eval() $= c_2 L^2 + c_1 L + c_0$.

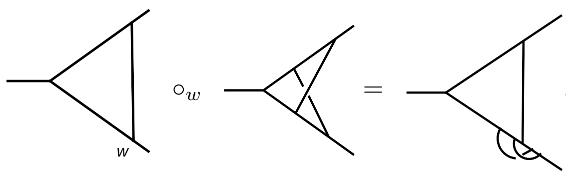
where the beginner unfamiliar with QFT need not wonder how these come about for the time being. In fact, there is an intricate story behind eval which involves the evaluations of integrals and subtractions to be unfolded in the lectures to come. For the moment, we content ourselves with noting that purely combinatorial criteria determine how these polynomials, their degrees and coefficients are related. The difference between the two graphs in (1.2) is that the second one has a 1PI subgraph *inserted*, which is the boxed one in



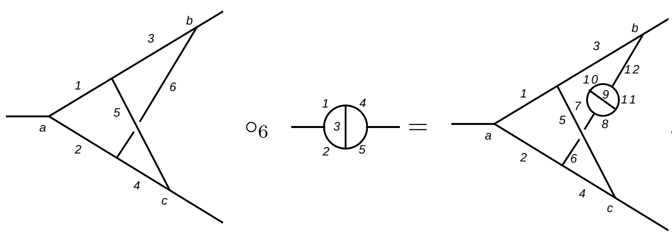
Subgraph insertions. On the set of graphs, we can define an insertion operation, in this particular case,

(1.3)


where $\circ_v$ instructs us to insert the graph following behind it at vertex v. Or, if we choose the lowermost vertex to be the insertion place, call it w, we find

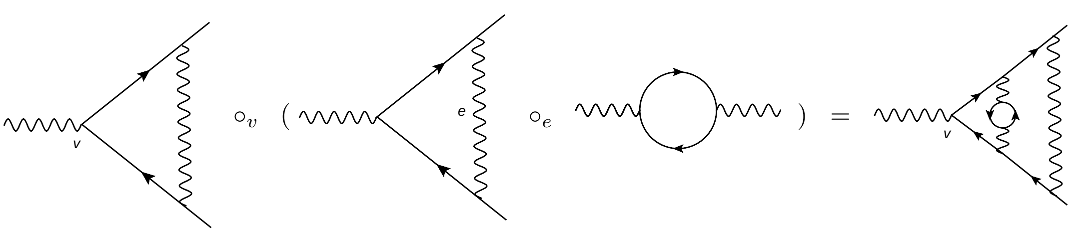
(1.4)


To see how the labelling changes upon insertion, consider the self-energy graph insertion

(1.5)


Notice that the labelling shifts by the number of internal edges which enter the vertex graph through insertion and that the external edges(which are half-edges) of the inserted self-energy graph are joined with the half-edges of edge 6 of the 'hosting' graph.

Here is an important fact: all graphs are made up of 1PI subgraphs. We can therefore, with these insertion operations at hand, construct all Feynman graphs with given a fixed 'skeleton': here is an example from quantum electrodynamics(QED) with wiggly lines

(1.6)


where the skeleton is the leftmost 1-loop graph and the insertions are carried out according to this: first, the self-enery graph is inserted into wiggly edge e, then the result is inserted at vertex v to yield the vertex graph on the right hand side.

1.2. Operads and tree diagrams

Operads. Consider the multiplication map $m : A \otimes A \to A$ on an algebra A. It is associative by definition, in the language of commutative diagrams,

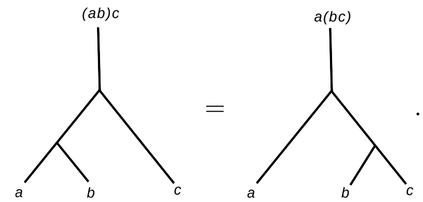
$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "id\otimes m"] \arrow[d, "m\otimes id"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
\tag{1.7}
$$

commutes. With the usual shorthand $ab := m(a \otimes b)$ for $a, b \in A$, this is nothing but

$$(ab)c = a(bc). \tag{1.8}$$

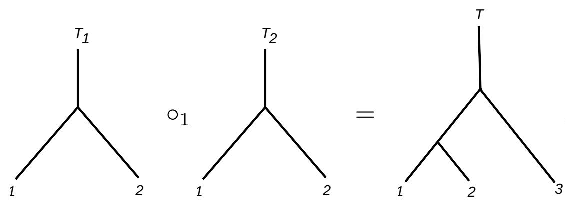
We can write this in the form a tree diagram:

(1.9)



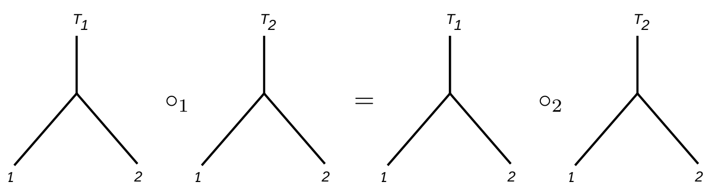
Seen as a 'multiplication machine', this tree has input slots, represented by its leaves at the bottom, and one output slot, given by the uppermost vertex. Every internal vertex represents a multiplication procedure. Just like we have done before with subgraphs, we can build trees by insertion operations:

(1.10)



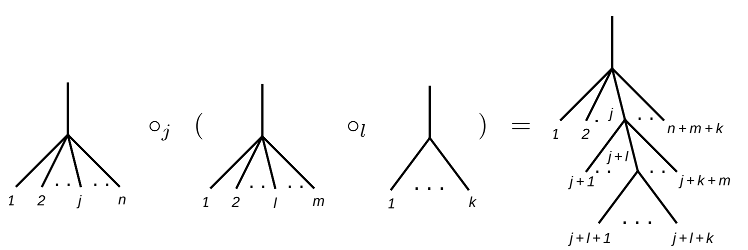
where $\circ_1$ says 'attach tree T_2 to leaf 1 of tree T_1 to obtain the final tree T'. Note how the leaves of the resulting tree are labelled. Note also that upon attaching a tree τ to another, say T, at a leaf of T, we merge the uppermost line of τ with this leaf. The operad equation corresponding to (1.9) takes the form

(1.11)



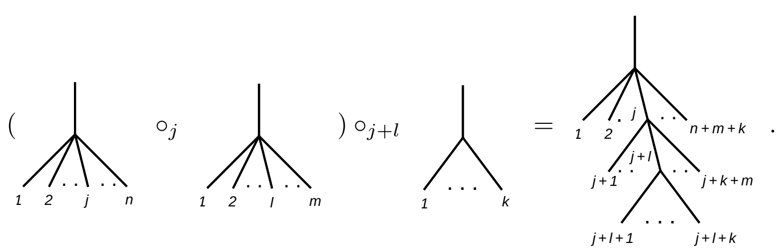
More generally, a repeated operad application might in tree diagram language look like

(1.12)



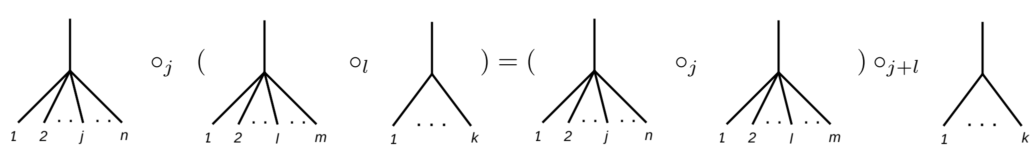
where the dots stand for the approriate number of lines. This is a more general situation as we are not restricted to strictly binary trees which arise in the context of the multiplication map(strictly binary tree means every node which is not a leaf has exactly two children). Note that setting the brackets around the first two trees and changing the insertion instruction from $\circ_l$ to $\circ j + l$

(1.13)



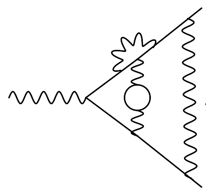
leads to the same tree as in (1.12). We thus have the operad equation

(1.14)

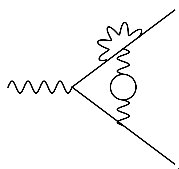


Loosely speaking and ignoring various other subtle aspects, an operad on the set of trees is a map that takes any number of trees and composes them as in (1.13) to form a single tree. For a precise definition see [**VAT04**].

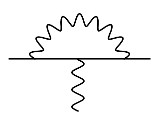
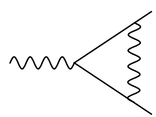
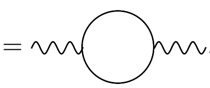
Subgraph insertions as operads. We can represent a Feynman graph by a tree diagram if we decorate the nodes of the tree appropriately with subgraphs. Instead of giving a precise definition of this bijection at this stage, we illustrate it by an example. Take the QED graph

(1.15) $\Gamma =$  ,

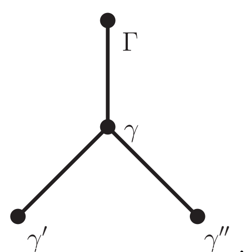
where we have omitted the orientation arrows which have no business to hang around in what follows. Γ contains the 1PI subgraph

(1.16) $\gamma =$ 

which itself habours the two 1PI subgraphs

(1.17) $\gamma' =$  $=$  and $\gamma'' =$  .

To represent Γ by a tree diagram, we use what is called a *decorated rooted tree*. This is a tree with decorated nodes and a 'root', a distinguished node that always has its place on top. The corresponding tree for Γ then takes the form



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The root, decorated with Γ, stands for the whole graph Γ. Every node below it represents a subgraph. The children of these nodes are the subgraphs of those subgraphs, and so forth.

The advantage of a Feynman graph's tree representation is that it makes its *subgraph structure* clearly visible. Moreover, it can be read as recipe of subgraph insertions that have to be made in order to obtain the graph.

It is these very subgraph insertions and the corresponding attachments of subtrees onto trees representing Feynman graphs which are of operadic nature. Therefore, we arrive at the conclusion that naturally *Feynman graphs come with an operad structure.*

1.3. The weight of a graph

Graph homology and labelling. Recall that although this might not always show up in its pictorial representation, a Feynman graph Γ has labelled vertices and oriented edges with source and target vertex. We do not consider self-loops, i.e. if $e \in \Gamma^{[1]}$ then we always have $s(e) \neq t(e)$. In addition to that, the vertices are ordered. Putting all this data together, we get an *oriented graph* in the sense of *graph homology.* Without taking too wide a detour, we briefly sketch this homology.

Consider a chain complex of $\mathbb{Q}$-vector spaces generated by Feynman graphs and indexed by the number of internal edges. There is a boundary operator d from one vector space to the next defined by

$$d\Gamma = \sum_{e \in \Gamma^{[1]}_{int}} \operatorname{sgn}(\Gamma_e)\Gamma_e, \tag{1.18}$$

where Γ_e is the graph we obtain when we shrink the internal edge e to a point and $\mathrm{sgn}(\Gamma_e) = \pm 1$ is such that $d \circ d = 0$. Then, roughly speaking, the quotient spaces with respect to the boundary operator yield the graph homology. For more on this, see [**ConVo03**].

The labelling of a graph Γ may for instance be given by the following maps. The momentum labelling $\zeta : \Gamma^{[1]} \to \mathbb{M}^4$ associates to each edge a 4-momentum in Minkowski space $\mathbb{M}^4 \simeq \mathbb{R}^4$. The edge variable labelling $A : \Gamma^{[1]}_{int} \to \mathbb{R}_+$ assigns a variable to each internal edge and, as we will see, constitutes what is known as the two *Kirchhoff* or *Symanzik graph polynomials*(for more on this, see chapter 7). Furthermore, let Γ^H be the set of half-edges of Γ, then $a : \Gamma^H \to \mathbb{R}_+$ is the map associating a variable to each half-edge. This map will be relevant in setting up the *corolla polynomial* of Γ, expounded in [**KrSS12**].

Assigning integers to graphs. We introduce a map $\omega : \Gamma^{[1]}_{int} \cup \Gamma^{[0]} \to \mathbb{Z}$ which assigns an integer to each internal edge and to each vertex of a graph Γ. Then we define the *weight* of the graph Γ by

$$\omega_D(\Gamma) := \sum_{e \in \Gamma^{[1]}_{int}} \omega(e) + \sum_{v \in \Gamma^{[0]}} \omega(v) - D \cdot h_1(\Gamma), \tag{1.19}$$

where D is the dimension of spacetime and $h_1(\Gamma)$ the first Betti number of Γ which is the number of independent loops. We set

$$\omega(\quad) = 2, \qquad \omega(\quad) = 0 \tag{1.20}$$

and have in $D = 6$ dimensions of spacetime

$$\omega_6(\quad) = 2\omega(\quad) + 2\underbrace{\omega(\quad)}_{=0} - 6 = 4 - 6 = -2, \tag{1.21}$$

and

$$\omega_6(\quad) = 5\omega(\quad) + 4\omega(\quad) - 6 \cdot 2 = 10 - 12 = -2, \tag{1.22}$$

where the Betti numbers are $h_1(\quad) = 1$ and $h_1(\quad) = 2$. Next, consider

$$\omega_6(\quad) = \omega_6(\quad) = \omega_6(\quad) = \omega_6(\quad) = 0, \tag{1.23}$$

where $h_1(\quad) = 3$. Another example is a vertex graph with four external legs, for which we find

$$\omega_6(\quad) = \omega_6(\quad) = \omega_6(\quad) = 2. \tag{1.24}$$

These calculations suggest that the weight of a graph is determined by the number of external legs, i.e. if

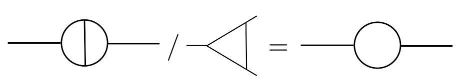
$$|\Gamma^{[1]}_{ext}| = |\widetilde{\Gamma}^{[1]}_{ext}| \tag{1.25}$$

for two graphs Γ and $\widetilde{\Gamma}$ with edge and vertex types of those in (1.20), one always has $\omega_6(\Gamma) = \omega_6(\widetilde{\Gamma})$. In fact, it is not difficult to show that if

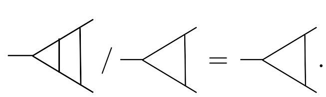
$$\omega_D(\Gamma) = \omega_D(\widetilde{\Gamma}) \qquad \forall \Gamma, \widetilde{\Gamma} : |\Gamma^{[1]}_{ext}| = |\widetilde{\Gamma}^{[1]}_{ext}| \tag{1.26}$$

then $D = 6$ follows.

Contractions. Graph insertions can be reversed by an operation called *contraction.* The contraction of a subgraph γ in a graph Γ is an operation yielding the so-called *cograph* Γ/γ, which is the graph one obtains by shrinking all internal edges of γ to a single point while the external leg structure remains untouched. For example,

(1.27) 

and

(1.28) 

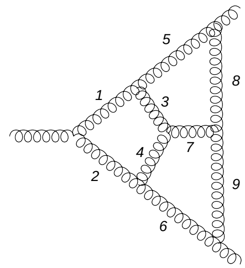
Here is an interesting fact: if for some dimension D we find that (1.26) holds for a certain species of graphs, i.e. with certain vertex and edge types, then

$$\omega_D(\Gamma) = \omega_D(\Gamma/\gamma) \qquad \forall \gamma \subset \Gamma : \omega_D(\gamma) \leq 0. \tag{1.29}$$

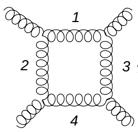
In other words, if (1.26) holds, we do not change the weight of a graph if we contract one of its subgraphs of non-positive weight. A graph γ of non-positive weight (in dimension D), i.e with $\omega_D(\gamma) \leq 0$ is also referred to as a *(superficially) divergent graph*. We define a *primitive graph* to be a 1PI graph that is void of any divergent proper 1 PI subgraph.

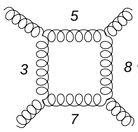
1.4. Forests of a graph

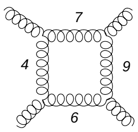
Consider the 7-corolla gluon graph

(1.30) $\Gamma :=$ 

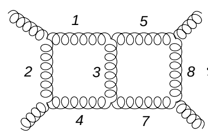
which has only gluon edges and trivalent corollas. It has the following proper subgraphs:

(1.31) $\gamma_1 =$ 

, $\gamma_2 =$ 

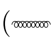
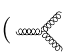
, $\gamma_3 =$ 

and their unions

(1.32) $\gamma_{12} := \gamma_1 \cup \gamma_2 =$ 

, $\gamma_{23} := \gamma_2 \cup \gamma_3$, $\gamma_{13} := \gamma_1 \cup \gamma_3$.

The weights of its vertex and edge types are

(1.33) $\omega($  $) = 2,$ $\omega($  $) = -1.$

Then we find for $D = 4$: $\omega_4(\Gamma) = -1$ and $\omega_4(\gamma_i) = \omega_4(\gamma_{ij}) = 0$ for all i, j. Hence all subgraphs are divergent, and so is Γ itself. Primitive subgraphs are γ_1, γ_2 and γ_3.

Definition 1.4.1. *Let Γ be a graph. A forest f of Γ is a collection of divergent 1 PI proper subgraphs $\gamma \subsetneq \Gamma$ such that for any $\gamma, \gamma' \in f$ one of the following conditions holds:*

$$(i)\ \gamma \subset \gamma', \qquad (ii)\ \gamma' \subset \gamma, \qquad (iii)\ \gamma \cap \gamma' = \emptyset, \tag{1.34}$$

i.e. either the subgraphs of f are disjoint or contained in each other.

The forests of our gluon graph Γ in (1.30) are:

$$\{\gamma_1, \gamma_{12}\}, \quad \{\gamma_2, \gamma_{12}\}, \quad \{\gamma_1, \gamma_{13}\}, \quad \{\gamma_3, \gamma_{13}\}, \quad \{\gamma_2, \gamma_{23}\}, \quad \{\gamma_3, \gamma_{23}\} \tag{1.35}$$

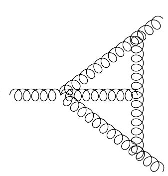
and every single subgraph by itself, $\{\gamma_j\}$, $\{\gamma_{ji}\}$ for all i, j.

Definition 1.4.2. *A forest f of a graph Γ is* maximal*, if the cograph*

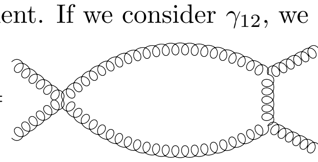
$$\Gamma/f := \Gamma/\cup_{\gamma \in f}\, \gamma$$

is primitive.

Consider the simple forest $\{\gamma_1\}$. If we contract it in Γ,

(1.36) $\Gamma/\gamma_1 =$  ,

where the 4-valent vertex has weight zero, we see that γ_1 itself is not a maximal forest since this cograph *does* have two 1PI proper subgraphs (of weight -5) and hence is not primitive. Neither are γ_2 and γ_3 maximal forests by the same argument. If we consider γ_{12}, we find the primitive cograph

(1.37) $\Gamma/\gamma_{12} =$ 

which tells us that γ_{12} by itself constitutes a maximal forest of Γ. The same goes for γ_{13} and γ_{23} as well as all forests in (1.35).

Definition 1.4.3. *A maximal forest f of a graph Γ is called* complete *if any $\gamma \in f$ is either primitive or there is a proper subgraph $\gamma' \in f$ of γ such that γ/γ' is primitive.*

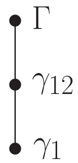
This means that only the forests in (1.35) are complete. The forest $\{\gamma_{12}\}$ has only one non-primitive subgraph for which there is no further subgraph in this forest that could be contracted to yield a primitive graph. Consider the maximal forest $\{\gamma_1, \gamma_{12}\}$. It is complete because γ_1 and the cograph γ_{12}/γ_1 are primitive, i.e. have no divergent proper 1PI subgraphs.

To display the nestedness of subgraphs, it makes sense to write complete forests as a sequence of subsets:

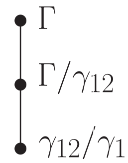
$$\gamma_1 \subsetneq \gamma_{12} \subsetneq \Gamma, \qquad \gamma_1 \subsetneq \gamma_{13} \subsetneq \Gamma, \qquad \gamma_2 \subsetneq \gamma_{12} \subsetneq \Gamma, \qquad \gamma_2 \subsetneq \gamma_{23} \subsetneq \Gamma, \tag{1.38}$$

and so on, taking into account every disjoint subgraph sequence, too.

Now that we have the notion of a graph's forest, we can specify the one-to-one correspondence between Feynman graphs and decorated rooted trees: take the complete forest $f = \{\gamma_1, \gamma_{12}\}$. The corresponding rooted tree now takes either the form



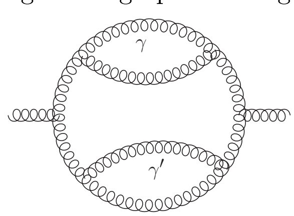
or



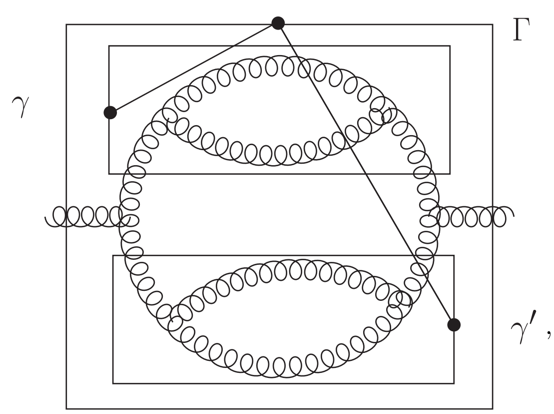
,

both decorations are possible. Now, we acknowledge that *a decorated rooted tree corresponds to a complete forest of a Feynman graph.*

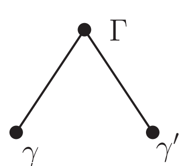
A final example with two disjoint divergent subgraphs is the gluon graph

(1.39) $\Gamma =$ 

with divergent subgraphs γ and γ'. The reader may check that $\omega_4(\Gamma) = \omega_4(\gamma) = \omega_4(\gamma') = -2$. The corresponding rooted tree can be read off from the box system



where each divergent subgraph corresponds to a leaf and the whole graph to the root. Thus, we have the simple tree



with two leaves decorated by the two subgraphs. The forests are γ, γ' and $\{\gamma, \gamma'\}$. Only the latter is complete.

CHAPTER 2

The Hopf Algebra of Rooted Trees

2.1. The route to a Hopf algebra

Our goal is to establish algebraic structures on the set of Feynman graphs which are those of what is known as a *Hopf algebra*. However, understanding how these structures work on Feynman graphs and that they do have the desired properties is anything but trivial. Our preferred route is this: we first acquaint ourselves with these structures on the much simpler set of *undecorated rooted trees* and see how their *pre-Lie structure* naturally gives rise to a *Lie algebra structure*. On account of the *Milnor-Moore theorem*, a Hopf algebra structure is then guaranteed. Because all these structures can also be found on the set of Feynman graphs, we will see that they do indeed form a *Hopf algebra*.

Pre-Lie structure. Let Γ be a Feynman graph and $\mathcal{I}(\gamma|\Gamma)$ the set of all possible insertion places for γ into Γ, i.e. a set of vertices or edges of Γ. Consider the bilinear operation

$$\Gamma \star \gamma = \sum_{i \in \mathcal{I}(\gamma|\Gamma)} \Gamma \circ_i \gamma. \tag{2.1}$$

In fact, it is *pre-Lie*, which means that it is not associative and deviates from associativity in a very specific way, namely

$$(\Gamma \star \gamma_1) \star \gamma_2 - \Gamma \star (\gamma_1 \star \gamma_2) = (\Gamma \star \gamma_2) \star \gamma_1 - \Gamma \star (\gamma_2 \star \gamma_1) \tag{2.2}$$

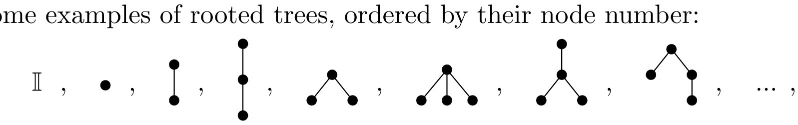
which is *not* zero in general. The commutator with respect to this operation satisfies the Jacobi identity and hence gives rise to a Lie algebra which then, by the theorem of Milnor and Moore, ensures a Hopf algebra structure.

2.2. Rooted trees

Unlike Feynman graphs, rooted trees are standard graphs as they are known in graph theory.

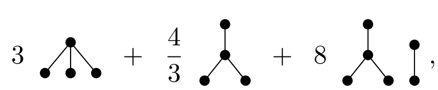
Definition 2.2.1. *A* tree T *is a connected and simply connected*[1] *graph with vertex set* $T^{[0]}$ *and edge set* $T^{[1]}$. *A* rooted tree *is a non-planar tree with a distinguished node* $r \in T^{[0]}$ *such that any egde is oriented away from it. By* $\mathbb{I}$ *we denote the empty tree.* $|T| := \#(T^{[0]})$ *is the number of nodes.*

Here are some examples of rooted trees, ordered by their node number:



where the root is always represented by the topmost node and we have refrained from drawing arrows to indicate the orientation of the edges. We introduce an algebra structure on the set of rooted trees by considering the free commutative $\mathbb{Q}$-algebra with generators labelled by rooted trees. These generators will be identified with their rooted trees such that we get expressions like

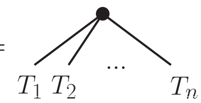
(2.3)



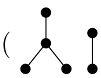
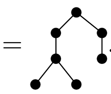
where the last term is a product of trees. In graph-theoretic terms, the multiplication is the disjoint union of trees. A product of trees is called *forest*(of rooted trees). The neutral element of multiplication is the empty tree $\mathbb{I}$(or 'empty forest') with $|\mathbb{I}| = 0$. As for trees, we set $|f|$ to be the number of nodes in the forest f. We denote this algebra by H.

[1]No loops.

Grafting operator. Let $\mathcal{T}$ be the set of all rooted trees and $\langle \mathcal{T} \rangle_{\mathbb{Q}}$ its linear span over the rationals. We introduce the *grafting operator* B_+ as a linear map $H \to \langle \mathcal{T} \rangle_{\mathbb{Q}}$ by $B_+(\mathbb{I}) = \bullet$ and for a forest of trees $T_1, ..., T_n$

(2.4) $B_+(T_1 \; ... \; T_n) :=$ 

mapping any forest to a single tree by attaching the roots to a single new node which then becomes the new root. A concrete example is

(2.5) $B_+($  $) =$ .

Note that the product of trees is commutative, which would cause us trouble at this point if the trees were planar. Thanks to their non-planarity, there is a unique forest X for every tree $T \in \mathcal{T}$ such that $T = B_+(X)$, a fact which is somewhat obvious from the definition of the operator B_+.

Grading. By the number of nodes, there is a natural grading on H. Let $\mathcal{F}_n$ be the set of all forests with n nodes, i.e. $\tau \in \mathcal{F}_n$ implies $|\tau| = n$. The grading is then given by their linear $\mathbb{Q}$-span

$$H_n := \langle \mathcal{F}_n \rangle_{\mathbb{Q}} \tag{2.6}$$

and hence $H = \bigoplus_{n \geq 0} H_n$, where $H_0 = \mathbb{Q}\mathbb{I}$. The elements of the subspaces H_n are called *homogeneous*: they are linear combinations of forests with the same number of nodes each.

More formally, our algebra H is a triple $(H, m, \mathbb{I})$, with multiplication[2] $m : H \otimes H \to H$ and unit map[3] $\mathbb{I} : \mathbb{Q} \to H_0$, the latter of which sends $q \in \mathbb{Q}$ to $q\mathbb{I}$. The product has the grading property

$$m(H_l \otimes H_k) \subset H_{l+k}. \tag{2.7}$$

Another important property of m is its associativity

$$m(a \otimes m(b \otimes c)) = m(m(a \otimes b) \otimes c) \qquad \forall a, b, c \in H \tag{2.8}$$

which can also be expressed in terms of a commutative diagram

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\mathrm{id} \otimes m} & H \otimes H \\ \Big\downarrow{\scriptstyle m \otimes \mathrm{id}} & & {\scriptstyle m}\Big\downarrow \\ H \otimes H & \xrightarrow{m} & H \end{array} \tag{2.9}$$

as we have already seen in section 1.2.

Coproduct. To promote our algebra H to a *bialgebra*, we additionally need two linear maps: the *coproduct* Δ and the *counit* $\hat{\mathbb{I}}$. The first linear map, the coproduct $\Delta : H \to H \otimes H$ must be *coassociative*: $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$, or, alternatively,

$$\begin{array}{ccc} H \otimes H \otimes H & \xleftarrow{\mathrm{id} \otimes \Delta} & H \otimes H \\ {\scriptstyle \Delta \otimes \mathrm{id}}\Big\uparrow & & \Big\uparrow{\scriptstyle \Delta} \\ H \otimes H & \xleftarrow{\Delta} & H \end{array} , \tag{2.10}$$

must commute, which is (2.9) with reversed arrows. It must also be compatible with the product in the sense of an algebra morphism: if we write $ab := m(a \otimes b)$, then this takes the simple form

$$\Delta(ab) = \Delta(a)\Delta(b), \tag{2.11}$$

i.e. the coproduct must be multiplicative. The product on $H \otimes H$ is defined by

$$(a \otimes b)(a' \otimes b') := aa' \otimes bb' \; . \tag{2.12}$$

The counit $\hat{\mathbb{I}}$ vanishes on all trees and $\hat{\mathbb{I}}(\mathbb{I}) = 1$. The set

$$\mathrm{Aug} := \bigoplus_{n \geq 1} H_n \tag{2.13}$$

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[2]The tensor product symbol $\otimes$ used here stands for the tensor product with respect to the rationals, i.e. for the symbol $\otimes_{\mathbb{Q}}$.

[3]Beware: this is a sleight abuse of notation.

comprises the kernel of the counit $\hat{\mathbb{I}}$. This makes $\ker \hat{\mathbb{I}} =$Aug it into an ideal named *augmentation ideal.*

In general, the coproduct maps a tree from $\mathcal{T}$ to a linear combination of elements in $H \otimes H$. One way to define the coproduct Δ on H is recursively by virtue of

$$\Delta \circ B_+ = B_+ \otimes \mathbb{I} + (\mathrm{id} \otimes B_+) \circ \Delta. \tag{2.14}$$

This works as follows. Given a tree $T = B_+(X)$, where X is its corresponding forest. Then,

$$\Delta(T) = \Delta \circ B_+(X) = B_+(X) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+) \circ \Delta(X). \tag{2.15}$$

First, we set $\Delta(\mathbb{I}) := \mathbb{I} \otimes \mathbb{I}$, whereby we have defined the coproduct on H_0. Next, we define it on H_1:

$$\Delta(\bullet) = \Delta \circ B_+(\mathbb{I}) = B_+(\mathbb{I}) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+)\Delta(\mathbb{I}) = \bullet \otimes \mathbb{I} + \mathbb{I} \otimes \bullet\,, \tag{2.16}$$

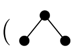
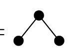
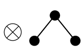
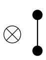
and then on H_2,

(2.17) $\Delta($  $) = \Delta \circ B_+(\bullet) = B_+(\bullet) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+)\Delta(\bullet) =$  $\otimes \mathbb{I} + \bullet \otimes \bullet + \mathbb{I} \otimes$  .

H_2 has another element: the forest $\bullet\,\bullet$. By multiplicativity, this is

$$\Delta(\bullet\,\bullet) = \Delta(\bullet)\Delta(\bullet) = (\bullet \otimes \mathbb{I} + \mathbb{I} \otimes \bullet)(\bullet \otimes \mathbb{I} + \mathbb{I} \otimes \bullet) = \bullet\,\bullet \otimes \mathbb{I} + \mathbb{I} \otimes \bullet\,\bullet + 2\ \bullet \otimes \bullet. \tag{2.18}$$

As an exercise, the reader may verify that

(2.19) $\Delta($  $) =$  $\otimes \mathbb{I} + \mathbb{I} \otimes$  $+ 2\ \bullet \otimes$  $+ \bullet\,\bullet \otimes \bullet.$

The fact that this really does define a genuine coproduct is worth a

Theorem 2.2.1. *The algebra morphism Δ, recursively defined by the identity* (2.14) *is coassociative, i.e. it satisfies*

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta. \tag{2.20}$$

Proof. By induction: starting with H_0 and then walking up the grading from H_n to H_{n+1}. On H_0, the identity holds trivially: both sides yield $\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I}$. The rest is left as an exercise for the reader. Hint: insert $T = B_+(X)$ and use (2.14) as well as coassociativity on both sides. $\square$

A standard notation for the coproduct of an element $\mathbb{I} \neq x \in H$ is

$$\Delta(x) = \mathbb{I} \otimes x + x \otimes \mathbb{I} + \sum_{(x)} x' \otimes x'' = \mathbb{I} \otimes x + x \otimes \mathbb{I} + \widetilde{\Delta}(x), \tag{2.21}$$

where $\widetilde{\Delta}(x)$ is called *reduced coproduct.* This motivates us to define a class of elements with a simple coproduct.

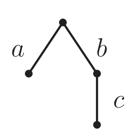
Definition 2.2.2. *An element $x \in H$ is called* primitive*, if $\widetilde{\Delta}(x) = 0$.*

The simplest example is the single root: from (2.16) we have $\widetilde{\Delta}(\bullet) = 0$. Another less trivial example is

(2.22) $x = 2$  $- \bullet\,\bullet$

to be verified by the reader, where use of (2.17) and (2.18) is highly recommended.

Admissible cuts. There is another way to define the coproduct. Let $v \in T^{[0]}$ be the node of a tree T and $P_v \subset T^{[1]}$ the set of paths from v up to the root of T. Then an *admissible cut* $C \subset T^{[1]}$ is a subset of edges such that $|p \cap C| \leq 1$ for all paths $p \in P_v$. That is, any path must cross no more than one edge of C. By $\mathcal{C}(T)$ we denote the set of all admissible cuts of a tree T. Take the tree

(2.23) $T =$ 

with edge set $T^{[1]} = \{a, b, c\}$. Admissible cuts are $C_1 = \{a\}$, $C_2 = \{a, b\}$, $C_3 = \{c\}$ and $C_4 = \{a, c\}$.

Given a cut $C \in \mathcal{C}(T)$, we take all edges $e \in C$ and remove them from T. What we are left with is a forest $P^C(T)$, called the 'pruned' part and $R^C(T)$ the connected component containing the root. For the cut C_2 in our example (2.23) this is

(2.24) $P^{C_2}(T) = \bullet$  $\qquad R^{C_2}(T) = \bullet$

The coproduct of a tree T can then be defined by

$$\Delta(T) = T \otimes \mathbb{I} + \mathbb{I} \otimes T + \sum_{C \in \mathcal{C}(T)} P^C(T) \otimes R^C(T). \tag{2.25}$$

The reader should check to find the same results as in (2.16) to (2.19) by employing this definition. It is now straightforward to show that the coproduct Δ has the grading property

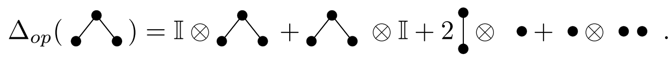
$$\Delta(H_n) \subset \bigoplus_{j=0}^{n} H_j \otimes H_{n-j}. \tag{2.26}$$

For the coproduct, this is easier to be seen in the definition of (2.25), while is it obvious for the multiplication map.

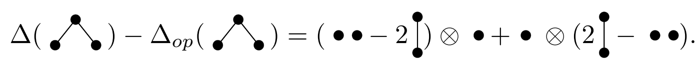
There is an additional property of interest that a coproduct can have: it can be *cocommutative*. This is the case when for all $x \in H$ one finds that $\Delta(x)$ is invariant under the 'flip', a linear map which interchanges the two elements in a tensor product, i.e. $a \otimes b \mapsto b \otimes a$. If we define

$$\Delta_{op} := \text{flip} \circ \Delta \tag{2.27}$$

('opposite') then $\Delta - \Delta_{op}$ vanishes if the coproduct is cocommutative. However, this is not the case for Δ on H:

(2.28) 

Comparing this result with (2.19) shows that Δ is not cocommutative. The difference is

(2.29) 

In this result, there is an interesting observation to made: the elements to the left and the right of the tensor symbol are primitive. This is a first sign of something looming on the horizon which we do not yet understand but will come to later.

Coinverse. What we have so far is the quadruple $(H, m, \mathbb{I}, \Delta, \hat{\mathbb{I}})$ which is all one needs for a bialgebra. To promote this further to a *Hopf algebra*, we need a *coinverse*, also called *antipode*, an algebra antimorphism $S : H \to H$, defined by the property

$$m \circ (S \otimes \text{id}) \circ \Delta = \mathbb{I} \circ \hat{\mathbb{I}} = m \circ (\text{id} \otimes S) \circ \Delta. \tag{2.30}$$

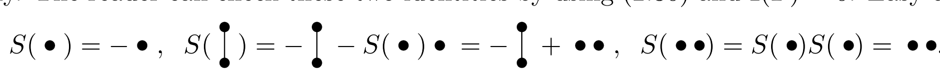
This definition immediately implies $S(\mathbb{I}) = \mathbb{I}$ and a recursion relation

$$S(T) = -T - \sum_{C \in \mathcal{C}(T)} S(P^C(T)) R^C(T) \tag{2.31}$$

for a non-trivial tree T or,

$$S(T) = -T - \sum_{C \in \mathcal{C}(T)} P^C(T) S(R^C(T)). \tag{2.32}$$

alternatively. The reader can check these two identities by using (2.30) and $\hat{\mathbb{I}}(T) = 0$. Easy examples are

(2.33) 

The coinverse can also be defined by

$$S(T) = -T - \sum_{C \subseteq T^{[1]}} (-1)^{|C|} P^C(T) R^C(T), \tag{2.34}$$

where $P^C(T)$ and $R^C(T)$ are as above but this time all possible subsets of edges are permitted. It is easy to check this for the trees in (2.33).

The defining property in (2.30) might strike those unacquainted with it as a bit odd at first sight. But in fact, it is rather natural in the following context. Given two linear maps $f, g : H \to H$ on the Hopf algebra H. Then the *convolution product*

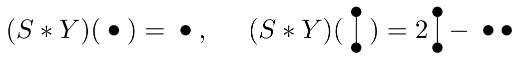
$$f * g := m \circ (f \otimes g) \circ \Delta \tag{2.35}$$

defines another linear map on H. One can show without much effort that $*$ qualifies as a group operation with neutral element $E := \mathbb{I} \circ \hat{\mathbb{I}}$ and inverse $f^{*-1} = f \circ S$ for any linear f on H, where the special role of the coinverse S becomes apparent at this point. In the light of this, (2.30) states that S be the $*$-inverse of the identity map id on H.

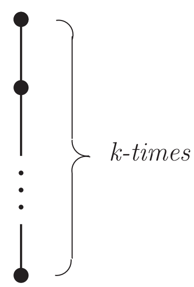
Grading operator. A linear map D on an algebra is called *derivation*, if

$$D(ab) = D(a)b + aD(b) \tag{2.36}$$

for all algebra elements a, b. One such derivation on H is the *grading operator* Y. It is given by $Y(T) := |T|T$ for a tree T, i.e. it simply multiplies a tree by its number of nodes. For a homogeneous $x \in H_n$, one has $Y(x) = |x|x = nx$, hence the name. If we take the convolution product with the coinverse, namely $S * Y$, we find

(2.37) 

It turns out, that $S * Y$ maps all (non-trivial) ladders to primitive elements. Ladders, denoted λ_k, are defined by $\lambda_0 := \mathbb{I}$ and $\lambda_{k+1} := B_+(\lambda_k)$, they take the form

(2.38) $\lambda_k =$ 

and their coproduct is $\Delta(\lambda_k) = \sum_{j=0}^{k} \lambda_j \otimes \lambda_{k-j}$.

Proposition 2.2.2. *$(S * Y)(\lambda_k)$ is primitive for all $k \geq 1$.*

Proof. Exercise. First check (2.37) to get aquainted with this operator. Then use

$$\Delta S = \mathrm{flip}(S \otimes S)\Delta \qquad \text{and} \qquad \Delta Y = (Y \otimes \mathrm{id} + \mathrm{id} \otimes Y)\Delta\ . \tag{2.39}$$

The first identity can be found in any book on Hopf algebras. Y's being a coderivative is a consequence of being a derivation in combination with being its own dual. It is not difficult to prove for ladder trees. The proof is also implicitly given in the appendix (see B.2) $\square$

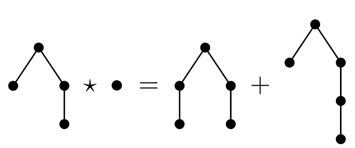
We shall see that understanding the Dynkin operator $D_Y = S * Y$ is already half the battle in understanding the renormalization group! The reader can find more on this interesting operator in appendix section B.2

2.3. Pre-Lie structure on the Hopf algebra of rooted trees

Analogous to what we have seen in section 2.1, we define a pre-Lie product on H. A *leaf* l of a tree T is a vertex of T with no children. We denote the set of all leaves of a tree T by $\mathcal{F}(T)$('foliage'). Let $\tau, \tau' \in \mathcal{T}$ be two trees and $l \in \mathcal{F}(\tau)$ a leaf of τ. Then let $T = \tau \circ_l \tau'$ be the tree which arises upon grafting the root of τ' to the leaf l of τ such that T has one more edge than τ and τ' together. Then

$$\tau \star \tau' := \sum_{l \in \mathcal{F}(\tau)} \tau \circ_l \tau' \tag{2.40}$$

defines a bilinear product on H which is pre-Lie, as we will see. An example is

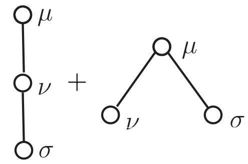
(2.41) 

Let us now adopt a more convenient notation: given two trees $\mu, \nu \in \mathcal{T}$, we use the shorthand

(2.42) 

$:= \mu \star \nu.$

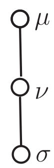
If we now take a third tree σ and right-$\star$-multiply it, we get[4]

(2.43) $$(\mu \star \nu) \star \sigma =$$  .

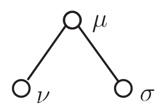
where the first term on the right-hand side stands for the sum over all grafting operations where σ is attached to the leaves of ν(as part of the trees in $\mu \star \nu$) and the second, where σ is grafted to the leaves of μ not parentally connected to ν. If we now shift the brackets and first calculate $\nu \star \sigma$, arriving at

(2.44) $$\nu \star \sigma =$$ 

and then left-multiply this sum by μ, we find

(2.45) $$\mu \star (\nu \star \sigma) =$$  .

Thus, the *associator* is

(2.46) $$X(\mu, \nu, \sigma) := (\mu \star \nu) \star \sigma - \mu \star (\nu \star \sigma) =$$  .

On account of the non-planarity of our rooted trees, we realize that it is symmetric with respect to an interchange between the latter two arguments, i.e. $X(\mu, \nu, \sigma) = X(\mu, \sigma, \nu)$ which means

(2.47) $$(\mu \star \nu) \star \sigma - \mu \star (\nu \star \sigma) = (\mu \star \sigma) \star \nu - \mu \star (\sigma \star \nu)$$

and thus the grafting $\star$-operation is pre-Lie. Question: If we had defined the $\star$-product by only allowing a specific subset of leaves in $\mathcal{F}(\tau)$, would this bilinear operation still be pre-Lie? Would the pre-Lie property be lost if we allowed the root to be a grafting place? We finally point out that

(2.48) $$[\mu, \nu] := \mu \star \nu - \nu \star \mu$$

defines a bilinear map which obeys the Jacobi identity by virtue of being pre-Lie which is easy to check. This makes H into a Lie algebra!

