An Étude in non-linear Dyson–Schwinger Equations*

Dirk Kreimer[a†]
Karen Yeats[b]

[a]Institut des Hautes Études Scientifiques,
91440 Bures-sur-Yvette, France

[b]Math. Dept., Boston Univ.,
Boston, MA02215, USA

We show how to use the Hopf algebra structure of quantum field theory to derive nonperturbative results for the short-distance singular sector of a renormalizable quantum field theory in a simple but generic example. We discuss renormalized Green functions $G_R(\alpha, L)$ in such circumstances which depend on a single scale $L = \ln q^2/\mu^2$ and start from an expansion in the scale $G_R(\alpha, L) = 1 + \sum_k \gamma_k(\alpha)L^k$. We derive recursion relations between the γ_k which make full use of the renormalization group. We then show how to determine the Green function by the use of a Mellin transform on suitable integral kernels. We exhibit our approach in an example for which we find a functional equation relating weak and strong coupling expansions.

1. The structure of Green functions

Our interest is the high energy sector of a renormalizable quantum field theory. We want to explore consequences of the underlying Hopf algebra structure to obtain non-perturbative results, using the self-similarity of Green functions.

Our approach is based on the Dyson–Schwinger equations for one-particle irreducible renormalized Green functions. As we want to obtain nonperturbative results the choice of a renormalization condition for us means simply the choice of a boundary condition for the full Green function.

In this short paper we want to exhibit the basic idea underlying such a program. We focus on the question how to treat the non-linearity of Dyson–Schwinger equations systematically. We assume we work in a renormalizable quantum field theory which provides a finite set $\mathcal{R}$ of amplitudes which need renormalization.

For a given superficially divergent amplitude

overall div. amplitude

$r \in \mathcal{R}$ we let Γ^r be the sum

$$\Gamma^r = \mathbb{I} + \sum_{\Gamma} \alpha^{|\Gamma|} \frac{\Gamma}{\mathrm{sym}(\Gamma)} \tag{1}$$

propagators?

over all 1PI graphs Γ contributing to that amplitude, with α a loop-counting small parameter. Projection onto suitable form factors $\phi(r)$ allows the sum to start with one, so that by the application of the Feynman rules $\phi(\Gamma^r)$ is the corresponding structure function and the Lagrangian L is given by

$$L = \sum_{r \in \mathcal{R}} \phi(r). \tag{2}$$

We can then write [1]

Hochschild 1-cocycles B_+

$$\Gamma^r = \mathbb{I} + B_+(\Gamma^r, Q(\{\Gamma^i\})),, \tag{3}$$

where $Q = Q(\{\Gamma^i\}_{r \in \mathcal{R}})$ evaluates to an invariant charge under the Feynman rules and the Hochschild one-cocycle

$$B_+^r(\Gamma^r, Q) = \sum_{k \geq 1} \alpha^k B_+^{k;r}(\Gamma^r Q^k) \tag{4}$$

is a sum of one-cocycles $B_+^{k,r}$ and Q is a monomial in the Γ^r. The uniqueness of Q implies

*work supported in parts by grant nsf-dms0401262 at Center for Mathematical Physics, Boston University
†supported by CNRS

the Slavnov–Taylor identities for the renormalized couplings [1].

The $B_+^{k,r}$ themselves are obtained from the skeleton graphs γ of the theory:

$$B_+^{k,r} = \sum_\gamma \frac{1}{\mathrm{sym}(\gamma)} B_+^\gamma, \tag{5}$$

where the sum is over all Hopf algebra primitives γ contributing to the amplitude r at k loops. These maps are defined to be closed Hochschild one-cocycles on the sub Hopf algebra generated by their concatentations and products [1,2].

Effectively, (3) reduces the sum (1) over all graphs to a sum over primitive ones, making use of the recursive structure of this fixpoint equation which determines the sum of graphs which contribute to a chosen amplitude. The sums involved typically reflect the universal law of [3] and will be discussed in detail in upcoming work.

