β function QED to two loops - traditionally and with Corolla polynomial

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The β function of quantum electrodynamics is computed to two loops using dimensional regularization in momentum space, and the Corolla differential developed by Kreimer. The one-loop coefficient is $\frac{4}{3}$, the two-loop coefficient is 4. The Corolla integrand is analyzed regarding its subgraph structure.

Keywords:
quantum electrodynamics, vacuum polarization, Corolla polynomial, dimensional regularization in momentum space

Die β-Funktion der Quantenelektrodynamik wird zu zwei Schleifen berechnet, sowohl mit dimensionaler Regularisierung im Impulsraum, als auch mit Hilfe des Korolla-Differenzials, das Kreimer entwickelte. Der Koeffizient der Ein-Schleifen-Rechnung ist $\frac{4}{3}$, der der Zwei-Schleifen-Rechnung ist 4. Der Korolla-Integrand wird bezüglich der Subgraphen-Struktur untersucht.

Schlagwörter:
Quantenelektrodynamik, Vakuumpolarisation, Corolla-Polynom, dimensionale Regularisierung im Impulsraum

Contents

1. Introduction

In my master thesis, I will discuss the β function of quantum electrodynamics, which is given by the photonic propagator, or vacuum polarization. The computation, which will be done to two loops, can be utilized in different ways. Here, we are showing dimensional regularization in momentum space, the standard textbook approach and the subject matter in most quantum field theory courses, and the computation using the Corolla polynomial, a graph polynomial which Dirk Kreimer has decribed in great detail in [1] and [2]. Even though quantum electrodynamics to two loops is nothing new, the computations are important because they enable the testing of new methods, in this case the transition from scalar to gauge amplitudes using the Corolla differential. Dimensional regularization in momentum space is very straight-forward and comparatively easy to follow, but has its mathematical stumbling blocks. The computation using the Corolla polynomial is a lot more rigorous and also has a lot more potential.
Doing the computations, a lot of help has come from [3], lecture notes from a QED and QCD lecture with a lot of detailed computations. During the traditional computations, we are going to use dimensional regularization, a technique in which the dimension of space-time, D, is altered by shifting it by a small value ε to $D = 4 - 2\varepsilon$ in order to isolate the poles of the amplitudes at $D = 4$. During this thesis, we will encounter single and double poles in ε.
In the first part of the thesis, the β function and renormalization in general are explained. After that, a short remark to the Ward identities is made. Next, the computation of the one-loop and the two-loop graphs contributing to vacuum polarization in done first in momentum space, then using the Corolla approach. At last, the results are discussed. In the appendix, the computation of traces of γ matrices and the one-loop master formula for dimensional regularization with a lot of explicit results is found, as well as the derivation of an important formula for two-loop computations, called the triangle relation, and the Feynman rules for QED.

1.1. The β-function

When dealing with quantum field theories, one usually uses a perturbative expansion of graphs for the computations, which are completed to a fixed order of perturbation, normally given by the loop order, or first Betti number, of a graph. However, perturbation theory bares some problems and cannot be the only approach to a complete quantum field theory.
Let us consider an amplitude involving momenta of order q, for example the vacuum polarization of quantum electrodynamics. Its n-loop contribution contains the perturbation parameter, the fine-structure constant α, to the n-th power, α^n. There are also up to n factors of $\ln\left(\frac{q^2}{m_e^2}\right)$. Consequently, perturbation theory has its limitations when $\alpha\left|\ln\left(\frac{q^2}{m_e^2}\right)\right|$ becomes large, even though α by itself might be small, which is the case in quantum electrodynamics. By introducing some scale μ, a renormalization

point is fixed and this point defines the renormalized coupling. The introduction of this renormalization scale leads to the appearance of logarithms $\ln\left(\frac{E}{\mu}\right)$, so perturbation theory is bound to break down whenever $E \gg \mu$ or $E \ll \mu$, even though the coupling constant might still be small at that energy.
To find remedy, one could think about introducing not a constant, but a "sliding" renormalization scale μ, which is not related to the particle masses in a fixed way. One could choose μ to be of the same magnitude as the energy E of the process in question, and thereby taking away the possible harm of the $\ln\left(\frac{E}{\mu}\right)$ terms. As long as the coupling constants g_μ defined at this sliding scale μ remain small, perturbation theory is still possible. [4]

1.1.1. The renormalization group

Think about the coupling constants defined at a given scale μ. Using perturbation theory, one could derive the physical amplitudes and from them the coupling constants at a new scale $\mu + \mathrm{d}\mu$. One would get a differential equation to integrate, which relates the coupling constants at different scales. The method of the renormalization group can also give insight on the asymptotic behaviour at very high or very low energies, even though the coupling constants might not be small enough at these scales to use perturbation theory anymore. In fact, it turns out that in order to perform calculations at some energy E, one must first eliminate the degrees of freedom of much higher energy.
One very intuitive way to do this is by simply using a finite cut-off. This goes along with ensuring that the physical quantities in the thoery remain cut-off-independent, eventually leading to having to introduce an infinite amount of interaction types allowed by the theory. Obviously, using a cut-off is not very convenient for renormalizable theories, especially because it violates Lorentz invariance.
In order to get around the large logarithms mentioned above, one needs to find an adequate way of defining renormalized coupling constants and operators. Renormalization group methods do exactly that.[4]

Renormalization group equations

Presume there is a coupling constant g_μ which depends on a sliding scale μ, but not on the masses of the particles in the theory. In ordner to calculate g_E at a given energy E, it would be of no help to use perturbation theory, since we are interested in the results where perturbation theory no longer holds. Instead, we proceed as follows: First, we understand the discrete steps where g_μ may be calculated in terms of the renormalized coupling g_R as long as $\frac{\mu}{m}$ is small enough, i.e. not much larger than unity. $g_{\mu'}$ can be computed in terms of g_μ just as long as $\frac{\mu'}{\mu}$ is small enough. Following these steps, one eventually reaches g_E. Next, we go from discrete to continuous calculations.
From dimensional analysis we can derive that the relation between any two couplings g_μ and $g_{\mu'}$ is of the form

$$g_{\mu'} = G\left(g_\mu, \frac{\mu'}{\mu}, \frac{m}{\mu}\right) \tag{1.1}$$

Now, we differentiate both sides with respect to μ' and then set $\mu' = \mu$. We get the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\mu'} g_{\mu'} \bigg|_{\mu'=\mu} = \frac{1}{\mu} \frac{\partial}{\partial f} G\left(g_\mu, f, \frac{m}{\mu}\right)\bigg|_{f=1}$$

where we have written $f := \frac{\mu'}{\mu}$ and used the chain rule for the derivative on the right-hand side. Bringing the μ to the other side and renaming $\mu' = \mu$ yields