[4] The $\star$-product is linear.

CHAPTER 3

Hopf Algebra Characters and Hochschild Cohomology

3.1. Prologue: A Hopf algebra induced by a Lie algebra

Hopf algebras arise in various contexts, one of which is non-commutative geometry where a Hopf algebra structure was discovered by Connes and Moscovici[**CM98**]. This so-called *Connes-Moscovici Hopf algebra* is intimately related to that discovered by the lecturer Dirk Kreimer himself[**Kr98**]. The connection between the two Hopf algebras is expounded in [**CoKr98**] which also provides the background for this lecture and is recommended as a reference for the eager student.

Connes-Moscovici Hopf algebra. Consider a vector space V over the field $\mathbb{Q}$ generated by the symbols X, Y and a countable collection $\{\delta_n : n \in \mathbb{N}\}$ of symbols. Let $T(V)$ be its tensor algebra. We write the product of two vectors $v, w \in T(V)$ as a juxtaposition vw of which we recall that it is *not commutative* by definition. Next, we additionally introduce a bilinear Lie bracket by

$$[X,Y] = X, \qquad [X,\delta_n] = \delta_{n+1}, \qquad [Y,\delta_n] = n\delta_n, \qquad [\delta_n,\delta_m] = 0 \tag{3.1}$$

for all $n, m \in \mathbb{N}$. If we establish an equivalence relation by identifying $[v,w] \sim vw - wv$ for $v, w \in T(V)$ and take the quotient $T(V)/\sim$, we obtain a *Lie algebra*, also known as *universal enveloping algebra* of V. To make it into a bialgebra, we define a coproduct by

$$\Delta(Y) := Y \otimes 1 + 1 \otimes Y, \qquad \Delta(X) := X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \qquad \Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1, \tag{3.2}$$

where $1 := \delta_0$ is the neutral element of multiplication. If we require

$$\Delta(hh') = \Delta(h)\Delta(h') \tag{3.3}$$

and hence $\Delta([h,h']) = [\Delta(h), \Delta(h')]$, $\Delta(\delta_n)$ is determined by $\delta_n = [X, \delta_{n-1}]$. The reader may verify that

$$\Delta(\delta_2) = \delta_2 \otimes 1 + 1 \otimes \delta_2 + \delta_1 \otimes \delta_1 \tag{3.4}$$

and

$$\Delta(\delta_3) = \delta_3 \otimes 1 + 1 \otimes \delta_3 + 3\delta_1 \otimes \delta_2 + \delta_2 \otimes \delta_1 + \delta_1^2 \otimes \delta_1 \tag{3.5}$$

follow[1]. One can show that Δ indeed is a coprodut on $H_T := T(V)/\sim$. The general form of the coproduct of the generators δ_n is

$$\Delta(\delta_n) = \delta_n \otimes 1 + 1 \otimes \delta_n + R_{n-1}, \tag{3.6}$$

where

$$R_n = n\delta_1 \otimes \delta_n + [\Delta(X), R_{n-1}] \tag{3.7}$$

is defined by this recursive relation. With the existence of an antipode S, which can also be proved, H_T is in fact a *Hopf algebra*. It is an example of a Connes-Moscovici Hopf algebra alluded to above.

What is so interesting about this Hopf algebra? It turns out, as we shall see, to be isomorphic to a *Hopf subalgebra* H_{CM} within our Hopf algebra of rooted trees H!

3.2. Connes-Moscovici Hopf subalgebra

Natural growth. Recall that a derivation is a linear map D on an algebra such that

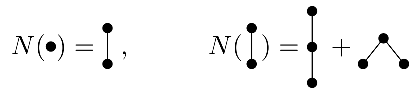
$$D(ab) = D(a)b + aD(b) \tag{3.8}$$

for any elements a and b of the algebra. We consider a linear map $N : H \to H$ defined as follows. For the empty tree, we set $N(\mathbb{I}) = \bullet$, whereas for a non-trivial tree $T \neq \mathbb{I}$ we set

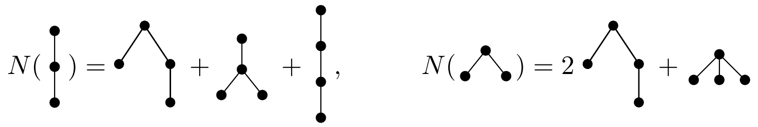
$$N(T) := \sum_{v \in T^{[0]}} T_v, \tag{3.9}$$

[1] $[a \otimes b, a' \otimes b'] = aa' \otimes bb' - a'a \otimes b'b$

where $T_v := T \circ_v \bullet$ is the tree we obtain when we graft a single leaf to the vertex v of T such that $|T_v| = |T| + 1$. To define N on forests, we require it to be a derivation on the augmentation ideal Aug$= \bigoplus_{k\geq 1} H_k$. Note that we exclude $H_0 = \mathbb{Q}\mathbb{I}$ from this since otherwise the derivation property (3.8) would imply $N(\mathbb{I}) = 0$. N is referred to as *natural growth*. Examples for its action are

(3.10) 

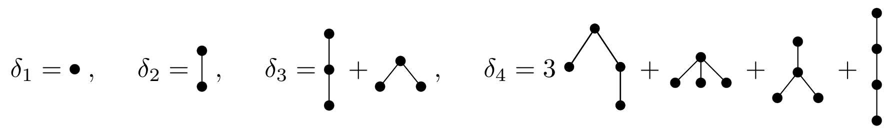
and

(3.11) 

Repeated application of the operator N on the single root yields interesting objects in H:

$$\delta_k := N^k(\mathbb{I}) = \underbrace{N \circ N \circ ... \circ N}_{k-\text{times}}(\mathbb{I}) \tag{3.12}$$

for $k \geq 0$. The first δ_k's read $\delta_0 = \mathbb{I}$,

(3.13) 

and so on. The prefactors before the trees are called *Connes-Moscovici weights*. Notice that it is no accident that we have used the same symbols δ_k as in the previous section.

Proposition 3.2.1. *The elements $\delta_k \in H$ defined in (3.12) generate a Hopf subalgebra, the so-called* Connes-Moscovici Hopf subalgebra $H_{CM} \subset H$. *More precisely, H_{CM} is the free commutative algebra of their $\mathbb{Q}$-linear span,*

$$\langle\{\delta_k : k \geq 0\}\rangle_{\mathbb{Q}}. \tag{3.14}$$

Proof. To prove this assertion, one only has to show that H_{CM} is closed under the coproduct and the antipode, i.e. that

$$\Delta(H_{CM}) \subset H_{CM} \otimes H_{CM}, \qquad S(H_{CM}) \subset H_{CM} \tag{3.15}$$

as all other properties are garanteed by H being a Hopf algebra. First note that by construction, δ_n is a sum of trees, $\delta_n = \sum_j T_j$, where a tree may appear several times in the sum, depending on its Connes-Moscovici weight. We proceed by induction. Assume that $\Delta(\delta_n) \in H_{CM} \otimes H_{CM}$. For $n = 1, 2$ this is trivial. For $n + 1$, we have

$$\Delta(\delta_{n+1}) = \delta_{n+1} \otimes \mathbb{I} + \mathbb{I} \otimes \delta_{n+1} + \sum_j \sum_{C \in \mathcal{C}(T'_j)} P^C(T'_j) \otimes R^C(T'_j), \tag{3.16}$$

where $\delta_{n+1} = \sum_j T'_j$ and linearity of Δ have been used. We now have to find a reason why the admissible cuts in (3.16) do not lead out of H_{CM}. Taking a closer look, we realize that the admissible cuts of the tree T'_j fall into two rough categories: either the new grown leaf is cut off directly above its edge(case 1), or it is not(case 2). In the latter case it is either part of the pruned part P^C(case 2a) or the root component R^C(case 2b). In case 1, the cut off leaf will always appear as a factor δ_1 on the lhs of the tensor product. This case has two subcategories: cuts of just a single edge with just one leaf cut off and nothing else(case 1a) and those cuts where the leaf is cut off along with some other vertices or trees(case 1b). Case 1a results in n equal terms of the form $\delta_1 \otimes \delta_n$. How many terms do we get from the second type, with other parts also cut off? Assume we want to 'carry out' this cut: first we cut off all other parts by removing all the corresponding edges and save the new grown leaf for last. Where can we find it? Answer: there are exactly as many possibilities as there are nodes on the remaining tree. All in all, finally, the reduced part of the coproduct on the rhs of (3.16) can be rewritten with $\delta_n = \sum_l T_l$ in the form

$$n\ \delta_1 \otimes \delta_n \quad + \quad \sum_l \sum_{C \in \mathcal{C}(T_l)} |R^C(T_l)|\ \delta_1\ P^C(T_l) \otimes R^C(T_l) + \sum_l \sum_{C \in \mathcal{C}(T_l)} N(P^C(T_l)) \otimes R^C(T_l)$$
$$+ \quad \sum_l \sum_{C \in \mathcal{C}(T_l)} P^C(T_l) \otimes N(R^C(T_l))$$

in which we cannot spot any element not in H_{CM}. Then closedness under the antipode follows immediately. □

3.3. Characters

Recall that the convolution product $*$ is a bilinear operation on the set of linear maps $H \to H$. More generally, a convolution can be defined for linear maps ψ, ϕ from H to a commutative algebra A with multiplication m_A by

$$\psi * \phi := m_A(\psi \otimes \phi)\Delta \tag{3.17}$$

where we have suppressed the composition sign: a habit we shall frequently allow ourselves to succumb to from now on. We choose $A := \mathbb{C}$ and define *characters* to be algebra homomorphisms $\phi : H_{CM} \to \mathbb{C}$ such that $\phi(\mathbb{I}) = 1$. Then the convolution is naturally given by

$$\phi * \psi = m_{\mathbb{C}}(\phi \otimes \psi)\Delta = (\phi \otimes \psi)\Delta, \tag{3.18}$$

such that for a tree T we have

$$(\phi * \psi)(T) = \phi(T) + \psi(T) + \sum_{C \in \mathcal{C}(T)} \phi(P^C(T))\psi(R^C(T)). \tag{3.19}$$

We will relate these characters to *formal diffeomorphisms*, which are formal power series in $\mathbb{C}[[x]]$ of the form

$$h(x) = x + a_2 x^2 + a_3 x^3 + \ldots\ , \tag{3.20}$$

with $h'(0) = 1$. Because of this latter property they are said to be *tangent to the identity.* Their derivatives

$$h'(x) = 1 + 2a_2 x + 3a_3 x^2 + \ldots \tag{3.21}$$

carry the same amount of information: we do not lose anything by differentiating[2]. Neither do we lose any information if we consider their logarithm

$$\log h'(x) = \log(1 + 2a_2 x + 3a_3 x^2 + \ldots) = 2a_2 x + (3a_3 - 2a_2^2)x^2 + \ldots \tag{3.22}$$

We associate to this so obtained formal power series a character ϕ_h on H_{CM} by setting

$$\phi_h(\delta_k) := \partial_x^k \log h'(x)\,|_{x=0}\ . \tag{3.23}$$

Then we have the interesting identity

$$\phi_{h \circ g} = \phi_h * \phi_g \tag{3.24}$$

which we shan't prove here. The reader is referred to [**CoKr98**] for details. We check this formula for a simple case. Given the two diffeomorphisms

$$h(x) = x + ax^2\ , \qquad g(x) = x + bx^2 \tag{3.25}$$

one has $\log h'(x) = 2ax - 2a^2x^2 + \ldots$ and $\log g'(x) = 2bx - 2b^2x^2 + \ldots$, which translates to

$$\phi_h(\delta_1) = 2a\ , \qquad \phi_h(\delta_2) = -4a^2\ , \qquad \phi_g(\delta_1) = 2b\ , \qquad \phi_g(\delta_2) = -4b^2. \tag{3.26}$$

On the one hand, we get for the composition $h \circ g$

$$\log(h \circ g)'(x) = 2(a+b)x + 2(ab - a^2 - b^2)x^2 + \ldots \tag{3.27}$$

which means $\phi_{h\circ g}(\delta_1) = 2(a+b)$ and $\phi_{h\circ g}(\delta_2) = 4(ab - a^2 - b^2)$ for the corresponding character. On the other,

$$(\phi_h * \phi_g)(\delta_1) = \phi_h(\delta_1)\phi_g(\mathbb{I}) + \phi_h(\mathbb{I})\phi_g(\delta_1) = \phi_h(\delta_1) + \phi_g(\delta_1) = 2a + 2b \tag{3.28}$$

and

$$(\phi_h * \phi_g)(\delta_2) = \phi_h(\delta_2)\phi_g(\mathbb{I}) + \phi_h(\mathbb{I})\phi_g(\delta_2) + \phi_h(\delta_1)\phi_g(\delta_1) = -4a^2 - 4b^2 + 4ab. \tag{3.29}$$

[2] Why is that?

3.4. The Group of Hopf Characters

Let H be a Hopf algebra over $\mathbb{Q}$ with unit $\mathbb{I}$ and a grading $\{H_n\}_{n\geq 0}$ such that $H_0 = \mathbb{Q}\mathbb{I}$. Moreover, let A be a commutative algebra over $\mathbb{Q}$ with multipilcation $m_A : A \otimes A \to A$. Then *Hopf (algebra) characters* are algebra morphisms from H to A which preserve the neutral element of multiplication. We will now see that they form a group with respect to the convolution product. The *convolution* of two characters $\phi, \psi : H \to A$ is given by

$$\phi * \psi = m_A(\phi \otimes \psi)\Delta_H, \tag{3.30}$$

where Δ_H is the coproduct on H. This operation is associative: let ϕ_1, ϕ_2 and ϕ_3 be characters. Then

$$\begin{aligned}
(\phi_1 * \phi_2) * \phi_3 &= m_A((\phi_1 * \phi_2) \otimes \phi_3)\Delta_H = m_A(m_A(\phi_1 \otimes \phi_2)\Delta_H \otimes \phi_3)\Delta_H && (3.31)\\
&= m_A(m_A \otimes \mathrm{id}_A)((\phi_1 \otimes \phi_2) \otimes \phi_3)(\Delta_H \otimes \mathrm{id}_H)\Delta_H && (3.32)\\
&= m_A(\mathrm{id}_A \otimes m_A)(\phi_1 \otimes (\phi_2 \otimes \phi_3))(\mathrm{id}_H \otimes \Delta_H)\Delta_H && (3.33)\\
&= m_A((\phi_1 \otimes m_A(\phi_2 \otimes \phi_3)\Delta_H)\Delta_H = \phi_1 * (\phi_2 * \phi_3). && (3.34)
\end{aligned}$$

In (3.33) we have used the associativity of m_A and the coassociativity of Δ_H. Given the neutral element of multiplication $1_A \in A$, one can define a unit map $\mathbb{I}_A : \mathbb{Q} \to A$ by $\mathbb{I}_A := \phi \circ \mathbb{I}$, where ϕ is any character. Then we have $\mathbb{I}_A(1) = 1_A$. The map $e := \mathbb{I}_A \circ \hat{\mathbb{I}}$ is in fact a character which, as the following calculation will reveal, is the neutral element of the convolution $*$-product: first check that for any character ψ

$$(\psi * e)(\mathbb{I}) = m_A(\psi \otimes e)(\mathbb{I} \otimes \mathbb{I}) = 1_A = \psi(\mathbb{I}) = ... = (e * \psi)(\mathbb{I}) \tag{3.35}$$

and for $\mathbb{I} \neq x \in H$

$$(\psi * e)(x) = \psi(x) + e(x) + \sum_{(x)} \psi(x')e(x'') = \psi(x), \tag{3.36}$$

where we have used $e(\mathrm{Aug}) = 0$. The same goes for $e * \psi$. If we take the antipode S of our Hopf algebra and define $\overline{\phi} := \phi \circ S$ for a character ϕ, we see that

$$\begin{aligned}
\phi * \overline{\phi} &= m_A(\phi \otimes \phi \circ S)\Delta_H = m_A(\phi \otimes \phi)(\mathrm{id}_H \otimes S)\Delta_H = \phi \circ m_H(\mathrm{id}_H \otimes S)\Delta_H && (3.37)\\
&= \phi \circ \mathbb{I} \circ \hat{\mathbb{I}} = \mathbb{I}_A \circ \hat{\mathbb{I}} = e && (3.38)
\end{aligned}$$

and also $\overline{\phi} * \phi = e$. Note that we have used the defining property of the antipode(see section 2.2) and the multiplicativity of characters

$$m_A(\phi \otimes \phi) = \phi \circ m_H \tag{3.39}$$

where m_H is the product on H. However, the upshot is that $\overline{\phi} = \phi \circ S$ is the *inverse* of ϕ with respect to the convolution product. It is left to the reader to verify that this multiplicativity property is preserved by the convolution. Given all this, we have proven the following

Proposition 3.4.1. *The set of characters G_A^H from H to A, i.e. unity-preserving algebra morphisms, forms a group with respect to the convolution product.*

3.5. Coradical Filtration

A *filtration* of an algebra A is a sequence of subspaces

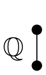
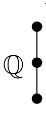
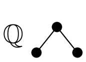
$$A^0 \subset A^1 \subset A^2 \subset ... \tag{3.40}$$

such that $A = \bigcup_{n\geq 0} A^n$ and $m_A(A^n \otimes A^m) \subset A^{n+m}$, where m_A is the multiplication on A. Note that we are not talking about subalgebras of A, but subspaces. For a bialgebra one has the additional requirement

$$\Delta(A^n) = \sum_{i+j=n} A^i \otimes A^j. \tag{3.41}$$

for the coproduct.

Let now again H be our Hopf algebra of rooted trees. We can use the grading in H to construct a filtration: $H^n := \bigoplus_{j=0}^n H_j$ is a filtration for H, where we have introduced H_k in section 2.2 as the subspace of $\mathbb{Q}$-linear combinations of forests in H with k nodes. The first subspaces in the grading are

(3.42) $H_0 = \mathbb{Q}\mathbb{I}$, $H_1 = \mathbb{Q}\bullet$, $H_2 \simeq \mathbb{Q}\bullet\bullet \oplus \mathbb{Q}$  , $H_3 \simeq \mathbb{Q}\bullet\bullet\bullet \oplus \mathbb{Q}$  $\oplus\, \mathbb{Q}\bullet$  $\oplus\, \mathbb{Q}$  , and so on[3]. There is another filtration which is key to us: the so-called *coradical filtration*. To define it, we need to introduce some more maps on H. First a projector $P : H \to \mathrm{Aug}$ onto the augmentation ideal. It is characterized by linearity, $P(\mathbb{I}) = 0$ and $P = P^2$. Note that also $P = \mathrm{id} - E$ with $E = \mathbb{I} \circ \hat{\mathbb{I}}$. Furthermore, we define a family of maps

$$\Delta^0 := \mathrm{id}\ , \quad \Delta^1 := \Delta\ , \quad \Delta^2 := (\Delta \otimes \mathrm{id})\Delta\ , \quad \Delta^{k+1} := (\Delta \otimes id^{\otimes k})\Delta^k \tag{3.43}$$

we may also write as

$$\Delta^k = (\Delta \otimes \mathrm{id}^{\otimes k-1})(\Delta \otimes \mathrm{id}^{\otimes k-2})...(\Delta \otimes \mathrm{id})\Delta \tag{3.44}$$

which makes explicit that this is tantamount to repeatedly applying the coproduct to the first element of the resulting tensor product. Now, finally, we consider an additional family of maps

$$\Delta_0 := P\ , \quad \Delta_1 := (P \otimes P)\Delta^1\ , \quad \Delta_k := P^{\otimes k+1}\Delta^k. \tag{3.45}$$

Loosely speaking, what these maps essentially do is this: Δ^k creates a sum of $(k+1)$-fold tensor products from a single element $x \in H$. Then $P^{\otimes k+1}$ annihilates all those terms in the sum that have at least one empty tree $\mathbb{I}$ in their tensor product. For example, we do not need to apply the coproduct to create an empty tree for $\mathbb{I} \in H$, hence $\Delta_0(\mathbb{I}) = 0$. For primitive elements x we have $\Delta_1(x) = 0$, since we must apply the coproduct at least once to get an empty tree in every term: $\Delta(x) = \mathbb{I} \otimes x + x \otimes \mathbb{I}$.

Definition 3.5.1. *The* coradical filtration $H^0 \subset H^1 \subset H^2 \subset ...$ *of the Hopf algebra of rooted trees* H *is given by*

$$H^n := \{x \in H \mid \Delta_n(x) = 0\ \} \tag{3.46}$$

How does this relate to the grading? Let us consider some more examples.

$$\Delta_0(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}) = \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\ , \qquad \Delta_1(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}) = (P \otimes P)\Delta(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}) = (P \otimes P)(\mathbb{I} \otimes \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} + \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} \otimes \mathbb{I} + \bullet \otimes \bullet) = \bullet \otimes \bullet. \tag{3.47}$$

The two-node ladder is in H^2, as the next calculation shows:

$$\begin{aligned}\Delta_2(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}) &= (P \otimes P \otimes P)(\Delta \otimes \mathrm{id})\Delta(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}) = (P \otimes P \otimes P)(\Delta \otimes \mathrm{id})(\mathbb{I} \otimes \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} + \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} \otimes \mathbb{I} + \bullet \otimes \bullet) \\ &= (P \otimes P \otimes P)(\mathbb{I} \otimes \mathbb{I} \otimes \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} + \Delta(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}) \otimes \mathbb{I} + \bullet \otimes \mathbb{I} \otimes \bullet + \mathbb{I} \otimes \bullet \otimes \bullet) = 0.\end{aligned}$$

The reader may check that

$$\Delta_1(\bullet\bullet) = 2\bullet \otimes \bullet\ , \quad \Delta_2(\bullet\bullet) = 0\ , \quad \Delta_1(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} - \frac{1}{2}\bullet\bullet) = 0\ , \tag{3.48}$$

which means that a linear combination of two elements in H^n may actually also be in the subset $H^{n-1} \subset H^n$. The *coradical degree* of an element $y \in H$ is defined as the number

$$\mathrm{cor}(y) := \min\{n \mid y \in H^n\}. \tag{3.49}$$

The relation between the elements of the coradical filtration H^n and those of the grading H_n is this: if $y \in H_n$, then $\mathrm{cor}(y) \leq n$. This can be made explicit by virtue of the coproduct's grading property:

$$\Delta(H_n) \subset \sum_{j+l=n} H_j \otimes H_l \tag{3.50}$$

from which

$$\Delta^k(H_n) \subset \sum_{j_1+...+j_{k+1}=n} H_{j_1} \otimes ... \otimes H_{j_{k+1}} \tag{3.51}$$

follows. How can you avoid $j_l = 0$ for at least one l in the sum if $k \geq n$? You cannot. $\square$

Proposition 3.5.1. *The growth operator* B_+ *increases the coradical degree.*

Proof. This follows from the identity

$$\Delta B_+ = B_+ \otimes \mathbb{I} + (\mathrm{id} \otimes B_+)\Delta, \tag{3.52}$$

or, more concretely for an element $x \in H$

$$\Delta B_+(x) = B_+(x) \otimes \mathbb{I} + \mathbb{I} \otimes B_+(x) + x \otimes B_+(\mathbb{I}) + \sum_{(x)} x' \otimes B_+(x''). \tag{3.53}$$

[3] Why choose these 'isomorphic to' symbols?

Let $\mathrm{cor}(x) = n$. The reader may check that by virtue of (3.53) one finds

$$\Delta_n(B_+(x)) = \Delta_{n-1}(x) \otimes B_+(\mathbb{I}) + \sum_{(x)} \Delta_{n-1}(x') \otimes B_+(x'') \tag{3.54}$$

which does not vanish because $\Delta_{n-1}(x) \neq 0$ by assumption. Hint: first prove

$$\Delta^n = (\Delta^{n-1} \otimes \mathrm{id})\Delta, \tag{3.55}$$

(inductively) and then use it. □

Coradical degree and Feynman rules. As we shall see later, the coradical degree of an element in $h \in H$ sets an upper bound to the degree of the polynomial that physics assigns to the sum of graphs represented by h. Feynman rules can for example be given by a character $\phi_L : H \to A$, where $A = \mathbb{C}[L]$ is a polynomial algebra. For the element h this yields something of the form

$$\phi_L(h) = \sum_{j=1}^{\mathrm{cor}(h)} c_j(h) L^j. \tag{3.56}$$

It will then be the renormalization group to dictate how the coefficients c_j are related to each other. Examples are

(3.57) $\phi_L($  $) = c_1 L + c_2 L^2$, $\qquad \phi_L(\bullet\bullet) = \phi_L(\bullet)^2 = (d_1 L)^2 = d_1^2 L^2$

and

(3.58) $\phi_L($  $- \frac{1}{2}\bullet\bullet) = c_1 L$.

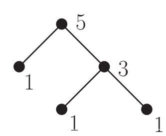
3.6. Tree factorials

Let T be a rooted tree and $v \in T^{[0]}$ one of its vertices. If $e_v \in T^{[1]}$ is the adjacent edge just above this vertex, the cut $C = \{e_v\} \in \mathcal{C}(T)$ yields a tree $T(v) := P^C(T)$ with v as its root. The number $\#T(v) = |T(v)|$ is called the *weight* of the vertex v. Then we have the following

Definition 3.6.1. *The* tree factorial $T!$ *of a tree* T *is given by* $T! := \prod_{v \in T^{[0]}} |T(v)|$. *For the empty tree we set* $\mathbb{I}! := 1$.

If we label the two trees

(3.59) $\lambda_4 =$ 

, $\quad T =$ 

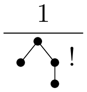
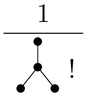
,

with their vertex weights, their tree factorials are $\lambda_4! = 4!$ and $T! = 1 \cdot 1 \cdot 1 \cdot 3 \cdot 5 = 15$. For ladder trees, i.e. trees without sidebranchings, $\lambda_k! = k!$ is obvious.

Let $l \in \mathcal{F}(T) \subset T^{[0]}$ be a leaf, i.e. a (childless) vertex with weight one. By T/l we denote the tree T with l removed. An interesting identity is

$$\frac{|T|}{T!} = \sum_{l \in \mathcal{F}(T)} \frac{1}{(T/l)!} \ . \tag{3.60}$$

For our two examples, this can be easily checked:

(3.61) $\dfrac{|\lambda_4|}{\lambda_4!} = \dfrac{4}{4!} = \dfrac{1}{3!} = \dfrac{1}{\lambda_3!}$, $\qquad \dfrac{|T|}{T!} = \dfrac{5}{15} = \dfrac{2}{8} + \dfrac{1}{12} = 2\dfrac{1}{(\;\;)!}$  $+ \dfrac{1}{(\;\;)!}$  .

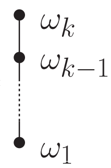
For a proof see [**Kr99**] or [**Lued**].

3.7. Iterated Integrals

Let $[a,b] \subset \mathbb{R}$ be an interval. Consider a collection of differential one-forms $\omega_j(x) = f_j(x)dx$, $j = 1,2,3,...$ on the real line $\mathbb{R}$. A family of so-called *iterated integrals* can be defined by

$$F_0(a;b) := 1 \ , \qquad F_n(a;\omega_1,...,\omega_n;b) := \int_a^b f_n(x)F_{n-1}(a;\omega_1,...,\omega_{n-1};x)dx \qquad n \geq 1. \tag{3.62}$$

We associate these to decorated ladder trees

(3.63) $\lambda_k =$  ,

where the *decoration* is a map D assigning a differential one-form to each vertex. Generally, the target set of a decoration can be anything, whatever is of interest. The integrals in (3.62) comprise 'nested' integrations. Note that the leaf of λ_k corresponds to the innermost integration, i.e. that of the one-form ω_1, whereas the outermost integration has the one-form ω_k as kernel.

Setting $a = 1$ and $b = x \geq 1$, we can choose the same one-form for all nested integrations and get for $\omega(x) = dx/x$ integrals of the form

$$F_k(1;\omega,...,\omega;x) = \int_1^x \frac{dy_k}{y_k} \int_1^{y_k} \frac{dy_{k-1}}{y_{k-1}} \ ... \int_1^{y_2} \frac{dy_1}{y_1} \ , \tag{3.64}$$

where this is typical physics notation, coming in handy here: the outermost integration is represented by the leftmost integration measure dy_k/y_k, with everything else depending on y_k to the right of it. The innermost integration is on the rightmost position. We invite the reader to verify by induction that

$$F_k(1;\omega,...,\omega;x) = \frac{(\ln x)^k}{k!} = \frac{(\ln x)^{|\lambda_k|}}{\lambda_k!}. \tag{3.65}$$

The identity

$$F_n(1;\omega,...,\omega;x) = \sum_{j=0}^{n} F_j(1;\omega,...,\omega;x')F_{n-j}(x';\omega,...,\omega;x) \tag{3.66}$$

for any $x' \in [1,x]$ is an instance of *Chen's lemma* about iterated integrals. For example, for $n = 2$ we have,

$$F_2(1;\omega,\omega;x) = \frac{1}{2}\ln^2 x \tag{3.67}$$

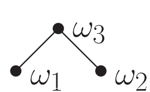
on the lhs and

$$\begin{aligned} \sum_{j=0}^{2} F_j(...)F_{2-j}(...) \quad &= \quad F_0(1;x')F_2(x';\omega,\omega;x) + F_1(1;\omega;x')F_1(x';\omega;x) + F_2(1;\omega,\omega;x')F_0(x';x) \\ &= \quad 1 \cdot \frac{1}{2}\ln^2(x/x') + \ln x' \ln(x/x') + \frac{1}{2}\ln^2 x' \cdot 1 \end{aligned}$$

on the rhs of (3.66). Are these results the same? Yes, they are: in contrast to what seems obvious, the rhs does not depend on x'.

Tree-terated Integrals. We now generalize this game to all types of rooted trees, this time including sidebranchings. The corresponding integrals are sometimes (more or less) jokingly referred to as *tree-terated integrals*.

The rules for a tree T are actually quite simple. Any vertex $v \in T^{[0]}$ corresponds to an integration with a measure induced by its decoration one-form ω_v and the nestedness of integrations is determined by the kinship relations of the vertices amongst each other: the root $r \in T^{[0]}$, being ancestor to any other vertex in T, corresponds to the outermost integration with one-form ω_r, while the children of a vertex v are represented by disjoint integrations nested inside the integral with ω_v. For example, we decorate a simple three-vertex tree

(3.68) $T =$ 

with one-forms

$$\omega_j(x) = f_j(x)dx,\ j = 1, 2, 3 \tag{3.69}$$

and translate this to

$$F_T(a; \omega_1, \omega_2, \omega_3; b) = \int_a^b \omega_3(x) \int_a^x \omega_1 \int_a^x \omega_2 = \int_a^b dx\ f_3(x) \int_a^x dx' f_1(x') \int_a^x dx'' f_2(x'')\ , \tag{3.70}$$

where we have in one integral suppressed integration variables when there is no necessity for them to appear. To define a tree-terated integral for a general tree, let T be a decorated rooted tree with root r and decoration D and

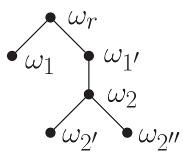
$$D(\tau) := \{\omega_v : v \in \tau^{[0]}\} \tag{3.71}$$

the set of the 'decor' one-forms associated to a subtree $\tau \subseteq T$(including $\tau = T$). Then there is a decorated forest $B_-(T)$ with its multiset of trees $\pi_0(B_-(T))$ which we obtain when we jettison all edges adjacent to the root r. The tree-terated integral we associate to T is then given by

$$F_T(a; D(T); b) := \int_a^b \omega_r(x) \prod_{\tau \in \pi_0(B_-(T))} F_\tau(a; D(\tau); x). \tag{3.72}$$

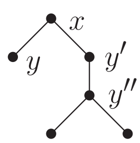
Here is one more example to make this definition clear:

(3.73)



$$= \int_a^b \omega_r(x) \int_a^x \omega_1(y) \int_a^x \omega_{1'}(y') \int_a^{y'} \omega_2(y'') \int_a^{y''} \omega_{2'} \int_a^{y''} \omega_{2''}\ .$$

It is helpful to additionally label the vertices with the integration variables of the associated integration:



,

where it is not necessary to mention the leaves' integration variables; for vertex 1 we have done it for clarity's sake, though. Here is how the kinship relations determine the integrations: $\int \omega_{2'}$ and $\int \omega_{2''}$ are disjoint integrations being nested inside the integration involving ω_2, as the vertices $2'$ and $2''$ are children of vertex 2, which is itself child of vertex $1'$ and therefore subject to the integration with $\omega_{1'}$ and so on.

With this more general definition, we can come back to our simple iterated integrals we constructed by iterating integrations of the one-form $\omega(x) = dx/x$ and ask whether there is a generalization of (3.65). In fact, we come full circle with the following

Proposition 3.7.1. *For trees decorated uniformly with the one-form $\omega(x) = dx/x$, the tree-terated integrals defined through (3.72) evaluate for $a = 1$ and $b = x \geq 1$ to*

$$F_T(1; \omega, ..., \omega; x) = \frac{(\ln x)^{|T|}}{T!}\ . \tag{3.74}$$

Proof. . Inductively all the way through the tree sets $\mathcal{T}_n = \{t \in \mathcal{T} : |t| = n\}$. First the reader may check that

$$\int_1^x \frac{dy}{y} (\ln y)^n = \frac{(\ln x)^{n+1}}{n+1}. \tag{3.75}$$

Getting the induction started on $\mathcal{T}_1$ is trivial. Try also $\mathcal{T}_2$ and $\mathcal{T}_3$ to get familiar with these integrals. Assume now the assertion holds on $\mathcal{T}_n$ and choose $T \in \mathcal{T}_{n+1}$, i.e. $|T| = n + 1$. Then, by definition we have

$$F_T(1; \omega, ..., \omega; x) = \int_1^x \frac{dy}{y} \prod_{\tau \in \pi_0(B_-(T))} \frac{1}{\tau!} (\ln y)^{|\tau|} \tag{3.76}$$

because, by assumption, the formula holds for a tree $\tau \in \bigcup_{j \leq n} \mathcal{T}_j$. The reader should ponder over

$$T! = |T| \prod_{\tau \in \pi_0(B_-(T))} \tau! \tag{3.77}$$

and write down neatly the complete proof as an exercise. $\square$

Chen's lemma in (3.66) suggests there might be some coproduct-like operation behind the scenes. This is indeed the case: if we use Sweedler's notation for the coproduct of a tree T,

$$\Delta(T) = \sum_j T'_j \otimes T''_j, \tag{3.78}$$

where the coproduct acknowledges the decoration, the more general version of (3.66) for the corresponding tree-terated integrals is

$$F_T(a; D(T); b) = \sum_j F_{T'_j}(a; D(T'_j); \zeta) F_{T''_j}(\zeta; D(T''_j); b) \tag{3.79}$$

with $a \le \zeta \le b$.

The coproduct *acknowlegdes* the decoration? Yes, for the tree in (3.68) this takes the form

$$\Delta\Big({}_{\bullet\omega_1}\overset{\bullet\omega_3}{\wedge}{}_{\bullet\omega_2}\Big) = {}_{\bullet\omega_1}\overset{\bullet\omega_3}{\wedge}{}_{\bullet\omega_2} \otimes \mathbb{I} + \mathbb{I} \otimes {}_{\bullet\omega_1}\overset{\bullet\omega_3}{\wedge}{}_{\bullet\omega_2} + \bullet\omega_1 \otimes \overset{\bullet\omega_3}{\underset{\bullet\omega_2}{|}} + \bullet\omega_2 \otimes \overset{\bullet\omega_3}{\underset{\bullet\omega_1}{|}} + \bullet\omega_1 \bullet\omega_2 \otimes \bullet\omega_3\,, \tag{3.80}$$

where the difference to the coproduct of this tree's undecorated cousin is that the third and fourth term are acknowledged to be unequal.

3.8. What is Hochschild cohomology?

The standard answer to this question is: Hochschild cohomology is the dual of a Lie algebra cohomology. For the latter, the coboundary operator d in this cohomology acts according to

$$d[a, b] = [da, b] + [a, db] \tag{3.81}$$

on the Lie bracket for Lie algebra elements a, b. If we take the universal enveloping algebra of this Lie algebra, a one-cocycle is a linear operator D such that for the product ab one has

$$D(ab) = D(a)\hat{\mathbb{I}}(b) + aD(b), \tag{3.82}$$

where $\hat{\mathbb{I}}$ is the counit, i.e. $\hat{\mathbb{I}}(b)$ is a scalar. The reader may verify that (3.82) is equivalent to

$$\Delta D' = D' \otimes \mathbb{I} + (id \otimes D')\Delta \tag{3.83}$$

for the dual operator D'. To work this out, consider, for example, the coproduct Δ. If we denote the dual of an element a by $\langle a, \cdot \rangle$, then $\langle a, m(b \otimes c)\rangle = \langle \Delta(a), b \otimes c\rangle$ expresses the dual relation between product and coproduct. Also, one should note that the counit $\hat{\mathbb{I}}$ is defined as the dual to the unit map $\mathbb{I}$. A nice reference for the dual relationship between Hopf and Lie algebras is the doctoral thesis [**Foi02**] (written in French).

Hochschild Cohomology of H. Regardless of the dual relation to a Lie algebra, we shall now define the Hochschild cohomology of our Hopf algebra of rooted trees H. Consider a cochain complex of the $\mathbb{Q}$-vector spaces $\mathrm{Hom}(H, H^{\otimes n})$ of linear maps $L : H \to H^{\otimes n}$, $n \in \mathbb{N}$ and $H^{\otimes 0} := \mathbb{Q}$. The vectors in $\mathrm{Hom}(H, H^{\otimes n})$ are referred to as n-cochains. By virtue of the coproduct, we define a map

$$\Delta_{(j)} : H^{\otimes n} \to H^{\otimes n+1}\ , \qquad \Delta_{(j)} := \mathrm{id}^{\otimes j-1} \otimes \Delta \otimes \mathrm{id}^{\otimes n-j} \tag{3.84}$$

which applies the coproduct to the j-th slot. Next, consider the linear operator

$$b : \mathrm{Hom}(H, H^{\otimes n}) \to \mathrm{Hom}(H, H^{\otimes n+1}) \tag{3.85}$$

defined by

$$bL := (\mathrm{id} \otimes L)\Delta + \sum_{j=1}^{n} (-1)^j \Delta_{(j)} L + (-1)^{n+1} L \otimes \mathbb{I} \tag{3.86}$$

for all $n \in \mathbb{N}$. To avoid confusion, $L \otimes \mathbb{I}$ is to be understood as the map

$$H \ni x \mapsto L(x) \otimes \mathbb{I} \in H^{\otimes n} \otimes H. \tag{3.87}$$

To also clarify the compositions of the form $\Delta_{(j)}L$, let now $L : H \to H \otimes H$ be a 2-cochain. For $a \in H$, the image under L takes in general the form

$$L(a) = \sum_j a'_j \otimes a''_j, \tag{3.88}$$

where the sum is assumed to be finite. Then the two examples

$$\Delta_{(1)}L(a) = \Delta_{(1)}(\sum_j a'_j \otimes a''_j) = \sum_j \Delta(a'_j) \otimes a''_j \tag{3.89}$$

and

$$\Delta_{(2)}L(a) = \Delta_{(2)}(\sum_j a'_j \otimes a''_j) = \sum_j a'_j \otimes \Delta(a''_j) \tag{3.90}$$

illustrate the action of these compositions. Note that the images on the rhs are all in $H^{\otimes 3}$.

It turns out that b as defined in (3.86) is a *coboundary operator* of the cochain complex, that is, it has the property $b \circ b = 0$. However, instead of going through the proof, which is more tedious than illuminating, we note that the pair

$$(\mathrm{Hom}(H, H^{\otimes \cdot}), b) \tag{3.91}$$

is a cochain complex. Cochains $L : H \to H^{\otimes n}$ with $bL = 0$ are called *closed* or n-cocycles. They form a subspace which we denote by $C^n(H)$. Within this space there are obviously those elements $L \in C^n(H)$ that vanish under the coboundary operator b because there is an $(n-1)$-cochain ϕ such that $L = b\phi$, i.e. $bL = 0$ simply on the grounds that $b \circ b = 0$. These so-called *exact* n-cocycles, or more common, *coboundaries*, establish yet another subspace $B^n(H) \subset C^n(H)$. Coboundaries are also referred to as *trivial* cocyles. They carry this name because they get degraded to zero maps in the quotient spaces

$$\mathcal{B}_H^{(n)} := C^n(H)/B^n(H) \ . \tag{3.92}$$

which constitute the *Hochschild cohomology* of H. Its n-th element $\mathcal{B}_H^{(n)}$ is called *n-th Hochschild cohomology.*

One-cocycles $L \in C^1(H)$ are characterized by the identity

$$\Delta L = (id \otimes L)\Delta + L \otimes \mathbb{I} \tag{3.93}$$

because $bL = (id \otimes L)\Delta - \Delta L + L \otimes \mathbb{I} = 0$, which is (3.86) for $n = 1$ since $\Delta_{(1)} = \Delta$. A prominent example of a non-trivial one-cocycle is the growth operator B_+ which satisfies this identity by definition if one defines the coproduct recursively by it as we have done in section 2.2. How can we tell B_+ is not trivial? This is because for $\alpha \in \mathrm{Hom}(H, \mathbb{Q})$ one has the coboundary

$$b\alpha = (id \otimes \alpha)\Delta - \alpha \otimes \mathbb{I} \tag{3.94}$$

by the above definition in (3.86) and consequently[4] $b\alpha(\mathbb{I}) = 0$. And this is certainly not the growth operators's behaviour which we recall to be $B_+(\mathbb{I}) = \bullet$.

3.9. Universal Property of connected commutative Hopf algebras

We now come to a very important result concerning the Hochschild cohomology of H which might at first glance seem abstract and of little practical use. However, as shall become apparent as soon as we consider an example and even more so as the lecture series progresses, it provides the mathematical underpinning for the *Hopf-algebraic structure of renormalization.*

This result holds more generally for *connected* commutative Hopf algebras, where a connected Hopf algebra is by definition equipped with a grading $H = \bigoplus_{j \geq 0} H_j$ starting with $H_0 \simeq \mathbb{Q}\mathbb{I}$.

