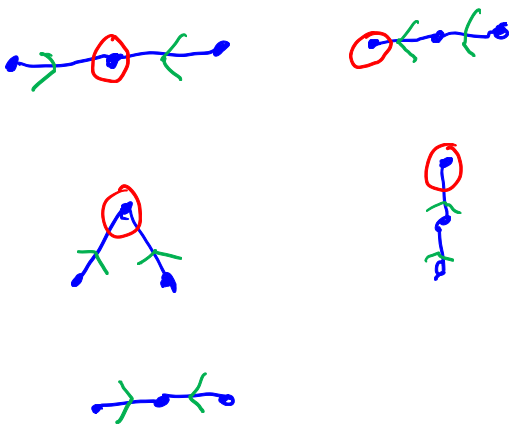
Rooted trees.

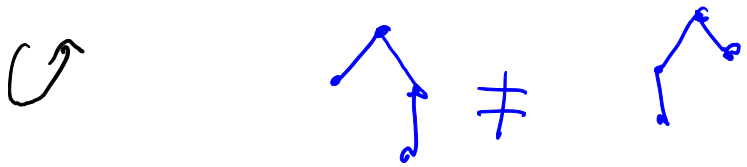
Graph made up from edges and vertices:

- every edge connects two vertices
- it is connected (if it is a tree)
- it is simply connected $\Leftrightarrow$ no loops
- it has a distinguished vertex
- orient all edges towards that vertex



root is uppermost vertex

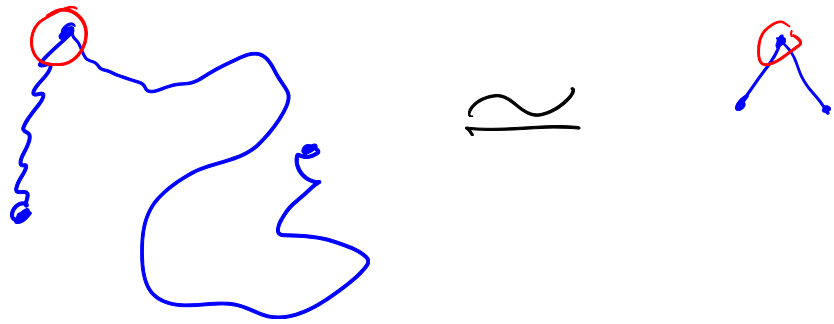
- we do not orient the plane



planar rooted trees

unoriented: " $=$ " <u>non-planar rooted trees</u>

- we work up to isomorphisms



So we have defined non-planar rooted trees.

$\mathbb{T} \simeq \emptyset$ no rooted tree $\quad |\mathcal{T}(0)| = 1$

just a root $\quad |\mathcal{T}(1)| = 1$

$|\mathcal{T}(2)| = 1$

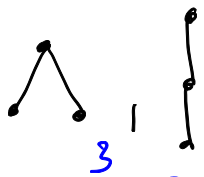
Let us consider forests: disjoint unions of rooted trees

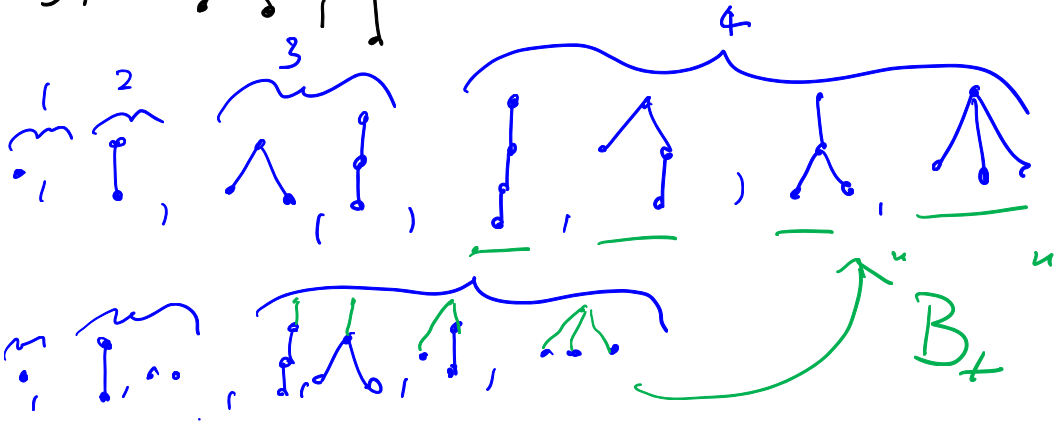
$\mathcal{F}(n)$ is the set of all forests on n vertices