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_\mu = \frac{\partial}{\partial f} G\left(g_\mu, f, \frac{m}{\mu}\right)\bigg|_{f=1} =: \beta\left(g_\mu, \frac{m}{\mu}\right) \ . \tag{1.2}$$

There is no zero mass singularity, so for $\mu \gg m$, Eq. (1.2) becomes

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_\mu = \beta\left(g_\mu, 0\right) \equiv \beta\left(g_\mu\right) \tag{1.3}$$

which is referred to as the Callan-Symanzik equation.
In order to calculate g_E, one would integrate Eq. (1.3), where an initial value $\mu_i = M$ has to be chosen such that it is large enough to neglect masses m compared with μ for $\mu \geq M$, but also small enough to avoid large logarithms $\ln\left(\frac{M}{m}\right)$ which would not allow the use of perturbation theory. Then we can calculate g_M by using the conventional renormalized coupling constant g_R. As long as $\beta(g)$ does not vanish between G_M and G_E, we can formally write

$$\ln\left(\frac{E}{M}\right) = \int_{g_M}^{g_E} \frac{\mathrm{d}g}{\beta(g)} \ . \tag{1.4}$$

Please note that the results derived do not depend on pertubative methods. However, perturbation theory may and will be used to produce results for the functions G and β. This is independent of the derivation of the formulas.[4]

Renormalizing operators

When renormalizing an operator $\mathcal{O}$, for example a field, one intoduces on N-factor,

$$\left(\mathcal{O}\right)_R = N^{(\mathcal{O})}\mathcal{O} \tag{1.5}$$

$N^{(\mathcal{O})}$ is chosen in such a way that $N^{(\mathcal{O})}F(p)$ becomes finite at a chosen renormalization point, where $F(p)$ is a divergent factor in the matrix elements of $\mathcal{O}(p)$. For example, one could define $\left(\mathcal{O}\right)_R$ in such a way that $N^{(\mathcal{O})}F(0) \stackrel{!}{=} 1$. This, however, leads to an infrared singularity.
To avoid singularities due to branch cuts, one option is to calculate off-shell matrix elements of operators since branch cuts lie on the real axis. For example, if a particle with momentum p decays into several particles with masses m_i, then there will be a branch cut on the positive real p^2-axis for $p^2 \geq \left(\sum_i m_i\right)^2$. There is no singularity if the initial particle has mass $M < \sum_i m_i$ because then, $p^2 = M^2$ is off the cut. Problematically, for massless theories, the value of p^2 and the branch point at which the cut starts both meet at the origin, resulting in a singularity. Therefore, it might make sense to analyze off-shell matrix elements. Then, we need to consider the N-factors that appear in the definition of the renormalized operators whose matrix

elements are finite. These N-factors can be defined in a way that the correction factors resulting from divergent subgraphs all cancel when the operator has zero momentum, or if a field is on its mass shell, or in any other convenient way. As discussed above, the formula for the N-factor holds zero-mass singularities which will result in large logarithms at energies $E \gg m$, and we will try to fix this problem by introducing a sliding scale μ. The matrix elements of the renormalized operator will read

$$\mathcal{O}_\mu = N_\mu^{(\mathcal{O})}\mathcal{O} \ . \tag{1.6}$$

The correction factor due to divergent subgraphs containing operator $\mathcal{O}$ are cancelled at a renormalization point which has energy scale μ, brought about by $N_\mu^{(\mathcal{O})}$.[4]

1.1.2. The β-function of quantum electrodynamics

In electrodynamics, two renormalization constants turn out to be dependent on just one function, referred to as Z_3 for historical reasons. $Z_3^{-\frac{1}{2}}$ renormalizes the electromagnetic field, whereas $Z_3^{\frac{1}{2}}$ renormalizes the electric charge of the electron, $e_R = Z_3^{\frac{1}{2}}e$. Please note that this is a very special case and that in general, each renormalized function carries an independent renormalization constant.
Naturally, one defines the renormalized electric charge and the renormalized electromagnetic field at a sliding scale μ such that

$$e_\mu = N_\mu^{(A)-1}e = Z_3^{-\frac{1}{2}}\frac{1}{N_\mu^{(A)}}e_R \ . \tag{1.7}$$

$$A_\mu^\nu = N_\mu^{(A)}A^\nu = Z_3^{\frac{1}{2}}N_\mu^{(A)}A_R^\nu \ . \tag{1.8}$$

Then, e_μ times the field $N_\mu^{(A)}A^\nu$ renormalized at scale μ is independent of the choice of μ,

$$e_\mu A_\mu^\nu = e_R A_R^\nu \ . \tag{1.9}$$

Please note that in this notation, ν denotes a Lorentz index whereas μ represents the sliding scale.
We see (for example in [4]) that for small e_μ, e_μ increases when μ increases. However, the asymptotic behaviour is still unknown and cannot be explored using perturbation theory.

Asymptotic behaviour of the β-function

Using the methods of the renormalization group, one finds that there are several possible behaviours of the β-function for large couplings, connected to the behaviour of g_μ for large μ. The following analysis deals with theories with only one coupling constant. For a discussion on a theory with more than one coupling constant, please refer to .
In theories like ϕ^4-theory or quantum electrodynamics, the β-function for small g is positive, so $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g_\mu \geq 0$ for small g. Quantum chromodynamics, on the other hand, has $\beta(g) < 0$ for small g. [4]

1.1.3. Ward-Takahashi identities

A more graphical approach

In order to get a feeling for the Ward-Takahashi identities, imagine an amplitude for some QED process which involves an external photon, which we will assign momentum k. As we know, the amplitude M can be written in the form

$$M(k) = \epsilon_\mu(k) M^\mu(k)$$

where M is a Fourier-transformed correlation function. Let us take a close look at the diagram with an external photon line. This diagram is a contribution to the amplitude $M(k)$, which consists of all diagrams with the external-leg structure of $M(k)$. If that photon line is removed, we obtain a simpler diagram which is part of a simpler amplitude, call it M_0. Next, we can reinsert the photon at any point allowed by the Feynman rules, the results are always contributions to $M(k)$. If one sums over all diagrams that contribute to M_0 (that is, over all graphs with the external leg structure of M_0), and then sums over all possible insertion places for the photon line, the result is $M(k)$.