Theorem 3.9.1 (Universal Property). *Let $B_+ \in \mathcal{B}_H^{(1)}$ be a non-trivial one-cocyle of a connected commutative Hopf algebra H. Then the pair (H, B_+) is unique up to Hopf algebra isomorphisms and universal among all such pairs $(\widetilde{H}, L)$. In other words, given any connected commutative Hopf algebra $\widetilde{H}$ and $L \in \mathcal{B}_{\widetilde{H}}^{(1)}$, then there exists a unique Hopf algebra isomorphism $\rho : H \to \widetilde{H}$ such that*

$$\rho \circ B_+ = L \circ \rho, \tag{3.95}$$

or, in terms of a commutative diagram, such that

$$\begin{array}{ccc} H & \xrightarrow{\rho} & \widetilde{H} \\ {\scriptstyle B_+}\big\downarrow & & \big\downarrow{\scriptstyle L} \\ H & \xrightarrow{\rho} & \widetilde{H} \ . \end{array} \tag{3.96}$$

commutes.

[4] Note that by $\alpha \otimes \mathbb{I}$ we mean the map $x \mapsto \alpha(x) \otimes \mathbb{I}$ from H to $\mathbb{Q} \otimes H \simeq H$.

PROOF. By induction up along the grading $H = \bigoplus_{j\geq 0} H_j$. The proof is not difficult and is strongly recommended to the reader as an exercise. The complete proof can be looked up in [**CoKr98**]. Before trying, the reader may have a look at the special case of H being the Hopf algebra of rooted trees, treated in [**Panz12**]. □

Algebra of formal integrals. Let now again H be our Hopf algebra of rooted trees. As it is connected and commutative, it has the universal property. We consider an application displaying some typical features of the problem of renormalization in QFT and take a first look at its underlying Hopf algebra structure. In section 3.7 we have encountered integrals with respect to differential forms like

$$\int_a^b \omega = \int_a^b \frac{dx}{x} \tag{3.97}$$

which we assigned to decorated rooted trees. Obviously, the case of uniformly decorated trees can be easily reduced to that of undecorated trees. However, we shall for now concern ourselves with ill-defined integrals of the sort in (3.97) where $b \to \infty$, that is, integrals like

$$\int_a^\infty \frac{dx}{x}\ . \tag{3.98}$$

This integral ill-defined on account of its logarithmic divergence, as is diverges as fast as $\ln(b)$ for $b \to \infty$. We let $a > 0$ to avoid yet another source of trouble. To still deal with (3.98) in a mathematically sound way, we view it as the formal pair $(\int_a^\infty, \omega)$, where the symbol $\int_a^\infty$ is seen as an element of the *Betti cohomology* (whatever that may be) and $\omega(x) = dx/x$ as an element of the *de Rham cohomology.*

We identify a formal pair $(\int_X, \omega)$ with the integral $\int_X \omega$ if it is well-defined as an integral of the differential form ω integrated over some interval $X \subset \mathbb{R}$ with $\inf X > 0$. These pairs form an algebra where the corresponding operations on this algebra are given as follows: the sum of two formal pairs is the analogue of the sum of the two corresponding integrals. The multiplication of two formal pairs is

$$(\int_X, \omega(x))(\int_Y, \omega(y)) = (\int_X \int_Y, \omega(x)\omega(y)) \tag{3.99}$$

where this evaluates to the product of two independent integrals $\int_X \omega(x)$ and $\int_Y \omega(y)$ in case they are well-defined. This algebra structure is in fact sufficient for what we are concerned with.

Corollary 3.9.2. *Suppose in the set-up of Theorem 3.9.1 that the target set $\widetilde{H}$ is a commutative algebra and $L : \widetilde{H} \to \widetilde{H}$ a linear operator. Then there exists a unique algebra isomorphism $\rho_L : H \to \widetilde{H}$ for L such that*

$$\rho_L \circ B_+ = L \circ \rho_L. \tag{3.100}$$

PROOF. Obvious from the proof of Theorem 3.9.1. □

We may therefore let $\widetilde{H}$ be the algebra of formal pairs. Recall that an algebra isomorphism like ρ_L is called *(Hopf) character.* For $a \in \mathbb{R}$ we consider a character ϕ_a from H to our formal pairs such that

$$\phi_a(\mathbb{I}) = (\emptyset, 1)\ , \qquad \phi_a \circ B_+(\mathbb{I}) = (\int_a^\infty, \omega), \tag{3.101}$$

where $(\emptyset, 1)$ is the neutral element of the multiplication we defined in (3.99). One can show that ϕ_a is the unique algebra morphism ρ_L for the linear operator L given by the identity

$$(\phi_a \circ B_+)(T) = (\int_a^\infty, \omega(x)\phi_x(T)), \tag{3.102}$$

for a tree T, where the rhs contains the formal integral of the form $\phi_x(T) = (\int_x^\infty, \omega_T)$ with integral kernel $\omega(x) = dx/x$ and ω_T the differential form associated to the forest of T. More explicitly, L acts on a formal pair as

$$L(\int_x^\infty, \omega_T)(a) = (\int_a^\infty \int_x^\infty, \omega(x)\omega_T(x)), \tag{3.103}$$

which corresponds to a formal integral operator on formal pairs. Note that the pairs are viewed as depending on an external parameter.

Renormalization. We aim to turn these formal pairs into convergent integrals, a procedure known as *renormalization*. Consider the antipode S and recall that it yields

$$S(B_+(\mathbb{I})) = -B_+(\mathbb{I}) \tag{3.104}$$

as $B_+(\mathbb{I}) = \bullet$ is the root. Let R be the evaluation map such that $R\phi_a = \phi_b$, that is, R changes the lower 'integration' bound from a to $b > 0$. We then consider

$$R\phi_a(S(B_+(\mathbb{I})) = -\phi_b(B_+(\mathbb{I})) = -(\int_b^\infty, \omega) \tag{3.105}$$

and add this to $\phi_a \circ B_+(\mathbb{I})$ to find

$$\phi_a(B_+(\mathbb{I})) + R\phi_a(S(B_+(\mathbb{I})) = (\int_a^\infty, \omega) + (-\int_b^\infty, \omega) = (\int_a^\infty - \int_b^\infty, \omega) = (\int_a^b, \omega) \tag{3.106}$$

which can be identified with the well-defined integral $\int_a^b \omega = \int_a^b dx/x = \ln(b/a)$. We define another character $S_R^\phi : H \to \widetilde{H}$ by the identity

$$S_R^\phi(h) = -R[S_R^\phi * (\phi_a \circ P)](h)\ , \qquad h \in \mathrm{Aug} \tag{3.107}$$

where P is the projection onto the augmentation ideal and $*$ is the convolution product as it has been already defined for characters from H to an algebra like $\widetilde{H}$. Then generally, the character

$$\phi_{a,R} := S_R^\phi * \phi_a \tag{3.108}$$

yields the well-defined integral. The simplest example of this is (3.106): first compute

$$\begin{aligned} S_R^\phi(\bullet) &= -R[S_R^\phi * (\phi_a \circ P)](\bullet) = -R[S_R^\phi(\mathbb{I})\phi_a(\bullet)] = -R[(\emptyset, 1)\phi_a(\bullet)] && (3.109)\\ &= -\phi_b(\bullet) = -(\int_b^\infty, \omega), && (3.110) \end{aligned}$$

where we recall that $\bullet \in H$ is primitive, i.e.

$$\Delta(\bullet) = \bullet \otimes \mathbb{I} + \mathbb{I} \otimes \bullet. \tag{3.111}$$

Then, finally, we evaluate (3.108):

$$\begin{aligned} \phi_{a,R}(\bullet) &= (S_R^\phi * \phi_a)(\bullet) = S_R^\phi(\mathbb{I})\phi_a(\bullet) + S_R^\phi(\bullet)\phi_a(\mathbb{I}) \\ &= (\emptyset, 1)(\int_a^\infty, \omega) + (-\int_b^\infty, \omega)(\emptyset, 1) = (\int_a^b, \omega)\ . \end{aligned}$$

As this is a well-defined integral, we identify

$$(\int_a^b, \omega) = \int_a^b \omega = \ln(b/a). \tag{3.112}$$

In summary, we have found the renormalized value $\phi_{a,R}(\bullet) = \ln(b/a)$ of the prior to renormalization ill-defined integral $\phi_a(\bullet) = \int_a^\infty \omega$. The parameter b corresponds to what is known in 'real-world' quantum field theories as *renormalization point*. It is the parameter associated to the *renormalization scheme* which corresponds to the map R.

CHAPTER 4

Hopf-Algebraic Renormalization

4.1. Rota-Baxter operator and characters

Let $(H, m_H, \mathbb{I}, \Delta_H, \hat{\mathbb{I}}, S)$ be a connected, commutative Hopf algebra and V an algebra equipped with an associative, commutative product m_V and a unit $1_V \in V$. We write the product of elements $v, w \in V$ simply as a juxtaposition vw. A linear map $R : V \to V$ is said to be a *Rota-Baxter operator* if

$$R[ab] + R[a]R[b] = R[R[a]b + aR[b]] \tag{4.1}$$

for all $a, b \in V$. One can interprete the rhs of this equation as a measure of how much this map deviates from anti-multiplicativity: if R was anti-multiplicative, the rhs would vanish. Next, we consider characters from H to V. Because V is an algebra, we can define a convolution product for characters $\phi, \psi : H \to V$ as usual by

$$\phi * \psi = m_V(\phi \otimes \psi)\Delta_H. \tag{4.2}$$

As we have already mentioned before, (4.2) defines a group law on the group of characters G_V^H. For a character $\phi \in G_V^H$ we define a linear map $S_R^\phi : H \to V$ by setting $S_R^\phi(\mathbb{I}) := 1_V$ and

$$S_R^\phi(h) = -R[(S_R^\phi * \phi P)(h)] \ , \tag{4.3}$$

for $h \in \mathrm{Aug}$. The map $P = \mathrm{id}_H - \mathbb{I} \circ \hat{\mathbb{I}} : H \to \mathrm{Aug}$ is a projector and $\phi P = \phi \circ P$ a shorthand notation. Uniqueness of S_R^ϕ is ensured by the coproduct's grading property

$$\Delta_H(H_n) \subset \bigoplus_{k+l=n} H_k \otimes H_l \tag{4.4}$$

from which follows that the identity in (4.3) defines S_R^ϕ recursively on H. One can check that by virtue of R having the Rota-Baxter property in (4.1), the map S_R^ϕ is multiplicative, i.e.

$$S_R^\phi(xy) = S_R^\phi(x) S_R^\phi(y) \tag{4.5}$$

which qualifies it to be a member of the character group G_V^H. We will refer to this character henceforth as the *counterterm*. The symbol S_R^ϕ has been chosen because the antipode S satisfies

$$S(h) = -(S * P)(h) \ , \qquad \forall h \in \mathrm{Aug} \tag{4.6}$$

where the convolution $*$ is that in H, i.e. $S * P = m_H(S \otimes P)\Delta_H$. (4.6) can be easily derived from the antipode's defining property $S * \mathrm{id}_H = \mathbb{I} \circ \hat{\mathbb{I}}$. The character $\overline{\phi}$, given by

$$S_R^\phi * \phi = S_R^\phi + S_R^\phi * \phi P =: S_R^\phi + \overline{\phi} \ . \tag{4.7}$$

is known as *Bogoliubov map* in physics. Another character, the map $\phi_R := S_R^\phi * \phi$ corresponds to what goes under the name *renormalized Feynman rules*. It can also be written as the result of a subtraction procedure

$$\phi_R(x) = (S_R^\phi * \phi)(x) = (\mathrm{id}_V - R)\overline{\phi}(x) \qquad x \in \mathrm{Aug}, \tag{4.8}$$

following straightforwardly from (4.3) and (4.7).

4.2. Feynman rules as a character

Feynman rules, as they arise in perturbative QFT, assign parameter-dependent integrals to Feynman graphs. However, this correspondence came about actually right the other way round: physicists doing perturbative calculations encountered complicated and nested integrals for which they devised mnemotechnically very convenient pictorial representations to help them organize their computations. The rules for drawing these graphical representations were later named after their inventor, Richard

Feynman. Mathematically, we view these rules as *characters from the Hopf algebra of Feynman graphs to an appropriate target algebra.*

Regularizing Integrals. Since Feynman graphs are in a one-to-one correspondence with Feynman graph-decorated rooted trees, we shall study Feynman rules characters on this Hopf algebra first. The Feynman integrals one encounters in a typical QFT are mostly ill-defined as they are divergent. For example, the integral

$$I = \int_0^\infty \frac{1}{1+x}\, dx \tag{4.9}$$

is logarithmically divergent. A possible first countermeasure is to *regularize* it. By inserting a convergence factor x^{-z} the modified integral

$$I(z) = \int_0^\infty \frac{x^{-z}}{1+x}\, dx \tag{4.10}$$

does indeed converge for certain $z \in \mathbb{C} \setminus \{0\}$. The function $I(z)$ has a Laurent series around $z = 0$ in which the pole term is of particular interest. Dropping it and taking the limit $z \to 0$ is one possible way to extract relevant information from the formerly ill-defined integral expression in (4.9).

Regularized Feynman rules. We consider a toy QFT model that has the same essential features as a real-world QFT. Let c be a 1PI primitive Feynman graph with loop number $|c|$ and B_+^c a Hochschild one-cocyle in the Hopf algebra H_D of decorated rooted trees. B_+^c takes a decorated forest X to the tree T obtained by grafting all trees of X to the single node decorated by c, which becomes T's root. Then we introduce our *regularized Feynman rules* for a toy model QFT in dimension $D = 2$ by the identity

$$\phi(B_+(X))\{q^2/\mu^2; z\} = \mu^{2z} \int d^D y\, \frac{f_c(|y|)}{y^2+q^2} \left(\frac{y^2}{\mu^2}\right)^{-(\frac{|c|}{2}-1)z} \phi(X)\{y^2/\mu^2; z\}\ , \tag{4.11}$$

which is an instance of the operator identity we have seen in the universality theorem. The linear operator on the target algebra corresponds to the integral operator on the rhs of (4.11). If we denote this integral operator by $\mathcal{L}_c$, (4.11) takes the form

$$(\phi \circ B_+^c)(X) = \mathcal{L}_c(\phi(X))\ . \tag{4.12}$$

The ingredients of all this are the following. First note that $q, y \in \mathbb{R}^2$ and $D = 2 - 2z$, where the integration measure is defined as

$$d^D y := |y|^{D-1}\Omega_{D-1}\ d|y| = |y|^{1-2z}\Omega_{1-2z}\ d|y| \tag{4.13}$$

obtained by modifying the dimension parameter D in the usual Lebesgue measure in two dimensions in terms of spherical coordinates by a complex number z with $|z| \ll 1$. Defining the spherical part Ω_{D-1} causes no trouble because the integrand will always be angle-independent and we can evaluate the angular part to give a well-defined expression

$$\int \Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)} = \frac{2\pi^{1-z}}{\Gamma(1-z)}\ , \tag{4.14}$$

for $D = 2 - 2z$. The real parameter $\mu > 0$ is kept fixed for the moment. The function f_c is supposed to be real-valued and approach a constant as $|y| \to \infty$. For simplicity, we let it be constant from the start, say f_c. We choose a shorthand notation for the integral kernel in (4.11) by defining the measure

$$dM_z^c(y,q) := \frac{2\pi^{1-z}}{\Gamma(1-z)} \frac{\mu^{2z} f_c}{y^2+q^2}\ |y|^{1-2z}\ d|y| \tag{4.15}$$

on $\mathbb{R}^2$. Then (4.11) can be recast in the form

$$(\phi \circ B_+^c)(X)\{q^2/\mu^2; z\} = \int dM_z^c(y,q)\ \left(\frac{y^2}{\mu^2}\right)^{-(\frac{|c|}{2}-1)z} \phi(X)\{y^2/\mu^2; z\}\ . \tag{4.16}$$

As strange as the factor μ^{2z} in (4.15) may seem, a simple substitution $y \to y/\mu = \xi$ makes clear that all integrals essentially depend only on the ratio q^2/μ^2.

We will be able to associate Laurent series around $z = 0$ in $\mathbb{C}$ to these so obtained integrals, just as one can do with the integral in (4.10). Therefore, the target algebra V consists of Laurent series in z. The coefficients of these series are smooth real-valued functions of the parameter $q^2/\mu^2 > 0$. It is important to note that these Laurent series have no essential singularity, only poles of finite order. Because the coefficients are in $\mathcal{C}^\infty(\mathbb{R}^+)$, we denote the target algebra by

$$\mathcal{C}^\infty(\mathbb{R}^+)[z^{-1}, z]] \ . \tag{4.17}$$

However, it will turn out that by the upcoming theorem the coefficients are polynomials in $L = \ln(q^2/\mu^2)$, i.e. we can be more precise by writing

$$V = \mathbb{C}[L][z^{-1}, z]] \ . \tag{4.18}$$

Analytic continuation. Consider the identity

$$\int \frac{(y^2)^{-u}}{y^2+q^2} d^D y = \frac{\pi^{D/2}}{\Gamma(D/2)} \Gamma(D/2-u)\Gamma(1+u-D/2) \ (q^2)^{D/2-u-1} \tag{4.19}$$

which the reader may verify[1]. In cases where the lhs does not converge but the rhs has an analytic continuation, we define the lhs integral by this analytic continuation of the rhs(a typical strategy applied in physics to not be hindered by such petty hurdles). This way we associate a Laurent series to

$$\phi(\bullet_{c})\{q^2/\mu^2; z\} = (\phi \circ B_+^c)(\mathbb{I}) = \int dM_z^c(y,q) \ \left(\frac{y^2}{\mu^2}\right)^{-(\frac{|c|}{2}-1)z} \phi(\mathbb{I})\{q^2/\mu^2; z\} \ , \tag{4.20}$$

i.e. the Feynman rules for the simple decorated root $\bullet_{c} = B_+^c(\mathbb{I})$. In terms of the introduced measure, (4.19) yields

$$\int dM_z^c(y,q) \ \left(\frac{y^2}{\mu^2}\right)^{-\alpha z} = f_c \ \frac{\pi^{1-z}}{\Gamma(1-z)} \ \Gamma(1-(1+\alpha)z) \ \Gamma((1+\alpha)z) \ \left(\frac{q^2}{\mu^2}\right)^{-(1+\alpha)z} \tag{4.21}$$

with $\alpha := |c|/2 - 1$. Note that by ϕ being a character, one has

$$\phi(\mathbb{I})\{q^2/\mu^2; z\} = 1 \ , \tag{4.22}$$

i.e. the 'Laurent series' $1_V = 1$. Let now c be a one-loop graph, i.e. $|c| = 1$ and set $f_c = 1$. Then (4.21) yields

$$\phi(\bullet_{c})\{q^2/\mu^2; z\} = \frac{\pi^{1-z}}{\Gamma(1-z)} \ \Gamma(1-z/2) \ \Gamma(z/2) \ \left(\frac{q^2}{\mu^2}\right)^{-z/2} =: f_\bullet(z) \ \left(\frac{q^2}{\mu^2}\right)^{-z/2} \tag{4.23}$$

where everything except $\Gamma(z/2)$ behaves benignly. To find the associated Laurent series we use

$$\Gamma(1+z) = \exp\left(-\gamma_E z + \textstyle\sum_{k\geq 2}(-1)^k \zeta(k) \frac{z^k}{k}\right) = 1 - \gamma_E z + \frac{1}{2}\left(\zeta(2) + \gamma_E^2\right) z^2 + \mathcal{O}(z^3) \tag{4.24}$$

for the gamma function. The constant γ_E is defined by the peculiar limit

$$\gamma_E := \lim_{N\to\infty} \left(\sum_{k=1}^{N} \frac{1}{k} - \ln N\right) \ , \tag{4.25}$$

and known as *Euler-Mascheroni constant*, a sort of 'renormalized' Riemann zeta value at 1, where $\zeta(1) = \sum_{k=1}^\infty k^{-1} = \infty$. Employing these formulas we find that[2]

$$\Gamma(z/2) = \frac{2}{z} \ \Gamma(1+z/2) = \frac{2}{z} - \gamma_E + \frac{1}{24}(\pi^2 + 6\gamma_E^2)z + \mathcal{O}(z^2) \tag{4.26}$$

obviously has a pole of first order. All together, the first few terms of the Laurent series of the function $f_\bullet(z)$ are

$$f_\bullet(z) = \frac{2\pi}{z} - 2\pi(\gamma_E + \ln\pi) + \frac{\pi}{12}(12[\ln\pi + \gamma_E]^2 - \pi^2)z + \mathcal{O}(z^2) \ . \tag{4.27}$$

Then, using $L = \ln(q^2/\mu^2)$ to rewrite $(q^2/\mu^2)^{-z/2} = \exp(-Lz/2)$ we get

$$\begin{aligned} \phi(\bullet_{c})\{q^2/\mu^2; z\} &= \frac{2\pi}{z} - \pi(L + 2\gamma_E + 2\ln\pi) \\ &+ \frac{\pi}{12}(3L^2 + 12[\ln\pi + \gamma_E]L + 12[\ln\pi+\gamma_E]^2 - \pi^2)z + \mathcal{O}(z^2) \ . \end{aligned} \tag{4.28}$$

---

[1] Hint:Look up the various representations of the Betafunction $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and use (4.14).

[2] $\zeta(2) = \pi^2/6$

To calculate the counterterm $S_R^\phi = -R\overline{\phi}$ we have to consider the Bogoliubov map $\overline{\phi}$. First recall that $\Delta(\bullet_c) = \bullet_c \otimes \mathbb{I} + \mathbb{I} \otimes \bullet_c$, as well as

$$S_R^\phi(\mathbb{I})\{q^2/\mu^2; z\} = 1 \qquad \text{and} \qquad (\phi P)(\bullet_c)\{q^2/\mu^2; z\} = \phi(\bullet_c)\{q^2/\mu^2; z\} \; . \tag{4.29}$$

Then we compute

$$\overline{\phi}(\bullet_c)\{q^2/\mu^2; z\} = (S_R^\phi * \phi P)(\bullet_c)\{q^2/\mu^2; z\} = \phi(\bullet_c)\{q^2/\mu^2; z\} \; , \tag{4.30}$$

in which the term $\bullet_c \otimes \mathbb{I}$ gets killed by the projector P.

For the Rota-Baxter operator R we choose the evaluation map setting $q^2/\mu^2 = 1$ in all coefficients of the Laurent series: given $\psi \in V$, applying R yields

$$R\psi\{q^2/\mu^2; z\} = \psi\{1; z\} \; , \tag{4.31}$$

which sets $L = 0$. This map is obviously Rota-Baxter, if we quickly revisit (4.1). Then, consequently, the counterterm is given by

$$\begin{aligned} S_R^\phi(\bullet_c)\{q^2/\mu^2; z\} &= -R[\overline{\phi}(\bullet_c)\{q^2/\mu^2; z\}] = -\overline{\phi}(\bullet_c)\{1; z\} = -f_\bullet(z) \\ &= -\frac{2\pi}{z} + 2\pi(\gamma_E + \ln\pi) - \frac{\pi}{12}(12[\ln\pi + \gamma_E]^2 - \pi^2)z + \mathcal{O}(z^2) \; . \end{aligned}$$

For the renormalized character $\phi_R = S_R^\phi * \phi$ we obtain

$$\begin{aligned} \phi_R(\bullet_c)\{q^2/\mu^2; z\} &= (S_R^\phi * \phi)(\bullet_c)\{q^2/\mu^2; z\} = S_R^\phi(\bullet_c)\{q^2/\mu^2; z\} + \phi(\bullet_c)\{q^2/\mu^2; z\} \\ &= -\phi(\bullet_c)\{1; z\} + \phi(\bullet_c)\{q^2/\mu^2; z\} \\ &= -\pi L + \frac{\pi}{4}(L^2 + 4[\ln\pi + \gamma_E]L)z + \mathcal{O}(z^2) \; , \end{aligned} \tag{4.32}$$

where everything in (4.28) not dependent on L has been subtracted. What is important to note at this point is that the *pole term has dropped out, rendering a pole-free renormalized character* ϕ_R.

Physical limit and locality. The limit

$$\phi_R(\bullet_c)\{q^2/\mu^2\} := \lim_{z\to 0} \phi_R(\bullet_c)\{q^2/\mu^2; z\} = -\pi L = -\pi \ln(q^2/\mu^2) \tag{4.33}$$

is known as the *physical limit* of the renormalized Feynman rules. For a real-world QFT (we are in a toy model, remember[3]), this would in principle be an observable quantity!

Pole terms that depend on the parameter $L = \ln(q^2/\mu^2)$ are called *non-local poles*. They may appear in the intermediate steps during a calculation but must drop out along the way so as to make sure the Bogoliubov map is purged of such poles. The trouble is, if the Bogoliubov map still contains non-local poles, they will not be cancelled by the Rota-Baxter subtraction: these terms vanish upon applying R and cannot be subtracted out! The physical limit would then not exist. A Bogoliubov map free of this pathology is said to be *local*.

Theorem 4.2.1. *Let $T \in H_D$ be a decorated rooted tree and ϕ the Feynman rules as given in (4.11). Assume that both the Bogoliubov map $\overline{\phi} = S_R^\phi * \phi P$ and the renormalized character $\phi_R = S_R^\phi * \phi$ have Laurent series around $z = 0$ with polynomials in $L = \ln(q^2/\mu^2)$ as coefficients such that the limits*

$$\lim_{z\to 0} \frac{\partial}{\partial L} \overline{\phi}(T)\{q^2/\mu^2; z\} \qquad \textit{('locality')} \tag{4.34}$$

and

$$\phi_R(T)\{q^2/\mu^2\} = \lim_{z\to 0} (S_R^\phi * \phi)(T)\{q^2/\mu^2; z\} \qquad \textit{('physical limit')} \tag{4.35}$$

exist, the latter being a polynomial in L. Then, if these assumptions hold for the tree $T = \bullet_c$ with any decoration c, they hold for all trees $T \in H_D$.

Proof. Inductively with respect to the grading. Let $T = B_+^c(X)$ be a tree and $X = \prod_k T_k$ a forest for whose trees T_k the assumptions hold. We abbreviate $\Delta(X) = \sum X' \otimes X''$ and consider the Bogoliubov map

$$\begin{aligned} \overline{\phi}(T) &= (S_R^\phi * \phi P)(T) = (S_R^\phi * \phi P)B_+^c(X) \\ &= m_V(S_R^\phi \otimes \phi P)(B_+^c(X) \otimes \mathbb{I} + (id \otimes B_+^c)\Delta(X)) = \sum S_R^\phi(X')\phi(B_+^c(X'')) \\ &= \sum S_R^\phi(X')\mathcal{L}_c(\phi(X'')) = \sum \mathcal{L}_c(S_R^\phi(X')\phi(X'')) = \mathcal{L}_c((S_R^\phi * \phi)(X)) = \mathcal{L}_c(\phi_R(X)) \; . \end{aligned}$$

[3]Real-World QFTs like the Standard Model do yield results of this form, though.

The counterterm never depends on anything other than the parameter z and can therefore be drawn under the integral. By assumption, the renormalized character ϕ_R for X has a Laurent series is of the form

$$\phi_R(X)\{y^2/\mu^2; z\} = \prod_k \phi_R(T_k)\{y^2/\mu^2; z\} = \sum_{j=0}^{\infty} u_j(\ln(y^2/\mu^2)) z^j = \sum_{j=0}^{\infty} u_j(\tilde{L})\ z^j \tag{4.36}$$

with polynomials $u_j(\tilde{L})$ in $\tilde{L} = \ln(y^2/\mu^2)$. Then, $\overline{\phi}(T) = \mathcal{L}_c(\phi_R(X))$ is

$$\overline{\phi}(T)\{q^2/\mu^2; z\} = \sum_{j=0}^{\infty} \int dM_z^c(y, q)\ \left(\frac{y^2}{\mu^2}\right)^{-\alpha z} u_j(\ln(y^2/\mu^2))\ z^j\ , \tag{4.37}$$

with $\alpha = |c|/2 - 1$. Plugging in $u_j(\tilde{L}) = \sum_{a=1}^{r_j} u_{j,a}\tilde{L}^a$ this turns into[4]

$$\overline{\phi}(T)\{q^2/\mu^2; z\} = \sum_{j=0}^{\infty}\sum_{a=1}^{r_j} u_{j,a} \int dM_z^c(y, q)\ \left(\frac{y^2}{\mu^2}\right)^{-\alpha z} (\ln(y^2/\mu^2))^a\ z^j\ , \tag{4.38}$$

where the integrals bring in pole terms of finite order. Taking the derivative in (4.38) with respect to $L = \ln(q^2/\mu^2)$, which acts only on the integral kernel and increases the polynomial degree in the denominator,

$$\frac{\partial}{\partial L}\overline{\phi}(T)\{q^2/\mu^2; z\} = q^2\frac{\partial}{\partial q^2}\overline{\phi}(T)\{q^2/\mu^2; z\} \tag{4.39}$$

yields convergent integrals also for $z = 0$. This means all poles are local! Therefore, the principle part of $\overline{\phi}(T)$, i.e. the poles, cannot depend on $L = \ln(q^2/\mu^2)$. This entails that

$$R\overline{\phi}(T)\{q^2/\mu^2; z\} = \overline{\phi}(T)\{1; z\} \tag{4.40}$$

still contains all pole terms. Upon substraction

$$\phi_R(T)\{q^2/\mu^2; z\} = \overline{\phi}(T)\{q^2/\mu^2; z\} - \overline{\phi}(T)\{1; z\} \tag{4.41}$$

the poles are *bound to drop out* and hence $\phi_R(T)\{q^2/\mu^2; z\}$ is pole-free.

To prove the assertion that the physical limit $\phi_R(T)\{q^2/\mu^2; z\}$ is a polynomial in L, it suffices to show that the Laurent series of $\overline{\phi}(T)\{q^2/\mu^2; z\}$ has polynomial coefficients with variable L. We inspect one of the integrals in (4.38): let us pick the integral of the coefficient $u_{j,a}$ and substitute $y = \mu\xi$ to get

$$\int dM_z^c(\mu\xi, q/\mu)\ (\xi^2)^{-\alpha z}(\ln \xi^2)^a = \int d|\xi|\ |\xi|^{1-2z}\frac{2f_c\pi^{1-z}}{\Gamma(1-z)}\frac{(\xi^2)^{-\alpha z}}{\xi^2 + (q^2/\mu^2)}(\ln \xi^2)^a\ .$$

We set $s := \sqrt{q^2/\mu^2}$ and rescale again $\xi = s\chi$ to find

$$\int dM_z^c(\mu\xi, q/\mu)\ (\xi^2)^{-\alpha z}(\ln \xi^2)^a = (s^2)^{-(\alpha+1)z}\int d|\chi|\ |\chi|^{1-2z}\frac{2f_c\pi^{1-z}}{\Gamma(1-z)}\frac{(\chi^2)^{-\alpha z}}{\chi^2 + 1}(\ln \chi^2 + \ln s^2)^a\ .$$

In the light of the exponential series

$$(s^2)^{-(1+\alpha)z} = \exp(-(1+\alpha)z\ln s^2) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(1+\alpha)^k(\ln s^2)^k z^k \tag{4.42}$$

one can see clearly that this integral is a Laurent series with polynomials in $L = \ln s^2$ as coefficients. It follows that this also holds for $\overline{\phi}(T)\{q^2/\mu^2; z\}$. $\square$

We study two more examples to see all this explicitly. Let for simplicity all nodes be uniformly decorated by c. Then we can suppress c in the notation. As before, we let $|c| = 1$, i.e. $\alpha = -1/2$ and $f_c = 1$. The renormalized value assigned to the tree $\overset{\bullet}{\underset{\bullet}{|}} = B_+(\bullet)$ is given by

$$\begin{aligned}\phi_R(\overset{\bullet}{\underset{\bullet}{|}}) &= (S_R^{\phi} * \phi)(\overset{\bullet}{\underset{\bullet}{|}}) = S_R^{\phi}(\overset{\bullet}{\underset{\bullet}{|}})\phi(\mathbb{I}) + S_R^{\phi}(\mathbb{I})\phi(\overset{\bullet}{\underset{\bullet}{|}}) + S_R^{\phi}(\bullet)\phi(\bullet)\\ &= S_R^{\phi}(\overset{\bullet}{\underset{\bullet}{|}}) + \phi(\overset{\bullet}{\underset{\bullet}{|}}) + S_R^{\phi}(\bullet)\phi(\bullet) = -R[\overline{\phi}(\overset{\bullet}{\underset{\bullet}{|}})] + \overline{\phi}(\overset{\bullet}{\underset{\bullet}{|}}) = (\mathrm{id}_V - R)[\overline{\phi}(\overset{\bullet}{\underset{\bullet}{|}})]\ .\end{aligned} \tag{4.43}$$

[4]Why does it start with $a = 1$?

We will tackle these characters one after the other. First we compute

$$
\begin{aligned}
\phi(\overset{\bullet}{\underset{\bullet}{|}})\{q^2/\mu^2;z\} &= \int dM_z^c(y,q)\ \left(\frac{y^2}{\mu^2}\right)^{z/2}\phi(\bullet)\{y^2/\mu^2;z\} = f_\bullet(z)\int dM_z^c(y,q) \\
&= f_\bullet(z)\frac{\pi^{1-z}}{\Gamma(1-z)}\Gamma(1-z)\Gamma(z)\left(\frac{q^2}{\mu^2}\right)^{-z} =: f_{\overset{\bullet}{\underset{\bullet}{|}}}(z)\left(\frac{q^2}{\mu^2}\right)^{-z} \\
&= f_{\overset{\bullet}{\underset{\bullet}{|}}}(z)e^{-Lz}\ .
\end{aligned}
\tag{4.44}
$$

This function has a first and a second order pole term,

$$
f_{\overset{\bullet}{\underset{\bullet}{|}}}(z) = \frac{2\pi^2}{z^2} - \frac{4\pi^2(\gamma_E+\ln\pi)}{z} + \frac{\pi^2}{12}(\pi^2+48[\gamma_E+\ln\pi]^2) + \mathcal{O}(z)\ ,
$$

which leads to a non-local pole term in[5]

$$
\begin{aligned}
\phi(\overset{\bullet}{\underset{\bullet}{|}})\{q^2/\mu^2;z\} \quad = \quad & \frac{2\pi^2}{z^2} - \frac{2\pi^2(L+2[\gamma_E+\ln\pi])}{z} \\
& + \frac{\pi^2}{12}(12L^2+48(\gamma_E+\ln\pi)L+\pi^2+48[\gamma_E+\ln\pi]^2) + \mathcal{O}(z)\ ,
\end{aligned}
$$

The Bogoliubov map includes a subtraction

$$
\begin{aligned}
\overline{\phi}(\overset{\bullet}{\underset{\bullet}{|}}) &= \phi(\overset{\bullet}{\underset{\bullet}{|}})\{q^2/\mu^2;z\} - R[\phi(\bullet)\{q^2/\mu^2;z\}]\phi(\bullet)\{q^2/\mu^2;z\} \\
&= \phi(\overset{\bullet}{\underset{\bullet}{|}})\{q^2/\mu^2;z\} - \phi(\bullet)\{1;z\}\phi(\bullet)\{q^2/\mu^2;z\} \\
&= f_{\overset{\bullet}{\underset{\bullet}{|}}}(z)\left(\frac{q^2}{\mu^2}\right)^{-z} - f_\bullet(z)f_\bullet(z)\left(\frac{q^2}{\mu^2}\right)^{-z/2} \\
&= f_{\overset{\bullet}{\underset{\bullet}{|}}}(z)e^{-zL} - f_\bullet(z)f_\bullet(z)e^{-Lz/2} \\
&= -\frac{2\pi^2}{z^2} - \frac{4\pi^2[\gamma_E+\ln\pi])}{z} + \frac{\pi^2}{12}(6L^2+5\pi^2-48[\gamma_E+\ln\pi]^2) + \mathcal{O}(z)\ .
\end{aligned}
\tag{4.45}
$$

The subtraction has cancelled the non-local pole. To see how this comes about, we have a look at the pole terms of the series

$$
f_\bullet(z)f_\bullet(z)e^{-zL/2} \quad = \quad \frac{4\pi^2}{z^2} - \frac{2\pi^2(L+4[\gamma_E+\ln\pi])}{z} + \frac{\pi^2}{2}(L^2+8[\ln\pi+\gamma_E]L) + \mathcal{O}(1)\ ,
$$

which makes explicit why the non-local pole has dropped out. One more subtraction

$$
\phi_R(\overset{\bullet}{\underset{\bullet}{|}})\{q^2/\mu^2;z\} = (\mathrm{id}_V - R)[\overline{\phi}(\overset{\bullet}{\underset{\bullet}{|}})\{q^2/\mu^2;z\}] = \frac{\pi^2}{2}L^2 + \mathcal{O}(z)
\tag{4.46}
$$

yields the pole-free renormalized value whose physical limit exists and is a polynomial in L:

$$
\phi_R(\overset{\bullet}{\underset{\bullet}{|}})\{q^2/\mu^2\} = \frac{\pi^2}{2}L^2 = \frac{\pi^2}{2}\ (\ln(q^2/\mu^2))^2\ .
\tag{4.47}
$$

Next, we briefly discuss the tree ${}_\bullet\!\overset{\bullet}{\wedge}\!{}_\bullet = B_+(\bullet\,\bullet)$. The character ϕ yields

$$
\phi({}_\bullet\!\overset{\bullet}{\wedge}\!{}_\bullet) = (\phi\circ B_+)(\bullet\,\bullet) = \mathcal{L}_c(\phi(\bullet\,\bullet)) = \mathcal{L}_c(\phi(\bullet)\phi(\bullet))\ ,
\tag{4.48}
$$

which amounts to

$$
\begin{aligned}
\phi({}_\bullet\!\overset{\bullet}{\wedge}\!{}_\bullet)\{q^2/\mu^2;z\} \quad &= \quad \int dM_z^c(y,q)\left(\frac{y^2}{\mu^2}\right)^{z/2}(f_\bullet(z))^2\left(\frac{y^2}{\mu^2}\right)^{-z} \\
&= \quad (f_\bullet(z))^2\int dM_z^c(y,q)\left(\frac{y^2}{\mu^2}\right)^{-z/2}\ .
\end{aligned}
$$

Employing (4.21), this is

$$
\begin{aligned}
\phi({}_\bullet\!\overset{\bullet}{\wedge}\!{}_\bullet)\{q^2/\mu^2;z\} \quad &= \quad (f_\bullet(z))^2\frac{\pi^{1-z}}{\Gamma(1-z)}\Gamma(1-3z/2)\Gamma(3z/2)\left(\frac{q^2}{\mu^2}\right)^{-3z/2} \\
&=: \quad f_{{}_\bullet\!\overset{\bullet}{\wedge}\!{}_\bullet}(z)\left(\frac{q^2}{\mu^2}\right)^{-3z/2}\ .
\end{aligned}
$$

---

[5]Here is the point to start using some computer algebra software!

The Bogoliubov map is

$$\begin{aligned}\overline{\phi}(\Lambda) &= (S_R^\phi * \phi P)(\Lambda) = \phi(\Lambda) + S_R^\phi(\bullet\bullet)\phi(\bullet) + 2\ S_R^\phi(\bullet)\phi(\overset{\bullet}{\underset{\bullet}{\vert}}) \\ &= \phi(\Lambda) + S_R^\phi(\bullet)^2\phi(\bullet) + 2\ S_R^\phi(\bullet)\phi(\overset{\bullet}{\underset{\bullet}{\vert}})\ ,\end{aligned}$$

where we remember that the coproduct yields

$$\Delta(\Lambda) = \Lambda \otimes \mathbb{I} + \mathbb{I} \otimes \Lambda + \bullet\bullet \otimes \bullet + 2\bullet \otimes \overset{\bullet}{\underset{\bullet}{\vert}}\ . \tag{4.49}$$

Then, remembering $S_R^\phi(\bullet)\{q^2/\mu^2; z\} = -\phi(\bullet)\{1; z\} = -f_\bullet(z)$, we have

$$\overline{\phi}(\Lambda) = f_\Lambda(z)\left(\frac{q^2}{\mu^2}\right)^{-3z/2} + f_\bullet(z)^3\left(\frac{q^2}{\mu^2}\right)^{-z/2} - 2f_\bullet(z)f_{\overset{\bullet}{\underset{\bullet}{\vert}}}(z)\left(\frac{q^2}{\mu^2}\right)^{-z}\ .$$

Finally, to obtain the renormalized character, we subject it to the Rota-Baxter subtraction

$$\begin{aligned}\phi_R(\Lambda)\{q^2/\mu^2; z\} &= (\mathrm{id}_V - R)\overline{\phi}(\Lambda) \\ &= f_\Lambda(z)(e^{-3zL/2} - 1) + f_\bullet(z)^3(e^{-zL/2} - 1) - 2f_\bullet(z)f_{\overset{\bullet}{\underset{\bullet}{\vert}}}(z)(e^{-zL} - 1)\end{aligned}$$

with $L = \ln(q^2/\mu^2)$. Its Laurent series has no pole

$$\phi_R(\Lambda)\{q^2/\mu^2; z\} = -\frac{\pi^3}{3}(L^3 + \pi^2 L) + \mathcal{O}(z) \tag{4.50}$$

telling us that the Bogolibov map was local, and the physical limit clearly is a polynomial in L .

Generally, for this simple model, in which every node is decorated with the one-loop graph c, one has for a tree T

$$\phi(T)\{q^2/\mu^2; z\} = \left(\frac{y^2}{\mu^2}\right)^{-\frac{|T|}{2}z} \prod_{v \in T^{[0]}} F_{t(v)}(z)\ , \tag{4.51}$$

where $t(v)$ is the subtree dangling down from vertex $v \in T^{[0]}$ and the function $F_{t(v)}(z)$ is given by

$$F_{t(v)}(z) := \frac{\pi^{1-z}}{\Gamma(1-z)}\ \Gamma(1 - |t(v)|z/2)\ \Gamma(|t(v)|z/2)\ . \tag{4.52}$$

By multiplicativity of ϕ, this formula defines ϕ also for a forest $w = \prod_k T_k$. The meromorphic function $F_\bullet$ deserves its own name: it is called the *Mellin transform.* Its Laurent coefficients are in this model actually sufficient to figure out the values of the renormalized character on the whole Hopf algebra H, once the appropriate combinatorial laws are known, that is. Question: what is the meaning of the factor

$$\frac{\pi^{1-z}}{\Gamma(1-z)}\ ? \tag{4.53}$$

However, easy question, let us have a look at the product

$$f_T(z) = \prod_{v \in T^{[0]}} F_{t(v)}(z)\ . \tag{4.54}$$

It is actually the function we have introduced during the above computations, this time for a general tree T. The identity in (4.51) is not difficult to prove: one only has to show that it satisfies (4.11) or (4.16) setting $|c| = 1$ and $f_c = 1$, but we will leave it there(see also [**Kr03**]).

4.3. Renormalized character

The renormalized character ϕ_R is also an algebra morphism in the sense of the universality theorem. The corresponding 'intertwining' equation is

$$\phi_R \circ B_+^c = (\mathrm{id}_V - R)\mathcal{L}_c \circ \phi_R\ , \tag{4.55}$$

where the linear operator on the target algebra V is given by $(\mathrm{id}_V - R)\mathcal{L}_c$. This identity is easy to prove and very instructive. Note that $\overline{\phi} \circ B_+^c = \mathcal{L}_c \circ \phi_R$, which is a byproduct of the proof of Theorem 4.2.1. Then the assertion is trivial:

$$\phi_R \circ B_+^c = (\mathrm{id}_V - R)\overline{\phi} \circ B_+^c = (\mathrm{id}_V - R)\mathcal{L}_c \circ \phi_R\ . \tag{4.56}$$

It is instructive because it tells us how renormalization works: each subintegration is cured of its divergence individually!