We set

$$\Gamma^r = \mathbb{I} + \sum_j c_j^r \alpha^j \tag{6}$$

and those c_j^r are the linear generators of a sub-Hopf algebra:

Theorem 1 *There exists maps $B_+^{k;r}$, polynomials $P_{k,j}^r$ in those linear generators and integers s_r such that*

$$\Gamma^r = \mathbb{I} + \sum_k \alpha^k B_+^{k;r}\left(\Gamma^r Q^k\right), \tag{7}$$

$$\Delta B_+^{k;r} = B_+^{k;r} \otimes \mathbb{I} + \left(\mathrm{id} \otimes B_+^{k;r}\right)\Delta, \tag{8}$$

$$Q = \alpha \prod_{r\in\mathcal{R}} [\Gamma^r(\alpha)]^{s_r}, \tag{9}$$

$$\Delta c_k^r = \sum_{j=0}^k P_{k,j}^r \otimes c_{k-j}^r, \tag{10}$$

which make the system $\{c_k^r\}$ into a sub Hopf algebra $H(\Delta, m, S, \epsilon)$ of the Feynman graph Hopf algebra.

The polynomials $P_{k,j}^r$ are easily determined and we refer the reader to [1] for details.

Feynman rules are then defined in accordance with the Hochschild cohomology:

$$\phi\left(B_+^\gamma(h)\right)(\{q\}) = \int d_\gamma(\{k\},\{q\})\phi(h)(\{k\}). \tag{11}$$

Here $d_\gamma(\{k\},\{q\})$ is a measure determined by the primitive γ which depends on internal loop momenta $\{k\}$ and external momenta $\{q\}$ such that the Hopf algebra primitive determines the kernel d_γ and hence the Feynman rules by

$$\phi(B_+^\gamma(\mathbb{I})) = \int d_\gamma(\{k\},\{q\}), \tag{12}$$

and an appropriate insertion of $\phi(X)(\{k\})$ in (11) into the integrand provided by d_γ in accordance with the pre-Lie structure of graphs is understood. Similarly we define renormalized Feynman rules by iterating in a subtracted kernel

$$\phi_R\left(B_+^\gamma(\mathbb{I})\right) = \int [d_\gamma(\{k\},\{q\}) - d_\gamma(\{k\},\{\mu\})], \tag{13}$$

where $\{\mu\}$ indicates a suitable renormalization point.

There exists a basis of graphs and external structures in the Hopf algebra such that

$$\phi_R(Q) = \phi_R(Q)(L), \tag{14}$$

where $L = \ln q^2/\mu^2$ is a single scale which determines the running of the invariant charge. The choice of such a basis disentangles internal subdivergences into divergent contributions which depend on a single scale and finite contributions which determine the set of primitive elements in such a base. In this base, short distance singularities are captured by Green functions which are functions of two dimensionless variables α, L, with a remarkable duality between these two variables first observed in [5].

In perturbation theory the Feynman rules now allow us to write

$$G_R^r(\alpha, L) = \phi_R(\Gamma^r) = 1 + \sum_k \alpha^k \phi_R(c_k^r)(L). \tag{15}$$

We can expand in a different manner   vs RGE

$$G_R^r(\alpha, L) = 1 + \sum_k \gamma_k^r(\alpha) L^k, \quad \text{DSE} \tag{16}$$

and the renormalization group dictates relations between the γ_k^r. We work them out in a moment.

First, we note that in the case of a linear DSE [2], we get

$$\partial_L \phi(Q)(L) = 0, \tag{17}$$

$P_{lin}(\ldots) = 0$

$P_{lin}(\ldots) = \ldots - \gamma(\alpha)$

Ex: $G(L,\alpha) = \left(\frac{t^2}{\mu^2}\right)^{-\gamma(\alpha)} = 1 - L\gamma(\alpha) + \frac{1}{2}\gamma^2(\alpha) L^2$

and hence a scaling solution

$$G(\alpha, L) = e^{-\gamma(\alpha)L} \tag{18}$$

solves the linear DSE so that

$$\gamma_k(\alpha) = \frac{\gamma_1(\alpha)^k}{k!}. \tag{19}$$

To proceed in general we consider the map

$$P_{\text{lin}}^{(n)} = \underbrace{P_{\text{lin}} \otimes \cdots \otimes P_{\text{lin}}}_{n \text{ times}} \Delta^{n-1} \tag{20}$$

where P_{lin} is the projector into the linear span of generators of the Hopf algebra. From [1,2] we have:

Theorem 2 *The linearized coproduct is obtained as*

$$P_{\text{lin}}^{(2)}\Gamma^r = P_{\text{lin}}\Gamma^r \otimes P_{\text{lin}}\Gamma^r + P_{\text{lin}}Q \otimes \alpha\partial_\alpha \Gamma^r,$$

where

$$P_{\text{lin}}Q = \sum_r s_r P_{\text{lin}}\Gamma^r. \tag{21}$$

This allows us to understand the iterative structure of the next-to… leading log expansion (16).