$\mathcal{F}(0) \sim \mathbb{T}$

$\mathcal{F}(1) \sim \bullet$

$\mathcal{F}(2) \sim$  $\quad |\mathcal{F}(2)| = 2$

$\mathcal{T}(3)$  $\quad |\mathcal{T}(3)| = 2$



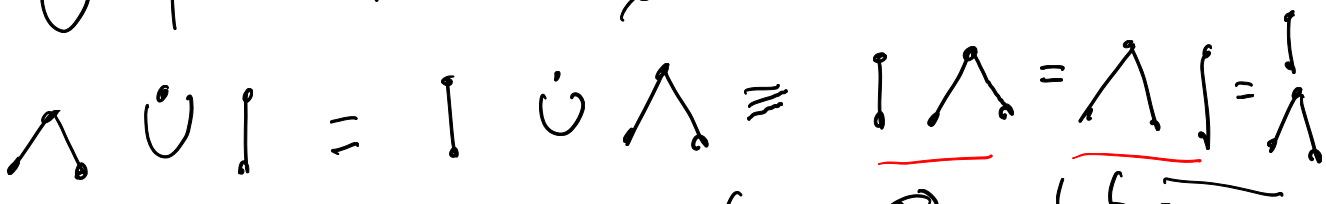
Prop. There are as many forests on n vertices as there are trees on $(n+1)$ vertices.

So fix, we have the set of all rooted trees and their forests by disjoint union.

Let us make this into an algebra.

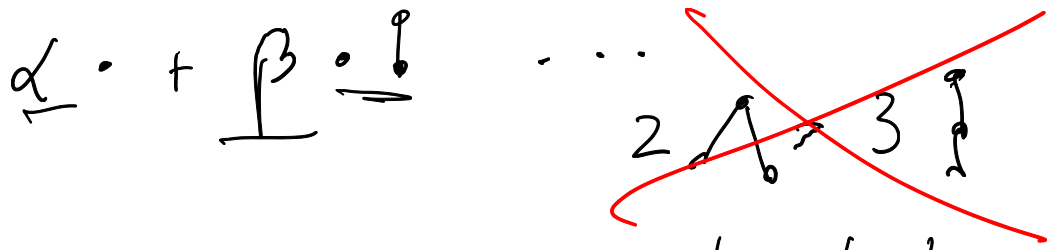
Take disjoint union:

$$\emptyset \,\dot\cup\, F = F \qquad \emptyset \simeq \mathbb{1}$$



Consider the free commutative $\mathbb{Q}$ algebra generated by all forests $\mathcal{F}(n)$ $\forall n$.

(Vector space over $\mathbb{Q}$ $\langle \mathcal{F}(n) \rangle$)



Make this into a bialgebra.

Need a map: $\Delta : H \to H \otimes H$

$$\begin{tikzcd}
H \otimes H \arrow[r, "m"] \arrow[d, shift right, "id \otimes m"'] & H \arrow[d, shift right, "\Delta"'] \\
H \otimes H \otimes H \arrow[r, "m \otimes id"] \arrow[u, shift right, "id \otimes \Delta"'] & H \otimes H \arrow[u, shift right, "m"']
\end{tikzcd}$$

associative algebra

co-associative algebra

Also, we need a co-unit map.