4.4. Weinberg's Theorem

The integrals associated to Feynman graphs, known as *Feynman integrals*, are in many cases divergent, a typical example is the integral

$$I_D(q^2) = \int \frac{d^D k}{(k^2+m^2)((k-q)^2+m^2)} \tag{4.57}$$

in $D = 6$ dimensions, with mass parameter $m > 0$. It is quadratically divergent because, by simple power counting, the integrand behaves asymptotically as $\sim |k|$ for very large $|k|$: for an upper integration boundary Λ, the value of the integral then growths as Λ^2 for $\Lambda \to \infty$. To render this integral convergent and extract the relevant information out of it, one first introduces a *regulator* in a similiar manner as in the toy model of [**Kr03**](see section 4.2), i.e. a parameter like z in $D = 6 - 2z$. The result is a Laurent series with poles. Then a well-chosen (Rota-Baxter) subtraction procedure R will make sure these poles are discarded and the physical limit $D \to 6$

$$I_R(q^2) = \lim_{D\to 6}(\mathrm{id} - R) I_D(q^2) \tag{4.58}$$

is finite. As the rhs of (4.58) has a convergent integral for every $D \neq 6$, one can reformulate the R-subtraction at the integrand level. Denoting the above integrand in (4.57) by $\mathrm{Int}(q,k)$, this takes the form

$$I_R(q^2) = \lim_{D\to 6} \int d^D k \; (\mathrm{id} - R)\mathrm{Int}(q,k) \tag{4.59}$$

which, using again the symbol R, is an admittedly sloppy notation. However, the point is: the integrand is now structured in such a way that

$$I_R(q^2) = \int d^6 k \; (\mathrm{id} - R)\mathrm{Int}(q,k) \tag{4.60}$$

perfectly converges. Therefore, we could have skipped the regularization procedure from the start: this type of renormalization scheme, known as *BPHZ renormalization* needs no regularization! It might be necessary to introduce a regulator for practical reasons, though. Anyway, given a nested Feynman integral of the form

$$J_D(q^2) = \int d^D k \int d^D k' \; \mathrm{Int}_1(q,k) \; \mathrm{Int}_2(k,k') \; , \tag{4.61}$$

with two multiplied integrands $\mathrm{Int}_1(q,k)$ and $\mathrm{Int}_2(q,k)$, the BPHZ-renormalized version of it, i.e.

$$J_R(q^2) = \int d^D k \; (\mathrm{id} - R)[\mathrm{Int}_1(q,k) \int d^D k' \; (\mathrm{id} - R)\mathrm{Int}_2(k,k')] \; , \tag{4.62}$$

can only be expected to yield a well-defined convergent integral if every subintegration including the outermost ('non-proper' sub)integration is convergent. In essence, this is the assertion of *Weinberg's theorem*, which says that *a Feynman graph gives rise to a convergent integral if it is convergent by power counting in all its sectors.* The reason we point this out is this: it is because of this maybe obvious fact that renormalization actually works. Weinberg's well-written paper [**Wein60**] be recommended to the reader at this point.

4.5. Feynman graphs and their Hopf algebra

We shall now endow the set of Feynman graphs with a Hopf algebra structure. Strictly speaking, we have already implicitly introduced these structures during the course of the previous chapters: given the one-to-one correspondence between Feynman graphs and decorated rooted trees, the Hopf algebra operations on Feynman graphs are almost obvious. First some definitions.

Definition 4.5.1. *A graph Γ is called n-PI if for any n internal edges $e_1, ..., e_n \in \Gamma^{[1]}_{int}$ one finds that*

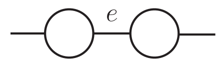
$$\Gamma' = \Gamma \setminus \{e_1, ..., e_n\} \tag{4.63}$$

is still a connected graph. A graph Γ is said to be divergent if its weight

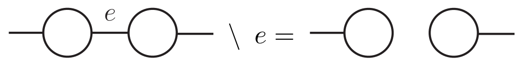
$$\omega_6(\Gamma) = -6|\Gamma| + \sum_{e\in\Gamma^{[1]}_{int}} \omega(e) + \sum_{v\in\Gamma^{[0]}} \omega(v) \tag{4.64}$$

is non-positive.

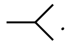
For example, the graph

(4.65) 

is not 1PI since

(4.66) 

is not connected.

Definition 4.5.2. *H_{FG} is the Hopf algebra of 1PI divergent Feynman graphs with edge type* *and 3-valent vertices of the form* .

The product of two graphs Γ_1 and Γ_2 is given by their disjoint union, i.e.

(4.67) $$m_{H_{FG}}(\Gamma_1 \otimes \Gamma_2) := \Gamma_1 \cup \Gamma_2$$

but for the most part denoted as a simple juxtaposition: $\Gamma_1\Gamma_2$. The neutral element of multiplication is the empty graph $\emptyset$, denoted by $\mathbb{I}$ just like the empty forest(or tree).

A *grading* $H_{FG} = \oplus_{j\geq 0} H^{(j)}$ is induced by the first Betti number of a graph:

(4.68) $$|\Gamma| := h_1(\Gamma) \qquad\qquad \Gamma \in H_{FG} \ ,$$

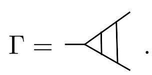
which is defined as the number of independent cycles of the graph Γ. The elements in $H^{(j)}$ are disjoint unions of graphs with a total number of j independent loops. The grading starts with $H^{(0)} = \mathbb{Q}\mathbb{I}$ and $\mathrm{Aug} = \bigoplus_{j\geq 1} H^{(j)}$ is the augmentation ideal, i.e. everything else than the span of the unit $\mathbb{I}$.

As there is no potential for confusion, we will write the counit as $\hat{\mathbb{I}}$, just like with the Hopf algebra of rooted trees and let it be the linear map $H_{FG} \to \mathbb{Q}$ such that $\hat{\mathbb{I}}(\mathbb{I}) = 1$ and $\hat{\mathbb{I}}(\mathrm{Aug}) = 0$.

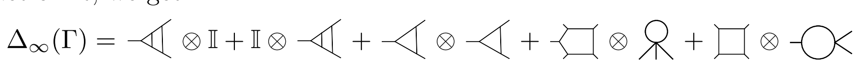
How can we define a coproduct? Since the coproduct must have the grading property $\Delta(H^{(k)}) \subset \bigoplus_{l=0}^{k} H^{(l)} \otimes H^{(k-l)}$ we need an operation that lowers the grading degree of a graph. One first guess is this: let $P(\Gamma)$ be the set of all proper subgraphs γ of a graph $\Gamma \in H_{FG}$ such that γ is a product of 1PI subgraphs of Γ. Then,

(4.69) $$\Delta_\infty(\Gamma) = \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I} + \sum_{\gamma \in P(\Gamma)} \gamma \otimes \Gamma/\gamma$$

may define a coproduct on H_{FG}. But it does not. Consider the graph

(4.70) 

If we let Δ_∞ act on it, we get

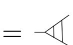
(4.71) 

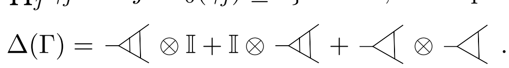
What are these funny animals in the latter two terms? We pick two of them and compute their weights,

(4.72) 

This tells us that they are not divergent, because their weights are positive. They do therefore not belong to the set H_{FG}! To avoid graphs like these two, we add the additional requirement that all image graphs be divergent. Thus, the coproduct had better be defined

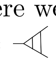
(4.73) $$\Delta(\Gamma) = \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I} + \sum_{\gamma \in \mathcal{P}(\Gamma)} \gamma \otimes \Gamma/\gamma \ ,$$

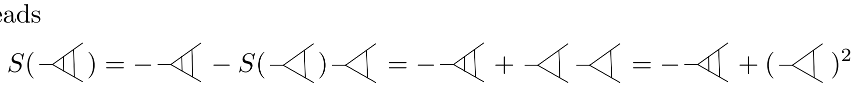
with $\mathcal{P}(\Gamma) := \{\gamma \in P(\Gamma) \mid \gamma = \prod_j \gamma_j \text{ s.t.} \forall j : \omega_6(\gamma_j) \leq 0\}$. Then, the coproduct of the graph $\Gamma =$  is

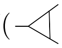
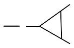
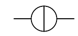
(4.74) 

By virtue of the grading property of Δ, the antipode $S : H_{FG} \to H_{FG}$ is completely determined by setting $S(\mathbb{I}) = \mathbb{I}$ and by the identity

(4.75) $$S(\Gamma) = -(S * P)(\Gamma) = -m_{H_{FG}}(S \otimes P)\Delta(\Gamma) = -\Gamma - \sum_{\gamma \in \mathcal{P}(\Gamma)} S(\gamma)\ \Gamma/\gamma \ ,$$

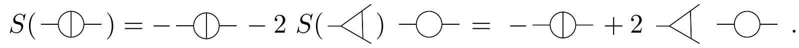
where we have written the product of the two graphs $S(\gamma)$ and the cograph Γ/γ as a juxtaposition. For $\Gamma =$  this reads

(4.76) 

because of $S($$) = -$ by (4.75). Another example is the graph :

(4.77) 

The antipode yields

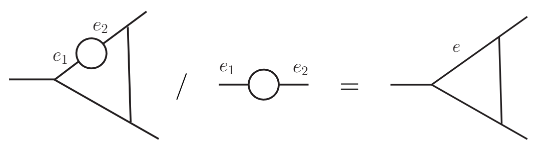
(4.78) 

Note that $S(h) = -(P * S)(h)$ for $h \in \mathrm{Aug}$ is an equation equivalent to (4.75), by definition of S.

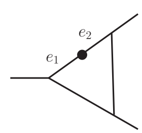
Contraction of a propagator graph. A remark concerning the contraction operation is in order. If $\gamma \subsetneq \Gamma$ is a propagator graph, i.e.

$$|\gamma_{ext}^{[1]}| = 2 \ , \tag{4.79}$$

then the cograph Γ/γ has lost all information about γ altogether in the following sense. In the first chapter we have defined the cograph as the graph we obtain if we shrink all internal edges of γ inside Γ into one point while keeping the external leg structure of γ. This was not to say that we are left with a 2-vertex in case γ is a propagator graph! Consider the contraction

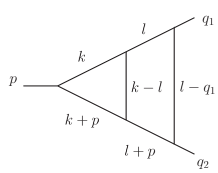
(4.80) 

The external edges e_1 and e_2 of γ, strictly speaking half-egdes, merge with their adjacent half-edges to leave behind one single edge e! In our example, the result is the graph on the rhs of (4.80) and not an animal like

(4.81)  !

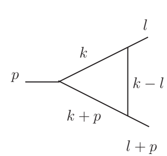
4.6. Hopf-algebraic renormalization

The good news is that we now do not just have a Hopf algebra of Feynman graphs H_{FG} but also have a character group $G_V^{H_{FG}}$ in much the same way we have seen for the Hopf algebra of rooted trees H! Consider the labelled Feynman graph and the corresponding divergent Feynman integral for $D = 6$:

(4.82) 

$$= \int \frac{d^D l}{l^2(l-q_1)^2(l+p)^2} \int \frac{d^D k}{k^2(k-l)^2(k+p)^2}$$

Assume we regularize it by setting $D = 6 - 2z$. Then we first take care of the subintegration, i.e. the Feynman graph subsector

(4.83) 

$$= \int \frac{d^D k}{k^2(k-l)^2(k+p)^2} - \int \frac{d^D k}{k^2(k-l)^2(k+p)^2}\bigg|_{l^2=p^2=\mu^2}$$

and renormalize it, where $l, p \in \mathbb{R}^6$ are the external parameters. We replace the subintegral in (4.82) by this term and get

$$I_D = \int \frac{d^D l}{l^2(l-q_1)^2(l+p)^2} \left(\int \frac{d^D k}{k^2(k-l)^2(k+p)^2} - \int \frac{d^D k}{k^2(k-l)^2(k+p)^2}\bigg|_{l^2=p^2=\mu^2} \right) . \tag{4.84}$$

However, this is not an expression for which the limit $D \to 6$ exists on account of the logarithmic divergence of the l-integration. We need yet another subtraction to achieve this aim, that is, add

$$-RI_D =$$
$$- \left[\int \frac{d^D l}{l^2(l-q_1)^2(l+p)^2} \left(\int \frac{d^D k}{k^2(k-l)^2(k+p)^2} - \int \frac{d^D k}{k^2(k-l)^2(k+p)^2}\bigg|_{l^2=p^2=\mu^2} \right) \right]\Bigg|_{p^2=q_1^2=\mu^2}$$

and the physical limit $\lim_{D\to 6}(I_D - RI_D)$ exists. The renormalized expression $(I_D - RI_D)$ consists of 4 terms with different integrals: if we write the renormalized subintegration in (4.83) as $(I'_D - RI'_D)$ and sloppily denote by A the kernel of the outer l-integration in front of the round brackets in (4.84), this takes the form

$$I_R = \int A(I'_D - RI'_D) - R[\int A(I'_D - RI'_D)] \tag{4.85}$$

$$= \int AI'_D - (RI'_D)(\int A) - R[\int AI'_D] + R[(\int A)(RI'_D)] \tag{4.86}$$

$$= \int AI'_D - (RI'_D)(\int A) - R[\int AI'_D] + (RI'_D)R[(\int A)] \ . \tag{4.87}$$

We compute the renormalized character $\phi_R = S_R^\phi * \phi$ to see how this relates to the underlying Hopf algebra structure:

$$\begin{aligned}
\phi_R(\ ) &= (S_R^\phi * \phi)(\ ) = S_R^\phi(\ ) + \phi(\ ) + S_R^\phi(\ )\ \phi(\ ) \\
&= -R[(S_R^\phi * \phi P)(\ )] + \phi(\ ) - R[(S_R^\phi * \phi P)(\ )]\ \phi(\ ) \\
&= -R[\phi(\ ) + S_R^\phi(\ )\ \phi(\ )] + \phi(\ ) - R[\phi(\ )]\ \phi(\ ) \\
&= -R[\phi(\ ) - R[\phi(\ )]\ \phi(\ )] + \phi(\ ) - R[\phi(\ )]\ \phi(\ ) \\
&= -R[\phi(\ )] + R[R[\phi(\ )]\ \phi(\ )] + \phi(\ ) - R[\phi(\ )]\ \phi(\ ) \\
&= -R[\phi(\ )] + R[\phi(\ )]\ R[\phi(\ )] + \phi(\ ) - R[\phi(\ )]\ \phi(\ )
\end{aligned}$$

We can reorder those terms to get

$$\phi_R(\ ) = \phi(\ ) - R[\phi(\ )]\ \phi(\ ) - R[\phi(\ )] + R[\phi(\ )]\ R[\phi(\ )] \tag{4.88}$$

The first subtraction eradicates the subdivergence assciated to the subgraph('subsector')

$$\gamma = \tag{4.89}$$

whereas the subtraction of the last two terms cures the remaining divergence. By comparing carefully, we identify

$$\int AI'_D = \phi(\ )\ , \qquad -RI'_D = -R[\phi(\ )]\ , \qquad \int A = \phi(\ )\ , \tag{4.90}$$

$$-R[\int AI'_D] = -R[\phi(\ )]\ , \qquad -R[(\int A)] = -R[\phi(\ )]\ . \tag{4.91}$$

There should be no confusion because by close inspection we see that $R[\int A] = RI'_D$.

4.7. One-cocycles and finitely generated Hopf algebras

One can take a Feynman graph and use it as a generator of a Hopf algebra. A simple example can be constructed from the primitive graph

$$\gamma = \ . \tag{4.92}$$

The freely generated commutative $\mathbb{Q}$-algebra has a linear basis simply consisting of monomials γ^n, $n \geq 1$. The coproduct does not bring in anything new

$$\Delta(\gamma) = \gamma \otimes \mathbb{I} + \mathbb{I} \otimes \gamma\ , \tag{4.93}$$

except for the neutral element $\mathbb{I}$. If we add this to our algebra, and take $\{\mathbb{I}, \gamma\}$ as the set of generators, we have an infinite dimensional Hopf subalgebra H_γ generated by just two elements $\gamma, \mathbb{I} \in H_{FG}$. In the same manner do we add all subgraphs γ of a graph Γ generated by the coproduct from it to this generator set. Let us denote the set of generators of the Hopf algebra H_Γ given rise to by a graph Γ in this way by $G(\Gamma)$. Examples are

$$G(\ ) = \left\{ \mathbb{I}, \ , \ \right\}\ , \qquad G(\ ) = \left\{ \mathbb{I}, \ , \ , \ \right\} \tag{4.94}$$

and

$$G(\ ) = \left\{ \mathbb{I}, \ , \ , \ , \ , \ \right\}\ . \tag{4.95}$$

These finitely generated Hopf algebras do even have a natural grading, defined by the loop number. Can we establish a Hochschild cohomology? The one-cocycles are linear maps L such that

$$\Delta L = (\mathrm{id} \otimes L)\Delta + L \otimes \mathbb{I} \; . \tag{4.96}$$

Consider the Hopf algebra $H_{-\circ-}$ generated by $G(-\!\bigcirc\!-) = \{\, \mathbb{I}, -\!\bigcirc\!- \}$. Let $B_+^{-\circ-}$ be a Hochschild one-cocycle. Consider

$$\Delta B_+^{-\circ-}(\mathbb{I}) = (\mathrm{id} \otimes B_+^{-\circ-})\Delta(\mathbb{I}) + B_+^{-\circ-}(\mathbb{I}) \otimes \mathbb{I} = \mathbb{I} \otimes B_+^{-\circ-}(\mathbb{I}) + B_+^{-\circ-}(\mathbb{I}) \otimes \mathbb{I} \; . \tag{4.97}$$

This suggests that $B_+^{-\circ-}(\mathbb{I}) = -\!\bigcirc\!-$ as this is the only primitive element in $G(-\!\bigcirc\!-)$. Next, consider

$$\begin{aligned}
\Delta B_+^{-\circ-}(-\!\bigcirc\!-) &= (\mathrm{id} \otimes B_+^{-\circ-})\Delta(-\!\bigcirc\!-) + B_+^{-\circ-}(-\!\bigcirc\!-) \otimes \mathbb{I} \\
&= \mathbb{I} \otimes B_+^{-\circ-}(-\!\bigcirc\!-) + -\!\bigcirc\!- \otimes B_+^{-\circ-}(\mathbb{I}) + B_+^{-\circ-}(-\!\bigcirc\!-) \otimes \mathbb{I} \\
&= \mathbb{I} \otimes B_+^{-\circ-}(-\!\bigcirc\!-) + B_+^{-\circ-}(-\!\bigcirc\!-) \otimes \mathbb{I} + -\!\bigcirc\!- \otimes -\!\bigcirc\!-
\end{aligned}$$

This requires

$$B_+^{-\circ-}(-\!\bigcirc\!-) = -\!\circledcirc\!- \notin H_{-\circ-} \tag{4.98}$$

which tells us that in $H_{-\circ-}$ there is no one-cocycle! We have seen that the only candidate proved disappointing. Let us look at $H_{-\Phi-}$. First, we note that (4.97) holds just as well as in $H_{-\circ-}$, hence we choose again $B_+^{-\circ-}(\mathbb{I}) = -\!\bigcirc\!-$. We could, of course, have chosen the other primitive graph available, but that wouldn't make a difference. Consider the graph

$$\Gamma := B_+^{-\circ-}(-\!\triangleleft\,) \; . \tag{4.99}$$

If we use the one-cocycle property (4.96) as we have done before, we find

$$\Gamma = \frac{1}{2} -\!\Phi\!- \tag{4.100}$$

which surely is in $H_{-\Phi-}$. However, the requirement in (4.98) is still valid on this Hopf algebra. It turns out that generally, *a Hopf subalgebra H_Γ generated by a graph Γ has no one-cocycle*. What's behind this is that the property (4.96) implies a one-cocycle increases the grading degree. The product does also increase the grading degree but in a different way. Question: can a one-cocycle map a primitive graph to a product of two primitive graphs?

CHAPTER 5

Lie algebraic Structures and Renormalization

5.1. Lie algebra of jets

Consider two smooth real-valued functions $f, g \in \mathcal{C}^\infty(\mathbb{R})$ on the line $\mathbb{R}$. For a fixed $x_0 \in \mathbb{R}$ let their Taylor polynomials of degree m be denoted by

$$(T_m f)(x) = \sum_{j=0}^{m} \frac{1}{j!} f^{(j)}(x_0)\ (x - x_0)^j\ , \qquad (T_m g)(x) = \sum_{j=0}^{m} \frac{1}{j!} g^{(j)}(x_0)\ (x - x_0)^j\ . \tag{5.1}$$

We declare f and g to be equivalent if these Taylor polynomials agree $T_m f = T_m g$ and write $f \sim g$. In particular $T_m f \sim f$ and $T_m g \sim g$. Let now $x_0 = 0$. The equivalence classes $\{[f] : f \in \mathcal{C}^\infty(\mathbb{R})\}$ span a linear space on which we may also define a multiplication through

$$(T_m f) \cdot (T_m g) := T_m(fg)\ . \tag{5.2}$$

Note that this definition implies that we take the quotient with respect to the polynomial ideal

$$(x^{m+1}) := x^{m+1}\mathbb{R}[x]\ , \tag{5.3}$$

being the space of all polynomials with vanishing coefficients up to the m-th. The elements of this quotient space are referred to as *jets of order* m at $x_0 = 0$. For $f \in \mathcal{C}^\infty(\mathbb{R})$, the corresponding jet is usually denoted by $J^m_{x_0} f$ and can be seen as represented by an abstract polynomial. Next, let us have a look at the differential operators

$$Z_k := -x^{k+1}\partial_x \tag{5.4}$$

on $\mathcal{C}^\infty(\mathbb{R})$. On account of the fact that the product of such differential operators is associative, their commutator

$$[Z_k, Z_l] = (k - l) Z_{k+l} \tag{5.5}$$

establishes a Lie algebra structure. This is a representation of what is known as *Witt algebra*. We combine these two at first glance disparate concepts by considering differential operators of the form

$$D_f = f(x)\partial_x\ , \qquad f \in \mathcal{C}^\infty(\mathbb{R})\ . \tag{5.6}$$

Then we apply the Taylor polynomial equivalence to the smooth prefactor functions, just as above. These differential operators can now be viewed as tangent vectors at the base point $x_0 = 0$ with $\mathbb{R}$ as a smooth one-dimensional manifold. Those readers who are not familiar with differential geometry may stick with the differential operator notion, it is not wrong[1].

However, the corresponding equivalence classes are m-th order jets in a differential geometric context. By $\mathcal{A}^1_n$ we denote the linear space of jets of order $m = (n + 1)$ given rise to by tangent vectors of the form[2]

$$D_f = f(x)\partial_x\ , \qquad f \in \mathcal{C}^\infty(\mathbb{R}) : f(0) = f'(0) = 0\ , \tag{5.7}$$

with the quotient taken with respect to the polynomial ideal (x^{n+2}). Equipped with the commutator in (5.5), one easily sees that $\mathcal{A}^1_n$ is a Lie algebra! We call it the *Lie algebra of jets* (of $(n + 1)$-th order at $x_0 = 0$). However, what is of interest to us is that its universal enveloping algebra $\mathcal{U}(\mathcal{A}^1_n)$ is dually related to the Connes-Moscovici Hopf subalgebra H_{CM} introduced in section 3.1.

[1] As derivations, the tangent vectors $f(x)\partial_x$ act on 'germs' of smooth functions at $x = 0$.

[2] The superscript '1' in $\mathcal{A}^1_n$ stands for 'vanishing derivatives up to 1-st order'.

5.2. Milnor-Moore theorem

We briefly recall the concept of the universal enveloping algebra of a Lie algebra. Let $\mathcal{L}$ be a Lie algebra. Its tensor algebra $T(\mathcal{L})$ is given by the direct sum of linear spaces $\mathcal{L}^{\otimes k}$

$$T(\mathcal{L}) = \bigoplus_{k=0}^{\infty} \mathcal{L}^{\otimes k} \ , \tag{5.8}$$

where the product of $a, b \in \mathcal{L}$ is written as $a \otimes b \in \mathcal{L} \otimes \mathcal{L}$ or as a simple juxtaposition ab and for the higher spaces accordingly. Taking the quotient with respect to the equivalence relation

$$ab - ba = a \otimes b - b \otimes a \sim [a, b] \tag{5.9}$$

one obtains the universal enveloping algebra of $\mathcal{L}$, denoted as $\mathcal{U}(\mathcal{L})$. Note that prior to establishing the equivalence relation, both sides of the $\sim$ sign in (5.9) are *not* the same: the lhs is an element of $\mathcal{L} \otimes \mathcal{L}$ and the rhs of $\mathcal{L}$.

Proposition 5.2.1. *Let H^n_{CM} be the Hopf subalgebra of H_{CM} generated by the set $\{\mathbb{I}, \delta_1, ..., \delta_n\}$. Then, we have*

$$H^n_{CM} \simeq \mathcal{U}(\mathcal{A}^1_n)^* \ . \tag{5.10}$$

Proof. We define linear forms L_k, $k \leq n$ on H^n_{CM} by

$$\langle L_k, P(\delta) \rangle := \hat{\mathbb{I}} \left(\frac{\partial}{\partial \delta_k} P(\delta) \right) \qquad\qquad \delta = (\delta_1, ..., \delta_n) \ , \tag{5.11}$$

which means that we formally differentiate the polynomial $P(\delta) \in H^n_{CM}$ with respect to the variable δ_k, and annihilate anything except terms proportional to $\mathbb{I}$, which are mapped to $\mathbb{R}$. This means for monomials

$$\langle L_k, \delta_l \rangle = \begin{cases} 1 & k = l \\ 0 & \text{else} \end{cases} \tag{5.12}$$

with the neutral element $L_0 = 1$ dual to $\delta_0 = \mathbb{I}$, i.e. $1 L_k = L_k$ and $\langle 1, \cdot \rangle = \hat{\mathbb{I}}$. One now has to show that these linear forms satisfy the Witt algebra commutator relation in (5.5). This amounts to showing that

$$(k - l)\langle L_{k+l}, P(\delta) \rangle = \langle L_k L_l - L_l L_k, P(\delta) \rangle = \langle L_k \otimes L_l - L_l \otimes L_k, \Delta P(\delta) \rangle \tag{5.13}$$

and that all stuctures are dual to each other. For example, the product on H^n_{CM} is dual to the coproduct on the Lie algebra:

$$\begin{aligned} \langle L_k, PQ \rangle &= \langle L_k, P \rangle \hat{\mathbb{I}}(Q) + \hat{\mathbb{I}}(P) \langle L_k, Q \rangle = \langle L_k, P \rangle \langle 1, Q \rangle + \langle 1, P \rangle \langle L_k, Q \rangle \\ &= \langle L_k \otimes 1, P \otimes Q \rangle + \langle 1 \otimes L_k, P \otimes Q \rangle = \langle L_k \otimes 1 + 1 \otimes L_k, P \otimes Q \rangle \\ &= \langle \Delta(L_k), P \otimes Q \rangle \ . \end{aligned}$$

Note that choosing the elements $L_k \simeq Z_k$ to be primitive in the Hopf algebra $\mathcal{U}(\mathcal{A}^1_n)$ is a necessity. For a complete proof the reader is referred to Proposition 3 in [**CoKr98**]. □

Milnor-Moore duality. In fact, this important result is a special instance of the following theorem, known as *Milnor-Moore theorem*:

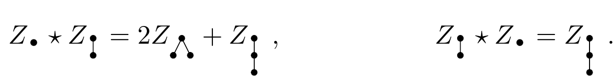
Theorem 5.2.2. *Let H be a graded, connected and commutative Hopf algebra. Then H is the dual of the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ of some Lie algebra $\mathcal{L}$.*

Proof. See [**Menc**]. □

As we already know, the Hopf algebra of rooted trees H is connected and commutative. What is the corresponding Lie algebra $\mathcal{L}$? Consider the set of symbols Z_T indexed by rooted trees $T \in \mathcal{T}$. We take their linear span over $\mathbb{Q}$ and moreover, we define a bilinear operation by

$$Z_{T_1} \star Z_{T_2} := \sum_{T \in \mathcal{T}} n(T_1, T_2; T) Z_T \ , \tag{5.14}$$

where $n(T_1, T_2; T)$ is the number of cuts $C \in \mathcal{C}(T)$ such that $P^C(T) = T_1$ and $R^C(T) = T_2$. Easy examples are

(5.15)


The latter differs from the former due to $n(\overset{\bullet}{\underset{\bullet}{|}}, \bullet; \overset{\bullet}{\bullet\!\wedge\!\bullet}) = 0$. This $\star$-operation is *pre-Lie* and thus the associator

$$A(T_1, T_2, T_3) := Z_{T_1} \star (Z_{T_2} \star Z_{T_3}) - (Z_{T_1} \star Z_{T_2}) \star Z_{T_3} \tag{5.16}$$

is symmetric with respect to interchanging the last two arguments, i.e.

$$A(T_1, T_2, T_3) = A(T_1, T_3, T_2) \ , \tag{5.17}$$

which garantees that the bilinear bracket

$$[Z_{T_1}, Z_{T_2}] := Z_{T_1} \star Z_{T_2} - Z_{T_2} \star Z_{T_1} \tag{5.18}$$

satisfies the Jacobi identity. Therefore, we have the following

Lemma 5.2.3. *The bracket defined in (5.18) satisfies the Jacobi identity.*

PROOF. Exercise: one may use (5.17). □

Proposition 5.2.4. *The linear space $\mathcal{L} := \langle Z_T, T \in \mathcal{T} \rangle_{\mathbb{Q}}$ is a Lie algebra with respect to the bracket (5.18) and its universal enveloping algebra $\mathcal{U}(\mathcal{L})$ is dual to the Hopf algebra of rooted trees H.*

PROOF. Analogous to that of Proposition (5.2.1). The corresponding linear forms are given by

$$\langle Z_T, \cdot \rangle = \hat{\mathbb{I}} \frac{\partial}{\partial T} \tag{5.19}$$

with a formal derivative with respect to the tree $T \in \mathcal{T}$ and the operation dual to the coproduct on H is the $\star$-product. The reader may check that

$$\langle [Z_\bullet, Z_{\overset{\bullet}{\underset{\bullet}{|}}}], \overset{\bullet}{\bullet\!\wedge\!\bullet} \rangle = \langle Z_\bullet \otimes Z_{\overset{\bullet}{\underset{\bullet}{|}}} - Z_{\overset{\bullet}{\underset{\bullet}{|}}} \otimes Z_\bullet, \Delta(\overset{\bullet}{\bullet\!\wedge\!\bullet}) \rangle \ . \tag{5.20}$$

□

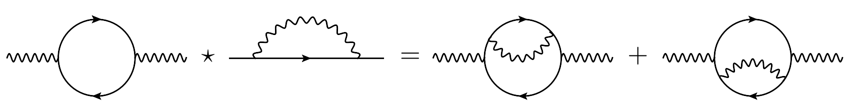
The analogous notion exists for the Hopf algebra of Feynman graphs H_{FG}. In section 2.1 we have introduced the pre-Lie product

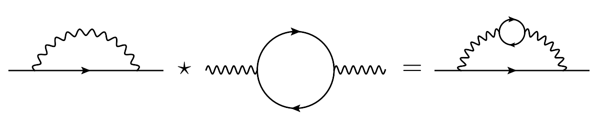
$$\Gamma \star \gamma = \sum_{i \in \mathcal{I}(\gamma|\Gamma)} \Gamma \circ_i \gamma \ , \tag{5.21}$$

where $\mathcal{I}(\gamma|\Gamma)$ is the set of insertion places for the graph γ into Γ. One can also write this as

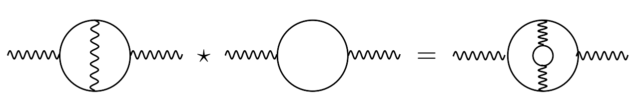
$$\Gamma \star \gamma = \sum_{\Gamma'} n(\gamma, \Gamma; \Gamma')\Gamma' \tag{5.22}$$

with $n(\gamma, \Gamma; \Gamma')$ being the number of possibilities to insert γ into Γ in such a way as to obtain the graph Γ'. Examples for QED graphs are

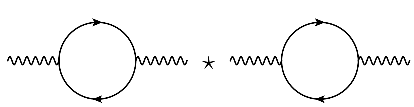
(5.23)  ,

(5.24) 

and, suppressing the arrows,

(5.25)  .

Note that

(5.26)  $= 0$

since there are no insertion places in the first for the second graph. Only *internal* (wiggly) photon lines are insertion places for a photon propagator graph.

5.3. The Riemann-Hilbert problem

Consider a smooth curve C in $\mathbb{C}$ and a map $\gamma : C \to G$ with values in a Lie group G. Let γ be analytic along C, i.e. for a parametrization $z : [0,1] \to C,\ t \mapsto z(t)$ of C, the derivative

$$\frac{d}{dt}\gamma(z(t)) = \lim_{\epsilon\to 0} \epsilon^{-1}\{\gamma(z(t+\epsilon)) - \gamma(z(t))\} \tag{5.27}$$

exists in G. Then there is a holomorphic continuation of γ on a tubular neighbourhood U_C of C. If we extend this neighbourhood so as to include the origin, we obtain a domain D on which there is a meromorphic continuation $D \to G$ which we might also call γ. Then, $\gamma(z)$ has Laurent series with coefficients in G. One version of the *Riemann-Hilbert problem* is to find two other such maps $\gamma_\pm$ factorizing γ in the form

$$\gamma(z) = \gamma_-(z)^{-1} \cdot \gamma_+(z) \qquad\qquad z \in D\ , \tag{5.28}$$

where γ_+ is holomorphic. This pair of maps $\gamma_\pm$ is called *Birkhoff decomposition* of γ. The product is that of the Lie group and $\gamma_-(z)^{-1} \in G$ is the inverse of $\gamma_-(z) \in G$ with respect to the Lie group product. We may visualize the curve C and the maps $\gamma, \gamma_\pm$ on the Riemann sphere $S^2 \simeq \mathbb{P}^1(\mathbb{C})$ as in Fig.1. If we

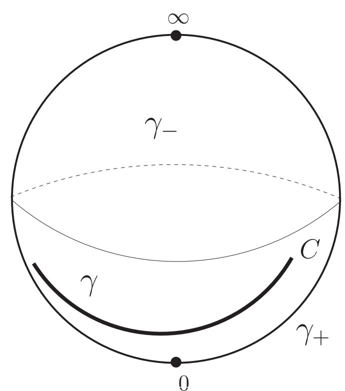


FIGURE 1. The Riemann sphere and the Lie group-valued maps $\gamma, \gamma_\pm : D \to G$.

assume that there is only one singularity at the origin, the domain where $\gamma_-(z)^{-1}$ is holomorphic tends to be on the upper half of the Riemann sphere. As a matter of fact, *renormalization with dimensional regularization in quantum field theory provides an example for a solution of the Riemann-Hilbert problem*!

Let us see how this comes about. First recall that the regularized Feynman rules are given in the form of characters $\phi : H_{FG} \to V$, where V is the target algebra. By fixing all external parameters like momenta $q \in \mathbb{R}^4$ we may choose this algebra to be $V = \mathbb{C}[z^{-1}, z]]$, i.e. the ring of Laurent series with finite principle part. How can this be related to renormalization? Where is the Lie group G and the maps γ, $\gamma_\pm$?

To answer these questions, we may take the view that the Hopf characters are maps from H_{FG} to $\mathbb{C}$. In other words, we replace the ring of Laurent series with $\mathbb{C}$ as target algebra. The assignment

$$z \mapsto \gamma(z) = \phi(\cdot)\{z\}\ . \tag{5.29}$$

yields a character in $G_{\mathbb{C}}^{H_{FG}}$ at every fixed point z on the domain $D \subset \mathbb{P}^1(\mathbb{C})$. Then, the Lie group G is this character group with Lie group product given through the character convolution

$$\phi * \psi = m_{\mathbb{C}}(\phi \otimes \psi)\Delta\ . \tag{5.30}$$

To see that this character group G really is a manifold, let $\mathcal{G}$ be an ordered set of all Feynman graphs, i.e. the countable generator set for the Hopf algebra H_{FG}. Now, note that a character ϕ can be characterized by its values on $\mathcal{G}$, i.e. by a sequence

$$(\phi(\Gamma)\{z\})_{\Gamma\in\mathcal{G}} \tag{5.31}$$

in $\mathbb{C}$ indexed by Feynman graphs. A character is therefore represented by an element in the infinite-dimensional manifold $\mathbb{C}^\infty$, where the global chart is given by the assignment

$$\phi \mapsto (\phi(\Gamma)\{z\})_{\Gamma\in\mathcal{G}}\ . \tag{5.32}$$

Consequently, sweeping the involved subtleties brought about by $\dim G = \infty$ under the carpet, G looks like $\mathbb{C}^\infty$ and thus G is an infinite-dimensional complex manifold(see [**CoKr00**] for more).

Recall that the renormalized character ϕ_R is given by $\phi_R = S_R^\phi * \phi$, which can also be written as

$$\phi = (S_R^\phi)^{*-1} * \phi_R = (S_R^\phi \circ S) * \phi_R \ . \tag{5.33}$$

This is the Birkhoff decomposition with $\gamma_+ = \phi_R$ and $\gamma_- = S_R^\phi$, where the former always maps to pole-free Laurent series.

5.4. Minimal subtraction renormalization scheme

We now consider a concrete example for a Birkhoff decomposition in a renormalization scheme known as *minimal subtraction*. This scheme differs from that introduced in sections 4.2 because it has a different Rota-Baxter subtraction. The Rota-Baxter operator is given by the projector

$$R : \mathbb{C}[z^{-1}, z]] \to \mathbb{C}[z^{-1}] \tag{5.34}$$

which maps a Laurent series free of essential singularities to its principle part, i.e. given a Laurent series $f = \sum_{k \geq -r_f} f_k z^k \in \mathbb{C}[z^{-1}, z]]$, one has

$$R[\sum_{k=-r_f}^{\infty} f_k z^k] = \sum_{k=-r_f}^{-1} f_k z^k \tag{5.35}$$

and thus

$$(\mathrm{id} - R)[\sum_{k=-r_f}^{\infty} f_k z^k] = \sum_{k \geq 0} f_k z^k \in \mathbb{C}[[z]] \tag{5.36}$$

is pole-free. Of course, not to forget, one has to prove the next

Lemma 5.4.1. *The projection operator defined in (5.35) is Rota-Baxter with respect to the usual product of Laurent series, i.e. for $f, g \in \mathbb{C}[z^{-1}, z]]$, we have*

$$R[fg] + R[f]R[g] = R[R[f]g + fR[g]] \ . \tag{5.37}$$

Proof. Exercise. □

Let now the regularized Feynman rules $\phi : H_{FG} \to \mathbb{C}[z^{-1}, z]]$ in a simple model be given by

$$\phi(\Gamma)\{z\} = \left(\frac{q^2}{\mu^2}\right)^{-|\Gamma| z} F_\Gamma(z) \ , \tag{5.38}$$

for a Feynman graph $\Gamma \in H_{FG}$. For a fixed complete forest $\mathfrak{F}(\Gamma)$ of Γ, the function F_Γ is given by

$$F_\Gamma(z) = \frac{c_{-1}}{|\Gamma| z} f(|\Gamma| z) \prod_{\gamma \in \mathfrak{F}(\Gamma)} \frac{c_{-1}}{|\gamma| z} f(|\gamma| z) \ , \tag{5.39}$$

and

$$f(z) = 1 + \sum_{l=0}^{\infty} \frac{c_l}{c_{-1}} z^{l+1} \ . \tag{5.40}$$

We write $L = \ln(q^2/\mu^2)$ and get for the graph $\Gamma = -\!\bigcirc\!- \in H_{FG}$ with $\mathfrak{F}(\Gamma) = \emptyset$

$$\phi(-\!\bigcirc\!-)\{z\} = \frac{c_{-1}}{z} e^{-Lz} f(z) = \frac{c_{-1}}{z} + c_0 - c_{-1} L + \frac{1}{2}(2c_1 - 2c_0 L + c_{-1} L^2) z + \mathcal{O}(z^2) \ . \tag{5.41}$$

The counterterm S_R^ϕ is

$$S_R^\phi(-\!\bigcirc\!-) = -R[(S_R^\phi * \phi P)(-\!\bigcirc\!-)] = -R[S_R^\phi(\mathbb{I})\phi(-\!\bigcirc\!-)] = -R[\phi(-\!\bigcirc\!-)] \tag{5.42}$$

and thus evaluates to the simple Laurent polynomial

$$S_R^\phi(-\!\bigcirc\!-)\{z\} = -\frac{c_{-1}}{z} \ . \tag{5.43}$$

The renormalized character ϕ_R then gives

$$\phi_R(-\!\bigcirc\!-)\{z\} = \frac{c_{-1}}{z} e^{-Lz} f(z) - \frac{c_{-1}}{z} = c_0 - c_{-1} L + \frac{1}{2}(2c_1 - 2c_0 L + c_{-1} L^2) z + \mathcal{O}(z^2) \tag{5.44}$$

with physical limit $z \to 0$

$$\phi_R(-\!\bigcirc\!-) = c_0 - c_{-1}L \ . \tag{5.45}$$

The L-independent term c_0 is an artfact of this special renormalization scheme. It does not appear in kinematical subtraction, where the Rota-Baxter operator sets $L = 0$. In this example, the Birkhoff decomposition is

$$\begin{aligned} ((S_R^\phi)^{*-1} * \phi_R)(-\!\bigcirc\!-) &= (S_R^\phi)^{*-1}(-\!\bigcirc\!-) + \phi_R(-\!\bigcirc\!-) = (S_R^\phi \circ S)(-\!\bigcirc\!-) + \phi_R(-\!\bigcirc\!-) \\ &= S_R^\phi(S(-\!\bigcirc\!-)) + \phi_R(-\!\bigcirc\!-) = -S_R^\phi(-\!\bigcirc\!-) + \phi_R(-\!\bigcirc\!-) \\ &= R[\phi(-\!\bigcirc\!-)] + \phi_R(-\!\bigcirc\!-) = \phi(-\!\bigcirc\!-) \ , \end{aligned}$$

i.e. the expected result. Next, consider the graph $-\!\circledcirc\!-$. The Feynman rules in (5.38) give

$$\begin{aligned} \phi(-\!\circledcirc\!-)\{z\} &= \frac{c_{-1}^2}{2z^2} e^{-2Lz} f(z) f(2z) \\ &= \frac{c_{-1}^2}{2z^2} + \frac{1}{2}(3c_{-1}c_0 - 2c_{-1}^2 L)z^{-1} + c_{-1}^2 L^2 - 3c_{-1}c_0 L + c_0^2 + \frac{5}{2}c_{-1}c_1 + \mathcal{O}(z) \ , \end{aligned}$$

where the complete forest is $\mathfrak{F}(\Gamma) = \{-\!\bigcirc\!-\}$. For the counterterm we find

$$S_R^\phi(-\!\circledcirc\!-) = -R[(S_R^\phi * \phi P)(-\!\circledcirc\!-)] = -R[\phi(-\!\circledcirc\!-) + S_R^\phi(-\!\bigcirc\!-)\phi(-\!\bigcirc\!-)] \ . \tag{5.46}$$

If we compute the term in square brackets, the Bogoliubov map $\overline{\phi}$, we get

$$\overline{\phi}(-\!\circledcirc\!-) = \frac{c_{-1}^2}{2z^2} e^{-2Lz} f(z) f(2z) - \frac{c_{-1}^2}{z^2} e^{-Lz} f(z) \tag{5.47}$$

with Laurent series

$$\overline{\phi}(-\!\circledcirc\!-) = -\frac{c_{-1}^2}{2z^2} + \frac{c_{-1}c_0}{2z} + \frac{1}{2}(c_{-1}^2 L^2 - 4c_{-1}c_0 L + 3c_1 c_{-1} + 2c_0^2) + \mathcal{O}(z) \ . \tag{5.48}$$

The Rota-Baxter projection yields the counterterm

$$S_R^\phi(-\!\circledcirc\!-) = -R[\overline{\phi}(-\!\circledcirc\!-)] = \frac{c_{-1}^2}{2z^2} - \frac{c_{-1}c_0}{2z} \ . \tag{5.49}$$

Finally, the renormalized character is

$$\phi_R(-\!\circledcirc\!-) = \frac{1}{2}(c_{-1}^2 L^2 - 4c_{-1}c_0 L + 3c_1 c_{-1} + 2c_0^2) + \mathcal{O}(z). \tag{5.50}$$

Grading operator. In analogy to the grading operator Y on the Hopf algebra H of rooted trees, there is also a grading operator Y on H_{FG}. It is defined as a derivation such that

$$Y(\Gamma) = |\Gamma|\Gamma \ , \tag{5.51}$$

where $Y(\mathbb{I}) = 0$ due to $|\mathbb{I}| = 0$ for the empty graph. Consider the results

$$\frac{\partial}{\partial L}\phi_R(-\!\bigcirc\!-)\{z\} = -c_{-1} + \mathcal{O}(z) \ , \qquad z\phi(S * Y)(-\!\bigcirc\!-)\{z\} = c_{-1} + \mathcal{O}(z) \tag{5.52}$$

and

$$\frac{\partial^2}{\partial L^2}\phi_R(-\!\circledcirc\!-)\{z\} = c_{-1}^2 + \mathcal{O}(z) \ , \qquad z^2\phi(S * Y^2)(-\!\circledcirc\!-)\{z\} = c_{-1}^2 + \mathcal{O}(z) \ . \tag{5.53}$$

We may therefore boldly assume that for $|\Gamma| = k$

$$\lim_{z\to 0} \frac{\partial^k}{\partial L^k}\phi_R(\Gamma)\{z\}\bigg|_{L=0} = (-1)^k \lim_{z\to 0} z^k \phi(S * Y^k)(\Gamma)\{z\} \ , \tag{5.54}$$

which says that the coefficient of the leading power in L of the physical limit $\phi_R(\Gamma)$ is related to the highest order pole of the regularized value of a linear combination of (proper) subgraphs of Γ. Examples for higher-loop order computations can be found in [**BroKr98**] and [**BroKr99**].