We define for $n > 1$

$$\sigma_n := \frac{1}{n!} m^{n-1} \underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{n \text{ times}} \Delta^{n-1}, \tag{22}$$

and σ_1 is the residue defined by

$$\sigma_1 = \partial_L \phi_R \left(S \star Y_{\text{aug}}\right)(L)|_{L=0}. \tag{23}$$

Actually, σ_n evaluates to the coefficient of the L^n term in the evaluation of a Hopf algebra element by the renormalized Feynman rules, by the scattering type formula [4].

We have

$$h \notin H_{\text{lin}} \Rightarrow \sigma_1(h) = 0, \tag{24}$$

so we can use Theorem 2 and, by the above definition (16) of $\gamma_k^r(\alpha)$,

$$\gamma_k^r(\alpha) = \sigma_k(\Gamma^r(\alpha)). \tag{25}$$

Projection onto the linear generators delivers the desired formula for the expansion in L:

$$\begin{aligned} \gamma_k^r(\alpha) \;&=\; \frac{1}{k}\Big[\gamma_1^r(\alpha)\gamma_{k-1}^r(\alpha) \\ &\quad + \sum_j s^j \gamma_1^j(\alpha)\alpha\partial_\alpha \gamma_{k-1}^r(\alpha)\Big]. \end{aligned} \tag{26}$$

With the the above choice of basis we can now introduce the Mellin transform

$$F_\gamma(\rho) = \int \phi(B_+^\gamma(\mathbb{I}))[k^2]^{-\rho} \tag{27}$$

(with obvious generalizations to the multivariate case as studied below for example) and the DSE turns into an equation which determines γ_1^r as we will see in a moment, while the further terms in the L expansion are determined from (26) above.

The Green function also has the usual expansion in α which is triangular wrt γ_k

$$\gamma_k^r(\alpha) = \sum_{j \geq k} \gamma_{k,j}^r \alpha^j. \tag{28}$$

We can hence proceed to work out the recursion relations which express the functions γ_k^r through the functions γ_1^r for $k > 1$, and turn the Dyson–Schwinger equations into an implicit equation which allows to determine the sole unknown functions $\gamma_1^r(\alpha)$ from the knowledge of the above Mellin transforms. A full discussion is given in future work. We now exhibit the approach in an example.

2. A simple example

For concreteness, we consider massless Yukawa theory and consider all self-iterations of the one-loop massless fermion propagator, with subtractions in the momentum scheme at $q^2 = \mu^2$. Our Green function is an inverse propagator with momentum q and a function of two variables a and $L = \ln q^2/\mu^2$. We ignore radiative corrections at the bosonic line and also at vertices, so the set $\mathcal{R}$ has a single element and the superscript r is suppressed henceforth. We rederive the results of [5] for this case.

We write the perturbative series for the Dyson–Schwinger equation as

$$X(a) = \mathbb{I} - aB_+^c\left(\frac{1}{X(a)}\right), \tag{29}$$

where $\int \phi(B_+^c(\mathbb{I}))$ provides the one-loop self-energy integral to be iterated. Note that upon setting $X(a) = \mathbb{I} - \underline{X}(a)$, this is the equation for the self-energy $\underline{X}(a) = -P_{\text{lin}}X(a)$ studied in [5].

$$G(L,a) = 1 - \alpha \int \frac{1}{G(\ln\frac{k^2}{\mu^2}, a)} \frac{1}{\cdots}$$

$$1 + \sum L^k \gamma_k$$

$$X(\alpha) = \mathbb{I} - a B_+\left(X(\alpha)\frac{1}{X^2(\alpha)}\right) \qquad \int d^4k \qquad \frac{k\cdot q}{(k^2)^{1+\rho}(k+q)^2}$$

With

$$Q = 1/X^2 \to P_{\text{lin}}(Q) = -2\underline{X}(a), \tag{30}$$

we find the linearized coproduct

$$P_{\text{lin}}^{(2)} X(a) = P_{\text{lin}} X(a) \otimes (P_{\text{lin}} - 2a\partial_a) X(a). \tag{31}$$

This is Proposition 1 of [5] and we also get

Theorem 3 *The next-to next-to... leading log expansion in L is given through the anomalous dimension $\gamma_1(a)$ as*

$$\gamma_k(a) = \frac{1}{k}\gamma_1(a)(1 - 2a\partial_a)\gamma_{k-1}(a). \tag{32}$$

This is Proposition 2 of [5].