The Feynman rules of QED only allow for one type of vertex, namely a photon coupling to a fermion and an antifermion. Therefore, a new external photon can only be attached on a fermionic line. This line can either be part of a closed loop, or part of a fermion running through the graph.

If the **electron runs between external points**, and the electron propagators have momenta, say, $p, p_1 = p + q_1, p_2 = p_1 + q_2, \ldots, p' = p_{n-1} + q_n$, the q_n are the momenta of the other photons coupling to the fermion line, and there are n vertices, that gives us $n + 1$ insertion points.

For an insertion after the i^{th} vertex, the momentum for any electron propagators p_j with $j \geq i$ is increased by k.

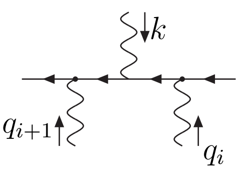
If we contract the new vertex with k_μ, we can rewrite the expression as

$$-iek_\mu\gamma^\mu = -ie\Big((\not{p}_i + \not{k}) - (\not{p}_i)\Big)$$

In the integrand corresponding with the amplitude, the vertex will be multiplied with the two adjacent propagators, from the left and right, respectively.

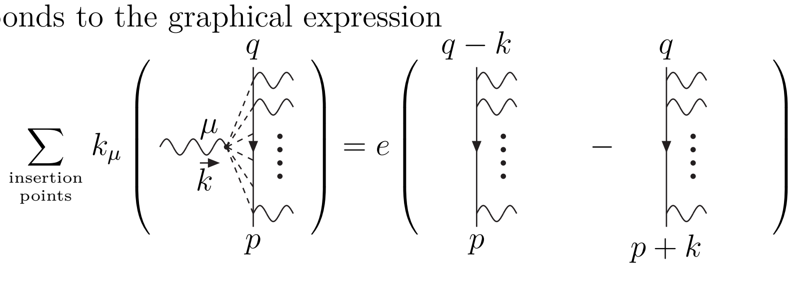
$$\frac{i}{\not{p}_i + \not{k}}(-ie\not{k})\frac{i}{\not{p}_i} = e\left(\frac{i}{\not{p}_i} - \frac{i}{\not{p}_i + \not{k}}\right)$$

Thus, when writing down the entire integrand, we will encounter terms like



$$= \ldots \left(\frac{i}{\not{p}_{i+1} + \not{k}}\right)\gamma^{\lambda_i+1}\left(\frac{i}{\not{p}_i} - \frac{i}{\not{p}_i + \not{k}}\right)\gamma^{\lambda_i}\left(\frac{i}{\not{p}_{i-1}}\right)\gamma^{\lambda_{i-1}}\ldots$$

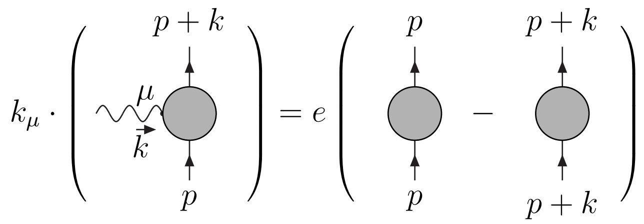
Now, if we sum over all terms given by inserting the photon in all $n + i$ possible spots, every term but the very first and last will cancel due to the changing signs. This corresponds to the graphical expression



If the photon attaches to an **internal electron loop**, then the very "first" term and the very "last" term will cancel because they are identical, so there will be no contribution. Thus, the diagrams in which a photon is attached along a closed fermion loop add up to zero when summing over all diagrams.
Since all external fermion lines must be running through, there is always an even number of external fermionic legs. Say, the amplitude $M(k)$ has $2n$ external fermions, the incoming ones labeled with momenta p_i, the outgoing ones labeled with momenta q_i. Then the amplitude M_0 is short one external photon $\gamma(k)$, but is otherwise identical to $M(k)$. Now, to extract $k_\mu M^\mu(k)$, we need to sum over all diagrams which give a contribution to M_0, and then sum over positions at which the photon could be inserted, and do this for each of these diagrams. The result is

$$\begin{aligned} &k_\mu M^\mu(k; p_1, \dots, p_n; q_1, \dots, q_n) = \\ &\qquad = e \sum_i \Big(M_0(p_1, \dots, p_n; q_1, \dots, q_i - k, \dots) - M_0(p_1, \dots, p_i + k, \dots; q_1, \dots, q_n) \Big) \end{aligned} \tag{1.10}$$

Eq. (1.10) is called the **Ward-Takahashi identity for correlation functions**. The simplest case, namely the $n = 1$ case, is a nice example:



(1.11)

The propagators we see on the right-hand side of Eq. (1.11) are the exact electron propagators $S(p)$ and $S(p+k)$, given by

$$S(p) = \frac{i}{\not{p} - \Sigma(p)}$$

where $\Sigma(p)$ is the sum of all 1PI graphs contributing to the electronic propagator (the self-energy).
We rewrite the full three-point function as a product of full fermionic propagators and an amputated scattering function whose vertex is denoted by $\Gamma^\mu(p+k, p)$. Then, we multiply with the inverse propagators from the left and right, respectively.

$$\begin{aligned} S(p+k)\Big(-iek_\mu \Gamma^\mu(p+k,p)\Big) S(p) &= e\Big(S(p) - S(p+k)\Big) \\ \Rightarrow \qquad -ik_\mu\Gamma^\mu(p+k,p) &= e\Big(S^{-1}(p+k) - S^{-1}(p)\Big) \end{aligned} \tag{1.12}$$