Momentum Scheme. In momentum scheme, (5.54) holds for all coefficients of the renormalized value of the graph Γ. To see this, we define a derivation $\theta_{-zL} : H_{FG} \to H_{FG}$ by

$$\theta_{-zL}(\Gamma) := e^{-zLY}(\Gamma) = e^{-zL|\Gamma|}\Gamma \tag{5.55}$$

for a Feynman graph Γ. Note that the exponential is the operator exponential of the grading operator Y and not the convolution exponential which appears in chapter 6. Assume now that the regularized Feynman rules are again given by (5.38). We rewrite it as

$$\phi_L(\Gamma)\{z\} = e^{-|\Gamma|zL}\phi_0(\Gamma)\{z\} \ , \tag{5.56}$$

where $L = \ln(q^2/\mu^2)$ is the momentum parameter. Written in this form, we can use the derivation θ_{-zL} and write

$$\phi_L(\Gamma)\{z\} = \phi_0(\theta_{-zL}(\Gamma))\{z\} \ , \tag{5.57}$$

by linearity of the character ϕ_0. In momentum scheme, the counterterm is $S_R^\phi = \phi_0 \circ S$, which allows us to represent the renormalized character in the form

$$\phi_{R,L} = (\phi_0 \circ S) * (\phi_0 \circ \theta_{-zL}) = \phi_0 \circ (S * \theta_{-zL}) = \phi_0 \circ (S * e^{-zLY}) \ , \tag{5.58}$$

where we have used that ϕ_0 is multiplicative. Then follows

$$\phi_{R,L}(\Gamma)\{z\} = \phi_0((S * e^{-zLY})(\Gamma))\{z\} = \sum_{l\geq 0} \frac{(-z)^l}{l!} L^l \phi_0 \circ (S * Y^l)(\Gamma) \ , \tag{5.59}$$

and thus,

$$c_l(\Gamma) := \lim_{z\to 0} \frac{(-z)^l}{l!} \phi_0 \circ (S * Y^l)(\Gamma)\{z\} \tag{5.60}$$

are the coefficients of the renormalized value of Γ in $\phi_{R,L}(\Gamma) = \sum_{l\geq 1} c_l(\Gamma)L^l$. Note that $c_0(\Gamma) = 0$ as $(S * Y^0)(\Gamma) = (S * \mathrm{id})(\Gamma) = 0$ by definition of the antipode S.

5.5. Virasoro algebras

Central extension of a Lie algebra. Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{Q}$ with Lie bracket $[\cdot,\cdot]$. A subspace $\mathfrak{a}$ is called (Lie algebra) *ideal* if $[\mathfrak{g},\mathfrak{a}] \subset \mathfrak{a}$. A Lie algebra is called *simple* if it has no nontrivial ideals, where the trivial ideals are the zero subspace $\{0\}$ and the Lie algebra itself.

For every $x \in \mathfrak{g}$ there is naturally a linear map $y \mapsto [x, y]$ denoted by ad_x. The assignment $x \mapsto \mathrm{ad}_x$ is a representation of $\mathfrak{g}$ on itself called *adjoint representation.* The reader may check by using the Jacobi identity that it is indeed a representation, i.e. for any $x, y \in \mathfrak{g}$

$$\mathrm{ad}_{[x,y]}(z) = [\mathrm{ad}_x, \mathrm{ad}_y](z) := (\mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x)(z) \qquad \forall z \in \mathfrak{g} \ . \tag{5.61}$$

The maps ad_x are derivations of the Lie bracket, i.e.

$$\mathrm{ad}_x([y, z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)] \ , \tag{5.62}$$

which the reader may also check quickly, again by employing the Jacobi identity. The kernel of the adjoint representation ad,

$$\mathcal{Z}(\mathfrak{g}) := \{ \ x \in \mathfrak{g} \mid \mathrm{ad}_x = 0 \ \} \tag{5.63}$$

is called the *centre* of $\mathfrak{g}$($\mathcal{Z}$ for german 'Zentrum'). These are all elements in $\mathfrak{g}$ that commute with all other elements. The centre is an ideal: if $a \in \mathcal{Z}(\mathfrak{g})$, then $[x, a] = 0 \in \mathcal{Z}(\mathfrak{g})$ for all $x \in \mathfrak{g}$. Because of this trivial commutator behaviour, *any* subspace $S \subset \mathcal{Z}(\mathfrak{g})$ is an ideal! The quotient vector space $\hat{\mathfrak{g}} := \mathfrak{g}/\mathcal{Z}(\mathfrak{g})$ is again a Lie algebra with Lie bracket

$$[x + \mathcal{Z}(\mathfrak{g}), y + \mathcal{Z}(\mathfrak{g})] := [x, y] + \mathcal{Z}(\mathfrak{g}) \ . \tag{5.64}$$

An *extension* of a Lie algebra $\mathfrak{c}$ is given by a short exact sequence of Lie algebras[3]

$$0 \to \mathfrak{a} \to \mathfrak{b} \to \mathfrak{c} \to 0 \tag{5.65}$$

where $\mathfrak{a}$ is an ideal of $\mathfrak{b}$ and $\mathfrak{c} \cong \mathfrak{b}/\mathfrak{a}$. One also says that $\mathfrak{b}$ is an extension of $\mathfrak{c}$ by $\mathfrak{a}$. Why 'extension'? Because one may say that $\mathfrak{b}$ arises when we add $\mathfrak{a}$ to $\mathfrak{c}$, i.e. $\mathfrak{b} \cong \mathfrak{c} \oplus \mathfrak{a}$.

The short exact sequence in (5.65) is called *central extension* of $\mathfrak{c}$ if $\mathfrak{a} \subset \mathcal{Z}(\mathfrak{b})$ is a subspace, i.e. an ideal from the centre of $\mathfrak{b}$. We may construct a very simple central extension of the Lie algebra $\hat{\mathfrak{g}}$ by

[3]See appendix for a very concise introduction.

putting a little subspace back in: take any element $c \in \mathcal{Z}(\mathfrak{g})$. Then $\mathfrak{a} := \mathbb{Q}c \subset \mathcal{Z}(\mathfrak{g})$ is a one-dimensional (sub)ideal. We set

$$\mathfrak{b} := \hat{\mathfrak{g}} \oplus \mathfrak{a} = \hat{\mathfrak{g}} \oplus \mathbb{Q}c \tag{5.66}$$

and have a central extension of $\hat{\mathfrak{g}}$ by $\mathfrak{a}$: it is given by $0 \to \mathfrak{a} \to \hat{\mathfrak{g}} \oplus \mathfrak{a} \to \hat{\mathfrak{g}} \to 0$.

Witt algebra. We recall from section 5.1, that a Lie algebra $\mathcal{W}$ with generators $\{L_n\}_{n\in\mathbb{Z}}$ and commutator

$$[L_m, L_n] = (m-n)L_{m+n} \tag{5.67}$$

is a so-called *Witt algebra*. Consider the derivation

$$\frac{d}{dx} : \mathbb{Q}[x^{-1}, x] \to \mathbb{Q}[x^{-1}, x] \tag{5.68}$$

on the ring of Laurent polynomials $\mathbb{Q}[x^{-1}, x]$ defined as usual by

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad\qquad n \in \mathbb{Z}\ . \tag{5.69}$$

Then the linear operators L_n

$$L_n = -x^{n+1}\frac{d}{dx} \tag{5.70}$$

generate a Witt algebra. In fact, by the Lie bracket (5.67), for any $n \in \mathbb{Z}$ the subspace

$$\mathcal{W}_n := \mathbb{Q}L_{-n} \oplus \mathbb{Q}L_0 \oplus \mathbb{Q}L_n \tag{5.71}$$

is a Lie subalgebra, i.e. a subspace of $\mathcal{W}$ closed under the Lie bracket. We may now extend $\mathcal{W}$ by a linear space $\mathbb{Q}c$ with a symbol c and arrive at the extended Lie algebra $\mathcal{V}$

$$\mathcal{V} := \mathcal{W} \oplus \mathbb{Q}c \tag{5.72}$$

with Lie bracket given by

$$[L_n, L_m]_\omega := (n-m)L_{n+m} + \omega(L_n, L_m)c\ , \qquad \text{and} \qquad [L_n, c]_\omega := 0 \tag{5.73}$$

for all $n, m \in \mathbb{Z}$, where $\omega : \mathcal{W} \otimes \mathcal{W} \to \mathbb{Q}$ is an antisymmetric bilinear form such that

$$(k-n)\omega(L_{k+n}, L_m) + (m-k)\omega(L_{m+k}, L_n) + (n-m)\omega(L_{n+m}, L_k) = 0 \tag{5.74}$$

for all $n, m, k \in \mathbb{Z}$. This condition garantees that (5.73) really is a Lie bracket. A simple possible choice for the bilinear form is

$$\omega(L_n, L_m) = \chi n(n^2-1)\delta_{n+m,0}\ , \tag{5.75}$$

where $\chi \in \mathbb{Q}$, and $\delta_{n,k}$ is the Kronecker delta, i.e. $\delta_{n,k} = 0$ for $k \neq n$ and $\delta_{n,n} = 1$. As an exercise, the reader may check that $\omega(\cdot,\cdot)$ in (5.75) really is antisymmetric and satisfies the Jacobi condition (5.74). This definition makes $\mathcal{W}_1 = \mathbb{Q}L_{-1} \oplus \mathbb{Q}L_0 \oplus \mathbb{Q}L_1$ into a trivial Lie subalgebra of $\mathcal{V}$ as

$$\omega \restriction_{\mathcal{W}_1 \otimes \mathcal{W}_1} = 0 \tag{5.76}$$

due to $\omega(L_n, \cdot) = 0 = \omega(\cdot, L_n)$ if $n \in \{-1, 0, 1\}$. Is this Lie subalgebra a Witt algebra? Yes. However, if we choose $\chi = 1/12$ the Lie algebra $\mathcal{V}$ is known as a *Virasoro algebra*, defined by (5.73). As we can see, this Lie algebra is a nontrivial one-dimensional central extension of the Witt algebra $\mathcal{W}$.

5.6. Insertion-Elimination Operators on Feynman graphs

We recall from section 5.2 the linear forms $\langle Z_T, \cdot\rangle$ indexed by trees on the Hopf algebra of rooted trees H defined by

$$\langle Z_T, T'\rangle = \begin{cases} 1 & T' = T \\ 0 & \text{else} \end{cases} \tag{5.77}$$

for trees $T, T' \in \mathcal{T}$ and $\langle Z_T, w\rangle = 0$ for any nontrivial forest $w \in \mathrm{Aug}^2 = \{\tau\tilde{\tau} : \tau, \tilde{\tau} \in \mathrm{Aug}\}$. We have learnt that the symbols Z_T span a Lie algebra $\mathcal{L}$ whose universal enveloping algebra $\mathcal{U}(\mathcal{L})$ is dual to H by the Milnor-Moore theorem. The analogous holds true for the Hopf algebra of Feynman graphs H_{FG} generated by symbols δ_Γ indexed by Feynman graphs Γ. To avoid awkward notational overload, we identify δ_Γ with Γ and write

$$\langle Z_\Gamma, \Gamma'\rangle = \begin{cases} 1 & \Gamma' = \Gamma \\ 0 & \text{else} \end{cases}\ . \tag{5.78}$$

We set them equal to zero on nontrivial products of Feynman graphs.

The reader shall be reminded of *derivations*, the space of which is denoted by $\mathrm{Der}(H_{FG})$: these are operators $D \in \mathrm{Der}(H_{FG})$ such that

$$D(XY) = D(X)Y + XD(Y) \tag{5.79}$$

for two Feynman graphs X, Y. This implies in particular $D(\mathbb{I}) = 0$. Because we may sometimes not appreciate this property, we shall relax our definition of derivations by only demanding (5.79) to hold for nontrivial products.

We define the *elimination operator* $Z^-_\Gamma : H_{FG} \to H_{FG}$ as a derivation by

$$Z^-_\Gamma(X) := (\langle Z_\Gamma, \cdot \rangle \otimes \mathrm{id})\Delta(X) = \sum_i \langle Z_\Gamma, X'_i \rangle X''_i \ , \tag{5.80}$$

for a Feynman graph X, where $\Delta(X) = \sum_i X'_i \otimes X''_i$ is a variation of Sweedler's notation. Note that only if the coproduct of the graph X picks out Γ as a subgraph of X does Z^-_Γ not vanish. In fact, the linear span

$$\mathcal{Z}^- := \langle Z^-_\Gamma, \ \Gamma \in H_{FG} \rangle_{\mathbb{Q}} \tag{5.81}$$

qualifies as a Lie algebra with the commutator $[Z^-_\Gamma, Z^-_{\Gamma'}] := Z^-_\Gamma Z^-_{\Gamma'} - Z^-_{\Gamma'} Z^-_\Gamma$ as bracket. It is the so-called *elimination Lie algebra* on the Hopf algebra of Feynman graphs H_{FG}. Another Lie algebra is given by derivations known as *insertion operators*

$$Z^+_\Gamma(X) := X \star \Gamma = \sum_{\Gamma'} n(\Gamma, X; \Gamma')\Gamma' \ , \tag{5.82}$$

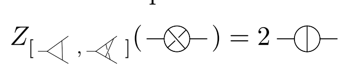
for a Feynman graph X, where the insertion of a graph into another has already been introduced in sections 2.1 and 5.2. Examples can be found there. These operators are part of a larger Lie algebra of derivations given by

$$Z_{[\Gamma_1,\Gamma_2]}(X) := \sum_i \langle Z_{\Gamma_2}, X'_i \rangle \ X''_i \star_{G_i} \Gamma_1 \ , \tag{5.83}$$

the so-called *insertion-elimination Lie algebra* on H_{FG}. The subscript bracket $[\Gamma_1, \Gamma_2]$ is to be read as a pair of data needed for the corresponding operator, without reference to a Lie bracket whatsoever. The symbol G_i stands for the *glueing data* of the i-th term of the coproduct of X: the operation $\star_{G_i}$ inserts the graph Γ_1 into where X'_i has been taken out. In particular, this means

$$X = X''_i \star_{G_i} X'_i \ . \tag{5.84}$$

Only if $X'_i = \Gamma_2$ for at least one i can $Z_{[\Gamma_1,\Gamma_2]}(X)$ be nonvanishing. For $X'_i = \mathbb{I}$ we have $X''_i = X$ and set $X''_i \star_{G_i} \Gamma_1 := X \star \Gamma_1$. We can therefore understand the action of $Z_{[\Gamma_1,\Gamma_2]}$ on $X \in H_{FG}$ as follows: it seeks out terms in the coproduct $\Delta(X)$ of the form $\Gamma_2 \otimes X/\Gamma_2$, inserts the graph Γ_1 into the cograph X/Γ_2 in place of Γ_1 and sums up all such terms. An example is

(5.85) 

where $\Gamma_2 =$  is cut out and replaced by $\Gamma_1 =$ , as it emerges in the coproduct

(5.86) 

However, the operator $Z_{[\Gamma_1,\Gamma_2]}$ may vanish on X for two reasons: Γ_2 is not a subgraph of X or Γ_1 and Γ_2 do not have the same external leg structure. The reader may ponder over this one: the latter case entails $Z_{[\Gamma_1,\Gamma_2]}(X) = 0$ for all Feynman graphs $X \in H_{FG}$.

The insertion and elimation operators partake of this insertion-elimination operator family due to

$$Z^+_\Gamma = Z_{[\Gamma,\mathbb{I}]} \ , \qquad\qquad Z^-_\Gamma = Z_{[\mathbb{I},\Gamma]} \ . \tag{5.87}$$

This is because inserting or eliminating the empty graph $\mathbb{I}$ is tantamount to not inserting or eliminating anything, respectively. The commutator can be shown to yield

$$\begin{aligned}[Z_{[\Gamma_1,\Gamma_2]}, Z_{[\Gamma_3,\Gamma_4]}] = {} & Z_{[Z_{[\Gamma_1,\Gamma_2]}(\Gamma_3),\Gamma_4]} - Z_{[\Gamma_3, Z_{[\Gamma_2,\Gamma_1]}(\Gamma_4)]} - Z_{[Z_{[\Gamma_3,\Gamma_4]}(\Gamma_1),\Gamma_2]} + Z_{[\Gamma_1, Z_{[\Gamma_4,\Gamma_3]}(\Gamma_2)]} \\ & + \delta_{\Gamma_1,\Gamma_4} Z_{[\Gamma_3,\Gamma_2]} - \delta_{\Gamma_2,\Gamma_3} Z_{[\Gamma_1,\Gamma_4]} \ , \end{aligned} \tag{5.88}$$

where

$$\delta_{\Gamma,\Gamma'} = \begin{cases} 1 & \Gamma = \Gamma' \\ 0 & \text{else} \end{cases} \tag{5.89}$$

is the Kronecker delta map for graphs. For a proof, see [**CoKr02**]. However messy this may look, there is pattern that the involved indices follow. It is there for the reader to be discovered.

Hopf algebra of words. Let A be a (possibly infinite) set of symbols which we call *alphabet*. If we take its elements and freely generate the noncommutative but associative algebra W over, say $\mathbb{Q}$, then any element is a linear combination of terms of the form

$$w = a_1 a_2 \,...\, a_n \ , \qquad\qquad a_j \in A \tag{5.90}$$

called *words*. It can be made into a Hopf algebra with coproduct

$$\Delta(a_1 a_2 ... a_n) = \mathbb{I} \otimes a_1 a_2 ... a_n + a_1 a_2 ... a_n \otimes \mathbb{I} + \sum_{j=1}^{n-1} a_1 ... a_j \otimes a_{j+1} ... a_n \ . \tag{5.91}$$

One can now introduce operators

$$Z_{w_1, w_2}(w) := \begin{cases} w_1 v & \text{if } \ w = w_2 v \\ 0 & \text{else} \end{cases} \tag{5.92}$$

which take out the subword w_2 and replace it by w_1. Note that in the coproduct of $w' = v w_2$ the subword w_2 will not appear on the lhs of the tensor sign on its own: therefore $Z_{w_1,w_2}(w') = 0$. This is completely analogous to what the corresponding insertion-elmination operators on H_{FG} do: only if the coproduct cuts out a subgraph(here: subword) and puts it to the lhs of the tensor sign does the operator not vanish. The commutator Lie bracket yields

$$\begin{aligned} [Z_{w_1,w_2}, Z_{w_3,w_4}] = {} & Z_{Z_{w_1,w_2}(w_3),w_4} - Z_{w_3,Z_{w_2,w_1}(w_4)} - Z_{Z_{w_3,w_4}(w_1),w_2} + Z_{w_1,Z_{w_4,w_3}(w_2)} \\ & + \delta_{w_1,w_4} Z_{w_3,w_2} - \delta_{w_2,w_3} Z_{w_1,w_4} \ , \end{aligned} \tag{5.93}$$

with an obviously equal index pattern as the insertion-elimination operators on H_{FG}(see also [**MeKr02**]).

5.7. Insertion-Elimination Lie algebra: the ladder case

An insertion-elimination Lie algebra can also be introduced on the Hopf algebra of rooted trees H. The insertion operators are given by

$$N_\tau(T) := \sum_{v \in T^{[0]}} T \cup_v \tau \tag{5.94}$$

for trees τ, T: the operation $T \mapsto T \cup_v \tau$ glues the root $r(\tau)$ of the tree τ to the vertex $v \in T^{[0]}$ of the tree T in such a way that

$$(T \cup_v \tau)^{[0]} = T^{[0]} \cup \tau^{[0]} \ , \qquad\qquad (T \cup_v \tau)^{[1]} = T^{[1]} \cup \tau^{[1]} \cup (v, r(\tau)) \ , \tag{5.95}$$

i.e. the additional edge $(v, r(\tau))$ connects the two trees. We have already encountered a special member of this family: the natural growth operator $N : H \to H$ in section 3.2. It is given by $N = N_\bullet$, which simply grafts a single leaf $\tau = \bullet$ to each vertex. As we have set $N_\bullet(\mathbb{I}) = \bullet$, this operator is also only a derivation in the weak sense on H as discussed above. The elimination operator is defined like that in (5.80)

$$M_\tau(T) := \sum_i \langle Z_\tau, T_i' \rangle T_i'' \tag{5.96}$$

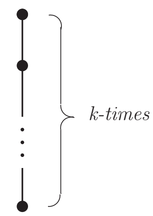
for a tree T and extended to a derivation on H. The general insertion-elimination operator is

$$Z_{[t_1,t_2]}(T) := \sum_i \langle Z_{t_2}, T_i' \rangle \ T_i'' \cup_{G_i} t_1 \ . \tag{5.97}$$

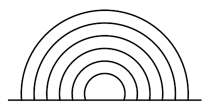
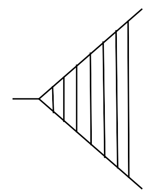
For the nasty case[4] $t_2 = \mathbb{I}$ we set $Z_{[t_1,\mathbb{I}]} := N_{t_1}$ and the nice case is $Z_{[\mathbb{I},t_2]} = M_{t_2}$.

[4]'Nasty' is the glueing data: an empty set void of any glueing directives.

Ladder Hopf algebra. We recall the ladder trees $\lambda_k \in H$

(5.98) $\lambda_k =$ 

with coproduct $\Delta(\lambda_k) = \sum_{j=0}^{k} \lambda_j \otimes \lambda_{k-j}$. They form a trivial Hopf subalgebra H_ℓ within H. These trees correspond to 'rainbow' or 'ladder' graphs like

(5.99)  and 

with a trivial subgraph structure. To obtain the corresponding insertion-elimination operators on H_ℓ we need to modify the insertion operator slightly. We do not have to modify the elimination operator M_{λ_m} as it already by definition satifies $M_{\lambda_m}(H_\ell) \subset H_\ell$ for any $m \in \mathbb{N}$. The insertion operator N_{λ_n} does not do us this favour, as it glues λ_n to *every* vertex of the argument tree, rendering sidebranchings. However, a simple choice is

$$Z^+_n(\lambda_k) := \lambda_{k+n} \qquad n \in \mathbb{N} \ , \tag{5.100}$$

which just grafts $\lambda_n \in H_\ell$ to λ_k at its only leaf down at the bottom. We denote the elimination operator by

$$Z^-_m(\lambda_k) = \theta(k-m)\lambda_{k-m} \qquad m \in \mathbb{N} \ , \tag{5.101}$$

where $\theta(n) = 1$ if $n \geq 0$ and vanishing otherwise. This means the ladder must be long enough as one cannot remove more rungs than are already there in the first place. The reader may check that indeed $Z^-_m = M_{\lambda_m}$ from (5.96). The more general insertion-elimination operators are then given by

$$Z_{n,m}(\lambda_k) := \theta(k-m)\lambda_{k-m+n} \ . \tag{5.102}$$

This coincides with $Z_{[\lambda_n,\lambda_m]}$ if $m \neq 0$(excluding the nasty case). These derivations comprise a doubly infinite family with at first glance messy commutator

$$\begin{aligned}[Z_{n,m}, Z_{l,s}] = \theta(l-m)Z_{n,l-m+s} - \theta(s-n)Z_{l,s-n+m} - \theta(n-s)Z_{n-s+l,m} \\ + \theta(m-l)Z_{n,m-l+s} - \delta_{m,l}Z_{n,s} + \delta_{n,s}Z_{m,l} \ .\end{aligned} \tag{5.103}$$

It is not messy, though: it is the analogon of (5.88), easy to check by replacing graphs by ladders. This Lie algebra, let it be denoted by $\mathfrak{L}_\ell$, has a grading: if we define the degree of an element by $\deg(Z_{n,m}) := n-m$ then the $\mathbb{Z}$-grading reads

$$\mathfrak{L}_\ell = \bigoplus_{j\in\mathbb{Z}} \ell_j \ , \qquad \ell_j := \mathrm{span}_{\mathbb{C}}\{Z_{n,m} \mid n,m \in \mathbb{N} : \deg(Z_{n,m}) = j \ \} \ . \tag{5.104}$$

Indeed, as an exercise the reader may check that $[\ell_k, \ell_i] \subset \ell_{k+i}$ really is satisfied by (5.103). The grading index has a straightforward interpretation: the elements in ℓ_j all effectively increase the length of a ladder by j. By the grading property, we have a decomposition of $\mathfrak{L}_\ell$ into Lie subalgebras

$$\mathfrak{L}_\ell = \mathfrak{L}_- \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_+ \tag{5.105}$$

where $\mathfrak{L}_- = \bigoplus_{j<0} \ell_j$, $\mathfrak{L}_0 = \ell_0$ and $\mathfrak{L}_+ = \bigoplus_{j>0} \ell_j$. In fact, $\mathfrak{L}_0$ is an abelian Lie subalgebra: all of its elements commute with each other by (5.103).

Classical Lie algebras. The classical infinite dimensional Lie algebra

$$\mathfrak{gl}(\infty) := \{ \ E_{i,j} \mid i,j \in \mathbb{Z} \ \} \tag{5.106}$$

of generators $E_{i,j}$ with Lie bracket

$$[E_{i,j}, E_{k,l}] = \delta_{j,k} E_{i,l} - \delta_{i,l} E_{k,j} \tag{5.107}$$

has a Lie subalgebra $\mathfrak{gl}_+(\infty) := \{E_{i,j} | i,j \geq 0\}$ which is isomorphic to an ideal of the insertion-elimination Lie algebra $\mathfrak{L}_\ell$: though tedious, one can show that the operators

$$E_{i,j} := Z_{i,j} - Z_{i+1,j+1} \tag{5.108}$$

obey (5.107) and form an ideal. We may identify this ideal with $\mathfrak{gl}_+(\infty)$ and consider the short exact sequence

$$0 \to \mathfrak{gl}_+(\infty) \longrightarrow \mathfrak{L}_\ell \longrightarrow C \to 0 \tag{5.109}$$

with $C := \mathfrak{L}_\ell / \mathfrak{gl}_+(\infty)$. It turns out that $\mathfrak{L}_\ell$ is not simple and that C allows for inifinitely many non-equivalent central extensions.

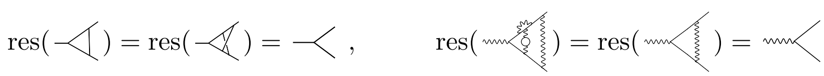
However, there is a certain chance that one may draw on the existing vast body of knowledge about infinite dimensional Lie algebras to further the understanding of the combinatorial structure of perturbation theory in QFT.

CHAPTER 6

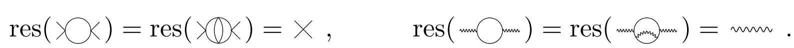
Renormalization Group

6.1. Formal power series and Green functions

Let $\Gamma \in H_{FG}$ be a Feynman graph. The *residue* of Γ is the graph $\mathrm{res}(\Gamma)$ obtained from Γ by shrinking *all* internal edges to a single point. Instead of residue, we shall also speak of the *external leg structure*. Examples are

(6.1) 

and

(6.2) 

By $\mathcal{R}$ we denote a set of such residues of interest for a given renormalizable theory. It is generally finite. The *valence* $\mathrm{val}(r)$ of the residue $r = \mathrm{res}(\Gamma)$ is defined as the number of external legs of the corresponding graph Γ.

We consider formal power series in one variable α with coefficients in H_{FG} for example of the form

$$\Gamma^r(\alpha) = \mathbb{I} \pm \sum_{\mathrm{res}(\Gamma)=r} \frac{\alpha^{|\Gamma|}}{\mathrm{Sym}(\Gamma)} \Gamma \tag{6.3}$$

where the sum is over all 1PI graphs with external leg structure r and $\mathrm{Sym}(\Gamma)$ is a symmetry factor associated to the graph Γ. If $\mathrm{val}(r) = 2$, then there is a minus sign in (6.3), and a plus sign in all other cases. We formally apply a character representing some given Feynman rules and get a perturbative expansion

$$G^r(\alpha, L, \theta) := \phi(\Gamma^r(\alpha))\{L, \theta\} = 1 \pm \sum_{\mathrm{res}(\Gamma)=r} \frac{\alpha^{|\Gamma|}}{\mathrm{Sym}(\Gamma)} \phi(\Gamma)\{L, \theta\} \ , \tag{6.4}$$

of what is known as a *Green function* $G^r(\alpha, L, \theta)$ in which L and θ are external scale and angle parameters or collections of such, respectively. If $\mathrm{val}(r) = 2$, we refer to G^r as *two-point function* and if $\mathrm{val}(r) \geq 3$ as *vertex function*. Strictly speaking, this Green function is the corresponding *structure function* for the amplitude $r \in \mathcal{R}$. The textbook Green function is then given by multiplication of G^r with a form factor such as p^2 or $\not{p} = p_\mu \gamma^\mu$ for an incoming momentum $p \in \mathbb{R}^4$, well-known to readers acquainted with QFT.

6.2. Combinatorial Dyson-Schwinger equations

The formal series $X(\alpha) = \sum_{k \geq 0} \alpha^k \lambda_k \in H_\ell[[\alpha]]$ with coefficients in the ladder Hopf subalgebra satisfies the equation

$$X(\alpha) = \mathbb{I} + \alpha B_+(X(\alpha)) \ , \tag{6.5}$$

which can be easily checked since $B_+(\lambda_k) = \lambda_{k+1}$ for all $k \in \mathbb{N}$. This equation is a simple example of a *Dyson-Schwinger equation*. Such equations do also exist for series with cofficients in the Feynman graph Hopf algebra H_{FG} like in (6.3). They are systems of equations of the form

$$\Gamma^r(\alpha) = \mathbb{I} + \mathrm{sgn}(s_r) B_+^r(\Gamma^r(\alpha), Q(\alpha)) \ , \qquad r \in \mathcal{R} \ , \tag{6.6}$$

where $Q(\alpha)$ is the so-called *invariant charge* given by

$$Q(\alpha) = \prod_{r \in \mathcal{R}} (\Gamma^r(\alpha))^{s_r} \tag{6.7}$$

with integers s_r. If $\mathrm{val}(r) = 2$ one has $s_r < 0$ and $s_r > 0$ otherwise. This ensures a minus sign in (6.6) for a propagator series. The operator $B_+^r(\cdot,\cdot)$ is defined as

$$B_+^r(\Gamma^r(\alpha), Q(\alpha)) = \sum_{k\geq 1} \alpha^k B_+^{k;r}(\Gamma^r(\alpha)Q(\alpha)^k) \tag{6.8}$$

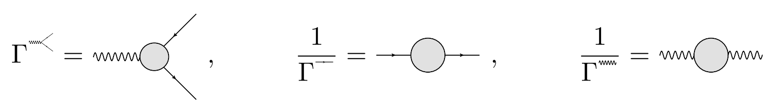
with one-cocycles $B_+^{k;r}$ which themselves are defined by

$$B_+^{k;r} = \sum_{\mathrm{res}(\gamma)=r,|\gamma|=k,\mathrm{prim.}} \frac{1}{\mathrm{Sym}(\gamma)} B_+^{\gamma} \tag{6.9}$$

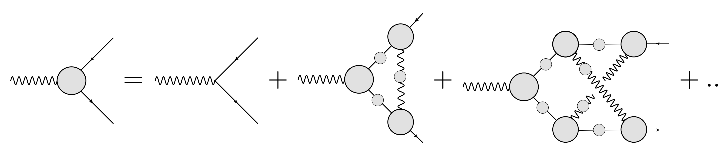
with one-cocyles B_+^{γ}. The sum extends over all 1PI primitive graphs γ with external leg structure r and loop number k. Recall that a graph γ is called primitive if $\Delta(\gamma) = \gamma \otimes \mathbb{I} + \mathbb{I} \otimes \gamma$. Notice that, in general, there are infinitely many primitive graphs and hence the sum in (6.8) is not finite. An example for the invariant charge $Q(\alpha)$ in QED is

$$Q(\alpha) = \frac{\Gamma^{\sim<}(\alpha)^2}{\Gamma^{\sim}(\alpha)\Gamma^{-}(\alpha)^2} \ . \tag{6.10}$$

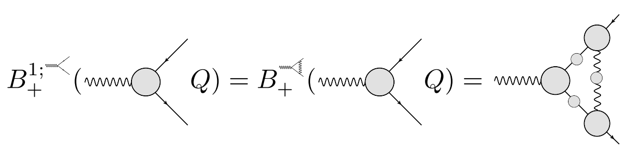
However cryptic these expressions may look, the product $\Gamma^r(\alpha)Q(\alpha)^k$ of formal power series has coefficients in H_{FG} which are exactly what one can glue into a 1PI primitive graph γ with k loops and external leg structure r. This glueing corresponds to what is known as vertex or propagator corrections in standard QFT where our formal series are generally depicted by graphs with blobs: for QED they take the form

(6.11) 

The Dyson-Schwinger equation for the QED vertex reads in this notation

(6.12) 

where the tree-level graph $\sim< = \mathbb{I}$ is what we count as an empty graph. To understand the action of the one-cocycles, consider the second term on the rhs of (6.12): it can be written as

(6.13) 

and has the following meaning: the growth operator $B_+^{\sim\triangleleft}$ uses the vertex series $\Gamma^{\sim<} = \sim<$ to provide for all radiative corrections at one vertex, say the leftmost one of the superscript skeleton graph $\gamma = \sim\triangleleft$. Then, it takes the invariant charge Q to glue in additional graphs so as to guarantee that every propagator is fully dressed and the remaining vertices are fully corrected. For the higher loop primitives, higher powers of Q are needed to dress all propagators and vertices which come with additional loops.

However, we come back to the general case and rewrite (6.6) into

$$\Gamma^r(\alpha) = \mathbb{I} + \mathrm{sgn}(s_r) \sum_{k\geq 1} \alpha^k B_+^{k;r}(\Gamma^r(\alpha)Q(\alpha)) \ , \qquad r \in \mathcal{R} \tag{6.14}$$

whose solution exists and may be written in the form

$$\Gamma^r(\alpha) = \mathbb{I} + \mathrm{sgn}(s_r) \sum_{k=1}^{\infty} \alpha^k c_k^r \ , \qquad r \in \mathcal{R} \ , \tag{6.15}$$

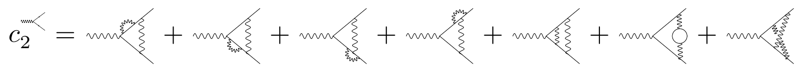
where $c_k^r \in H_{FG}$ is a linear combination of 1PI graphs with k loops and external leg structure r. These coefficients generate a Hopf subalgebra with coproduct

$$\Delta(c_k^r) = \sum_{j=0}^{k} P_{k,j}^r \otimes c_{k-j}^r \ , \tag{6.16}$$

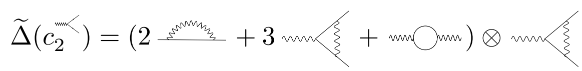
where $P^r_{k,j}$ is a polynomial in these generators(see also [**KrY06**]). For example, in QED one has

(6.17) 

and

(6.18) 

The reduced coproduct of the latter is

(6.19) 

which is, in terms of the generators,

(6.20)
$$\widetilde{\Delta}(c_2) = (2\ c_1 + 3\ c_1 + c_1) \otimes c_1 = P_{2,1} \otimes c_1 \ .$$

The other polynomials are $P_{2,0} = \mathbb{I}$ and $P_{2,2} = c_2$ for the trivial part of the coproduct.

6.3. The structure of Green functions

If we apply the renormalized Feynman rules ϕ_R to (6.15) as in (6.4), the corresponding Green function reads

(6.21)
$$G^r_R(\alpha, L, \theta) = \phi_R(\Gamma^r(\alpha))\{L, \theta\} = 1 + \operatorname{sgn}(s_r) \sum_{k=1}^{\infty} \alpha^k \phi_R(c^r_k)\{L, \theta\} \ .$$

The individual coefficients $\phi_R(c^r_k)$ are polynomials in the external scale parameter L which is why we can rewrite (6.21) to obtain

(6.22)
$$G^r_R(\alpha, L, \theta) = 1 + \sum_j \gamma^r_j(\alpha, \theta) L^j \ ,$$

where j may be a multi-index and $\gamma^r_j(\alpha, \theta)$ is a function of the loop parameter α and the angle parameter θ. In a very simple linear case, where $Q(\alpha) = \mathbb{I}$ and the operators in (6.9) are simplified significantly to yield the analogon of (6.5) for $H_{FG}[[\alpha]]$, the two-point Green function in (6.22) takes the form

(6.23)
$$G(\alpha, L) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \gamma(\alpha)^j L^j = \exp(-\gamma(\alpha) L) \ ,$$

i.e. $\gamma_j(\alpha) = (-1)^j \gamma(\alpha)^j / j!$, where $\gamma(\alpha)$ is known as the anomalous dimension.

The Dyson-Schwinger equations in (6.6) for the Hopf algebra of Feynman graphs H_{FG}, henceforth abbreviated by DSE, correspond to a system of integral equations for the Green functions in the target algebra $\mathcal{A}$ of the Feynman rules. This is on account of the universal property of graded Hopf algebras with Hochschild one-cocycles according to which the operators B^γ_+ in (6.9) translate to integral operators on the target algebra of the Feynman rules. This may take the form

(6.24)
$$(\phi \circ B^\gamma_+)(X)\{q\} = \int d_\gamma(k, q)\ \phi(X)\{k, q\}$$

for a graph X with some integration measure $d_\gamma(k, q)$ associated to the graph γ. The renormalized version of (6.24) is

(6.25)
$$(\phi_R \circ B^\gamma_+)(X)\{q\} = \int d_\gamma(k, q)\ (\phi(X)\{k, q\} - \phi(X)\{k, q_0\})$$

where q_0 is an external momentum such that $q_0^2 = \mu^2$, with μ being the renormalization point. To distinguish between these two different types of DSE we refer to the system of integral equations as *analytic* DSE and those in (6.6) as *combinatorial* DSE.

Infinitesimal characters. There is an interesting way to obtain the coefficient functions $\gamma_j^r(\alpha)$ in (6.22), where we suppress the angle-dependence in the notation for the moment. First we define a linear map $Y^{-1}: H_{FG} \to H_{FG}$ by $Y^{-1}(\mathbb{I}) = 0$ and

$$Y^{-1}(\gamma) = \frac{1}{|\gamma|}\gamma \tag{6.26}$$

for a product of Feynman graphs $\gamma = \prod_j \gamma_j$, where $|\gamma| := \sum_j |\gamma_j|$ counts the loops. This choice of notation is justified as Y^{-1} really is the inverse of the grading operator Y on the augmentation ideal Aug. Next, we introduce a family of linear maps $\sigma_n : H_{FG} \to \mathbb{C}$ by

$$\sigma_1 := \partial_L \phi_R Y^{-1}(S * Y)|_{L=0} \tag{6.27}$$

and

$$\sigma_n := \frac{1}{n!}\sigma_1^{*n} := \frac{1}{n!}\underbrace{\sigma_1 * \dots * \sigma_1}_{n-\text{times}} = \frac{1}{n!}m^{n-1}\sigma_1^{\otimes n}\Delta^{n-1} \tag{6.28}$$

for $n \geq 2$, where m is the usual multiplication in $\mathbb{C}$ and $*$ is the convolution product

$$\sigma_1 * \sigma_1 = m(\sigma_1 \otimes \sigma_1)\Delta\ . \tag{6.29}$$

Note that the map σ_1 is a so-called *infinitesimal character* on H_{FG} which means

$$\sigma_1(xy) = \sigma_1(x)\hat{\mathbb{I}}(y) + \hat{\mathbb{I}}(x)\sigma_1(y) \tag{6.30}$$

for all $x, y \in H_{FG}$. This implies $\sigma_1(\mathbb{I}) = 0$ and that it vanishes on nontrivial products, i.e.

$$\sigma_1(h) = 0 \tag{6.31}$$

if $h = h_1h_2$ with $h_1, h_2 \in$ Aug.