We can now work out with ease the recursions which express γ_k, $k > 1$ through the Taylor coefficients of γ_1, for example

$$\begin{aligned}
\gamma_2 &= \frac{1}{2}\left(\gamma_1^2 - 2\gamma_1 a\partial_a \gamma_1\right) && (33)\\
&= \frac{1}{2}\left[-\gamma_{1,1}^2 a^2 - 4\gamma_{1,1}\gamma_{1,2}\alpha^3 + \cdots\right], && (34)\\
\gamma_3 &= \frac{1}{6}\left(\gamma_1(1-2a\partial_a)\gamma_1(1-2a\partial_a)\gamma_1\right) && (35)\\
&= \frac{1}{6}\left(3a^3\gamma_{1,1}^3 + \cdots\right), && (36)
\end{aligned}$$

and so on.

Such recursions are obtained for any non-linear DSE by iterating Theorem 2. Also, we observe that we actually only need the cocommutative part in the determination of the coproduct as is evident from the very cocommutative definition (22) of σ_k, $k > 1$. The non-cocommutative part is always of lower degree in L in the obvious filtration by L.

It remains to understand how to compute $\gamma_1(\alpha)$. Instead of an explicit analysis of non-linear ODEs as in [5] we proceed here by the Mellin transform, as promised. Here, it reads

$$\begin{aligned}
F(\rho) &= \frac{1}{q^2}\int d^4k \frac{k\cdot q}{[k^2]^{1+\rho}(k+q)^2}\Big|_{q^2=1}\\
&=: \frac{r}{\rho} + \sum_{i\geq 0} f_i \rho^i. && (37)
\end{aligned}$$

In our simple example we have

$$F(\rho) = \frac{-1}{\rho(2-\rho)}. \tag{38}$$

Let us introduce a short hand notation:

$$\gamma \cdot U = \sum_{k=1}^{\infty} \gamma_k(\alpha) U^k. \qquad \gamma\cdot L = G(L,a) \tag{39}$$

Then, the Dyson–Schwinger equation becomes

$$\gamma \cdot L = \alpha(1 + \gamma\cdot\partial_{-\rho})^{-1}[e^{-L\rho} - 1]F(\rho)|_{\rho=0}, \tag{40}$$

where we evaluate the rhs at $\rho = 0$ after taking derivatives. The functional dependence of the non-linear DSE on $XQ = X^{-1}$ reflects itself on the rhs.

The only unknown quantities in this equation are the Taylor coefficients $\gamma_{1,j}$ which are implicitly defined through the Taylor coefficients of the Mellin transform (38) above.

Taking a derivative of (40) wrt L and setting L to zero allows us to read them off:

$$\begin{aligned}
\gamma_1 &= \alpha(1+\gamma\cdot\partial_{-\rho})^{-1}\rho F(\rho)|_{\rho=0} && (41)\\
&= \alpha r + \alpha\left(\sum_{k\geq 1}[\gamma\cdot\partial_{-\rho}]^k\right) \times\\
&\quad \times \left[\sum_{k=1}^{\infty}\rho^k f_{k-1}\right]|_{\rho=0}, && (42)
\end{aligned}$$

so $\gamma_{1,1} = r$ universally.

In our example $Q = X^{-2}$ we furthermore find

$$\begin{aligned}
\gamma_{1,2} &= r f_0, && (43)\\
\gamma_{1,3} &= r f_0^2 + r^2 f_1, && (44)
\end{aligned}$$

and so on.

In this manner one confirms the results of [5] which serve here as a mere example for a much more general approach. Note that working with the toy Mellin transform $r = f_i = 1$ reproduces the generating functions which counts the graphs contributing at each loop order confirming the count of Wick contractions in [6], in terms of Catalan numbers

$$1, 1, 2, 5, 14, 42, \cdots.$$

For the anomalous dimension itself we indeed also confirm the series A000699 of [7],

$$\text{A000699} = 1, 1, 4, 27, 248, 2830, \cdots, \tag{45}$$

from Eq.(42) above, in accordance with [6].