Eq. (1.12) is also sometimes referred to as Ward-Takahashi identity. This identity gives us the opportunity to retrieve a general relation between renormalization factors:

$$\Gamma^\mu(p+k,p) \to Z_1^{-1}\gamma^\mu \quad \text{as} \quad k \to 0$$

Let Z_2 be the residue of the pole in $S(p)$:

$$S(p) \sim \frac{iZ_2}{\not{p}}$$

Setting p near mass shell and expanding Eq. (1.12) about $k = 0$ gives to first order:

$$\begin{aligned} -iZ_1^{-1}\not{k} &= -iZ_2^{-1}\not{k} \\ \Rightarrow \qquad Z_1 &= Z_2 \end{aligned}$$

This relation is also sometimes called the Ward-Takahashi identity. Anyway it guarantees the exact cancellation of infinite rescaling factors in the electronic scattering amplitude. Because of the Ward-Takahashi identity, it suffices to compute the photon propagator in order to obtain the β function of quantum electrodynamics. [5]

A more analytic approach

The previous derivation of the Ward-Takahashi identity was rather graph-oriented, in this paragraph we will try a more analytic approach. Recall that in gauge theory, the Lagrangian density $\mathcal{L}$ (involving some matter fields ψ_l and the bosonic field A_μ) is invariant under a global gauge transformation,

$$\psi_l \to e^{iq_l\alpha}\psi_l \ , \qquad \alpha = \text{const.}$$

Because of this invariance, there must exist a current which is conserved (a Noether corrent):

$$\begin{aligned} \exists J^\mu &= -\, i\sum_l \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_l)} q_l\psi_l \\ \partial_\mu J^\mu &= 0 \end{aligned}$$

A Noether current always comes with a Noether charge:

$$Q := \int \mathrm{d}^3x J^0$$

Q is time-independent ($[Q, H] = 0$), translation-invariant ($[\vec{P}, Q] = 0$), and invariant under homogeneous Lorentz transformations ($[J^{\mu\nu}, Q] = 0$). Furthermore, Q acting on the true vacuum state ψ_0 must be proportional to the vacuum again, $Q\psi_0 = q_0\psi_0$. Because of Lorentz invariance, we can require that $\langle\psi_0\,|J_\mu\psi_0\rangle = 0$ and therefore $Q\psi_0 = 0$.
Q acting on a one-particle state $\psi_{\vec{p},\sigma,n}$, where $\vec{p}$ denotes its momentum, σ its spin, and n the kind of particle in the state, must give the same state again with the same momentum, spin, and Lorentz transformation properties: $Q\psi_{\vec{p},\sigma,n} = q_{(n)}\psi_{\vec{p},\sigma,n}$. $q_{(n)}$ is independent of $\vec{p}$ and σ. We call $q_{(n)}$ the **electric charge** of the one-particle state, where that charge is nothing more than the quantum number associated with the conserved current.
The canonical commutation relations are:

$$\begin{aligned} \left[J^0(\vec{x}, t), \psi_l(\vec{y}, t)\right] &= -q_l\psi_l(\vec{y}, t)\delta^{(3)}(\vec{x}, \vec{y}) \\ [Q, \psi_l(\vec{y}, t)] &= -q_l\psi_l(\vec{y}, t) \end{aligned}$$

But even more so, we can define some F to be a (local) function of the fields, field derivatives, and their adjoints, and the canonical commutation relations give

$$[Q, F(y)] = -q_F F(y)$$

with q_F the sum of all q_l of the fields and field derivatives minus the sum of all q_l of the adjoints. Therefore,

$$\langle\psi_0 \left|[Q, F(y)]\right| \psi_{\vec{p},\sigma,n}\rangle \Rightarrow \langle\psi_0 \left|F(y)\right| \psi_{\vec{p},\sigma,n}\rangle \left(q_F - q_{(n)}\right) = 0$$

Consequently, $q_F = q_{(n)}$ as long as $\langle\psi_0 \left|F(y)\right| \psi_{\vec{p},\sigma,n}\rangle \neq 0$.
This condition takes care that momentum space Green functions that involve F will have poles at the corresponding one-particle state values for the respective $\psi_{\vec{p},\sigma,n}$.
Thus, for a one-particle state, where $F = \psi_l$, we have $q_F = q_l$.
We call q_l the electric charge. However, this need not be the physical electric charge. q_l is a parameter in the Lagrangian $\mathcal{L}$, but the condition that $\mathcal{L}$ be invariant unter the global gauge transformation $\psi_l \rightarrow e^{iq_l\alpha}\psi_l$ does not fix the overall scale of the q_l. So what is the physical charge? Well, the physical electric charge describes the response of a matter fields to a given renormalized electromagnetic field A^μ. The scale of the q_l is fixed as soon as we require that this renormalized electromagnetic field shall appear in the matter Lagrangian $\mathcal{L}_M$, namely in the linear combinations resulting from the local gauge transformation, the covariant derivative $(\partial_\mu - iq_l A_\mu)\psi_l$, such that for our conserved Noether current, $J^\mu = \frac{\delta\mathcal{L}_M}{\delta A_\mu}$ holds.
Still, the renormalized electromagnetic field does not equal the bare field, and neither do the physical and the bare charge.

$$A^\mu \neq A^\mu_B \quad , \quad q_l \neq q_{Bl}$$

Let us take a close look at the simplest form the Lagrangian can take.

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_{B\nu} - \partial_\nu A_{B\mu})(\partial^\mu {A_B}^\nu - \partial^\nu {A_B}^\nu) + \mathcal{L}_M\Big(\psi_l, [\partial_\mu - iq_{Bl}A_{B\mu}]\psi_l\Big)$$

- A^μ, the renormalized electromagnetic field, is the complete propagator with pole at $p^2 = 0$ with unit residue: $A^\mu = Z_3^{-\frac{1}{2}} {A_B}^\mu$
- q_l is the physical response of charged particles to the renormalized electromagnetic field A^μ (and not ${A_B}^\mu$): $q_l = Z_3^{\frac{1}{2}} q_{Bl}$

It is obvious that the relation between the bare charge and renormalized charge is achieved by the use of the same scaling constant for all particles!
So, charge renormalization results merely from radiative corrections to the propagator of the photon. As a consequence, there appear numerous cancellations among other radiative corrections to the propagators and electromagnetic vertices of charged particles. These cancellations are called the **Ward idendities**. [6]

2. Textbook Approach

In this chapter, we are going to compute the one- and two-loop contributions to the photon propagator of quantum electrodynamics. For now, the standard textbook approach, in which the Feynman rules of QED are written down and the integrals are solved using dimensional regularization, is executed. At the end of the thesis, both approaches will be evaluated.
Regardless of the loop number, the ansatz for the amplitude is always the same.