Lemma 6.3.1. *$S * Y$ is an infinitesimal character.*

Proof. Let $x, y \in$ Aug. Then

$$\begin{aligned}
(S * Y)(xy) &= \sum_{(x)}\sum_{(y)} S(x'y')Y(x''y'') \\
&= \sum_{(x)}\sum_{(y)}[S(x')S(y')Y(x'')y'' + S(x')S(y')x''Y(y'')] \\
&= \sum_{(x)} S(x')Y(x'')\sum_{(y)} S(y')y'' + \sum_{(x)} S(x')x''\sum_{(y)} S(y')Y(y'') = 0
\end{aligned} \tag{6.32}$$

on account of $\sum_{(x)} S(x')x'' = (\mathrm{id} * S)(x) = (S * \mathrm{id})(x) = 0$ which holds by definition of the antipode S. □

The next assertion makes clear why these maps are of particular interest to us.

Proposition 6.3.2. *The linear map σ_n evaluates a graph Γ to its n-th order coefficient of $\phi_R(\Gamma)$ with respect to the variable L, i.e.*

$$\sigma_n(\Gamma) = \frac{1}{n!}\frac{\partial^n}{\partial L^n}\phi_R(\Gamma)\{L\}\Big|_{L=0}\ . \tag{6.33}$$

Proof. We have to use the fact that the set $\mathfrak{g}$ of infinitesimal characters is the Lie algebra generating the Lie group of characters G on H_{FG} in the sense that $G = \exp_*(\mathfrak{g})$, i.e. for every character ϕ, there exists an infinitesimal character $\sigma \in \mathfrak{g}$ such that

$$\phi = \exp_*(\sigma) := \sum_{n=0}^{\infty}\frac{\sigma^{*n}}{n!} \tag{6.34}$$

and vice versa with $\sigma_1^{*0} := \hat{\mathbb{I}}$ being the neutral element of the convolution product $*$. The inverse map of $\exp_*$ is given by

$$\log_*(\phi) = -\sum_{n=1}^{\infty}\frac{1}{n}(\hat{\mathbb{I}} - \phi)^{*n} = \sigma\ . \tag{6.35}$$

For more on this, see Appendix section A.3 or [**Man06**]. This is but a small step away from realizing that $\exp_*(L\mathfrak{g})$ for a variable L is the character group with target algebra $\mathbb{C}[L]$, i.e. for our character ϕ_R we have

$$\phi_R = \exp_*(L\sigma_R) \tag{6.36}$$

with some generator σ_R(see Appendix A.3). Then, clearly, we find

$$\partial_L \phi_R = \sigma_R * \phi_R \qquad \Rightarrow \qquad \partial_L \phi_R|_{L=0} = \sigma_R \ . \tag{6.37}$$

To prove (6.33) it suffices to show that $\sigma_R = \sigma_1$. To this end, we take a Feynman graph Γ and first calculate

$$\begin{aligned} \phi_R Y^{-1}(S*Y)(\Gamma) &= \left(\hat{\mathbb{I}} + L\sigma_R + \frac{L^2}{2!}(\sigma_R * \sigma_R) + ...\right) Y^{-1}(S*Y)(\Gamma) \\ &= L\sigma_R Y^{-1}(S*Y)(\Gamma) + \mathcal{O}(L^2) = \frac{L}{|\Gamma|}\sigma_R\left(\sum_j S(\Gamma'_j)Y(\Gamma''_j)\right) + \mathcal{O}(L^2) \\ &= \frac{L}{|\Gamma|}\sigma_R(S(\mathbb{I})Y(\Gamma)) + \mathcal{O}(L^2) = L\sigma_R(\Gamma) + \mathcal{O}(L^2) \ . \end{aligned} \tag{6.38}$$

□

A nice consequence is the following

Corollary 6.3.3. *The coefficient functions of the Green function G^r are given by*

$$\gamma_j^r(\alpha) = \sigma_j(\Gamma^r(\alpha)) \qquad \textit{and} \qquad G_R^r(\alpha, L) = \exp_*(L\sigma_1)(\Gamma^r(\alpha)) \ , \tag{6.39}$$

where the $*$-exponential is defined as in (6.34).

6.4. Renormalization Group Equation

The coefficient functions γ_k^r of the Green function G^r satisfy

$$\gamma_k^r(\alpha) = \frac{1}{k}\left(\gamma_1^r(\alpha) + \sum_{u\in\mathcal{R}} s_u \gamma_1^u(\alpha)\alpha\partial_\alpha\right)\gamma_{k-1}^r(\alpha) \ , \qquad r \in \mathcal{R} \ , \tag{6.40}$$

which is a consequence of

$$(\mathcal{P}_{lin}\otimes\mathcal{P}_{lin})\Delta(\Gamma^r(\alpha)) = \mathcal{P}_{lin}\Gamma^r(\alpha)\otimes\mathcal{P}_{lin}\Gamma^r(\alpha) + \mathcal{P}_{lin}Q(\alpha)\otimes\alpha\partial_\alpha\mathcal{P}_{lin}\Gamma^r(\alpha) \ , \tag{6.41}$$

where $\mathcal{P}_{lin}$ is the projector onto the linear span of the Hopf algebra's generators, i.e. the Feynman graphs, but excluding $\mathbb{I}$. It is a fairly easy exercise to derive the so-called *renormalization group equation*

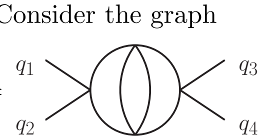
$$\left(-\frac{\partial}{\partial L} + \alpha\beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_1^r(\alpha)\right) G^r(\alpha, L) = 0 \tag{6.42}$$

from (6.40) with $G^r(\alpha, L) = 1 + \sum_{k=1}^{\infty}\gamma_k^r(\alpha)L^k$ and the function

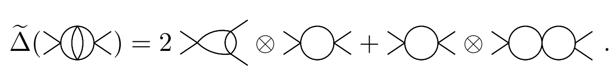
$$\beta(\alpha) := \partial_L\phi_R(Q(\alpha))|_{L=0} = ... = \sum_{u\in\mathcal{R}} s_u\gamma_1^u(\alpha) \tag{6.43}$$

known as β-function of the corresponding theory. A proof of both (6.40) and (6.42) can be found in Appendix section A.5, where the reader will also be introduced to a slightly stronger version of (6.41) and see how to fill the void ... in (6.43). Further relevant references are [**KrSui06**] and [**Y11**].

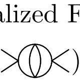
Example: a scalar 3-loop graph. Consider the graph

(6.44) $\Gamma =$ 

with reduced coproduct

(6.45) 

Say, the physical limit of some renormalized Feynman rules ϕ_R is

(6.46) $\phi_R($$) = c_1L + c_2L^2 + c_3L^3$, where $L = \ln(q^2/\mu^2)$ with $q := q_1 + q_2 = q_3 + q_4$, by momentum conservation. Given that we have

$$\phi_R(X)\{L\} = \sum_{j=1}^{\operatorname{cor}(X)} \sigma_j(X) L^j = \sum_{j=1}^{\operatorname{cor}(X)} \frac{1}{j!} \sigma_1^{*j}(X) L^j \tag{6.47}$$

for a Feynman graph X and the infinitesimal characters $\sigma_j : H_{FG} \to \mathbb{C}$ introduced in the previous section, we want to see how the coefficients c_1, c_2 and c_3 relate to those of its subgraphs. The coradical degree of a graph X is defined by

$$\operatorname{cor}(X) = \min\{\ n \mid \mathcal{P}_{lin}^{(n+1)}(X) = 0\ \}\ , \tag{6.48}$$

with $\mathcal{P}_{lin}^{(n+1)} := \mathcal{P}_{lin}^{\otimes n+1} \Delta^n$, analogous to the definitions for the coradical filtration of the Hopf algebra of rooted trees H in section 3.5. Let now for the subgraphs

$$\phi_R(\ ) = e_1 L + e_2 L^2\ , \qquad \phi_R(\ ) = d_1 L\ , \qquad \phi_R(\ ) = d_1^2 L^2 \tag{6.49}$$

be the case. The infinitesimal character $Y^{-1}(S * Y)$ yields

$$Y^{-1}(S * Y)(\ ) = \ - \frac{2}{3} \ \ + \frac{2}{3} (\ )^3 - \frac{2}{3} \ \tag{6.50}$$

which evaluates to

$$\phi_R(Y^{-1}(S * Y)(\ )) = c_1 L + \frac{1}{3}(3c_2 - 2e_1 d_1) L^2 + \frac{1}{3}(3c_3 - 2e_2 d_1) L^3\ . \tag{6.51}$$

Not surprisingly, the map σ_1 picks out the term

$$\sigma_1(\ ) = c_1\ . \tag{6.52}$$

The next map $\sigma_2 = (\sigma_1 * \sigma_1)/2!$ yields

$$\sigma_2(\ ) = \frac{2}{2!} \sigma_1(\ ) \sigma_1(\ ) + \frac{1}{2!} \sigma_1(\ ) \underbrace{\sigma_1(\ )}_{=0} = e_1 d_1\ . \tag{6.53}$$

For the third coefficient we have

$$\sigma_3(\ ) = \frac{2}{3!} \sigma_1(\ ) \sigma_1(\ ) \sigma_1(\ ) = \frac{1}{3} d_1^3 \tag{6.54}$$

since

$$P_{lin}^{(3)}(\ ) = P_{lin}^{\otimes 3}(\Delta \otimes \mathrm{id})\Delta(\ ) = 2\ \ \otimes \ \otimes \ \ . \tag{6.55}$$

All higher σ_n for $n \geq 4$ evaluate to zero, which is no suprise as the coradical degree of Γ is

$$\operatorname{cor}(\ ) = 3\ . \tag{6.56}$$

We conclude that the leading log coefficient $c_3 = d_1^3/3$ and the next-to-leading log coefficient $c_2 = e_1 d_1$ of $\phi_R(\Gamma)$ are determined by the value of σ_1 on the subgraphs of Γ. This is not surprising if we write ϕ_R as $*$-exponential

$$\phi_R = \exp_*(L\sigma_1) = \hat{\mathbb{I}} + L\sigma_1 + \frac{L^2}{2!} \sigma_1 * \sigma_1 + \frac{L^3}{3!} \sigma_1 * \sigma_1 * \sigma_1 + \ldots \tag{6.57}$$

with infinitesimal character σ_1: all terms of higher order than $k = 1$ contain only values of σ_1 on *proper subgraphs* and *cographs* of Γ since the trivial part of the coproduct of Γ evaluates to zero on account of $\sigma_1(\mathbb{I}) = 0$:

$$(\sigma_1 \otimes \sigma_1)(\mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I}) = \sigma_1(\mathbb{I})\sigma_1(\Gamma) + \sigma_1(\Gamma)\sigma_1(\mathbb{I}) = 0\ . \tag{6.58}$$

6.5. Renormalization Group Flow

We define be a family of derivations $\{\theta_t\}_{t\geq 0}$ on H_{FG} by setting $\theta_t(\Gamma) := e^{|\Gamma|t}\Gamma$ for a Feynman graph Γ, which is related to the grading operator Y according to

$$Y(\Gamma) = \frac{d}{dt}\theta_t(\Gamma)\bigg|_{t=0} \ . \tag{6.59}$$

Both Y and θ can also be defined as maps acting on linear maps $\psi : H_{FG} \to \mathbb{C}$ through

$$(Y\psi)(\Gamma) := \psi(Y(\Gamma)) \ , \qquad\qquad (\theta_t\psi)(\Gamma) := \psi(\theta_t(\Gamma)) \ . \tag{6.60}$$

Recall that regularized Feynman rules ϕ yield parameter-dependent functions $\phi(\Gamma)\{z,\mu\}$, where $z \in \mathbb{C}$ and $\mu > 0$ are the regulator and the renormalization scale parameter, respectively. In the following, we consider Feynman rules ϕ on H_{FG} such that

$$\theta_{tz}\phi(\Gamma)\{z,\mu\} = \phi(\Gamma)\{z,\mu e^t\} \ . \tag{6.61}$$

This is for example the case if the graph Γ is mapped to terms proportional to factors like

$$\left(\frac{q^2}{\mu^2}\right)^{-z|\Gamma|/2} = e^{-z|\Gamma|L/2} \ . \tag{6.62}$$

Each choice of $\mu > 0$ corresponds to a fixed renormalization scheme. Continuously changing it by $t \mapsto \mu e^t$ amounts to 'flowing' through this set of renormalization schemes. We are interested in the map

$$t \mapsto S_R^\phi * \theta_{tz}(S_R^\phi)^{*-1} \tag{6.63}$$

and, in particular, in the limit

$$F_t = \lim_{z\to 0} S_R^\phi * \theta_{tz}(S_R^\phi)^{*-1} \ . \tag{6.64}$$

It can be shown to exist and moreover, $F_{t+s} = F_t * F_s$ establishes a semi-group structure[**CoKr01**]. The map

$$\beta = \partial_t F_t|_{t=0} \tag{6.65}$$

turns out to be the β-function (of the corresponding theory) in physics(see next section). Now, note that infinitesimal characters $\psi : H_{FG} \to \mathbb{C}$ define a Lie algebra $\mathfrak{g}$ with bracket

$$[\psi,\psi']_* = \psi * \psi' - \psi' * \psi \qquad\qquad \psi,\psi' \in \mathfrak{g} \ . \tag{6.66}$$

Let $Z_0 \in \mathfrak{g}$ be a map of this type defined by

$$[Z_0,\psi]_* = Y\psi \tag{6.67}$$

for all $\psi \in \mathfrak{g}$. Then we have the interesting 'scattering' formula[**CoKr01**]

$$S_R^\phi = \lim_{t\to\infty} \exp_*(-t(\beta/z + Z_0)) \exp_*(tZ_0) \ , \tag{6.68}$$

where we remind the reader that $\exp_*$ is the $*$-convolution exponential[1] given by

$$\exp_*(\sigma) = \sum_{k=0}^{\infty} \frac{\sigma^{*n}}{n!} \ , \tag{6.69}$$

for an infinitesimal character $\sigma \in \mathfrak{g}$, where $\sigma^{*0} = \hat{\mathbb{I}}$. This exponential always evaluates to a finite sum on any element in H_{FG}, on account of $\sigma(\mathbb{I}) = 0$ and the coradical filtration.

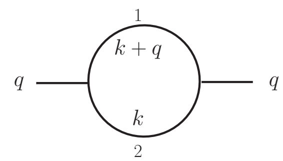
[1]Some authors omit the $*$-sign altogether, as it is generally clear from the context.

CHAPTER 7

Parametric Renormalization

7.1. Parametric Space

Consider the graph

(7.1) $\Gamma =$ 

in ϕ_6^3-theory with incoming momentum $q \in \mathbb{R}^4$ and internal momenta $k+q$ and k of particle 1 with mass m_1 and particle 2 with mass m_2, respectively. Up to prefactors, the usual euclidean momentum space Feynman rules associate the divergent integral

$$D_\Gamma = \frac{1}{\pi^2} \int_{\mathbb{R}^4} \frac{d^4k}{((k+q)^2 + m_1^2)(k^2 + m_2^2)} \tag{7.2}$$

to this graph. It has an *ultraviolett divergence*, or more precisely, it is logarithmically divergent because the integrand decreases asymptotically as $1/|k|$ for $k \to \infty$. This integral resembles the one in section 4.4. Instead of massaging it into a convergent integral, we apply the so-called *Schwinger trick*

$$\frac{1}{x} = \int_0^\infty dA \; e^{-xA} \;, \qquad x \in \mathbb{C} : \Re(x) > 0 \;, \tag{7.3}$$

to each propagator separately to get

$$D_\Gamma = \frac{1}{\pi^2} \int_0^\infty dA \int_0^\infty dB \; e^{-(m_1^2 A + m_2^2 B)} \int_{\mathbb{R}^4} d^4k \; e^{-[(k+q)^2 A + k^2 B]} \;, \tag{7.4}$$

where we have changed the order of integration. This is possible as the result remains infinite, a case of conservation of ill-definedness. However, the Schwinger trick itself, as applied to the integrand in (7.2), is a mathematically sound operation. If we now just focus on the innermost integration over k, we discover it to be a convergent Gaussian integral: the exponent of the last exponential can be rewritten

$$(k+q)^2 A + k^2 B = (A+B)\left(k + \frac{A}{A+B} q\right)^2 + \frac{AB}{A+B} q^2 \tag{7.5}$$

by completing the square in the first term on the rhs. After a simple shift of the integration variable k, this integral yields

$$e^{-\frac{AB}{A+B}q^2} \int_{\mathbb{R}^4} d^4k \; e^{-(A+B)k^2} = \pi^2 \; \frac{e^{-\frac{AB}{A+B}q^2}}{(A+B)^2} \;. \tag{7.6}$$

Inserting this back into (7.4), we get

$$D_\Gamma = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dA dB \; \frac{e^{-\frac{AB}{A+B}q^2 - (m_1^2 A + m_2^2 B)}}{(A+B)^2} \;, \tag{7.7}$$

an equally divergent integral. The only difference is: by virtue of the Schwinger trick in (7.3), we have transformed the ultraviolett divergence in momentum space $\mathbb{R}^4$ into an *infrared divergence* in *parametric space* $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}_+$, the space of the two Schwinger variables A, B. The divergence is in this case caused by the integrand's singular behaviour at $0 \in \mathbb{R}_+^2$, where both variables jointly vanish, i.e. $A = 0 = B$. The integral in (7.7) is called *parametric representation* of the Feynman integral (7.2) with *Schwinger parameters*.

An alternative parametric representation of (7.2) makes use of so-called *Feynman parameters*, which we shall not discuss here. Most QFT textbooks cover this topic, see for example [**PesSchr**].

7.2. Graph Polynomials

The integrand in (7.7) features the two polynomials that have a name:

$$\psi_\Gamma = A + B \ , \qquad \varphi_\Gamma = q^2 AB \tag{7.8}$$

are the two graph polynomials associated to the graph Γ, known as *first and second Symanzik polynomials*, or *Kirchhoff polynomials*. The integral in (7.7) then takes the form

$$D_\Gamma = \int_{\mathbb{R}^2_+} dAdB \ \frac{e^{-\frac{\varphi_\Gamma}{\psi_\Gamma} - (m_1^2 A + m_2^2 B)}}{\psi_\Gamma^2} = \int_{\mathbb{R}^2_+} \omega_\Gamma \ , \tag{7.9}$$

where ω_Γ is the corresponding (singular) differential form. We shall now define these polynomials for a general scalar graph Γ, i.e. a graph with a single edge and vertex type as that in (7.1). In principle, they can also be defined for other theories.

Let Γ be a scalar graph in D dimensions of spacetime. Then the integrand reads

$$I_\Gamma = \frac{e^{-\frac{\varphi_\Gamma}{\psi_\Gamma} - M\cdot A}}{\psi_\Gamma^{D/2}} \ , \qquad \text{where} \qquad M \cdot A := \sum_{e \in \Gamma^{[1]}_{int}} m_e A_e \tag{7.10}$$

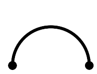
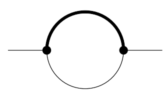
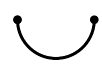
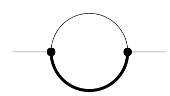
is a shorthand notation with Schwinger parameters $\{A_e : e \in \Gamma^{[1]}_{int}\}$. The two Symanzik polynomials are given as follows. Let Γ be a scalar graph.

Definition 7.2.1. *A connected and simply connected subgraph $T \subset \Gamma$ is called* spanning tree *of Γ if $T^{[0]} = \Gamma^{[0]}$. Then the* first Symanzik polynomial *is defined as*

$$\psi_\Gamma = \sum_T \prod_{e \notin T^{[1]}} A_e \ , \tag{7.11}$$

where the sum is over all spanning trees of Γ.

The graph in (7.1) has the two spanning trees

(7.12) $T_1 =$  $\subset$  and $T_2 =$  $\subset$  .

For the tree T_1 we have edge #2 with edge variable B that is not an edge of T_1, whereas for the tree T_2, there is edge #1 with variable A not being an edge of T_2. Thus, the products for each tree consist of only one factor and we get the two terms $A + B = \psi_\Gamma$.

Definition 7.2.2. *A* spanning two-forest *is a pair of connected and simply connected subgraphs $T_1, T_2 \subset \Gamma$ such that*

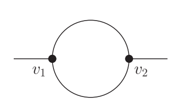
$$T_1 \cap T_2 = \emptyset \qquad \textit{and} \qquad T_1^{[0]} \cup T_2^{[0]} = \Gamma^{[0]} \ . \tag{7.13}$$

The second Symanzik polynomial *is then given by*

$$\varphi_\Gamma = - \sum_{T_1 \cup T_2} Q(T_1) \cdot Q(T_2) \prod_{e \notin T_1^{[1]} \cup T_2^{[1]}} A_e \tag{7.14}$$

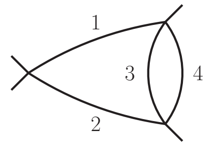
with the sum extending over all spanning two-forests of Γ and $Q(T_j)$ the sum of all euclidean momenta flowing into the tree T_j, where the momenta flowing out of it are included as flowing into it with a minus sign.

The product $Q(T_1) \cdot Q(T_2)$ is the usual euclidean scalar product. Note that by momentum conservation, we always have $-Q(T_1) \cdot Q(T_2) > 0$. This minus sign, however, is a convention. If dropped, one must also drop the one in the exponential in (7.10) in front of $\varphi_\Gamma / \psi_\Gamma$. For our graph Γ in (7.1) there is only one spanning two-forest:

(7.15) $T_1 = \bullet_{v_1} \ , \quad T_2 = \bullet_{v_2} \subset$  .

For systematic derivation of these polynomials the reader is referred to the original paper [**NoNa61**] and the more recent [**BoWei03**]. There are not many QFT textbooks with a thorough derivation. A nice exception is [**LeBe**].

Dunce's cap. We consider another, more complicated example, the so-called *Dunce's cap graph*:

(7.16) $\gamma =$  .

Its spanning trees can be characterized by their edges:

$$T_1^{[1]} = \{1,2\}, \quad T_2^{[1]} = \{1,3\}, \quad T_3^{[1]} = \{1,4\}, \quad T_4^{[1]} = \{2,4\}, \quad T_5^{[1]} = \{2,3\} \ . \tag{7.17}$$

This yields

$$\psi_\gamma = A_3A_4 + A_2A_4 + A_2A_3 + A_1A_3 + A_1A_4 \tag{7.18}$$

for the first Symanzik polynomial. In fact, this is the determinant of the *loop matrix*

$$N_\gamma = \begin{pmatrix} A_1 + A_2 + A_3 & A_1 + A_2 \\ A_2 + A_1 & A_1 + A_2 + A_4 \end{pmatrix} \ , \tag{7.19}$$

which is obtained as follows. Instead of spanning trees, let us consider the independent loops of γ. They may be written in terms of their participating edges. If we choose

$$l_1 = \{1,2,3\} \ , \qquad l_2 = \{1,2,4\} \ , \tag{7.20}$$

the components of the matrix N_γ are then given by

$$(N_\gamma)_{ij} = \sum_{e \in l_i \cap l_j} A_e \ , \qquad i,j = 1,2 \ . \tag{7.21}$$

However, this formula defines in general an $n \times n$ matrix N_Γ associated to a scalar graph Γ with the property that

$$\psi_\Gamma = \det N_\Gamma \tag{7.22}$$

for any choice of independent loops, which can be proven by algebraic methods(see [**KrSS12**], [**BloKr10**] and references therein). For the second Symanzik polynomial, one considers the block matrix

$$M_\Gamma = \left(\begin{array}{c|c} N'_\Gamma & (\sum_{e \in l_j} \mu_e A_e) \\ \hline (\sum_{e \in l_j} \mu_e A_e)^t & \sum_{e \in \Gamma^{[1]}_{int}} \overline{\mu}_e \mu_e A_e \end{array}\right) \ , \tag{7.23}$$

with the following entries. First, the symbols μ_e stand for a 2×2 matrix given by

$$\mu_e = p_e^0 1_{2\times 2} - ip_e^1\sigma^1 - ip_e^2\sigma^2 - ip_e^3\sigma^3 \ , \tag{7.24}$$

which we associate to the edge $e \in \Gamma^{[1]}_{int}$: $\{\sigma^j\}$ are the usual Pauli matrices and $p_e = (p_e^0, p_e^1, p_e^2, p_e^3)$ is the euclidean 4-momentum flowing along the oriented egde e. The notation

$$(\sum_{e \in l_j} \mu_e A_e) \tag{7.25}$$

stands for a column of 2×2 matrices with n components, one for each loop l_j. This amounts to a $2n \times 2$ matrix, whereas its transpose in the lower left block is of type $2 \times 2n$. N'_Γ is the $2n \times 2n$ matrix that one obtains from N_Γ when every entry a is replaced by the 2×2 matrix $a1_{2\times 2}$. This leads to M_Γ being an $(2n+2) \times (2n+2)$ matrix. The second Symanzik polynomial φ is then given by its Pfaffian determinant[**KrSS12**]

$$\varphi_\Gamma = \mathrm{Pf}(M_\Gamma) \ , \tag{7.26}$$

where the Pfaffian determinant is defined for a $(2m \times 2m)$ matrix A by

$$\mathrm{Pf}(A) := \sum_{\pi \in S_{2m}} \mathrm{sgn}(\pi) A_{\pi(1),\pi(2)} \ldots A_{\pi(2m-1),\pi(2m)} \ , \tag{7.27}$$

the sum being over all elements in the permutation group S_{2m}. However complicated this may seem, it is in general easier to identify a set of independent loops and construct the loop matrix for a graph, than to find all possible spanning trees.

7.3. Angles and Scales

Let Γ be a Feynman graph with external leg structure $\mathrm{res}(\Gamma) = r$ and external euclidean momenta $p_j \in \mathbb{R}^4$, $j = 1, ..., |r|$, where we denote by $|r|$ the number of external edges('legs'). To each edge $e \in \Gamma^{[1]}$ we associate a mass $m_e > 0$. Then, in general, the renormalized Feynman rules yield a function $\phi_R(\Gamma)$ depending on the variables $p_i \cdot p_j$ $(i, j = 1, ..., |r|)$ and mass parameters $\{m_e^2 | e \in \Gamma^{[1]}\}$. We may define the *scale* of the graph by

$$S := \sum_{j=1}^{|r|} p_j^2 \tag{7.28}$$

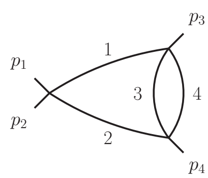
and introduce the new *scaled* variables $\theta_{ij} := p_i \cdot p_j / S$ and mass parameters $\theta_e := m_e^2 / S$. This allows us to define the rescaled Feynman rules

$$\phi'_R(\Gamma)\{S, \theta_{ij}, \theta_e\} := \phi_R(\Gamma)\{S\theta_{ij}, S\theta_e\} \tag{7.29}$$

mapping the graph Γ to a function of the *scale variable* $S > 0$ and the *angle variables* $\{\theta_{ij}, \theta_e\}$. We will denote the collection of the latter two by $\{\Theta\}$. We may wish to subject our renormalized Feynman rules to certain boundary conditions and therefore introduce some modified renormalized Feynman rules Φ_R like

$$\Phi_R(\Gamma)\{S, S_0, \Theta, \Theta_0\} := \phi'_R(\Gamma)\{S, \Theta\} - \phi'_R(\Gamma)\{S_0, \Theta_0\} \ , \tag{7.30}$$

where $\{S_0, \Theta_0\}$ is some reference (renormalization) point. The reader may find (7.30) slightly peculiar. However, it is nothing but to a change of the renormalization point. Take Dunce's cap

(7.31) $\Gamma =$  ,

for example. The Feynman rules (7.30) yield something of the form

$$\Phi_R(\ )\{S, S_0, \Theta, \Theta_0\} = c_0(\Theta, \Theta_0) + c_1(\Theta, \Theta_0)\ln(S/S_0) + c_2(\Theta, \Theta_0)\ln^2(S/S_0) \ . \tag{7.32}$$

For the only subgraph we get

$$\Phi_R(\ )\{S, S_0, \Theta, \Theta_0\} = c_0(\Theta, \Theta_0) + c_1(\Theta, \Theta_0)\ln(S/S_0) \ . \tag{7.33}$$

By the renormalization group we have

$$c_2(\Theta, \Theta_0) = \frac{1}{2} c_1(\Theta, \Theta_0) c_1(\Theta, \Theta_0) \ . \tag{7.34}$$

We now come back to parametric Feynman integrals which we have introduced in the previous section. Let

$$E_\Gamma := |\Gamma_{int}^{[1]}| \tag{7.35}$$

be the number of internal edges of a graph Γ and $\omega_\Gamma\{S, \Theta\}$ be the (singular) differential form such that

$$\omega_\Gamma\{S, \Theta\} = I_\Gamma\{S, \Theta\}(A_1, ..., A_{E_\Gamma})\ dA_1 \wedge ... \wedge dA_{E_\Gamma} \tag{7.36}$$

with Schwinger variables A_j and

$$\Phi(\Gamma)\{S, \Theta\} = \int_{\mathbb{R}_+^{E_\Gamma}} \omega_\Gamma\{S, \Theta\} \tag{7.37}$$

yields the rescaled unrenormalized Feynman integral in parametric representation. Suppose now that Γ is primitive. Then, this integral ceases to be of purely formal nature as soon as we replace (7.37) by

$$\Phi_R(\Gamma)\{S, S_0, \Theta, \Theta_0\} = \int_{\mathbb{R}_+^{E_\Gamma}} (\omega_\Gamma\{S, \Theta\} - \omega_\Gamma\{S_0, \Theta_0\}) \ , \tag{7.38}$$

which is a convergent and hence well-defined integral.

Symanzik polynomials. We recall the two Symanzik polynomials

$$\psi_\Gamma = \sum_T \prod_{e \notin T^{[1]}} A_e \qquad \text{and} \qquad \varphi_\Gamma = -\sum_{T_1 \cup T_2} Q(T_1) \cdot Q(T_2) \prod_{e \notin T_1^{[1]} \cup T_2^{[1]}} A_e \ . \tag{7.39}$$

The notation has been introduced in the foregoing section. The rescaling affects only the second polynomial φ_Γ as it depends on the kinematic variables $Q(T_1) \cdot Q(T_2)$ whereas the first Symanzik polynomial ψ_Γ does not depend on anything other than the Schwinger variables. To streamline the notation, we set

$$\varphi_\Gamma(\Theta) := \varphi_\Gamma / S \ , \qquad \phi_\Gamma(\Theta) := \varphi_\Gamma(\Theta) + \psi_\Gamma \sum_{j=1}^{E_\Gamma} A_j \theta_j \tag{7.40}$$

with the rescaled variables introduced above. Then the integrand I_Γ in (7.36) takes the form

$$I_\Gamma\{S, \Theta\} = \frac{e^{-S\frac{\phi_\Gamma(\Theta)}{\psi_\Gamma}}}{\psi_\Gamma^2} \tag{7.41}$$

in which we have supressed the integration (Schwinger) variables A_j and $D = 4$. We shall now rewrite

$$\Phi_R(\Gamma)\{S, S_0, \Theta, \Theta_0\} = \int_{\mathbb{R}_+^{E_\Gamma}} \frac{e^{-S\frac{\phi_\Gamma(\Theta)}{\psi_\Gamma}} - e^{-S_0\frac{\phi_\Gamma(\Theta_0)}{\psi_\Gamma}}}{\psi_\Gamma^2} \, dA_1 \wedge \ldots \wedge dA_{E_\Gamma} \tag{7.42}$$

into a projective integral and show that it exists for a primitive graph Γ. First we transform the Schwinger variables by $A_j = t a_j$, where $t := (A_1^2 + \ldots + A_{E_\Gamma}^2)^{1/2}$ and get

$$\begin{aligned} dA_1 \wedge \ldots \wedge dA_{E_\Gamma} &= (a_1 dt + t da_1) \wedge \ldots \wedge (a_{E_\Gamma} dt + t da_{E_\Gamma}) \\ &= t^{E_\Gamma - 1} dt \wedge (a_1 da_2 \wedge \ldots \wedge da_{E_\Gamma} - \ldots + (-1)^{E_\Gamma} a_{E_\Gamma} da_1 \wedge \ldots \wedge da_{E_\Gamma - 1}) \\ &=: t^{E_\Gamma - 1} dt \wedge \Omega_\Gamma \ , \end{aligned} \tag{7.43}$$

where Ω_Γ is the volume form in projective space $\mathbb{P}_\Gamma := \mathbb{P}^{E_\Gamma - 1}(\mathbb{R}_+)$. Due to

$$\psi_\Gamma\{A_j\} = t^{|\Gamma|} \psi_\Gamma\{a_j\} \tag{7.44}$$

and $\mathbb{R}^{E_\Gamma} \cong \mathbb{R}_+ \times \mathbb{P}_\Gamma$ the integral in (7.37) takes the form[1]

$$\int_{\mathbb{R}_+} \int_{\mathbb{P}_\Gamma} \frac{e^{-t \ S\frac{\phi_\Gamma(\Theta)}{\psi_\Gamma}}}{\psi_\Gamma^2} \, \frac{dt}{t} \wedge \Omega_\Gamma \tag{7.45}$$

which is ill-defined due to the integral over $\mathbb{R}_+$: the form dt/t is singular at $t = 0$, where all Schwinger variables collectively vanish. However, this integral can be regularized. We will now use the formula

$$\int_c^\infty \frac{dt}{t} e^{-tX} = -\ln c - \ln X - \gamma_E + \mathcal{O}(c \ln c) \tag{7.46}$$

with regulator $c > 0$ and fixed $X > 0$(see appendix for a proof). Subtracting this integral at X_0 allows us to take the limit $c \to 0$ to obtain

$$\int_0^\infty \frac{dt}{t} (e^{-tX} - e^{-tX_0}) = -\ln(X/X_0) \ . \tag{7.47}$$

This can be used to carry out the t-integration in (7.42) with transformed integration variables as in (7.45) yielding the projective integral

$$\begin{aligned} \Phi_R(\Gamma)\{S, S_0, \Theta, \Theta_0\} &= -\int_{\mathbb{P}_\Gamma} \frac{\ln(S/S_0) + \ln(\phi_\Gamma(\Theta)/\phi_\Gamma(\Theta_0))}{\psi_\Gamma^2} \, \Omega_\Gamma \\ &= (-\int_{\mathbb{P}_\Gamma} \frac{\Omega_\Gamma}{\psi_\Gamma^2}) \ln(S/S_0) - \int_{\mathbb{P}_\Gamma} \frac{\ln(\phi_\Gamma(\Theta)/\phi_\Gamma(\Theta_0))}{\psi_\Gamma^2} \, \Omega_\Gamma \ . \end{aligned} \tag{7.48}$$

Comparing this with (7.33) we identify

$$c_0^\Gamma(\Theta, \Theta_0) = -\int_{\mathbb{P}_\Gamma} \frac{\ln(\phi_\Gamma(\Theta)/\phi_\Gamma(\Theta_0))}{\psi_\Gamma^2} \, \Omega_\Gamma, \qquad c_1^\Gamma(\Theta, \Theta_0) = -\int_{\mathbb{P}_\Gamma} \frac{\Omega_\Gamma}{\psi_\Gamma^2} \ . \tag{7.49}$$

These numbers are *periods*: interesting numbers which we shall come back to later.

[1] Note that $2|\Gamma| = E_\Gamma$ for a vertex graph in ϕ^4-theory.

7.4. Forest Formula

We extend the definition of the two Symanzik polynomials for a product of graphs $\gamma = \prod_j \gamma_j$ by setting

$$\varphi_\gamma := \sum_j \varphi_{\gamma_j} \prod_{l \neq j} \psi_{\gamma_l}, \qquad \psi_\gamma := \prod_j \psi_{\gamma_j} \ . \tag{7.50}$$

Then, one has the following

Proposition 7.4.1. *Let $\gamma \subset \Gamma$ be a subgraph which is a product of 1PI divergent subgraphs. Then,*

$$\psi_\Gamma = \psi_{\Gamma/\gamma}\psi_\gamma + R^\Gamma_\gamma, \qquad \varphi_\Gamma = \varphi_{\Gamma/\gamma}\psi_\gamma + \overline{R}^\Gamma_\gamma \ , \tag{7.51}$$

with polynomials R^Γ_γ and $\overline{R}^\Gamma_\gamma$ such that

$$|\overline{R}^\Gamma_\gamma|_\gamma \geq |R^\Gamma_\gamma|_\gamma = |\psi_\gamma|_\gamma + 1, \tag{7.52}$$

where $|...|_\gamma$ is the polynomial degree in the edge variables of γ.

Proof. The proof makes use of the definition of spanning trees and two-forests. □

The well-known *forest formula* of QFT yields terms of the form

$$\frac{e^{-S_0 \frac{\phi_f(\Theta_0)}{\psi_f}}}{\psi_f^2} \frac{e^{-S \frac{\phi_{\Gamma/f}(\Theta)}{\psi_{\Gamma/f}}}}{\psi_{\Gamma/f}^2} - \frac{e^{-S_0 \frac{\phi_f(\Theta_0)}{\psi_f}}}{\psi_f^2} \frac{e^{-S_0 \frac{\phi_{\Gamma/f}(\Theta_0)}{\psi_{\Gamma/f}}}}{\psi_{\Gamma/f}^2} \ , \tag{7.53}$$

which, in the notation of (7.41) corresponds to

$$(\mathrm{id} - R_0) I_f\{S_0, \Theta_0\} I_{\Gamma/f}\{S, \Theta\} = I_f\{S_0, \Theta_0\} I_{\Gamma/f}\{S, \Theta\} - I_f\{S_0, \Theta_0\} I_{\Gamma/f}\{S_0, \Theta_0\}, \tag{7.54}$$

where R_0 evaluates the integrand at the renormalization point $\{S_0, \Theta_0\}$. If we apply the same procedure to the corresponding integral as in the previous section which lead to the projective integral in (7.48), we arrive at the projective form

$$M^f_\Gamma\{S, S_0, \Theta, \Theta_0\} = -\frac{\ln\left(\frac{S\phi_{\Gamma/f}(\Theta)\psi_f + S_0\phi_f(\Theta_0)\psi_{\Gamma/f}}{S_0\phi_{\Gamma/f}(\Theta_0)\psi_f + S_0\phi_f(\Theta_0)\psi_{\Gamma/f}}\right)}{\psi_{\Gamma/f}^2 \psi_f^2} \Omega_\Gamma \ . \tag{7.55}$$

Finally, the renormalized Feynman rules can be written as the projective form

$$\omega_\Gamma\{S, S_0, \Theta, \Theta_0\} = \sum_{f \in \mathcal{F}(\Gamma)} (-1)^{|f|} M^f_\Gamma\{S, S_0, \Theta, \Theta_0\} \ , \tag{7.56}$$

where the sum is over all forests of Γ including the empty one $f = \emptyset$, for which the graph polynomials are defined as $\psi_\emptyset := 1$ and $\varphi_\emptyset := 0$.

7.5. Decomposing Feynman rules

An interesting result of [**BrowKr11**] is

$$\Phi_R(\Gamma)\{S, S_0, \Theta, \Theta_0\} = \Phi^{*-1}_{\mathrm{fin}}(\Gamma)\{\Theta_0\} * \Phi_{1s}(\Gamma)\{S/S_0\} * \Phi_{\mathrm{fin}}(\Gamma)\{\Theta\} \tag{7.57}$$

which says that the Feynman rules can be decomposed with respect to the convolution product for characters into angle and scale-dependent parts[2].

Let us again consider a primitive graph Γ which is evaluated to the projective integral in (7.48). If we rewrite it in the form

$$\begin{aligned} \Phi_R(\Gamma)\{S, S_0, \Theta, \Theta_0\} = &\int_{\mathbb{P}_\Gamma} \frac{\ln \phi_\Gamma(\Theta_0)}{\psi_\Gamma^2}\ \Omega_\Gamma + (-\int_{\mathbb{P}_\Gamma} \frac{\Omega_\Gamma}{\psi_\Gamma^2}) \ln(S/S_0) \\ &- \int_{\mathbb{P}_\Gamma} \frac{\ln \phi_\Gamma(\Theta)}{\psi_\Gamma^2}\ \Omega_\Gamma \ , \end{aligned} \tag{7.58}$$

then the decomposition formula in (7.57) is nearly there. To see this, we take the coproduct of a primitive graph twice

$$\Delta^2(\Gamma) = \Gamma \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \Gamma \ , \tag{7.59}$$

[2]The index $1s$ stands for 'one-scale', 'fin' for 'finite', as these characters are finite from the start.

as this is what the three characters in (7.57) are applied to. As the antipode yields $S(\Gamma) = -\Gamma$ we see that

$$\Phi_{\text{fin}}^{*-1}(\Gamma)\{\Theta_0\} = \Phi_{\text{fin}}(S(\Gamma))\{\Theta_0\} = \Phi_{\text{fin}}(-\Gamma)\{\Theta_0\} = -\Phi_{\text{fin}}(\Gamma)\{\Theta_0\}\ . \tag{7.60}$$

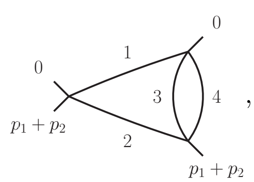
Note that (7.59) tells us that (7.57) can only deliver three terms and must be of the form

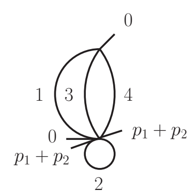
$$\Phi_R(\Gamma)\{S, S_0, \Theta, \Theta_0\} = \Phi_{\text{fin}}^{*-1}(\Gamma)\{\Theta_0\} + \Phi_{1s}(\Gamma)\{S/S_0\} + \Phi_{\text{fin}}(\Gamma)\{\Theta\}\ . \tag{7.61}$$

However, comparing this with (7.58) misleads us to erroneous assumptions: to find the characters in the decomposition formula (7.57), one has to introduce auxiliary Feynman graphs. We smuggle in the first Symanzik polynomial $\psi_{\Gamma^{2\bullet}}$ of an auxiliary graph $\Gamma^{2\bullet}$ and arrive at the correct terms which read

$$\Phi_{\text{fin}}(\Gamma)\{\Theta\} = -\int_{\mathbb{P}_\Gamma} \frac{\ln \frac{\phi_\Gamma(\Theta)}{\psi_{\Gamma^{2\bullet}}}}{\psi_\Gamma^2}\,\Omega_\Gamma, \qquad \Phi_{1s}(\Gamma)\{S/S_0\} = \Big(-\int_{\mathbb{P}_\Gamma} \frac{\Omega_\Gamma}{\psi_\Gamma^2}\Big)\ln(S/S_0)\ . \tag{7.62}$$

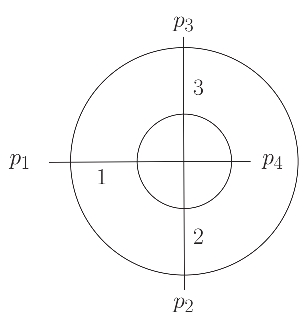
Although the contribution of this graph drops out in (7.58), it is necessary for a coherent definition. The auxiliary graph is obtained from Γ in two steps. First, one makes Γ into a so-called *single-scale* graph Γ^2: all internal masses are set to zero, all external momenta except two are set to zero and the remaining momenta 'carry' the whole flow of momenta. The superscript says that the graph has only 2 external vertices[3]. Identifying these two remaining external edges yields the graph $\Gamma^{2\bullet}$. As an example, consider again Dunce's cap in (7.31): if we apply this scheme, we get

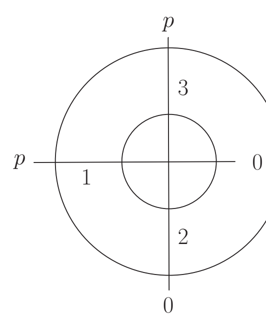
(7.63) $\Gamma^2 =$ 

, $\Gamma^{2\bullet} =$ 

.