Before we finish this paper by discussing a double Mellin transform we mention one particular nice feature of this example which we very much hope to work out in general in the future.

3. A functional equation

It is our hope that eventually a non-perturbative renormalized Green function can be related to suitably defined ζ-functions. The existence of a combinatorial Euler product underlying the decomposition of graphs into primitive graphs is a first hint [8].

Here we report on another such hint based on the possibility to reformulate the result of [5] such that a functional equation is obtained. Inspection of the solution in [5] (which is also suggested by the functional equation of the complementary error function) shows

$$\begin{aligned}\widetilde{\Sigma}(a,p) &= -\frac{\sqrt{a/(2\pi)}}{\exp(p^2)\mathrm{erfc}(p)} \times \\ &\quad \times \widetilde{\Sigma}\left(\frac{(\exp(p^2)\mathrm{erfc}(p))^4}{a/(2\pi)^2}, p\right), \qquad (46)\end{aligned}$$

where a is now the loop counting parameter and p is another variable such that with $z = e^{2L}$,

$$p = \frac{d}{dz}\sqrt{\frac{2}{a}}\left(z - z\widetilde{\Sigma}\left(\mu^2\sqrt{z}\right)\right). \qquad (47)$$

Note that on the lhs of (46) we have a weak coupling expansion for a, on the rhs we have a strong coupling expansion, hence an expansion in $1/a$.

With $T = a/(2\pi)$, $u = (\exp(p^2)\mathrm{erfc}(p))^{-4}$, and $Z(T,u) = \widetilde{\Sigma}(a,p)$ we get a functional equation reminiscent of a functional equation for a ζ-function in two variables for the function field case [9,10] for the non-perturbative renormalized Green function

$$Z(T,u) = -u^{\frac{5}{4}-1}T^{2(\frac{5}{4}-1)}Z\left(\frac{1}{Tu}, u\right). \qquad (48)$$

The propagator coupling duality of this Green function can now be expressed with $u = \exp(s+t)$, $T = \exp(-t)$, and $\zeta(s,t) = \exp(\frac{t-s}{8})Z(T,u)$ as

$$\zeta(s,t) = -\zeta(t,s), \qquad (49)$$

which we report here for motivation to think further about the connection between quantum field theory and ζ-functions.

4. The appearance of transcendentals

How do we continue in the case where we have several insertion places? Nothing changes in the above derivation apart from the fact that we now have to work with our double Mellin transform due to the fact that now both propagators obtain logarithmic corrections. As the primitive self-energy $B_+^c(\mathbb{I})$ now obtains corrections at the internal fermionic and bosonic line, we are led to consider a coupled system including also the bosonic propagator.

If we are hence to consider the system (based now on two elements in $\mathcal{R}$, fermion and boson self-energies)

$$\begin{aligned}X(a) &= \mathbb{I} - aB_+^c\left(\frac{1}{X(a)Y(a)}\right) \\ &= \mathbb{I} - aB_+^c\left(X(a)Q(a)\right), \qquad (50)\\ Y(a) &= \mathbb{I} - B_+^b\left(\frac{1}{X(a)^2}\right) \\ &= \mathbb{I} - aB_+^b\left(Y(a)Q(a)\right), \qquad (51)\end{aligned}$$

with $Q^{-1}(a) = X^2(a)Y(a)$ and

$$P_{\mathrm{lin}}Q(a) = -2P_{\mathrm{lin}}X(a) - P_{\mathrm{lin}}Y(a), \qquad (52)$$

describing all possible iterations of massless fermion and scalar one-loop graphs, the corresponding functions $\gamma_k^X(a), \gamma_k^Y(a)$ are determined in terms of the anomalous dimensions $\gamma_1^X(a), \gamma_1^Y(a)$. The latter are obtained from the system of Dyson–Schwinger equations

$$\begin{aligned}\gamma_1^X &= \alpha(1+\gamma^X\cdot\partial_{-\rho_1})^{-1}(1+\gamma^Y\cdot\partial_{-\rho_2})^{-1}\times \\ &\quad \times(\rho_1+\rho_2)F_a(\rho_1,\rho_2)|_{\rho_i=0}, \qquad (53)\\ \gamma_1^Y &= \alpha(1+\gamma^X\cdot\partial_{-\rho_1})^{-1}(1+\gamma^X\cdot\partial_{-\rho_2})^{-1}\times \\ &\quad \times(\rho_1+\rho_2)F_b(\rho_1,\rho_2)|_{\rho_i=0}. \qquad (54)\end{aligned}$$