2.1. Ansatz for the Amplitude

Regardless of the number of γ matrices in the trace, or of the number of loop momenta to be integrated out, the Feynman amplitude is of the form

$$\Phi_\Gamma = \int \mathrm{d}^D\bar{k} I_\Gamma$$

where I_Γ denotes the Feynman integrand of the graph Γ, D is the dimension of space time, and $\bar{k}$ is used as an abbreviation for all loop momenta. Dimensional analysis shows that Φ_Γ is quadratically divergent, but it is also known that the photon propagator is transversal, or in other words: It is possible to extract a factor $(q^2 g_{\mu\nu} - q_\mu q_\nu)$ from the expression, where q is the external momentum, and μ and ν are the Lorentz indices of the vertices the external photons couple to. These two factors of q which can be extracted leave a scalar expression $F(q^2)$ which is only logarithmically divergent! [7]

$$\Phi_\Gamma = (q^2 g_{\mu\nu} - q_\mu q_\nu) F(q^2) \tag{2.1}$$

The problem is reduced to finding the function $F(q^2)$ since the overall structure of the amplitude is already known. In order to achieve this, contract both sides of Eq. (2.1) with $g^{\mu\nu}$ and solve for $F(q^2)$:

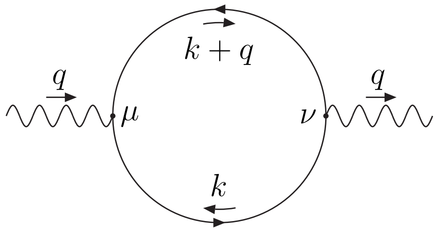
$$\begin{aligned} & g^{\mu\nu}\Phi_\Gamma = 3q^2 F(q^2) \\ \Rightarrow \qquad & F(q^2) = \frac{1}{3q^2} g^{\mu\nu}\Phi_\Gamma \end{aligned} \tag{2.2}$$

Therefore, it suffices to compute not Φ_Γ, but a much easier expression, namely $g^{\mu\nu}\Phi_\Gamma$. The result for Φ_Γ can then be read off using Eq. (2.1).

Remark Since $F(q^2)$ is logarithmically divergent, a simple subtraction at some reference scale $q^2 = \mu^2$ will be necessary. Therefore, the notation $F\left(\frac{q^2}{\mu^2}\right)$ will also be used. It indicates that the subtraction would already have taken place.

2.2. The One-Loop Graph

Let us start this chapter by computing the one-loop photon propagator of abelian gauge theory, also known as quantum electrodynamics, the traditional way we learn it in every introductory quantum field theory course. We take the graph



and write down its unrenormalized Feynman integrand according to the Feynman rules of QED (see Chapter D), and integrate the expression using dimensional regularization. We use the standard Feynman slash notation, where $\not{y} \equiv \gamma_\mu y^\mu$. The Feynman integrand of $\text{-}\!\bigcirc\!\text{-}$ is given by

$$\begin{aligned} I_{\text{-}\bigcirc\text{-}} &= -\operatorname{Tr}\left(ie\gamma_\mu i \frac{\not{k} + \not{q}}{(k+q)^2} ie\gamma_\nu i \frac{\not{k}}{k^2} \right) \\ &= -e^2 \operatorname{Tr}\left(\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta\right) \frac{(k+q)^\alpha k^\beta}{(k+q)^2 k^2} \end{aligned}$$

The loop momentum of the only loop is k, so there will be a D-dimensional integration of the Feynman integrand over k. The Feynman amplitude of $\text{-}\!\bigcirc\!\text{-}$ yields

$$\Phi_{\text{-}\bigcirc\text{-}} := \Phi(\text{-}\bigcirc\text{-}) = \int \mathrm{d}^D k \; I_{\text{-}\bigcirc\text{-}}$$

Because the incoming and outgoing photons are only attached to the vertices at μ and ν, Eq. (2.1) yields that $\Phi(\text{-}\!\bigcirc\!\text{-})$ is of the following form:

$$\Phi(\text{-}\bigcirc\text{-}) = \left(q^2 g_{\mu\nu} - q_\mu q_\nu\right) F\left(\frac{q^2}{\mu^2}\right)$$

where subtraction at $q^2 = \mu^2$ is understood. Keeping Eq. (2.2) in mind and computing the contraction of the integrand with $g^{\mu\nu}$, we get

$$\begin{aligned} g^{\mu\nu} I_{\text{-}\bigcirc\text{-}} &= -e^2 \operatorname{Tr}\left(\gamma_\mu \gamma_\alpha \gamma^\mu \gamma_\beta\right) \frac{(k+q)^\alpha k^\beta}{(k+q)^2 k^2} \\ &= 2e^2 \operatorname{Tr}\left(\gamma_\alpha \gamma_\beta\right) \frac{(k+q)^\alpha k^\beta}{(k+q)^2 k^2} \\ &= 8e^2 \frac{(k+q)\cdot k}{(k+q)^2 k^2} \\ &= 8e^2 \frac{k^2 + k\cdot q}{(k+q)^2 k^2} \\ \Rightarrow \qquad g^{\mu\nu} \Phi_{\text{-}\bigcirc\text{-}} &= 8e^2 \int \frac{k^2 + k\cdot q}{(k+q)^2 k^2} \, \mathrm{d}^D k \end{aligned} \tag{2.3}$$

In the second line, we use the γ matrix identity $\gamma_\mu \gamma_\alpha \gamma^\mu = -2\gamma_\alpha$. In the third line, we used the trace identity $\operatorname{Tr}\left(\gamma^\alpha \gamma^\beta\right) = 4g^{\alpha\beta}$.