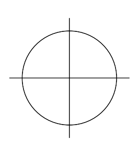
A more challenging example is the graph G

(7.64) $G =$ 

$\rightarrow$ $G^2 =$ 

,

where G^2 is *not* angle-dependent, if we choose p such that $p^2 = S$. G has the subgraph

(7.65) $\gamma =$  .

The double coproduct of G reads

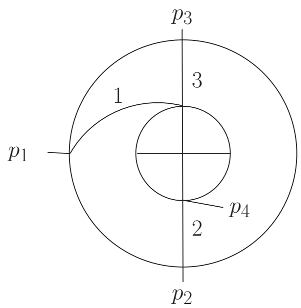
$$\Delta^2(G) = G \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes G \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes G + \gamma \otimes G/\gamma \otimes \mathbb{I} + \mathbb{I} \otimes \gamma \otimes G/\gamma + \gamma \otimes \mathbb{I} \otimes G/\gamma\ . \tag{7.66}$$

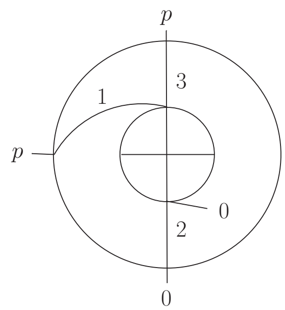
Applying the map $\Phi_{\text{fin}}^{*-1} \otimes \Phi_{1s} \otimes \Phi_{\text{fin}}$ gives

$$\begin{aligned}&\Phi_{\text{fin}}^{*-1}(G)\{\Theta_0\} + \Phi_{1s}(G)\{S/S_0\} + \Phi_{\text{fin}}(G)\{\Theta\} + \Phi_{\text{fin}}^{*-1}(\gamma)\{\Theta_0\}\Phi_{1s}(G/\gamma)\{S/S_0\}\\ &+ \Phi_{1s}(\gamma)\{S/S_0\}\Phi_{\text{fin}}(G/\gamma)\{\Theta\} + \Phi_{\text{fin}}^{*-1}(\gamma)\{\Theta_0\}\Phi_{\text{fin}}(G/\gamma)\{\Theta\}\ .\end{aligned} \tag{7.67}$$

[3] A vertex is called *external*, if it is adjacent to at least one external leg.

By G_γ we denote the graph in which the subgraph γ is made into a single-scale graph by rearranging its external edges in G(and setting all internal masses to zero):

(7.68) $G_\gamma =$ 

$\rightarrow$ $G^2_\gamma =$ 

.

These auxiliary graphs are either single-scale or have single-scale subgraphs and are the ingredients needed in constructing the decomposition in (7.67). For example, G^2_γ, when subjected to renormalized Feynman rules, yields an angle-independent term: both the whole graph and its subgraph are single-scale and therefore not angle-dependent. This graph hence contributes to $\Phi_{1s}(G)\{S/S_0\}$, where, in this example, a convenient scale S is given by $S := (p_1 + p_2 + p_3)^2$. For details concerning this example and the decomposition of Feynman rules as in (7.57), the reader is referred to [**BrowKr11**]. For Dunce's cap in (7.31), see [**BrowKr12**].

7.6. Periods as RG-Invariants

Let Γ be a primitive Feynman graph. In the previous lecture we have seen that the associated renormalized Feynman integral can be rewritten as the projective integral

$$\Phi_R(\Gamma)\{S, S_0, \Theta, \Theta_0\} = -\int_{\mathbb{P}_\Gamma} \frac{\ln(S/S_0) + \ln(\phi_\Gamma(\Theta)/\phi_\Gamma(\Theta_0))}{\psi_\Gamma^2}\, \Omega_\Gamma\ , \tag{7.69}$$

which is a polynomial in the scale variable $L = \ln(S/S_0)$

$$\Phi_R(\Gamma)\{S, S_0, \Theta, \Theta_0\} = c_0^\Gamma(\Theta, \Theta_0) + c_1^\Gamma(\Theta, \Theta_0)\ln(S/S_0) \tag{7.70}$$

with coefficients

$$c_0^\Gamma(\Theta, \Theta_0) = -\int_{\mathbb{P}_\Gamma} \frac{\ln(\phi_\Gamma(\Theta)/\phi_\Gamma(\Theta_0))}{\psi_\Gamma^2}\, \Omega_\Gamma, \qquad c_1^\Gamma(\Theta, \Theta_0) = -\int_{\mathbb{P}_\Gamma} \frac{\Omega_\Gamma}{\psi_\Gamma^2} =: p_\Gamma\ . \tag{7.71}$$

A closer look reveals that the highest order coefficient p_Γ is not at all angle-dependent, and moreover, does not depend on any kinematical data of the graph. It is a constant number which is *renormalization scheme independent* in the sense that it is invariant with respect to a change of the renormalization point $\{S_0, \Theta_0\}$: we refer to such numbers as *RG-invariants*. On account of the form of the integral, the number p_Γ is a *period*(see appendix for a short introduction).

Let now Γ be a graph such that $\widetilde{\Delta}(\Gamma) = \gamma \otimes \Gamma/\gamma$ with primitive sub- and cograph γ and Γ/γ, respectively, i.e.

$$\widetilde{\Delta}(\gamma) = 0, \qquad \widetilde{\Delta}(\Gamma/\gamma) = 0\ . \tag{7.72}$$

The renormalized Feynman rules evaluate this to

$$\Phi_R(\Gamma)\{S, S_0, \Theta, \Theta_0\} = c_0^\Gamma(\Theta, \Theta_0) + c_1^\Gamma(\Theta, \Theta_0)\ln(S/S_0) + p_\Gamma \ln^2(S/S_0)\ , \tag{7.73}$$

where again, the highest log coefficient p_Γ is scheme-independent. By the renormalization group, we know that it is given by the first order log coefficients of its subgraph and cographs: in this simple case, this means

$$p_\Gamma = \frac{1}{2} p_\gamma p_{\Gamma/\gamma}\ . \tag{7.74}$$

Given that these numbers are RG-invariants, and they are by the same argument as in (7.71), so is p_Γ. But there is another way to prove this. By the forest formula, we have

$$\Phi_R(\Gamma) = -\sum_{f\in\mathcal{F}(\Gamma)} (-1)^{|f|} \int_{\mathbb{P}_\Gamma} \frac{\ln\left(\frac{S\phi_{\Gamma/f}(\Theta)\psi_f + S_0\phi_f(\Theta_0)\psi_{\Gamma/f}}{S_0\phi_{\Gamma/f}(\Theta_0)\psi_f + S_0\phi_f(\Theta_0)\psi_{\Gamma/f}}\right)}{\psi_{\Gamma/f}^2\psi_f^2}\Omega_\Gamma\ . \tag{7.75}$$

As the set of forests $\mathcal{F}(\Gamma)$ has only one nontrivial forest, namely $f_1 = \gamma$ and the trivial one $f_2 = \emptyset$, this integral takes the form

$$\Phi_R(\Gamma) = -\int_{\mathbb{P}_\Gamma} \left[\frac{\ln \frac{S\phi_\Gamma(\Theta)}{S_0\phi_\Gamma(\Theta_0)}}{\psi_\Gamma^2} - \frac{\ln\left(\frac{S\phi_{\Gamma/\gamma}(\Theta)\psi_\gamma + S_0\phi_\gamma(\Theta_0)\psi_{\Gamma/\gamma}}{S_0\phi_{\Gamma/\gamma}(\Theta_0)\psi_\gamma + S_0\phi_\gamma(\Theta_0)\psi_{\Gamma/\gamma}}\right)}{\psi_{\Gamma/\gamma}^2\psi_\gamma^2} \right] \Omega_\Gamma \ . \tag{7.76}$$

It is in fact convergent: when all edge variables collectively tend to zero, the denominator polynomials approach each other quickly because

$$\psi_\Gamma = \psi_{\Gamma/\gamma}\psi_\gamma + R_\gamma^\Gamma \sim \psi_{\Gamma/\gamma}\psi_\gamma \ , \tag{7.77}$$

for vanishing edge variables, where $R_\gamma^\Gamma \to 0$ faster than all other terms. Furthermore, the numerators of the two fractions do also approach one another. All this happens in such a way that the integrand vanishes although each term individually has a singularity. To compute the coefficient p_Γ in (7.73), we apply the differential operator

$$\frac{\partial}{\partial \ln(S/S_0)} = S\frac{\partial}{\partial S} \tag{7.78}$$

twice to the rhs of (7.76), i.e. compute

$$-S\frac{\partial}{\partial S}S\frac{\partial}{\partial S}\int_{\mathbb{P}_\Gamma} \left[\frac{\ln \frac{S\phi_\Gamma(\Theta)}{S_0\phi_\Gamma(\Theta_0)}}{\psi_\Gamma^2} - \frac{\ln\left(\frac{S\phi_{\Gamma/\gamma}(\Theta)\psi_\gamma + S_0\phi_\gamma(\Theta_0)\psi_{\Gamma/\gamma}}{S_0\phi_{\Gamma/\gamma}(\Theta_0)\psi_\gamma + S_0\phi_\gamma(\Theta_0)\psi_{\Gamma/\gamma}}\right)}{\psi_{\Gamma/\gamma}^2\psi_\gamma^2} \right] \Omega_\Gamma \tag{7.79}$$

and show that this monstrous expression really is independent of the renormalization point. The result is

$$\int_{\mathbb{P}_\Gamma} \frac{(S_0/S)\phi_\gamma\phi_{\Gamma/\gamma}}{\psi_{\Gamma/\gamma}\psi_\gamma(\phi_{\Gamma/\gamma}\psi_\gamma + (S_0/S)\phi_\gamma\psi_{\Gamma/\gamma})^2}\Omega_\Gamma \ . \tag{7.80}$$

It is not obvious that this expression really is independent of $x = S/S_0$. However, it can be scaled away: set $a_e =: xa'_e$ for the edge variables of γ. The Symanzik polynomials and the projective form scale as

$$\Omega_\Gamma \to x^{E_\gamma}\Omega_\Gamma, \qquad \psi_\gamma \to x^{|\gamma|}\psi_\gamma, \qquad \phi_\gamma \to x^{|\gamma|+1}\phi_\gamma \ . \tag{7.81}$$

Together with $E_\gamma = 2|\gamma|$ for a graph in ϕ^4-theory, we get

$$\int_{\mathbb{P}_\Gamma} \frac{\phi_\gamma\phi_{\Gamma/\gamma}}{\psi_{\Gamma/\gamma}\psi_\gamma(\phi_{\Gamma/\gamma}\psi_\gamma + \phi_\gamma\psi_{\Gamma/\gamma})^2}\Omega_\Gamma \ , \tag{7.82}$$

which is independent of any external variables. To see that this decomposes into p_γ and $p_{\Gamma/\gamma}$, we use

$$\Omega_\Gamma = t^{E_\gamma - 1}dt \wedge \Omega_\gamma \wedge \Omega_{\Gamma/\gamma} \tag{7.83}$$

which involves the same type of transformation applied to the edge variables of γ as we have done in the last lecture. The integral then takes the form

$$\int_{\mathbb{P}_\Gamma} \frac{\phi_\gamma\phi_{\Gamma/\gamma}}{\psi_{\Gamma/\gamma}\psi_\gamma(\phi_{\Gamma/\gamma}\psi_\gamma + t\phi_\gamma\psi_{\Gamma/\gamma})^2}dt \wedge \Omega_\gamma \wedge \Omega_{\Gamma/\gamma} \ , \tag{7.84}$$

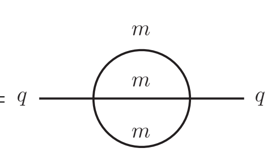
in which we can carry out the t-integration and finally arrive at

$$\int_{\mathbb{P}_\Gamma} \frac{\Omega_\gamma \wedge \Omega_\gamma}{\psi_\gamma^2\psi_{\Gamma/\gamma}^2} = (\int_{\mathbb{P}_\gamma} \frac{\Omega_\gamma}{\psi_\gamma^2})(\int_{\mathbb{P}_{\Gamma/\gamma}} \frac{\Omega_{\Gamma/\gamma}}{\psi_{\Gamma/\gamma}^2}) = p_\gamma p_{\Gamma/\gamma} \ . \tag{7.85}$$

7.7. Quadratic Divergences in BPHZ

We consider the scalar graph

(7.86) $\Gamma =$



in ϕ^4-theory in $D = 4$ dimensions with internal particles of equal mass m. The corresponding Symanzik polynomials are

$$\psi_\Gamma = A_2A_3 + A_1A_3 + A_1A_2 \ , \qquad \varphi_\Gamma = q^2A_1A_2A_3 \ . \tag{7.87}$$

Abbreviating

$$\phi_\Gamma(q^2,m^2) := q^2 A_1 A_2 A_3 + m^2(A_1+A_2+A_3)\psi_\Gamma, \tag{7.88}$$

the singular differential form in parametric representation for the graph Γ reads

$$\omega_\Gamma(q^2,m^2) = \frac{e^{-\phi_\Gamma(q^2,m^2)/\psi_\Gamma}}{\psi_\Gamma^2} dA_1\wedge dA_2\wedge dA_3 \ . \tag{7.89}$$

If we again carry out the transformation of variables $A_j = t a_j$ as before, we end up with[4]

$$\omega_\Gamma(q^2,m^2) = \frac{e^{-t\phi_\Gamma(q^2,m^2)/\psi_\Gamma}}{\psi_\Gamma^2} \frac{dt}{t^2}\wedge \Omega_\Gamma \ . \tag{7.90}$$

The identity

$$\int_c^\infty \frac{dt}{t^2}\ e^{-tX} = \frac{1}{c} - X - X\int_c^\infty \frac{dt}{t}\ e^{-tX} + \mathcal{O}(c) \tag{7.91}$$

helps us understand the behaviour of the Feynman integrand in (7.90) upon BPHZ subtraction at $q^2 = m^2$('on-shell') if we set

$$X := \phi_\Gamma(q^2,m^2)/\psi_\Gamma =: q^2 A + m^2 B \tag{7.92}$$

and plug this in (7.91) to get the projective form

$$\int_c^\infty e^{-tX}\frac{dt}{t^2}\wedge\frac{\Omega_\Gamma}{\psi_\Gamma^2} = \frac{\Omega_\Gamma}{c\psi_\Gamma^2} - \frac{X}{\psi_\Gamma^2}\Omega_\Gamma - \int_c^\infty e^{-tX}(q^2A+m^2B)\frac{dt}{t}\wedge\frac{\Omega_\Gamma}{\psi_\Gamma^2} + \mathcal{O}(c) \ . \tag{7.93}$$

If we set $X_0 := X|_{q^2=m^2} = m^2(A+B)$ and apply a BHZP subtraction,

$$\begin{aligned}\int_c^\infty \left[e^{-tX} - e^{-tX_0}\right]\frac{dt}{t^2}\wedge\frac{\Omega_\Gamma}{\psi_\Gamma^2} &= -(X-X_0)\frac{\Omega_\Gamma}{\psi_\Gamma^2} - \int_c^\infty (q^2e^{-tX} - m^2 e^{-tX_0})A\ \frac{dt}{t}\wedge\frac{\Omega_\Gamma}{\psi_\Gamma^2}\\ &\quad - m^2\int_c^\infty (e^{-tX}-e^{-tX_0})B\ \frac{dt}{t}\wedge\frac{\Omega_\Gamma}{\psi_\Gamma^2}\end{aligned} \tag{7.94}$$

all terms except one yield finite and well-defined expressions. The only misbehaving term in (7.94) is

$$-\int_c^\infty (q^2e^{-tX} - m^2e^{-tX_0})A\ \frac{dt}{t}\wedge\frac{\Omega_\Gamma}{\psi_\Gamma^2} \tag{7.95}$$

which leads to a logarithmically divergent contribution as $c\to 0$. We need to get rid of it by yet another subtraction: if we deduct

$$-\int_c^\infty (q^2e^{-tX_0} - m^2e^{-tX_0})A\ \frac{dt}{t}\wedge\frac{\Omega_\Gamma}{\psi_\Gamma^2} \tag{7.96}$$

from the misbehaving expression (7.95) we have tamed it since

$$-q^2\int_c^\infty (e^{-tX} - e^{-tX_0})A\ \frac{dt}{t}\wedge\frac{\Omega_\Gamma}{\psi_\Gamma^2} \tag{7.97}$$

keeps finite even in the limit $c\to 0$. This seems like a rather idiosyncratic surgical operation. However, the reader may check that the subtraction term (7.96) is given by

$$(q^2-m^2)\ \frac{\partial}{\partial q^2}\bigg|_{q^2=m^2}\int_c^\infty e^{-tX}\frac{dt}{t^2}\wedge\frac{\Omega_\Gamma}{\psi_\Gamma^2} \ . \tag{7.98}$$

If we set

$$\mathrm{Int}_\Gamma^c(q^2,m^2) := \int_c^\infty e^{-tX}\frac{dt}{t^2}\wedge\frac{\Omega_\Gamma}{\psi_\Gamma^2}\ , \tag{7.99}$$

then the total subtraction procedure we have employed is

$$\mathrm{Int}_\Gamma^c(q^2,m^2) - \mathrm{Int}_\Gamma^c(q^2,m^2)\big|_{q^2=m^2} - (q^2-m^2)\ \frac{\partial}{\partial q^2}\bigg|_{q^2=m^2}\mathrm{Int}_\Gamma^c(q^2,m^2) \tag{7.100}$$

which can be written in terms of a Rota-Baxter operator R given by

$$R[\mathrm{Int}_\Gamma^c(q^2,m^2)] := \mathrm{Int}_\Gamma^c(q^2,m^2)\big|_{q^2=m^2} + (q^2-m^2)\ \frac{\partial}{\partial q^2}\bigg|_{q^2=m^2}\mathrm{Int}_\Gamma^c(q^2,m^2) \ . \tag{7.101}$$

[4] For a propagator graph in ϕ^4-theory one has: $2|\Gamma| = E_\Gamma - 1$.

Then, we can take the limit and obtain the renormalized value of the graph Γ:

$$\begin{aligned}\lim_{c\to 0}\int_{\mathbb{P}_\Gamma}(\mathrm{id}-R)[\mathrm{Int}^c_\Gamma(q^2,m^2)] &= -\int_{\mathbb{P}_\Gamma}(X-X_0)\frac{\Omega_\Gamma}{\psi_\Gamma^2}+\int_{\mathbb{P}_\Gamma}X\ln(X/X_0)\frac{\Omega_\Gamma}{\psi_\Gamma^2}\\ &= -\int_{\mathbb{P}_\Gamma}[\phi_\Gamma(q^2,m^2)-\phi_\Gamma(m^2,m^2)]\frac{\Omega_\Gamma}{\psi_\Gamma^3}+\int_{\mathbb{P}_\Gamma}\phi_\Gamma(q^2,m^2)\ln\left(\frac{\phi_\Gamma(q^2,m^2)}{\phi_\Gamma(m^2,m^2)}\right)\frac{\Omega_\Gamma}{\psi_\Gamma^3}\ .\end{aligned}\tag{7.102}$$

7.8. Linear Dyson-Schwinger Equation

Recall from Lecture 13 that in the Hopf algebra of rooted trees H, the combinatorial DSE

$$X(\alpha)=\mathbb{I}+\alpha B_+(X(\alpha))\tag{7.103}$$

has a unique solution in the ladder Hopf algebra $H_\ell[[\alpha]]$ given by $X(\alpha)=\mathbb{I}+\sum_{k\geq 1}\lambda_k\alpha^k$, where λ_k is the ladder tree with k rungs.

Log-divergent case. In a simple toy model, we define the unregularized Feynman rules on the ladder Hopf algebra H_ℓ by the intertwining equation of the universality theorem[5]

$$\phi(B_+(\cdot),u)=\int_0^\infty\frac{dx}{x+u}\phi(\cdot,x)\ ,\tag{7.104}$$

where $u>0$ is an external parameter representing some kinematic variable. Sadly, if we take the simplest ladder $\lambda_1=\bullet=B_+(\mathbb{I})$, we find

$$\phi(\bullet,u)=\phi(B_+(\mathbb{I}),u)=\int_0^\infty\frac{dx}{x+u}\phi(\mathbb{I},x)=\int_0^\infty\frac{dx}{x+u}=\infty\tag{7.105}$$

on account of the log-divergence of the integral. We may cure this pathology by introducing a *cut-off regulator* Λ and get a finite regularized value

$$\phi_\Lambda(\bullet,u)=\int_0^\Lambda\frac{dx}{x+u}=\ln[1+\Lambda/u]\ .\tag{7.106}$$

The renormalized Feynman rules then give

$$\phi_R(\bullet,u/u_0):=\lim_{\Lambda\to\infty}[\phi_\Lambda(\bullet,u)-\phi_\Lambda(\bullet,u_0)]=-\ln[u/u_0]=\int_0^\infty dx\left[\frac{1}{x+u}-\frac{1}{x+u_0}\right]\ .\tag{7.107}$$

with renormalization point u_0. We can now recast (7.104) into the well-defined identity

$$\phi_R(B_+(\cdot),u/u_0)=\int_0^\infty dx\left[\frac{1}{x+u}-\frac{1}{x+u_0}\right]\phi_R(\cdot,x/u_0)\tag{7.108}$$

for the renormalized character. Applying this to (7.103), yields the integral equation

$$G(\alpha,\ln(u/u_0))=1+\alpha\int_0^\infty dx\left[\frac{1}{x+u}-\frac{1}{x+u_0}\right]G(\alpha,\ln(x/u_0)),\tag{7.109}$$

i.e. a Dyson-Schwinger Equation (DSE) for the renormalized Green function

$$G(\alpha,\ln(u/u_0)):=\phi_R(X(\alpha),u/u_0)\ .\tag{7.110}$$

Fortunately, the analytic DSE in (7.109) can actually be solved by virtue of the ansatz

$$G(\alpha,\ln(u/u_0))=\left(\frac{u}{u_0}\right)^{-\gamma(\alpha)}=e^{-\gamma(\alpha)\ln(u/u_0)}=\sum_{k=0}^\infty(-1)^k\frac{\gamma(\alpha)^k}{k!}\ln^k(u/u_0)\tag{7.111}$$

with a function $\gamma(\alpha)$. Plugging this into (7.109), one finds that (7.111) is a solution, if $\gamma(\alpha)$ obeys

$$1=\alpha F(\gamma(\alpha))\tag{7.112}$$

with Mellin transform

$$F(\rho)=\int_0^\infty\frac{x^{-\rho}}{1+x}=\Gamma(\rho)\Gamma(1-\rho)=\rho^{-1}+\frac{\pi^2}{6}\rho+\mathcal{O}(\rho^3).\tag{7.113}$$

The DSE in (7.109) can be formally rewritten in the form

$$G(\alpha,\ln(u/u_0))=Z(\alpha,u_0)+\alpha\int_0^\infty\frac{dx}{x+u}G(\alpha,\ln(x/u_0))\tag{7.114}$$

[5] See Lecture 7.

where $Z(\alpha, u_0) = S_R^\phi(X(\alpha), u_0)$ is the counterterm. This expression is strictly only valid, if the integrations are regularized, preferably by analytic regularization as in (7.113). However, we shall suppress the regulator in what follows. The counterterm $Z(\alpha, u_0)$ encapsulates the 'infinity' caused by the overall divergence of the integral kernel in (7.114). It is derived as follows. First note that the solution $X(\alpha)$ of the combinatorial DSE is grouplike, i.e. $\Delta(X(\alpha)) = X(\alpha) \otimes X(\alpha)$ and therefore, we have

$$\begin{aligned} \phi_R(X(\alpha), u/u_0) &= S_R^\phi(X(\alpha), u_0)\phi(X(\alpha), u) = S_R^\phi(X(\alpha), u_0)[1 + \alpha \int_0^\infty \frac{dx}{x+u}\phi(X(\alpha), x)] \\ &= S_R^\phi(X(\alpha), u_0) + \alpha \int_0^\infty \frac{dx}{x+u} S_R^\phi(X(\alpha), u_0)\phi(X(\alpha), x) \\ &= S_R^\phi(X(\alpha), u_0) + \alpha \int_0^\infty \frac{dx}{x+u}\phi_R(X(\alpha), u/u_0)\ . \end{aligned} \tag{7.115}$$

Comparing this with (7.109) yields the assertion.

Linearly divergent Green function. If the integral kernel of the Feynman rules in (7.104) is modified so as to take the form

$$K(x, u) = \frac{x}{x+u}\ , \tag{7.116}$$

one contracts a linear divergence: the integrand behaves like a constant as x goes to infinity. If we regularize it by a cut-off, we get

$$\phi_\Lambda(\bullet, u) = \int_0^\Lambda dx\ K(x, u) = \int_0^\Lambda dx\ \frac{x+u-u}{x+u} = \underbrace{\int_0^\Lambda dx}_{\text{lin. div.}} \ -\ u \underbrace{\int_0^\Lambda \frac{dx}{x+u}}_{\text{log-div.}}\ . \tag{7.117}$$

This suggests that if we add a counterterm of the form

$$Z_0(\Lambda) + uZ_1(\Lambda) := -\int_0^\Lambda dx + u\int_0^\Lambda \frac{dx}{x+u_0} = -\Lambda + u\ln(1 + \Lambda/u_0)\ , \tag{7.118}$$

i.e. decomposed into a linearly divergent term Z_0 and a log-divergent term Z_1, they jointly cure the integral $\int_0^\infty dxK(x, u)$ of its linear divergence in the sense that the limit

$$\phi_R(\bullet, u) = \lim_{\Lambda\to\infty}[\phi_\Lambda(\bullet, u) + Z_0(\Lambda) + uZ_1(\Lambda)] = -u\ln[u/u_0] \tag{7.119}$$

exists and is the renormalized value. If we write the kernel as

$$K(x, u) = \frac{x}{x+u} = 1 - \frac{u}{x+u} =: 1 - uC(x, u) \tag{7.120}$$

then the renormalized kernel $K_R(x, u)$ is given by

$$K_R(x, u) = -u[C(x, u) - C(x, u_0)] = \frac{u}{x+u_0} - \frac{u}{x+u} \tag{7.121}$$

which satisfies the boundary condition $K_R(x, u_0) = 0$. A kernel of the sort in (7.116) will appear in the analytic DSE of a toy model given by the unrenormalized (and hence ill-defined) identity

$$\Sigma(\alpha, u) = u + \alpha\int_0^\infty \frac{dx}{x+u}\Sigma(\alpha, x) \tag{7.122}$$

where $\Sigma(\alpha, u) = u\ G(\alpha, \ln(u/u_0))$ satisfies $\Sigma(\alpha, u_0) = u$. To renormalize it, we write it in terms of the corresponding (yet unknown) infinite counterterms and get

$$\Sigma(\alpha, u) = Z_0(\alpha, u_0) + uZ_1(\alpha, u_0) + \alpha\int_0^\infty \frac{dx}{x+u}\ \Sigma(\alpha, u). \tag{7.123}$$

Inserting the ansatz

$$u\ G(\alpha, \ln(u/u_0)) = u\left(\frac{u}{u_0}\right)^{-\gamma(\alpha)} = u\ e^{-\gamma(\alpha)\ln(u/u_0)} \tag{7.124}$$

yields

$$u\left(\frac{u}{u_0}\right)^{-\gamma(\alpha)} = Z_0(\alpha, u_0) + uZ_1(\alpha, u_0) + \alpha\int_0^\infty dx\ \frac{x}{x+u}\ \left(\frac{x}{u_0}\right)^{-\gamma(\alpha)}\ , \tag{7.125}$$

where we recognize our linearly divergent integral kernel $K(x,u)$ from (7.116). If we again decompose this kernel into $K(x,u) = 1 - uC(x,u)$ we get

$$\begin{aligned} u\left(\frac{u}{u_0}\right)^{-\gamma(\alpha)} &= Z_0(\alpha,u_0) + \alpha\int_0^\infty dx\left(\frac{x}{u_0}\right)^{-\gamma(\alpha)} \\ &+ uZ_1(\alpha,u_0) - \alpha\ u\int_0^\infty dx\ C(x,u)\ \left(\frac{x}{u_0}\right)^{-\gamma(\alpha)} \ . \end{aligned} \tag{7.126}$$

Wisely choosing the counterterms

$$Z_0(\alpha,u_0) = -\alpha\int_0^\infty dx\left(\frac{x}{u_0}\right)^{-\gamma(\alpha)}, \tag{7.127}$$

for the linear divergence and for the log-divergent piece

$$Z_1(\alpha,u_0) = 1 + \alpha\int_0^\infty dx\ C(x,u_0)\ \left(\frac{x}{u_0}\right)^{-\gamma(\alpha)} = 1 + \alpha F(\gamma(\alpha)) \tag{7.128}$$

with Mellin transform F as in (7.113), we find

$$u\left(\frac{u}{u_0}\right)^{-\gamma(\alpha)} = u + u\left(1 - \left(\frac{u}{u_0}\right)^{-\gamma(\alpha)}\right)\alpha F(\gamma(\alpha)). \tag{7.129}$$

It follows that the ansatz is a solution as long as $\gamma(\alpha)$ fulfills

$$-1 = \alpha F(\gamma(\alpha))\ . \tag{7.130}$$

APPENDIX A

Renormalization Group of Hopf Algebra Characters

Within this part of the Appendix, we shall use the following notation which is independent of all other parts of these lecture notes: C is a coassociative coalgebra, B will denote a connected bialgebra with grading $B = \bigoplus_{j\geq 0} B_j$, where $B_0 = \mathbb{Q}\mathbb{I}$. P_B is the projector onto the augmentation ideal of B. H will denote a connected Hopf algebra $H = \bigoplus_{j\geq 0} H_j$ and P the augmentation ideal projector. A is an algebra. Except for H, all structures on C, B, H, A are indexed by the corresponding letter. For example, Δ_C denotes the coproduct on C, whereas Δ_B and Δ are those of B and H, respectively.

A.1. Convolution Group

Let $\mathcal{L}(C, A)$ be the set of linear maps from C to A. By virtue of the structures on both spaces, the *convolution* of two linear maps $f, g \in \mathcal{L}(C, A)$, given by

$$f * g := m_A(f \otimes g)\Delta_C \ , \tag{A.1}$$

is an associative bilinear operation on $\mathcal{L}(C, A)$. The map $e := u_A \circ \epsilon_C$ is the neutral element with respect to $*$.

Proposition A.1.1. *The pair $(\mathcal{L}(C, A), *)$ is a monoid, i.e. a set equipped with an associative operation and a neutral element with respect to it.*

Proof. See section 3.4. □

Naturally, one can define $*$-powers by setting $f^{*0} := e$, $f^{*1} := f$ and $f^{*n+1} := f * f^{*n}$ recursively. Even exponentials

$$\exp_*(f) := \sum_{n\geq 0} \frac{f^{*n}}{n!} \tag{A.2}$$

may exist. However, let us first see whether one can find an inverse for a linear map f. For this to exist, we must make sure that the von Neumann series

$$f^{*-1} = (e - (e - f))^{*-1} = \sum_{n\geq 0} (e - f)^{*n} \tag{A.3}$$

can be given some sense. This is possible if we replace the coalgebra C by a connected bialgebra B and restrict ourselves to such linear maps that preserve the unit map. i.e. $f(\mathbb{I}_B) = 1_A$. Then the grading property of the coproduct

$$\Delta(H_n) \subset \bigoplus_{j=0}^{n} H_j \otimes H_{n-j} \tag{A.4}$$

garantees that for every element $x \in B$ there exists an $N > 0$ such that

$$(e - f)^{*n}(x) = 0 \qquad\qquad \forall n > N. \tag{A.5}$$

This is due to $(e - f)(\mathbb{I}_B) = e(\mathbb{I}_B) - f(\mathbb{I}_B) = 0$. Consequently, we have

Proposition A.1.2. *The subset $\mathcal{G}(B, A) := \{f \in \mathcal{L}(B, A) | f(\mathbb{I}_B) = 1_A\} \subset \mathcal{L}(B, A)$ is a* group*, called the* convolution group*, i.e. for every map $f \in \mathcal{G}(B, A)$ there exist a linear map f^{*-1} such that*

$$f * f^{*-1} = f^{*-1} * f = e \tag{A.6}$$

and $f^{*-1}(\mathbb{I}_B) = 1_A$.

PROOF. Take $x \in B$. Then there is an $N > 0$ such that $(e-f)^{*n} = 0$ for all $n > N$. Then, using the shorthand $\Delta(x) = x' \otimes x''$, we compute

$$\begin{aligned}(f^{*-1} * f)(x) &= \sum_{n\geq 0}(e-f)^{*n}(x')f(x'') = \sum_{n\geq 0}(e-f)^{*n}(x')(e(x'') - [e(x'') - f(x'')]) \\ &= \sum_{n\geq 0}(e-f)^{*n}(x')e(x'') - \sum_{n\geq 0}(e-f)^{*n}(x')[e(x'') - f(x'')] \\ &= \sum_{n\geq 0}(e-f)^{*n}(x) - \sum_{n\geq 0}(e-f)^{*n+1}(x) = (e-f)^{*0}(x) = e(x)\ .\end{aligned} \tag{A.7}$$

This works equally well with $f * f^{*-1}(x)$. □

Let us consider the subspace(!)

$$\mathfrak{g}(B,A) := \{\sigma \in \mathcal{L}(B,A) | \sigma(\mathbb{I}_B) = 0\} \tag{A.8}$$

of linear maps for which the convolution exponential surely exists, as $\exp_*(\sigma)(x)$ is a finite sum for all $x \in B$. We observe

$$\exp_*(\sigma)(\mathbb{I}_B) = e(\mathbb{I}_B) + \underbrace{\sigma(\mathbb{I}_B)}_{=0} + \frac{1}{2!}\sigma(\mathbb{I}_B)\sigma(\mathbb{I}_B) + ... = e(\mathbb{I}_B) = 1_A, \tag{A.9}$$

i.e. $\exp_*(\sigma) \in \mathcal{G}(B,A)$. We call σ the generator of $f = \exp_*(\sigma)$. Are all elements of $\mathcal{G}(B,A)$ generated by the elements in $\mathfrak{g}(B,A)$? The answer is yes.

Proposition A.1.3. $\exp_*(\mathfrak{g}(B,A)) = \mathcal{G}(B,A)$.

PROOF. The convolution exponential $\exp_* : \mathfrak{g}(B,A) \to \mathcal{G}(B,A)$ is a bijection and the convolution logarithm

$$\log_*(f) := -\sum_{n\geq 1}\frac{1}{n}(e-f)^{*n} \tag{A.10}$$

is its inverse by the same combinatorics and arguments as for the classical calculus logarithm. It clearly yields a finite sum for all $x \in B$ and $f \in \mathcal{G}(B,A)$. $\log_*(f)(\mathbb{I}_B) = 0$ is straightforward. □

A.2. Algebraic Birkhoff Decomposition and Convolution Group

Let $f \in \mathcal{G}(B,A)$ and $A = A_- \oplus A_+$ be a decomposition into linear subspaces. A pair of maps $f_\pm \in \mathcal{G}(B,A)$ is called *algebraic Birkhoff decomposition* of f with respect to the decomposition $A_\pm$ if

$$f_\pm(\ker \epsilon_B) \subset A_\pm \qquad \text{and} \qquad f = f_-^{*-1} * f_+\ . \tag{A.11}$$

Given two subspaces, the Birkhoff decomposition always exists and is unique.

Theorem A.2.1. *Let $f \in \mathcal{G}(B,A)$ and $A = A_- \oplus A_+$ be a decomposition into subspaces with projector $R : A \to A_-$. Then, the Birkhoff decomposition $f_\pm \in \mathcal{G}(B,A)$ is uniquely defined by the recursive relations*

$$f_-(x) = -R[(f_- * fP_B)(x)], \tag{A.12}$$

for every $x \in \ker \epsilon_B$ and $f_+ := f_- * f$.

PROOF. First existence. We define the linear map by setting $f_-(\mathbb{I}_B) := 1_A$ and using (A.12) which determines f_- uniquely due to

$$(f_- * fP_B)(x) \in f(x) + m_A(f_- \otimes f)(\textstyle\bigoplus_{j=1}^{n-1} B_j \otimes B_{n-j})\ . \tag{A.13}$$

$f_-(\ker \epsilon_B) \subset A_-$ is satisfied by definition. $f_+(\mathbb{I}_B) = f_{-1}(\mathbb{I}_B)f(\mathbb{I}_B) = 1_A$ is trivial. On account of

$$\begin{aligned}f_+(x) &= (f_- * f)(x) = (f_- * fP_B)(x) + f_-(x) = (f_- * fP_B)(x) - R[(f_- * fP_B)(x)] \\ &= [\mathrm{id}_B - R](f_- * fP_B)(x) \in A_+\end{aligned} \tag{A.14}$$

for $x \in \ker \epsilon_B$ we have $f_+(\ker \epsilon_B) \subset A_+$, because $[\mathrm{id}_B - R]$ projects onto A_+. Now Uniqueness: any Birkhoff decomposition $f_\pm$ satisfies (A.12): take any $x \in \ker \epsilon_B$, then

$$-R[(f_- * fP_B)(x)] = -R[(f_- * f)(x) - f_-(x)] = -R[f_+(x) - f_-(x)] = R[f_-(x)] = f_-(x)\ . \tag{A.15}$$

Because this recursive relation determines a map f_- uniquely, the Birkhoff decomposition is unique. □

A.3. Character Group

If we replace the connected bialgebra B by a connected Hopf algebra H, the convolution group has a subset

$$\widetilde{\mathcal{G}}(H,A) := \{f \in \mathcal{G}(H,A) \mid f(xy) = f(x)f(y)\ \forall x,y \in H\} \tag{A.16}$$

of multiplicative maps in which the inverse f^{*-1} of an element $f \in \widetilde{\mathcal{G}}(H,A)$ is given by $f \circ S$:

$$(fS * f)(x) = f(S(x'))f(x'') = f(S(x')x'') = f(e(x)) = f(u\epsilon(x)) = f(\epsilon(x)\mathbb{I}) = \epsilon(x)1_A = e(x), \tag{A.17}$$

where $e = u_A \circ \epsilon$ is the neutral element with respect to $*$. Note that fS is not necessarily in this subset! This shows the following calculation:

$$fS(xy) = f(S(xy)) = f(S(y)S(x)) = fS(y)\ fS(x) \tag{A.18}$$

which may not be equal to $fS(x)fS(y)$, only if the target algebra A is commutative, is this in general the case. Let now A be commutative. Then, we find for $f, g \in \widetilde{\mathcal{G}}(H,A)$

$$\begin{aligned}(f*g)(xy) &= f(x'y')g(x''y'') = f(x')f(y')g(x'')g(y'') = f(x')g(x'')f(y')g(y'') \\ &= (f*g)(x)(f*g)(y)\ ,\end{aligned} \tag{A.19}$$

which is worth a

Proposition A.3.1. *Let A be commutative. Then, $\widetilde{\mathcal{G}}(H,A) \subset \mathcal{G}(H,A)$ is a subgroup.*

This subgroup is named *character group*. Its elements are called *Hopf algebra characters* or *(Hopf) characters*. How can we characterize the generator set in $\mathfrak{g}(H,A)$ of this subgroup?

Proposition A.3.2. *The characters in $\widetilde{\mathcal{G}}(H,A)$ are generated by the linear space*

$$\widetilde{\mathfrak{g}}(H,A) = \{\sigma \in \mathfrak{g}(H,A) \mid \sigma(xy) = \sigma(x)e(y) + e(x)\sigma(y)\ \forall x,y \in H\} \tag{A.20}$$

of infinitesimal characters*. They form a Lie algebra with Lie bracket*

$$[\sigma,\omega]_* := \sigma * \omega - \omega * \sigma\ . \tag{A.21}$$

Proof. First one has to prove by induction that $\sigma^{*n}(xy) = \sum_{j=0}^{n}\binom{n}{j}\sigma^{*j}(x)\sigma^{*n-j}(y)$. The cases $n = 0,1,2$ are trivial but should be checked by the interested reader to be prepared for the induction step which is done by the same trick jumbling with summation indices as in the proof of the binomial formula of undergraduate calculus. The case $n = 2$ shows that in general $\sigma * \omega$ is not an infinitesimal character but their Lie bracket is. Then, making use of this result,

$$\exp_*(\sigma)(xy) = \exp_*(\sigma)(x)\exp_*(\sigma)(y) \tag{A.22}$$

is straightforward and completely analogous to the classical case of the exponential function. So far, this proves that every infinitesimal character generates a character. It remains to be shown that the convolution logarithm $\log_* \phi$ of a character ϕ is an infinitesimal character. We leave it to the reader to show that

$$\log_*(xy - e(x)y - xe(y)) = 0. \tag{A.23}$$

□

Fig.1 shows in terms of Venn diagrams how the convolution monoid, the convolution group, Hopf characters and their generators are situated within the space of linear maps $\mathcal{L}(H,A)$. We know from section A.2 that maps in the convolution group $\mathcal{G}(H,A)$ have a unique Birkhoff decomposition for a given decomposition of the target algebra A into subsets $A_\pm$. What about Hopf characters, are the maps $\phi_\pm$ of the Birkhoff decomposition of a Hopf character $\phi \in \widetilde{\mathcal{G}}(H,A)$ also Hopf characters?

Proposition A.3.3. *In the setup of Theorem A.2.1, let ϕ be a Hopf character and the projector R be a Rota-Baxter operator, i.e. such that*

$$R[ab] + R[a]R[b] = R[aR[b] + R[a]b] \tag{A.24}$$

for all $a, b \in A$. This is garanteed if $A_\pm$ are subalgebras[1]*. Then the Birkhoff decomposition maps $\phi_\pm$ are Hopf characters.*

[1] Not necessarily unital algebras!

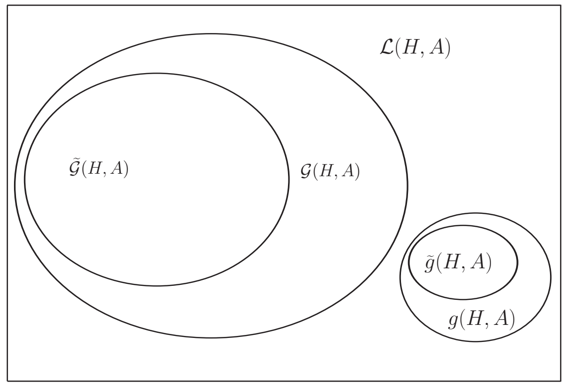


FIGURE 1. Convolution monoid, group and Hopf characters and their generator spaces.