$\mathbb{I}$ is the unit, but also (by abuse of notation) a map: $\mathbb{I}: \mathbb{Q} \to H$, $q \to q\,\mathbb{I}$

$\hat{\mathbb{I}}$ is the co-unit: $\hat{\mathbb{I}}: H \to \mathbb{Q}$

$\hat{\mathbb{I}}(f) = 0 \quad \forall f \in \mathcal{F}(n),\ n \geq 1$

$\hat{\mathbb{I}}(\mathbb{I}) = 1$ and $\hat{\mathbb{I}}$ is $\mathbb{Q}$-linear

For a bialgebra, we need co-product

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \quad \text{co-associativity}$$

The co-product of a product is the product of the co-products.

$$\Delta(f_1 \dots f_k) = \Delta(f_1) \cdot \ldots \cdot \Delta(f_k)$$

But what is Δ?

$$\Delta f_i := \sum_j f_i^{(j)'} \otimes f_i^{(j)''}$$

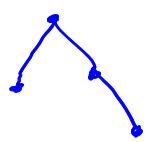
$$\Delta(f_1 f_2) = \sum_j (f_1 f_2)^{(j)'} \otimes (f_1 f_2)^{(j)''}$$

$$= \Big(\sum_{jk} f_1^{(j)'} f_2^{(k)'} \otimes f_1^{(j)''} f_2^{(k)''} \Big)$$

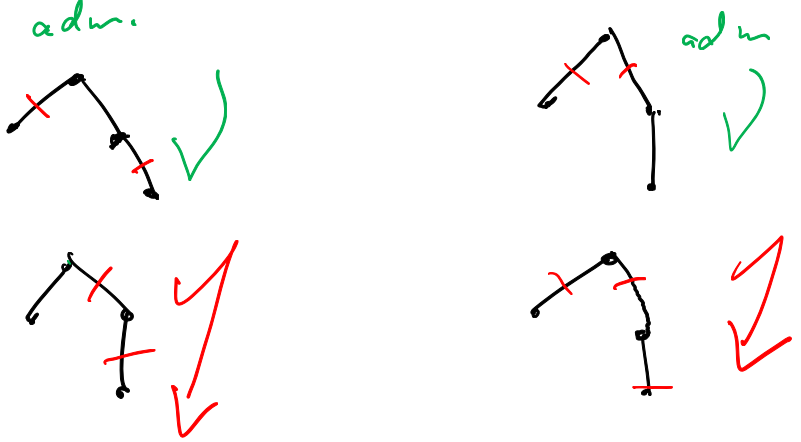
$$= \sum_j (\ldots \otimes \ldots) \sum_k (\ldots \otimes \ldots)$$

Δ? Two answers:

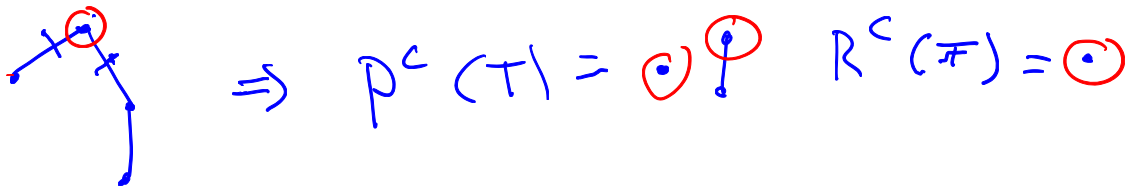
Introduce admissible cuts:

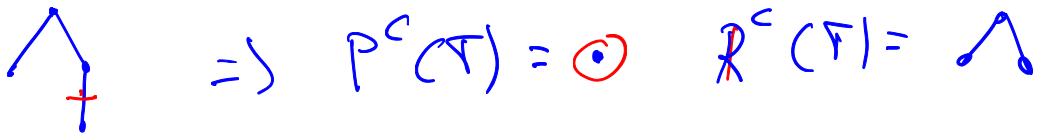


a cut is a set of edges of T. It is admissible if every path from a vertex to the root traverses at most one element of the set.



For C an admissible cut,
let $P^C(T)$ be the component of $T \setminus C$ not connected to the root
and $R^C(T)$ the single component connected to the root.

