Rooted trees.

Graph made up from edges and vertices:

- every edge connects two vertices
- it is connected (if it is a tree)
- it is simply connected $\Rightarrow$ no loops
- it has a distinguished vertex
- orient all edges towards that vertex

We do not orient the plane

$\uparrow$  $\leftarrow$  $\uparrow$  $\cong$  $\uparrow$  $\cong$  $\uparrow$

planar rooted trees

unoriented:  $\equiv$  $\cong$  non-planar rooted trees

we work up to isomorphisms
So we have the final non-planar rooted trees.

\[ \overline{\Pi} \neq \emptyset \]

no rooted tree
\[ |\overline{\Upsilon}(0)| = 1 \]

just a root
\[ |\overline{\Upsilon}(1)| = 1 \]

\[ |\overline{\Upsilon}(2)| = 1 \]

Let us consider forests: disjoint unions of rooted trees.
\[ \mathcal{T}(n) \text{ is the set of all forests on } n \text{ vertices} \]

\[ \mathcal{T}(0) \sim \overline{\Pi} \]

\[ \mathcal{T}(1) \sim \{} \]

\[ \mathcal{T}(2) \sim \{} \]

\[ |\mathcal{T}(2)| = 2 \]

\[ |\overline{\Upsilon}(3)| = 2 \]

\[ \mathcal{T}(3) \]

\[ |\overline{\Upsilon}(3)| = 2 \]

\[ B_4 \]
Prop. There are as many forests on $n$ vertices as there are trees on $(n+1)$ vertices.

So far, we have to sort of all rooted trees and span forests by disjoint union.

Let us make this into an algebra.

Take disjoint union:

$$\emptyset \cup F = F$$

$$\bigwedge \emptyset = 1$$

Consider the commutative ring $Q$, a free $\mathbb{Z}$-module generated by all forests $F$ in $\mathcal{G}$.

We call the space over $Q$ $\mathcal{G}$.

Need a map: $\Delta : H \to H \otimes H$

$$\begin{align*}
H \otimes H & \xrightarrow{\text{id} \otimes \text{id}} H \otimes H \\
\text{id}_H & \xrightarrow{\Delta} H \otimes H \\
\text{id}_H & \xrightarrow{\text{co-associative}} H \otimes H
\end{align*}$$

$$\Delta \otimes \text{id} : H \to H \otimes H$$

$$\text{id} \otimes \Delta : H \to H \otimes H$$

$$\Delta \otimes \Delta : H \to H \otimes H \otimes H$$

$$\text{id} \otimes \text{id} \otimes \text{id} : H \to H \otimes H \otimes H$$
Also, we need a co-unit map.

\[ \hat{\eta} \] is the unit, but also (by abuse of notation)

\[ \hat{\eta} : \eta \rightarrow \hat{\eta}, \quad \eta ightarrow \eta \]

\[ \hat{\eta} \] is the co-unit:

\[ \hat{\eta} : \hat{\eta} \rightarrow \eta \]

\[ \hat{\eta}(\xi) = 0 \quad \forall \xi \in \xi(n), \ n \geq 1 \]

\[ \hat{\eta}(\eta) = 1 \quad \text{and} \quad \hat{\eta} \text{ is } \mathcal{O} - \text{linear} \]

For a bialgebra, we need co-product

\[ (\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta \]

\[ \text{co-associativity} \]

The co-product of a product is the product of the co-products.

\[ \Delta(\xi_1 \cdots \xi_k) = \Delta(\xi_1) \cdots \Delta(\xi_k) \]

But what is \( \Delta \)?

\[ \Delta(\xi) = \sum_i \xi \otimes \xi_i \]

\[ \Delta(\xi_1 \xi_2) = \sum_{i,j} (\xi_1 \xi_i) \otimes (\xi_1 \xi_j) \]

\[ = \left( \sum_{i,j,k} \xi \xi_i \xi_j \right) \otimes \left( \sum_{i,j,k} \xi \xi_i \xi_j \right) \]

\[ \Delta \]

Two answers:

Introduce admissible coads:

A coad is a set of edges of \( \hat{\eta} \). It is admissible if every path from a vertex to the root fringes at most one element of \( \eta \) so far.
For $C$ an admissible cut, let $P^C(T)$ be the component of $T \setminus C$ not connected to the root and $R^C(T)$ the single component connected to the root.

\[
\Rightarrow P^C(T) = \text{•} \quad R^C(T) = \varnothing
\]

\[
\Rightarrow P^C(T) = \emptyset \quad R^C(T) = \bigwedge
\]

Define $\Delta(T) = \sum_{C \in \text{admissible}} P^C(T) \oplus R^C(T) + T \oplus T$

$\Delta : H \to H \otimes H$ by construction

and define $\Delta(T_r \ldots T_k) = \Delta(T_r) \ldots \Delta(T_k)$

$\Delta(A) = \otimes \bigwedge + \otimes \bigvee + \otimes 1 + \otimes 1 + \otimes 1 + \otimes 1$
Another example:

\[ \Delta (\cdot 1) = \Delta (\cdot) \Delta (1) \]

\[ = (\otimes I + I \otimes \cdot) (I \otimes I + I \otimes 1 + \cdots) \]

\[ = I \otimes I + \cdots I + \cdots I \otimes I + \cdots \]

So we have a definition. Still need to prove co-associativity.

First, 2nd definition:

\[ B_+ (T_1, \ldots, T_k) \Rightarrow T_1, \ldots, T_k X^+ \]

\[ \Delta B_+ (X) = B_+ (X) \otimes I + (\text{id} \otimes B_+) \cdot \Delta (X) \]

Lemma: the two definitions of \( \Delta \) are equivalent and co-associativity to be treated in exercise/class.

A (as found in text):

\[ H = \bigoplus_{i \geq 0} H (c_i) \]

\[ H (c) \Rightarrow H (c) \]

Proof of equivalence and co-associativity.