In order to solve Eq. (2.3), one uses a trick by splitting the sum and rewriting $k \cdot q$ using Eq. (A.4). After that, all summands are in a form that allow the use of the one-loop Master integral for dimensional regulariz ation, cf. Eq. (B.2), and Chapter B in general.

$$
\begin{aligned}
g^{\mu\nu}\Phi_{-\!\circ\!-} &= 4e^2 \int \frac{k^2 + (k+q)^2 - q^2}{(k+q)^2 k^2}\, \mathrm{d}^D k \\
&= 4e^2 \Big(\mathcal{M}(0,1,D,q^2) + \mathcal{M}(1,0,D,0) - q^2 \mathcal{M}(1,1,D,q^2)\Big) \qquad (2.4)\\
&= -4e^2 q^2 \mathcal{M}(1,1,D,q^2) \\
&= -4e^2 (q^2)^{\frac{D}{2}-1} \tilde{\Gamma}^{1,1}_D \\
Eq.\ (B.8) \Rightarrow \quad &= -4e^2 (q^2)^{\frac{D}{2}-1} \frac{\Gamma^2\left(\frac{D}{2}-1\right)\Gamma\left(3-\frac{D}{2}\right)}{\Gamma(D-2)} \cdot \frac{1}{2-\frac{D}{2}} \\
Eq.\ (B.9) \Rightarrow \quad &= -4e^2 (q^2)^{1-\varepsilon} \left(\frac{1}{\varepsilon} + \mathcal{O}(1)\right) \qquad (2.5)
\end{aligned}
$$

In the second line, we used the master integral. Because of the properties of the master integral, $M(0,\beta,D,q^2) = M(\alpha,0,D,q^2) \equiv 0$, two of the three terms in Eq. (2.4), vanish. In the last line, we used dimensional regularization where we set $D = 4 - 2\varepsilon$.

Let us continue the computation of the full contribution to the amplitude, since Eq. (2.5) still has a pole at $\varepsilon \to 0$. Without subtraction at some reference momentum μ, we would get

$$
\begin{aligned}
Eq.\ (2.2) \quad \Rightarrow \quad F(q^2) &= -\frac{4}{3} e^2 (q^2)^{-\varepsilon} \left(\frac{1}{\varepsilon} + \mathcal{O}(1)\right) \\
&= -\frac{4}{3} e^2 \left[1 - \varepsilon \ln(q^2) + \mathcal{O}(\varepsilon^2)\right] \left(\frac{1}{\varepsilon} + \mathcal{O}(1)\right)
\end{aligned}
$$

and the expression would diverge when the limit $\varepsilon \to 0$ were to be taken. However, when the subtraction at $q^2 = \mu^2$ is acted out, we get

$$
\begin{aligned}
F\left(\frac{q^2}{\mu^2}\right) &= -\frac{4}{3} e^2 \Big[\Big(1 - \varepsilon \ln(q^2) + \mathcal{O}(\varepsilon^2)\Big) - \Big(1 - \varepsilon \ln(\mu^2) + \mathcal{O}(\varepsilon^2)\Big)\Big] \left(\frac{1}{\varepsilon} + \mathcal{O}(1)\right) \\
&= -\frac{4}{3} e^2 \left[-\varepsilon \ln\left(\frac{q^2}{\mu^2}\right) + \mathcal{O}(\varepsilon^2)\right] \left(\frac{1}{\varepsilon} + \mathcal{O}(1)\right) \\
&= \frac{4}{3} e^2 \left[\ln\left(\frac{q^2}{\mu^2}\right) + \mathcal{O}(\varepsilon)\right] \Big(1 + \mathcal{O}(\varepsilon)\Big) \\
\Rightarrow \quad \lim_{\varepsilon \to 0} F\left(\frac{q^2}{\mu^2}\right) &= \underline{\underline{\frac{4}{3} e^2 \ln\left(\frac{q^2}{\mu^2}\right)}}
\end{aligned}
$$

The coefficient $\frac{4}{3}$ of the function $F\left(\frac{q^2}{\mu^2}\right)$ is the one-loop β function of quantum electrodynamics.

We have computed the one-loop β function of QED using the standard physics approach with subtraction and dimensional regularization. However, this approach is very weak on a mathematical level because it is not very rigorous, especially because integrals are written down that are non-existent because they diverge. To gain a more general knowledge of the structure of the β function, let us have a look at a more combinatorical level in Chapter 3.

2.3. The Two-Loop Graphs

For the two-loop graphs, the application of Feynman rules proves slightly more tricky than in the one-loop case. A lot of the computations can be attributed to the one-loop computations, but there are a few subtleties to be taken into account that we did not have to pay attention to before.
Because we are going to use dimensional regularization, we will need to rethink our use of γ matrices in traces and contractions in D instead of 4 dimensions. We did not have to do that before because the pole we encountered in Section 2.2 was only of order ε.
The Clifford algebra demands that $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ be fulfilled at all times. It is conventional to choose $\operatorname{Tr}(\mathbb{I}) = 4$ (and not D), but is is only a matter of choice because the trace of Unity may be any smooth function of D which satisfies $\operatorname{Tr}(\mathbb{I})|_{D=4} = 4$.
For contractions of γ matrices, we use [7]

$$\gamma_\rho \gamma_\mu \gamma^\rho = (2 - D)\gamma_\mu \tag{2.6}$$

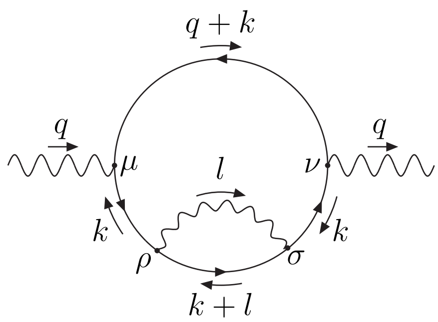
$$\gamma_\rho \gamma_\mu \gamma_\nu \gamma_\sigma \gamma^\rho = (D-6)\gamma_\sigma \gamma_\nu \gamma_\mu - 2(D-4)\Big(g_{\mu\nu}\gamma_\sigma - g_{\mu\sigma}\gamma_\nu + g_{\nu\sigma}\gamma_\mu\Big) \tag{2.7}$$

Keep in mind that even though $\operatorname{Tr}(\mathbb{I}) = 4$, $g_{\mu\nu}g^{\mu\nu} = D = 4 - 2\varepsilon$ during our computations.