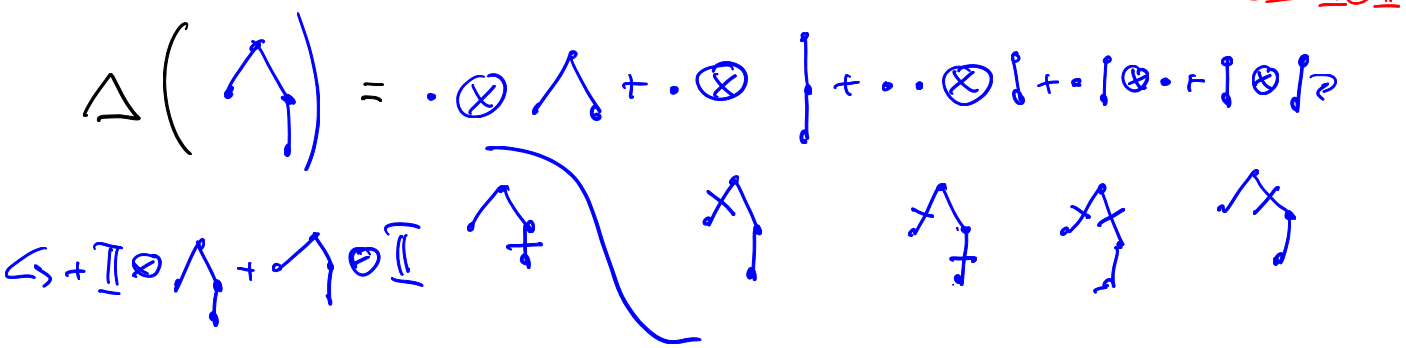




Define: $$\Delta(T) = \sum_{\substack{C \\ \text{admissible} \\ C \neq \emptyset}} P^C(T) \otimes R^C(T) + \underline{T \otimes \mathbb{1} + \mathbb{1} \otimes T}$$

$\Delta : H \to H \otimes H$ by construction
and define $\underline{\Delta(T_1 \dots T_k) = \Delta(T_1) \dots \Delta(T_k)}$

$\Delta \mathbb{1} = \mathbb{1} \otimes \mathbb{1}$



Another example:

$$\Delta(\cdot\,|) = \Delta(\cdot)\,\Delta(|)$$

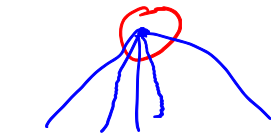
$$= (\cdot \otimes \mathbb{1} + \mathbb{1} \otimes \cdot)(| \otimes \mathbb{1} + \mathbb{1} \otimes | + \cdot \otimes \cdot)$$

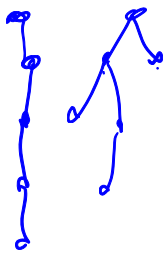
$$= \cdot\,| \otimes \mathbb{1} + \cdot \otimes | + \cdot\cdot \otimes \cdot + | \otimes \cdot\, \mathbb{1} \otimes \cdot\, | + \cdot \otimes \cdot\cdot$$

So we have a definition. Still need to prove co-associativity.

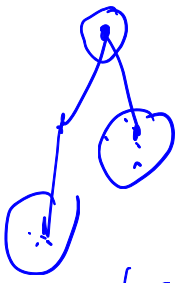
$\mathrm{tw}_j : \bigoplus_{i=1}^{\infty} H^{(i)}$

First, 2nd definition:





$$B_+(T_1 \ldots T_k) \to T_1 \ldots T_k \mathbb{1} \otimes X + \ldots$$



$$\Delta B_+(X) = B_+(X) \otimes \mathbb{1} + (\mathrm{id} \otimes B_+) \circ \Delta(X)$$

$\mathbb{1} \otimes B_+(X)$

Lemma: the two definitions of Δ are equiv.

Proof of equivalence and of co-associativity to be treated in exercise/class.

A last remark:

$$H = \bigoplus_{j \geq 0}^{\infty} H^{(j)}$$

a graded comm. Hopf alg.

not co-commutative

$\Delta(\Lambda) \neq \Delta^{op}(\Lambda)$

$H^{(j)} \simeq \mathbb{Z}^{(j)}$

Δ^{op}:

$H^{(0)} \simeq \mathbb{Q}\,\mathbb{1}$

$H' \otimes H'' \to H'' \otimes H'$