We thus have to consider two Mellin transforms in two variables each. They read for the fermion self-energy $B_+^c(\mathbb{I})$

$$\begin{aligned}F_c(\rho_1,\rho_2) &= \frac{1}{q^2}\int d^4k\frac{k\cdot q}{[(k+q)^2]^{1+\rho_1}[k^2]^{1+\rho_2}} \\ &= \frac{-1+\rho_2}{(2-\rho_1-\rho_2)}\frac{e^{\sum_{k=1}^{\infty}-2\zeta(2k+1)f_k(\rho_1,\rho_2)}}{(1-\rho_1-\rho_2)(\rho_1+\rho_2)}, \qquad (55)\end{aligned}$$

and for the boson self-energy $B_+^b(\mathbb{I})$

$$F_b(\rho_1,\rho_2) = \frac{1}{q^2}\int d^4k \frac{k\cdot(k+q)}{[(k+q)^2]^{1+\rho_1}[k^2]^{1+\rho_2}}$$
$$= \frac{1-(\rho_1+\rho_2)+\rho_1\rho_2}{(2-\rho_1-\rho_2)}\times$$
$$\times\frac{e^{\sum_{k=1}^{\infty}-2\zeta(2k+1)f_k(\rho_1,\rho_2)}}{(1-\rho_1-\rho_2)^2(\rho_1+\rho_2)}, \quad (56)$$

where traces have been taken to obtain the relevant structure functions and where $f_k(\rho_1,\rho_2)$ are the two-variable symmetric polynomials given by

$$f_k(\rho_1,\rho_2) = \sum_{j=1}^{2k}\frac{2k!}{j!(2k+1-j)!}\rho_1^j\rho_2^{2k+1-j}. \quad (57)$$

We observe that the appearance of transcendentals is utterly generated from the presence of a second insertion place.

Eqs.(53–56) completely determine the anomalous dimensions in question and together with (26,52) the two Green functions. To what extent this leads to further functional equations is under current investigation.

Acknowledgment

Both authors thank David Broadhurst for discussions and proofreading the ms.

REFERENCES

1. D. Kreimer, *Anatomy of a gauge theory*, Annals of Physics, in press, hep-th/0509135.
2. C. Bergbauer, D. Kreimer, *Hopf algebras in renormalization theory: Locality and Dyson–Schwinger equations from Hochschild cohomology*, Proc. 75 Renc. Phys. & Math. V. Turaev, L. Nyssen, eds., Eur. Math. Soc., in press, hep-th/0506190.
3. J. Bell, S. Burris, and K. Yeats, *Counting Rooted Trees: The Universal Law $t(n)C\rho^{-n}n^{-3/2}$.*, Elect. J. Comb., to appear, math.CO/0512432;
   D. Kreimer, K. Yeats, in preparation.
4. A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. II: The beta-function, diffeomorphisms and the renormalization group,* Commun. Math. Phys. **216** (2001) 215, hep-th/0003188.
5. D. J. Broadhurst and D. Kreimer, *Exact solutions of Dyson-Schwinger equations for iterated one-loop integrals and propagator-coupling duality,* Nucl. Phys. B **600** (2001) 403, hep-th/0012146.
6. D. J. Broadhurst and D. Kreimer, *Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Pade-Borel resummation,* Phys. Lett. B **475** (2000) 63, hep-th/9912093.
7. N. J. A. Sloane, (2006), The On-Line Encyclopedia of Integer Sequences, published electronically at www.research.att.com/ njas/sequences/.
8. D. Kreimer, *What is the trouble with Dyson-Schwinger equations?*, Nucl. Phys. Proc. Suppl. **135** (2004) 238, hep-th/0407016.
9. C. Deninger, *Two-variable zeta functions and regularized products*, math.NT/0210269.
10. R. Pellikaan, *On special divisors and the two variable zeta function of algebraic curves over finite fields,* in Arithmetic, Geometry and Coding Theory 4, Luminy 1993, pp. 175-184, (R. Pellikaan, M. Perret and S.G. Vladut eds.), Walter de Gruyter & Co, Berlin 1996.