The Feynman integrals of the two-loop graphs contain a trace of eight γ matrices and contractions with several momenta. There are two independent loop-momenta which are integrated out, one for each loop.

2.4. The Graph with a Fermionic Subgraph

In this section, the one-particle irreducible[1] two-loop graphs of the graphical expansion of the vacuum polarization will be taken into account where the divergent (one-loop) subgraph, , is inserted into one fermionic edge of the one-loop photon propagator, . We are facing two graphs of the kind

$\Gamma =$ 

The second graph would have the fermionic subgraph inserted in the upper fermionic line. Since there are two fermionic edges, there are exactly two possible insertion places. However, both graphs produce the same results when computing their amplitudes, which will be shown later in Section 2.4.3.
According to the Feynman rules of QED (see Chapter D), the Feynman integrand

[1] One-particle irreducible (1PI) means that the graph will not fall apart into two disjoint graphs if any one internal edge is removed.

of is given by

$$
\begin{aligned}
I &= -\operatorname{Tr}\left(ie\gamma_\mu i \frac{\not{k}+\not{q}}{(k+q)^2} ie\gamma_\nu i \frac{\not{k}}{k^2} ie\gamma_\sigma i \frac{\not{k}+\not{l}}{(k+l)^2} ie\gamma_\rho i \frac{\not{k}}{k^2} \right) i \frac{g^{\rho\sigma} - \xi \frac{l^\mu l^\nu}{l^2}}{l^2} \\
&=: I^g - \xi I^l
\end{aligned}
\tag{2.8}
$$

Obviously, the amplitude of is the sum of two independent parts due to the internal photon propagator which depends on the gauge parameter ξ. This did not appear in the one-loop case because there was no internal photon line. The value of ξ depends on the choice of gauge. Very common are the Feynman gauge, $\xi = 0$, where half the computation is obsolete, and the Landau gauge, $\xi = 1$, where the internal photon propagator becomes transveral. We will see that this is quite a nice choice for quantum electrodynamics because it will dispose of subdivergences: All graphs will only have simple poles in the dimensional regularization parameter ε.
As discussed in Section 2.1, the amplitude is given by the transversality of the photon, and a scalar function, $F\left(\frac{q^2}{\mu^2}\right)$. To determine $F\left(\frac{q^2}{\mu^2}\right)$, one contracts Eq. (2.8) with $g^{\mu\nu}$.
In the next subsections, we are going to compute the amplitudes I^g and I^l, with

$$
I^g = -ie^4 \frac{\operatorname{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\sigma\gamma_\delta\gamma^\sigma\gamma_\eta\right\}(k+q)^\alpha k^\beta (k+l)^\delta k^\eta}{(k+q)^2 (k^2)^2 l^2 (l+k)^2} \tag{2.9}
$$

$$
I^l = -ie^4 \frac{\operatorname{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\sigma\gamma_\delta\gamma_\rho\gamma_\eta\right\}(k+q)^\alpha k^\beta (k+l)^\delta k^\eta l^\sigma l^\rho}{(k+q)^2 (k^2)^2 (l^2)^2 (l+k)^2} \tag{2.10}
$$

2.4.1. The $g^{\mu\nu}$ part

Let us start the computation of the amplitude of the gauge-independent part, I^g. Take Eq. (2.9) and use the identity $\gamma_\mu\gamma_\nu\gamma^\mu = (2-D)\gamma_\nu$ (see Eq. (2.6)).

$$
I^g = -ie^4 (2-D) \frac{\operatorname{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\delta\gamma_\eta\right\}(k+q)^\alpha k^\beta (k+l)^\delta k^\eta}{(k+q)^2 (k^2)^2 l^2 (l+k)^2}
$$

In Eq. (A.3), the trace of the six γ matrices is given. The trace and the momenta in the numerator are contracted using mathematica (cf. Chapter E). Next, the result is contracted with $g^{\mu\nu}$.

$$
g^{\mu\nu} I = -4ie^4 (D-2)^2 \frac{k^2\Big((k\cdot l) + (k\cdot q) - (q\cdot l)\Big) + 2(k\cdot l)(k\cdot q) + (k^2)^2}{(k+q)^2 (k^2)^2 l^2 (l+k)^2} \tag{2.11}
$$

Now, rewriting the scalar products using Eq. (A.4) yields

$$
g^{\mu\nu} I = -2ie^4 (D-2)^2 \frac{\Big((k+q)^2 - q^2\Big)\Big((l+k)^2 - l^2\Big) + (k^2)^2 - 2k^2\, q\cdot l}{(k+q)^2 (k^2)^2 l^2 (l+k)^2} \tag{2.12}
$$

Note that the scalar product $q \cdot l$ has not been rewritten because there is no factor $(l \pm q)^2$ in the denominator.
According to Chapter B, the integral will vanish if there is only one factor with k or l in the denominator, regarless of its power. This means that every term in

Eq. (2.12) which contains a factor l^2 or $(l+k)^2$ can be omitted because its integral will vanish.[2] This leaves us with

$$g^{\mu\nu} I = -2ie^4(D-2)^2 \frac{k^2 - 2\, l \cdot q}{(k+q)^2 k^2 l^2 (l+k)^2}$$

So, our final integral is

$$\begin{aligned}
g^{\mu\nu}\Phi^g &= \iint \mathrm{d}^D k\, \mathrm{d}^D l\, g^{\mu\nu} I \qquad (2.13)\\
&= -2ie^4(D-2)^2 \iint \mathrm{d}^D k\, \mathrm{d}^D l\, \frac{k^2 - 2\, q_\mu l^\mu}{(k+q)^2 k^2 l^2 (l+k)^2}\\
&= -2ie^4(D-2)^2 \left\{ \int \frac{\mathrm{d}^D k}{(k+q)^2} \int \frac{\mathrm{d}^D l}{l^2(l+k)^2} - 2\int \frac{\mathrm{d}^D k}{(k+q)^2 k^2} q_\mu \int \mathrm{d}^D l \frac{l^\mu}{l^2(l+k)^2} \right]\Bigg\}\\
&= -ie^4 \left(I^{g,1} + I^{g,2} \right) \qquad (2.14)
\end{aligned}$$