PROOF. To understand that a projector onto subalgebras is always Rota-Baxter operator, one can easily check that (A.24) is fulfilled in the possible cases $a \in \ker R$, $b \in \operatorname{im} R$, and so on. The proof is inductive with respect to the grading of H. For $H_0 = \mathbb{Q}\mathbb{I}$. Assume $\phi_\pm$ are multiplicative on $\bigoplus_{j=0}^n H_j$. Then, choose $x, y \in H$ such that $xy \in H_{n+1}$. We use the abbreviation

$$\overline{\phi} := \phi_- * \phi P \tag{A.25}$$

in the following computation. Then,

$$\begin{aligned}
\phi_-(xy) &= -R[(\phi_-(x'y')\phi P(x''y'')] \overset{(*)}{=} -R[\phi_-(x')\phi_-(y')\phi P(x''y'')] \\
&= -R[\phi_-(x')\phi_-(y')\phi(x''y'') - \phi_-(x)\phi_-(y)] = -R[\phi_-(x')\phi_-(y')\phi(x'')\phi(y'') - \phi_-(x)\phi_-(y)] \\
&= -R[\phi_-(x')\phi(x'')\phi_-(y')\phi(y'') - \phi_-(x)\phi_-(y)] = -R[(\phi_- * \phi)(x)(\phi_- * \phi)(y) - \phi_-(x)\phi_-(y)] \\
&= -R[(\overline{\phi}(x) + \phi_-(x))(\overline{\phi}(y) + \phi_-(y)) - \phi_-(x)\phi_-(y)] \\
&= -R[\overline{\phi}(x)\overline{\phi}(y) + \overline{\phi}(x)\phi_-(y) + \phi_-(x)\overline{\phi}(y)] = -R[\overline{\phi}(x)\overline{\phi}(y) - \overline{\phi}(x)R\overline{\phi}(y) - R\overline{\phi}(x)\overline{\phi}(y)] \\
&= R[\overline{\phi}(x)]R[\overline{\phi}(y)] = \phi_-(x)\phi_-(y),
\end{aligned} \tag{A.26}$$

where we have used in $(*)$ that $x'y' \in H_{n+1}$ only if $x' = x, y' = y$, i.e. only if $x''y'' = \mathbb{I}$, which does not appear in the sum due to the presence of the projector P. Hence ϕ_- is multiplicative. Then so is $\phi_+ = \phi_- * \phi$(by Proposition A.3.1). □

Let us consider a nice example. Take the Hopf algebra of polynomials $\mathbb{C}[X]$ in one variable with coproduct

$$\Delta(X) := 1 \otimes X + X \otimes 1 \qquad \Rightarrow \qquad \Delta(X^n) = \sum_{j=0}^n \binom{n}{j} X^j \otimes X^{n-j}. \tag{A.27}$$

If we choose $A = \mathbb{C}$ as target algebra, the characters are given by the evaluation maps

$$\widetilde{\mathcal{G}}(\mathbb{C}[X], \mathbb{C}) = \{\mathrm{ev}_a : a \in \mathbb{C}\}. \tag{A.28}$$

This is because any character ϕ is determined completely by its value $\lambda := \phi(X)$ on the monomial X. Then, for any polynomial $p(X) \in \mathbb{C}[X]$ we get

$$\phi(p(X)) = p(\phi(X)) = p(\lambda) = \mathrm{ev}_\lambda(p(X)). \tag{A.29}$$

Let us see how the convolution acts,

$$(\mathrm{ev}_a * \mathrm{ev}_b)(X^n) = [(\mathrm{ev}_a * \mathrm{ev}_b)(X)]^n = [\mathrm{ev}_a(X)\mathrm{ev}_b(1) + \mathrm{ev}_a(1)\mathrm{ev}_b(X)]^n = \mathrm{ev}_{a+b}(X^n). \tag{A.30}$$

The identity $\mathrm{ev}_a * \mathrm{ev}_b = \mathrm{ev}_{a+b}$ makes everything explicit: the neutral element is ev_0 and the inverse for any evaluation character ev_a naturally is simply given by ev_{-a}. The generator set for these evaluations is just as easy. Consider the linear map $\partial_0 : \mathbb{C}[X] \to \mathbb{C}$ defined monomialwise by

$$\partial_0 X^n := n\ \mathrm{ev}_0(X^{n-1}) = \delta_{n,0} := \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{else} \end{cases} \tag{A.31}$$

This operator takes the derivative and evaluates the resulting polynomial at zero. Then, the infinitesimal character $\sigma_a := a\partial_0$ generates ev_a:

Proposition A.3.4. *For* $\sigma_a = a\partial_0 \in \widetilde{\mathfrak{g}}(\mathbb{C}[X],\mathbb{C})$ *one has* $\exp_*(\sigma_a) = \mathrm{ev}_a \in \widetilde{\mathcal{G}}(\mathbb{C}[X],\mathbb{C})$.

Proof. First, one has to show by induction that $\partial_0^{*n}(X^k) = k(k-1)...(k-n+1)\delta_{k,n}$ which means that the n-fold convolution of ∂_0 takes the n-fold derivative and evaluates it at zero. This is a nice exercise. With this result at hand, finishing the proof is straightforward. □

A.4. Renormalization Group of Hopf Characters

The results of the previous sections set the stage to define the *renormalization group* $\mathfrak{RG}$ as a group of characters.

Proposition A.4.1. *Let* $\phi \in \widetilde{\mathcal{G}}(H,\mathbb{C}[X])$ *be a coalgebra morphism, i.e.* $\Delta\phi = (\phi\otimes\phi)\Delta$. *Then its generator is given by the infinitesimal character*

$$\log_* \phi = X\partial_0\phi, \tag{A.32}$$

i.e. $\phi = \exp_*(X\partial_0\phi)$.

Proof.

$$\mathrm{ev}_a \log_* \phi = \log_*(\mathrm{ev}_a \circ \phi) = \log_*(\mathrm{ev}_a)\circ\phi = a\partial_0\circ\phi = \sigma_a\circ\phi. \tag{A.33}$$

Notice that we have sloppily used the same convolution sign $*$ for different spaces of maps. □

This is actually relevant to physics: if we have Feynman rules $\phi_L : H_{FG} \to \mathbb{C}[L]$ assigning polynomials in the kinematic variable L to Feynman graphs, its generator is given by the infinitesimal character $\sigma := \partial_0\phi_L$ in $\widetilde{\mathcal{G}}(H_{FG},\mathbb{C})$ and one has $\phi_L = \exp_*(L\sigma)$. Then, clearly

$$\phi_L * \phi_{L'} = \phi_{L+L'}\ , \tag{A.34}$$

which may be called *renormalization group equation*, where the renormalization group $\mathfrak{RG}$ is defined as the set of characters indexed by L:

$$\mathfrak{RG} = \{\ \phi_L \mid L\in\mathbb{R},\ \phi_L \in \widetilde{\mathcal{G}}(H_{FG},\mathbb{C}[L])\ \}. \tag{A.35}$$

This has the following meaning in physics: the kinematic variable is $L = \ln(q^2/\mu^2)$ with renormalization point μ. If we fix the external momentum parameter q^2, we can change L by varying the renormalization point μ. In doing so, we change the renormalization scheme. As we see, a change in L entails a change in Feynman rules according to (A.34). If we vary L infinitesimally, we get

$$\phi_{L+\varepsilon} = \phi_L * \phi_\varepsilon = \phi_L + \varepsilon\phi_L * \partial_0\phi_L + \mathcal{O}(\varepsilon^2) \tag{A.36}$$

Then, $\partial_L\phi = \phi_L * \partial_0\phi_L = \partial_0\phi_L * \phi_L$ is the infinitesimal version of the RG equation (A.34).

A.5. Proof of the Renormalization Group Equation

Let in the following $X^r(g)$ be a formal series in one parameter g with coefficients in H_{FG} such that it satisfies the combinatorial Dyson-Schwinger equation

$$X^r = \mathbb{I} + \mathrm{sgn}(s_r)\sum_{l\geq 1} g^l B_+^{l;r}(X^r Q^l), \tag{A.37}$$

where we will sometimes suppress the argument g. We shall prove the RG equation

$$[-\partial_L + g\beta(g)\partial_g + \gamma_1^r(g)]G^r(g,L) = 0 \tag{A.38}$$

for the Green function $G^r(g,L)$ and

$$k\gamma_k^r(g) = (\gamma_1^r(g) + g\beta(g)\partial_g)\gamma_{k-1}^r(g) \tag{A.39}$$

for the coefficient functions of its log-expansion

$$G^r(g,L) = 1 + \sum_{j\geq 1}\gamma_j^r(g)L^j. \tag{A.40}$$

We use the notation $X|_k := [g^k]X$ which denotes the k-th coefficient of the formal series X. The DSE implies $X|_0 = \mathbb{I}$ and

$$X^r|_k = \mathrm{sgn}(s_r)\sum_{l=1}^{k} B_+^{l;r}(X^r Q^l|_{k-l}) \tag{A.41}$$

for $k \geq 1$. Note that trivially $\Delta(X|_k) = \Delta(X)|_k$ and that we can write $X^r = \mathbb{I} + \mathcal{P}_{lin}X^r$ to separate the scalar part $\mathbb{I}$.

Proposition A.5.1. *Let X be a solution of the combinatorial DSE (A.37). Then, X satisifies the identity*

$$\Delta(X^r)|_n = \sum_{j=0}^{n}(X^rQ^{n-j})|_j \otimes (X^r)|_{n-j} \tag{A.42}$$

for all $n \geq 0$. Or, equivalently

$$\Delta X^r = \sum_{j\geq 0} X^rQ^j \otimes g^j x_j^r, \tag{A.43}$$

where $x_j^r = X^r|_j$, i.e. $X^r = \sum_{j\geq 0} x_j^r \alpha^j$.

Proof. First note that (A.43) is obtained by multiplying (A.42) with g^n and then summing over all n. We proceed by induction. The case $n = 0$ is trivial. Assume the assertion (A.42) holds for all indices $\leq n$. The following computation will be carried out with the series in (A.43), tacitly assuming that we take only the n-th partial sum. Alternatively, we can replace the series' coefficients by coefficients that vanish beyond the n-th. However, taking the coproduct, we get

$$\Delta X = X \otimes \mathbb{I} + \sum_{j>0} XQ^j \otimes x_j g^j = \mathbb{I}\otimes\mathbb{I} + (X-\mathbb{I})\otimes\mathbb{I} + \sum_{j>0} XQ^j \otimes x'_j, \tag{A.44}$$

where $X = X^r$ for notational convenience[2]. We will also use the shorthand notation $x'_j := x_j g^j$ and $X|'_j := x'_j$. Then, by (A.44), for any integer $s \in \mathbb{Z}$

$$\begin{aligned}
\Delta(X^s) = (\Delta X)^s &= \sum_{n\geq 0}\binom{s}{n}(\Delta X - \mathbb{I}\otimes\mathbb{I})^n \\
&= \sum_{n\geq 0}\sum_{m=0}^{n}\sum_{j_1>0}\cdots\sum_{j_m>0}\binom{s}{n}\binom{n}{m}(X-\mathbb{I})^{n-m}X^mQ^{j_1+\ldots+j_m}\otimes x'_{j_1}\ldots x'_{j_m} \\
&= \sum_{j>0}\sum_{n\geq 0}\sum_{m=0}^{n}\binom{s}{m}\binom{s-m}{n-m}\sum_{j_1+..+j_m=j}(X-\mathbb{I})^{n-m}X^mQ^j\otimes x'_{j_1}\ldots x'_{j_m} \\
&= \sum_{j>0}\sum_{n\geq 0}\sum_{m=0}^{n}\binom{s}{m}\binom{s-m}{n-m}(X-\mathbb{I})^{n-m}X^mQ^j\otimes (X-\mathbb{I})^m|'_j \\
&= \sum_{j\geq 0}\sum_{\nu\geq 0}\sum_{m\geq 0}\binom{s}{m}\binom{s-m}{\nu}(X-\mathbb{I})^{\nu}X^mQ^j\otimes (X-\mathbb{I})^m|'_j.
\end{aligned} \tag{A.45}$$

Note that in the last line we have changed the lower summation bound from $j = 1$ to $j = 0$ which is possible due to $(X-\mathbb{I})^m|'_0 = 0$. Finally, using $\sum_{i\geq 0}\binom{p}{i}(X-\mathbb{I})^i = X^p$ twice, we arrive at

$$\Delta(X^s) = \sum_{j\geq 0} X^sQ^j \otimes X^s|_j \ . \tag{A.46}$$

Next, we write the charge Q in the form

$$Q = \prod_{j=1}^{t}(X^{r_j})^{s_j}, \tag{A.47}$$

where $t := |R|$ is the number of residues and X^{r_j} the combinatorial perturbation series for the j-th residue. We are now in a position to compute its coproduct:

$$\begin{aligned}
\Delta(Q) = \prod_{j=1}^{t}(\Delta X^{r_j})^{s_j} &= \sum_{l_1\geq 0}\cdots\sum_{l_t\geq 0}(X^{r_1})^{s_1}\ldots(X^{r_t})^{s_t}Q^{l_1+\ldots+l_t}\otimes (X^{r_1})^{s_1}|'_{l_1}\cdots(X^{r_t})^{s_t}|'_{l_t} \\
&= \sum_{l_1\geq 0}\cdots\sum_{l_t\geq 0}Q^{l_1+\ldots+l_t+1}\otimes (X^{r_1})^{s_1}|'_{l_1}\cdots(X^{r_t})^{s_t}|'_{l_t} \\
&= \sum_{l\geq 0}\sum_{l_1+\ldots+l_t=l}Q^{l+1}\otimes (X^{r_1})^{s_1}|'_{l_1}\cdots(X^{r_t})^{s_t}|'_{l_t} \\
&= \sum_{l\geq 0}Q^{l+1}\otimes\sum_{l_1+\ldots+l_t=l}(X^{r_1})^{s_1}|'_{l_1}\cdots(X^{r_t})^{s_t}|'_{l_t} = \sum_{l\geq 0}Q^{l+1}\otimes Q|'_l
\end{aligned} \tag{A.48}$$

[2]We suppress the superscript r whenever there is no potential for confusion.

which looks surprisingly simple. So does

$$\Delta(Q^l) = \sum_{k_1\geq 0} \cdots \sum_{k_l\geq 0} Q^{k_1+\ldots+k_l+l} \otimes Q|'_{k_1} \ldots Q|'_{k_l} = \sum_{k\geq 0} Q^{l+k} \otimes Q^l|'_k. \tag{A.49}$$

The nice thing is, these latter two identities are (A.43) for Q and Q^l (instead of X). Now we can compute

$$\Delta(Q^l X) = (\sum_{k\geq 0} Q^{l+k} \otimes Q^l|_k)(\sum_{j\geq 0} Q^j X \otimes X|'_j) = \sum_{\nu\geq 0} Q^{l+\nu} X \otimes (Q^l X)|'_\nu. \tag{A.50}$$

which we have only proved up to the n-th coefficient:

$$\Delta(Q^l X)|_j = \sum_{\nu=0}^{j} (Q^{l+\nu} X)|_{j-\nu} \otimes (Q^l X)|_\nu \qquad j \leq n. \tag{A.51}$$

To finish the proof, consider the coproduct of the $(n+1)$-th coefficient

$$\Delta(X)|_{n+1} = \mathrm{sgn}(s_r) \sum_{l=1}^{n+1} \Delta B_+^{l;r}(X^r Q^l|_{n+1-l}), \tag{A.52}$$

where we have used (A.41). Applying the one-cocycle property of $B_+^{l;r}$, we find

$$\Delta(X)|_{n+1} = \mathrm{sgn}(s_r) \sum_{l=1}^{n+1} (\mathrm{id} \otimes B_+^{l;r}) \Delta(X^r Q^l|_{n+1-l}) + X|_{n+1} \otimes \mathbb{I}. \tag{A.53}$$

Because $l \geq 1$, we have $n+1-l \leq n$ which is why we can employ (A.51) and find

$$\begin{aligned} \Delta(X)|_{n+1} &= \mathrm{sgn}(s_r) \sum_{l=1}^{n+1} \sum_{\nu=0}^{n+1-l} (Q^{l+\nu} X)|_{n+1-l-\nu} \otimes B_+^{l;r}(Q^l X|_\nu) + X|_{n+1} \otimes \mathbb{I} \\ &= \mathrm{sgn}(s_r) \sum_{l=1}^{n+1} \sum_{\nu=l}^{n+1} (Q^\nu X)|_{n+1-\nu} \otimes B_+^{l;r}(Q^l X|_{\nu-l}) + X|_{n+1} \otimes \mathbb{I}. \end{aligned} \tag{A.54}$$

A change of summation order gives

$$\begin{aligned} \Delta(X)|_{n+1} &= \mathrm{sgn}(s_r) \sum_{\nu=1}^{n+1} \sum_{l=1}^{\nu} (Q^\nu X)|_{n+1-\nu} \otimes B_+^{l;r}(Q^l X|_{\nu-l}) + X|_{n+1} \otimes \mathbb{I} \\ &= \mathrm{sgn}(s_r) \sum_{\nu=1}^{n+1} (Q^\nu X)|_{n+1-\nu} \otimes \sum_{l=1}^{\nu} B_+^{l;r}(Q^l X|_{\nu-l}) + X|_{n+1} \otimes \mathbb{I} \\ &= \sum_{\nu=1}^{n+1} (Q^\nu X)|_{n+1-\nu} \otimes X|_\nu + X|_{n+1} \otimes \mathbb{I} = \sum_{\nu=0}^{n+1} (Q^\nu X)|_{n+1-\nu} \otimes X|_\nu \ . \end{aligned} \tag{A.55}$$

This concludes the induction step. □

Let $\mathcal{P}_{lin}$ be the projector onto the linear span of all Feynman graphs excluding $\mathbb{I}$, i.e. $\mathcal{P}_{lin}(h) = h$ if h is a Feynman graph and vanishing otherwise. Note that this projector differs from the augmentation ideal projector P.

Then, an immediate consequence is the following

Corollary A.5.2. *If we apply the projector $\mathcal{P}_{lin}$ on both sides of (A.43), we get*

$$(\mathcal{P}_{lin} \otimes \mathrm{id})\Delta(X^r) = \mathcal{P}_{lin} X^r \otimes X^r + \mathcal{P}_{lin} Q \otimes g\partial_g X^r. \tag{A.56}$$

This result already implies the RG equation in chapter 6: let σ denote the Lie algebra generator of the renormalized Feynman rules(denoted σ_1 in section 6.3), i.e. $\phi_R = \exp_*(L\sigma)$, then take $\sigma \otimes \sigma^{*n}$ and apply this to both sides of (A.56). The result is

$$\sigma^{*n+1}(X^r) = (\sigma \otimes \sigma^{*n})\Delta(X^r) = \sigma(X^r)\sigma^{*n}(X^r) + g\sigma(Q)\partial_g \sigma^{*n}(X^r). \tag{A.57}$$

We define the coefficient functions $\gamma_j^r(g) := \sigma_j(X^r(g))$, where $j!\sigma_j := \sigma^{*j}$. Then,

$$(n+1)\gamma_{n+1}^r(g) = \gamma_1^r(g)\gamma_n^r(g) + g\beta(g)\partial_g \gamma_n^r(g) = [\gamma_1^r(g) + g\beta(g)\partial_g]\gamma_n^r(g), \tag{A.58}$$

where

$$\beta(g) := \sigma(Q(g)) = \partial_0 \phi_R(Q(g)) = \partial_L \phi_R(Q(g))|_{L=0} \tag{A.59}$$

is the β-function. Let us compute it. Note that σ is an infinitesimal character which vanishes on $\mathbb{I}$ and nontrivial products. Then,

$$\begin{aligned}\sigma\left(\prod_{r\in\mathcal{R}}(X^r)^{s_r}\right) &= \sigma\left(\prod_{r\in\mathcal{R}}(\mathbb{I}+\mathcal{P}_{lin}X^r)^{s_r}\right) = \sigma\left(\prod_{r\in\mathcal{R}}(\mathbb{I}+s_r\mathcal{P}_{lin}X^r)\right)\\ &= \sigma\left(\sum_{r\in\mathcal{R}}s_r\mathcal{P}_{lin}X^r\right) = \sum_{r\in\mathcal{R}}s_r\sigma(\mathcal{P}_{lin}X^r) = \sum_{r\in\mathcal{R}}s_r\sigma(X^r) = \sum_{r\in\mathcal{R}}s_r\gamma_1^r.\end{aligned} \tag{A.60}$$

Thus, we have proven the identity

$$(n+1)\gamma_{n+1}^r(g) = \left(\gamma_1^r(g) + \sum_{t\in\mathcal{R}}s_t\gamma_1^t(g)g\partial_g\right)\gamma_n^r(g), \tag{A.61}$$

which are the RG equations for the coefficient functions $\gamma_n^r(g)$, $n\in\mathbb{N}$. If we multiply both sides by the n-th power of the variable L, we find

$$(n+1)\gamma_{n+1}^r(g)L^n = \left(\gamma_1^r(g) + \sum_{t\in\mathcal{R}}s_t\gamma_1^t(g)g\partial_g\right)\gamma_n^r(g)L^n, \tag{A.62}$$

and summing over all n, we arrive at the RG equation

$$\frac{\partial}{\partial L}G^r(g,L) = \left(\gamma_1^r(g) + \sum_{t\in\mathcal{R}}s_t\gamma_1^t(g)g\partial_g\right)G^r(g,L), \tag{A.63}$$

for the Green function $G^r(g,L) = 1 + \sum_{n\geq 1}\gamma_n^r(g)L^n$. Another possibility is to take (A.36) and apply the infinitesimal character

$$\partial_L\phi_L = \partial_0\phi_L * \phi_L = \sigma * \phi_L \tag{A.64}$$

to the series $X^r(g)$ and use (A.56) again. $\square$

APPENDIX B

The Dynkin Operator

Let H be a Hopf algebra with antipode S, counit ϵ and unit map u. We set $e := u \circ \epsilon$. Recall that the defining property of the antipode is

$$S * \mathrm{id} = e = \mathrm{id} * S \tag{B.1}$$

or, if we write $\Delta(x) = x' \otimes x''$, this takes the form $S(x')x'' = e(x) = x'S(x'')$.

B.1. Grouplike and Primitive Elements

An element $x \in H$ is called *grouplike*, if

$$\Delta(x) = x \otimes x \tag{B.2}$$

and *primitive* if

$$\Delta(x) = \mathbb{I} \otimes x + x \otimes \mathbb{I}. \tag{B.3}$$

Grouplike elements form a Hopf subalgebra and are related to primitive elements by formal exponentials: a *formal exponential* of $x \in H$ is given by the formal series $\exp(x) = \sum_{n\geq 0} x^n/n!$ which we identify with a sequence $a_n = \sum_{k=0}^{n} x^k/k!$ in H. Note that we do not ask if it converges to anything. What counts is that any element of this sequence is in H. The same goes for the logarithmic series

$$\log(x) := -\sum_{n\geq 1} \frac{1}{n}(x - \mathbb{I})^n. \tag{B.4}$$

It turns out that exp and log establish a relation between grouplike elements and primitive sequences and vice versa.

Proposition B.1.1. *Consider $x, y \in H$, where x is primitive and y grouplike. Then* $\exp(x)$ *is grouplike and* $\log(y)$ *is primitive.*

Proof.

$$\begin{aligned} \Delta(\exp(x)) &= \exp(\Delta(x)) = \exp(\mathbb{I} \otimes x + x \otimes \mathbb{I}) \overset{(*)}{=} \exp(\mathbb{I} \otimes x)\exp(x \otimes \mathbb{I}) \\ &= (\mathbb{I} \otimes \exp(x))(\exp(x) \otimes \mathbb{I}) = \exp(x) \otimes \exp(x). \end{aligned} \tag{B.5}$$

In $(*)$ we have used that the two terms $\mathbb{I} \otimes x$ and $x \otimes \mathbb{I}$ commute. This is also used in

$$\begin{aligned} \Delta \log(y) &= \log(\Delta y) = \log(y \otimes y) = \log((y \otimes \mathbb{I})(\mathbb{I} \otimes y)) \overset{(*)}{=} \log(y \otimes \mathbb{I}) + \log(\mathbb{I} \otimes y) \\ &= \log(y) \otimes \mathbb{I} + \mathbb{I} \otimes \log(y). \end{aligned} \tag{B.6}$$

□

B.2. Dynkin Operator and Projector

Let now H be commutative. For a grading operator Y on H, we define the *Dynkin operator*

$$D_Y := S * Y \tag{B.7}$$

and a map $\pi_Y := Y^{-1}(S * Y)$. Some of its properties will now be investigated. First note that the Dynkin operator fulfills $D_Y Y^{-1} = Y^{-1} D_Y$.

Proposition B.2.1. *D_Y and π_Y are infinitesimal characters with*

$$\ker D_Y = \ker \pi_Y = \mathbb{Q}\mathbb{I} \oplus (\ker \epsilon)^2, \tag{B.8}$$

where $(\ker \epsilon)^2 := \{xy \in H | x, y \in \ker \epsilon\}$.

PROOF. A quick computation shows $D_Y(xy) = D_Y(x)e(y) + e(x)D_Y(y)$, see Lemma 6.3.1. This implies $\mathbb{Q}\mathbb{I} \oplus (\ker \epsilon)^2 \subseteq \ker D_Y$. Then, to see that the reverse inclusion is true, we write the coproduct of $x \neq \mathbb{I}$ as $\Delta(x) = \mathbb{I} \otimes x + x \otimes \mathbb{I} + \sum_x x' \otimes x''$ and take any $x \in \ker D_Y$ to obtain

$$0 = D_Y(x) = Y(x) + \sum_x S(x')Y(x'') \tag{B.9}$$

and hence $Y(x) = -\sum_x S(x')Y(x'') \in (\ker \epsilon)^2$. $\square$

Note that any element $x \in H$ can be written as a finite linear combination $x = \sum_j \alpha_j x_j$ of homogeneous elements $x_j \in H$. The grading operator Y maps this to

$$Y(x) = \sum_j \alpha_j |x_j| x_j. \tag{B.10}$$

If $D_Y(x) = 0$, we can be sure that this linear combination is a linear combination of nontrivial products. This can only be the case if x was of this type in the first place. The next assertion shows that π_Y is a projector.

Proposition B.2.2. *$\pi_Y^2 = \pi_Y$ is a projection and for a primitive $x \in H$ one has $\pi_Y(x) = x$.*

PROOF. Take any $x \in H$ and compute $D_Y D_Y(x) = D_Y(Y(x) + \sum_x S(x')Y(x'')) = D_Y Y x$. This implies that π_Y is a projector(why?). Let x now be primitive. Then

$$\pi_Y(x) = Y^{-1}(S(x)Y(\mathbb{I}) + S(\mathbb{I})Y(x)) = Y^{-1}(S(\mathbb{I})Y(x)) = Y^{-1}(Y(x)) = x\ . \tag{B.11}$$

$\square$

For the next assertion, we will write $\Delta(x) = x' \otimes x''$ for convenience. An element $x \in H$ is called *cocommutative*, if $\Delta(x) = \text{flip}\Delta(x)$, i.e.

$$\Delta(x) = x' \otimes x'' = x'' \otimes x', \tag{B.12}$$

with the sum implicit, i.e. x' is not necessarily equal to x'', only for grouplike elements which trivially are cocommutative. Are primitive elements cocommutative? Slightly less trivial are ladder trees in the Hopf algebra of rooted trees. What about the polynomial Hopf algebra $\mathbb{C}[X]$? Is $p(X) \in \mathbb{C}[X]$ cocommutative? An interesting result is the last

Proposition B.2.3. *Let $x \in H$ be cocommutative. Then $D_Y(x)$ is primitive.*

PROOF. The map $\tau_{n,m}$ interchanges the n-th and the m-th element in a tensor product $x_1 \otimes ... \otimes x_k$, where, of course $n, m \leq k$. For example, to describe the multiplication map on $H \otimes H$, one needs a flip map: to express

$$(a \otimes b)(c \otimes d) = ac \otimes bd \tag{B.13}$$

in terms of the multiplication $m_{H\otimes H} : H \otimes H \otimes H \otimes H \to H \otimes H$, one must employ $\tau_{2,3}$ to get the result:

$$m(m(a \otimes c) \otimes m(b \otimes d)) = m(m \otimes m)(a \otimes c \otimes b \otimes d) = m(m \otimes m)\tau_{2,3}(a \otimes b \otimes c \otimes d) \tag{B.14}$$

Thus, we are 'coerced' to define $m_{H\otimes H} := m(m \otimes m)\tau_{2,3}$. Then,

$$\begin{aligned}
\Delta D_Y(x) &= \Delta m(S \otimes Y)\Delta(x) = (m \otimes m)\tau_{2,3}(\Delta \otimes \Delta)(S \otimes Y)\Delta(x) \\
&= (m \otimes m)\tau_{2,3}(\Delta S \otimes \Delta Y)\Delta(x) = (m \otimes m)\tau_{2,3}((S \otimes S)\Delta \otimes (Y \otimes \text{id} + \text{id} \otimes Y)\Delta)\Delta(x) \\
&= (m \otimes m)\tau_{2,3}(S \otimes S \otimes Y \otimes \text{id} + S \otimes S \otimes \text{id} \otimes Y))(\Delta \otimes \Delta)\Delta(x) \\
&= (m \otimes m)(S \otimes Y \otimes S \otimes \text{id} + S \otimes \text{id} \otimes S \otimes Y)(\Delta \otimes \Delta)\Delta(x) \\
&= (m(S \otimes Y) \otimes m(S \otimes \text{id}) + m(S \otimes \text{id}) \otimes m(S \otimes Y))(\Delta \otimes \Delta)\Delta(x) \\
&= (m(S \otimes Y)\Delta \otimes m(S \otimes \text{id})\Delta + m(S \otimes \text{id})\Delta \otimes m(S \otimes Y)\Delta)\Delta(x) \\
&= ((S * Y) \otimes (S * \text{id}) + (S * \text{id}) \otimes (S * Y))\Delta = ((S * Y) \otimes e + e \otimes (S * Y))\Delta(x)
\end{aligned} \tag{B.15}$$

This is what primitive means:

$$\begin{aligned}
\Delta D_Y(x) &= (S * Y)(x') \otimes e(x'') + e(x') \otimes (S * Y)(x'') = (S * Y)(x) \otimes e(\mathbb{I}) + e(\mathbb{I}) \otimes (S * Y)(x) \\
&= (S * Y)(x) \otimes \mathbb{I} + \mathbb{I} \otimes (S * Y)(x)
\end{aligned} \tag{B.16}$$

as $e(x') = 0 = e(x'')$ if $x' \neq \mathbb{I} \neq x'$ and $e(\mathbb{I}) = \mathbb{I}$. $\square$

APPENDIX C

Miscellanies

C.1. Exact sequences

Let $I \subset \mathbb{Z}$. A sequence of linear spaces $\{V_n\}_{n\in I}$ equipped with linear maps $f_n : V_n \to V_{n+1}$,

$$\text{(C.1)} \qquad ... \xrightarrow{f_{n-1}} V_n \xrightarrow{f_n} V_{n+1} \xrightarrow{f_{n+1}} ...$$

is called *exact sequence* if $\operatorname{im} f_i = \ker f_{i+1}$. Note that for

$$\text{(C.2)} \qquad 0 \to V \xrightarrow{f} V'$$

to be exact, f must be injective and in

$$\text{(C.3)} \qquad V \xrightarrow{f} V' \to 0$$

it must be surjective(the unmentioned maps are zero maps). An exact sequence with $|I| = 5$ of the form

$$\text{(C.4)} \qquad 0 \to V \xrightarrow{f} W \xrightarrow{h} Z \to 0$$

is called *short exact sequence*. It is characterized by three properties: f is injective, h surjective and $\operatorname{im} f = \ker h$. We can view V as a subspace of W under f since $\operatorname{im} f = \ker h$ is a subspace in W and by f's injectivity we have $V \cong \ker h$. Furthermore, if $\pi : W \to \overline{W}$ is the canonical projection onto $\overline{W} := W/\ker h$, then $\overline{h} := h \circ \pi^{-1} : \overline{W} \to Z$ is injective and, as h is surjective, also an isomorphism. Thus

$$\text{(C.5)} \qquad Z \cong \overline{W} = W/\ker h \cong W/V \ .$$

C.2. Integral identity

We will prove

$$\text{(C.6)} \qquad \int_c^\infty \frac{dt}{t} e^{-tX} = -\ln c - \ln X - \gamma_E + \mathcal{O}(c \ln c)$$

for $c > 0$ and $X > 0$. First, we do a partial integration

$$\text{(C.7)} \qquad \int_c^\infty \frac{dt}{t} e^{-tX} = \int_{cX}^\infty \frac{d\tau}{\tau} e^{-\tau} = -\ln(cX)e^{-cX} + \int_{cX}^\infty d\tau \ \ln\tau \ e^{-\tau} \ .$$

The latter integral can be rewritten into

$$\text{(C.8)} \qquad \int_{cX}^\infty d\tau \ \ln\tau \ e^{-\tau} = \int_0^\infty d\tau \ \ln\tau \ e^{-\tau} - \int_0^{cX} d\tau \ \ln\tau \ e^{-\tau} = -\gamma_E - \int_0^{cX} d\tau \ \ln\tau \ e^{-\tau} \ ,$$

where $\partial_z\Gamma(1+z) = \int_0^\infty d\tau \ \tau^z \ln\tau \ e^{-\tau}$ has been used. Since we have

$$\text{(C.9)} \qquad \int_0^{cX} d\tau \ \ln\tau \ e^{-\tau} \approx cX \ln(cX) e^{-cX}$$

for suffiently small $c > 0$, the result follows. $\square$

C.3. Periods

First some definitions. A number $a \in \mathbb{C}$ is called *algebraic* if it is the zero of a non-zero polynomial with rational coefficients. We denote the set of algebraic numbers by $\overline{\mathbb{Q}}$. If a is not algebraic, it is called *transcendental*. Obviously, we have the hierarchy

$$\text{(C.10)} \qquad \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \overline{\mathbb{Q}} \subset \mathbb{C} \ .$$

We will see that the set of periods has its place *in between* the latter two sets in the hierarchy.

A subset $S \subset \mathbb{R}^n$ is referred to as *algebraic*, if there are polynomials $P_1, ..., P_k \in \mathbb{Q}[X_1, ..., X_n]$ such that $x \in S$ implies $P_j(x) = 0$ for all j. Note that all polynomials can be trivial zero polynomials and thus $S = \mathbb{R}^n$. A subset $S' \subset \mathbb{R}^n$ is called *semialgebraic* if there are polynomials $P_1, ..., P_k, P'_1, ..., P'_l \in \mathbb{Q}[X_1, ..., X_n]$ such that $x \in S'$ means $P_j(x) = 0$ and $P'_r(x) > 0$ for all j, r. In other words, an algebraic subset is a subset of $\mathbb{R}^n$ defined by polynomial equalities. For semialgebraic sets some or all of these may be inequalities.

Definition C.3.1. *A number $p \in \mathbb{C}$ is called* period *if there are semialgebraic sets $S, S' \in \mathbb{R}^n$ and rational functions $u, v : \mathbb{R}^n \to \mathbb{R}$ with coefficients in $\mathbb{Q}$ such that*

$$p = \int_S u(x)dx + i \int_{S'} v(x)dx \ , \tag{C.11}$$

where dx is the usual Lebesgue measure on $\mathbb{R}^n$.

We denote the set of periods by $\mathcal{P}$. A simple example is the algebraic number

$$\sqrt{2} = \int_{2x^2<1} dx \ , \tag{C.12}$$

where $S = \{x \in \mathbb{R} \mid 2x^2 < 1\}$. It turns out that there is an equivalent definition in which u and v are only required to be *algebraic functions* and the algebraic sets S, S' defined by polynomials with *algebraic* coefficients. An algebraic function $f : \mathbb{R}^n \to \mathbb{R}$ is a function for which there is non-zero polynomial $P \in \mathbb{Q}[X_1, ..., X_n, Y]$ such that

$$P(x, f(x)) = 0 \tag{C.13}$$

for all $x \in \mathbb{R}^n$. This means that there are polynomials $a_0, ..., a_l \in \mathbb{Q}[X_1, ..., X_n]$ such that

$$a_0(x) + a_1(x)f(x) + ... + a_n(x)f(x)^n = 0 \ . \tag{C.14}$$

Take the simple case $n = 1$. Because a rational function f satisfies

$$q(x)f(x) - p(x) = 0 \tag{C.15}$$

we see that rational functions are algebraic. An example for a non-rational but algebraic function is

$$g(x) = \sqrt{1 + x^2} \tag{C.16}$$

which fulfils

$$1 + x^2 - g(x)^2 = 0 \ , \tag{C.17}$$

i.e. $P(x, y) = 1 + x^2 - y^2$. However, this equivalent definition implies immediately that $\mathcal{P}$ contains all algebraic numbers, i.e.

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \overline{\mathbb{Q}} \subset \mathcal{P} \subset \mathbb{C} \ . \tag{C.18}$$

is the new hierarchy that we have. Does $\mathcal{P}$ also contain transcendentals? Obviously, because

$$\int\int_{x^2+y^2<1} dx \ dy \ = \pi = 2 \int_{\mathbb{R}^+} \frac{dx}{1 + x^2} \ . \tag{C.19}$$

Further examples are *logarithms of algebraic numbers*:

$$\log(2) = \int_1^2 \frac{dx}{x} \ . \tag{C.20}$$

What is also worth mentioning is that they are *countable* just as is $\overline{\mathbb{Q}}$! Therefore $\mathcal{P} \setminus \overline{\mathbb{Q}}$ is a countable set of transcendentals. However, by conjecture, the numbers

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \ , \qquad \gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{2} + ... + \frac{1}{n} - \ln(n)\right) \tag{C.21}$$

are *not* periods.

Bibliography

[BloKr10] S.Bloch, D.Kreimer : *Feynman amplitudes and Landau singularities for 1-loop graphs.* arXiv: hep-th/1007.0338v2

[BoWei03] C.Bogner, S.Weinzierl: *Feynman graph polynomials*, arXiv: hep-th/1010.1667

[BroKr98] D.Broadhurst, D.Kreimer: *Renormalization automated by Hopf algebra*, (1998) arxiv: hep-th/9810087

[BroKr99] D.Broadhurst, D.Kreimer: *Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Padé-Borel resummation*, (1999) arxiv: hep-th/9912093

[BrowKr11] F.Brown, D.Kreimer: *Angles, Scales and Parametric Renormalization*, arXiv: hep-th/1112.1180

[BrowKr12] F.Brown, D.Kreimer: *Decomposing Feynman Rules*, arXiv: hep-th/1212.3923

[CM98] A.Connes, H.Moscovici: *Hopf algebras, cyclic homology and the transverse index theory*, Comm. Math. Phys. 198 (1998), 198-246.

[ConVo03] J.Conant, K.Vogtmann: *On a theorem of Kontsevich*, Algebr.Geom.Topol. 3(2003) 1167-1224, arXive: math/0208169v2.

[CoKr98] A.Connes, D.Kreimer: *Hopf algebras, Renormalization and Noncommutative Geometry*, Comm. Math. Phys. 199 (1998), 203-242, arXive: hep-th/9808042

[CoKr00] D.Kreimer: *Renormalization in quantum field theory and the Riemann-Hilbert Problem I: the Hopf algebra structure of graphs and the Main Theorem.* Comm.Math.Phys.210(2000), 249-273

[CoKr01] A.Connes, D.Kreimer: *Renormalization in Quantum Field Theory and the Riemann-Hilbert Problem II: The β- function, Diffeomorphisms and the Renormalization Group.* Comm.Math.Phys.216, 215-241(2001), arXiv: hep-th/003188

[CoKr02] A.Connes, D.Kreimer: *Insertion and Elimination: the Doubly Infinite Lie Algebra of Feynman graphs*, Ann.H.Poincaré 3 (2002), 411-433

[Foi02] Loic Foissy: *Les algébres de Hopf des arbrés enracinés décorés*, PhD thesis, 2002, University of Reims, France, available online on Loic Foissy's webpage

[Fren88] I.Frenkel, J.Lepowsky, A.Meurman: *Vertex operator algebras and the Monster*, Academic Press, Inc.(1988)

[Kr98] D.Kreimer: *On the Hopf algebra structure of perturbative quantum field theory*, Adv. Theor. Math. Phys. 2 (1998), 303-334, arXive: q-alg/9707029

[Kr99] D.Kreimer: *Chen's iterated integrals represents the operator product expansion*, Adv. Theor.Math.Phys. 3 (1999) 627-670, arXive: hep-th/9901099

[Kr03] D.Kreimer: *Factorization in quantum field theory: an exercise in Hopf algebras and local singularities*, arXiv: hep-th/0306020

[KrSS12] D.Kreimer, M.Sars,W.D.van Suijlekom: *Quantization of Gauge Fields, Graph Polynomials and Graph Homology*, arXiv: hep-th/1208.6477

[KrSui06] D.Kreimer, W.D.van Suijlekom: *Recursive relations in the core Hopf algebra*, arxiv: hep-th/09032849

[KrY06] D.Kreimer: *An étude in non-linear Dyson-Schwinger equations*, arXiv: hep-th/0605096

[LeBe] M.Le Bellac : *Quantum and Statistical Field theory.* Clarendon Press Oxford, 2002

[Lued] M.Lüders: *Baumfakultäten und kombinatorische Dyson-Schwinger Gleichungen*, Master Thesis, available online on the webpages of Dirk Kreimer's group

[Man06] D.Manchon: *Hopf algebras, from basics to applications to renormalization*, 2006, arxiv: math/0408405v2

[Menc] Igor Mencattini's PhD thesis. Available at this lecture's webpages.

[MeKr02] I.Mencattini, D.Kreimer: *The Structure of the Ladder Insertion-Elimination Lie Algebra*, Comm. Math. Phys. 259 (2005), 413-432

[NoNa61] Noborn, Nakanishi: *Parametric Integral formulas and analytic properties in perturbation theory*, supplement of Prog.in Theor.Phys. 18 (1961)

[Panz12] Erik Panzer: *Hopf-algebraic Renormalization in Kreimer's Toy Model*, Master thesis 2012, available on the webpages Dirk Kreimer's group or on arXive: hep-th/9808042

[PesSchr] M.E.Peskin, D.V.Schroeder: *An Introduction to Quantum Field Theory.* West View Press (1995)

[VAT04] Jon Eivind Vatne: *Introduction to operads*, lecture notes 2004, available online (see Dirk Kreimer's webpage of this lecture course).

[Wein60] S.Weinberg: *High Energy Behavoir in Quantum Field Theory*, Phys.Rev.118 (1960), p.838-849

[Y11] K.Yeats: *Rearranging Dyson-Schwinger equations*, AMS, Vol. 211, 995, arxiv: 'Growth estimates for Dyson-Schwinger equations', hep-th/08102249

[Zag01] M.Kontsevich, D.Zagier: *Periods*, see internet