$I^{g,1}$ is simply solved using the master formula for one-loop graphs, as in Chapter B. It yields

$$\begin{aligned}
I^{g,1} &= 2(D-2)^2 \int \frac{\mathrm{d}^D k}{(k+q)^2} \int \frac{\mathrm{d}^D l}{l^2(l+k)^2}\\
&= 2(D-2)^2 \int \frac{\mathrm{d}^D k}{(k+q)^2} \underbrace{\mathcal{M}(1,1,D,k^2)}_{=\tilde{\Gamma}_D^{1,1}(k^2)^{\frac{D}{2}-2}}\\
&= 2(D-2)^2 \tilde{\Gamma}_D^{1,1} \underbrace{\int \frac{\mathrm{d}^D k}{(k+q)^2 (k^2)^{2-\frac{D}{2}}}}_{=\mathcal{M}\left(1,2-\frac{D}{2},D,q^2\right)}\\
&= 2(D-2)^2 \tilde{\Gamma}_D^{1,1} \tilde{\Gamma}_D^{1,2-\frac{D}{2}} (q^2)^{D-3}
\end{aligned}$$

In Section B.2.5, we find the results for $\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{1,2-\frac{D}{2}}$. First, we insert the result in terms of D, in the next line we have set the space-time dimension D to its regularized value $D = 4 - 2\varepsilon$.

$$\begin{aligned}
I^{g,1} &= 2(D-2)^2 (q^2)^{D-3} \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{1}{(4-D)} \cdot \frac{1}{(3-D)\left(\frac{3D}{2}-4\right)}\\
&= 2(2-2\varepsilon)^2 (q^2)^{1-2\varepsilon} \frac{\Gamma^3(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(2-3\varepsilon)} \cdot \frac{1}{2\varepsilon} \cdot \frac{1}{(-1+2\varepsilon)(2-3\varepsilon)}\\
&= \big(8 + \mathcal{O}(\varepsilon)\big)\left((q^2)^{1-2\varepsilon}\right)\left(-\frac{1}{4\varepsilon} + \mathcal{O}(1)\right) \qquad (2.15)\\
&= \underline{\underline{(q^2)^{1-2\varepsilon}\left(-\frac{2}{\varepsilon} + \mathcal{O}(1)\right)}} \qquad (2.16)
\end{aligned}$$

The solving of $I^{g,2}$ is a little bit more tricky. The second integrand has a Lorentz index, so we cannot blindly use the master integral as we did before. However,

[2] In the future, instead of referencing Chapter B every time we use it, we will simply call this the "properties of the master integral".

because of this Lorentz index, the integral must be a four-vector which is then contracted with q_μ. Since l is the loop momentum in this case and it is integrated out, the result cannot be dependent on l^μ, only on k^μ, and a scalar function of k^2. So we make the ansatz

$$\int \mathrm{d}^D l \frac{l^\mu}{l^2(l+k)^2} \overset{!}{=} k^\mu F(k^2) \ . \tag{2.17}$$

In order to obtain $F(k^2)$, we contract Eq. (2.17) with k_μ and solve for $F(k^2)$, using Eq. (A.4).

$$\begin{aligned} F(k^2) &= \frac{1}{k^2}\int \mathrm{d}^D l \frac{l\cdot k}{l^2(l+k)^2} \\ &= \frac{1}{2k^2}\int \mathrm{d}^D l \frac{(l+k)^2 - l^2 - k^2}{l^2(l+k)^2} \\ &= -\frac{1}{2}\int \mathrm{d}^D l \frac{1}{l^2(l+k)^2} \\ &= -\frac{1}{2}\mathcal{M}(1,1,D,k^2) \end{aligned} \tag{2.18}$$

$$\Rightarrow \qquad \int \mathrm{d}^D l \frac{l^\mu}{l^2(l+k)^2} = -\frac{k^\mu}{2}\mathcal{M}(1,1,D,k^2) \tag{2.19}$$

Therefore, we get for $I^{g,2}$:

$$\begin{aligned} I^{g,2} &= -4(D-2)^2 \int \frac{\mathrm{d}^D k}{(k+q)^2 k^2} q_\mu \int \mathrm{d}^D l \frac{l^\mu}{l^2(l+k)^2} \\ &= -4(D-2)^2 \int \frac{\mathrm{d}^D k}{(k+q)^2 k^2} q_\mu \left(-\frac{1}{2}k^\mu \mathcal{M}(1,1,D,k^2)\right) \\ &= 2(D-2)^2 \tilde{\Gamma}^{1,1}_D \int \mathrm{d}^D k \frac{q\cdot k}{(k+q)^2 k^2}(k^2)^{\frac{D}{2}-2} \\ &= 2(D-2)^2 \tilde{\Gamma}^{1,1}_D \cdot \frac{1}{2}\int \mathrm{d}^D k \frac{(k+q)^2 - k^2 - q^2}{(k+q)^2 (k^2)^{3-\frac{D}{2}}} \\ &= -(D-2)^2 \tilde{\Gamma}^{1,1}_D \left\{ \mathcal{M}\left(1, 2-\frac{D}{2}, D, q^2\right) + q^2 \mathcal{M}\left(1, 3-\frac{D}{2}, D, q^2\right) \right\} \\ &= -(D-2)^2 (q^2)^{D-3} \left\{ \tilde{\Gamma}^{1,1}_D \tilde{\Gamma}^{1,2-\frac{D}{2}}_D + \tilde{\Gamma}^{1,1}_D \tilde{\Gamma}^{1,3-\frac{D}{2}}_D \right\} \end{aligned}$$

In the fourth line, $q \cdot k$ was rewritten using Eq. (A.4) and all factors of k^2 were put in the denominator. Next, the $(k+q)^2$ term was dropped since it would lead to a vanishing integral due to the properties discussed above and in Chapter B. The remaining integrals have been expressed using the one-loop master integral. Now, using Section B.2.5 and Section B.2.6, the products of Γ functions need evaluation.

$$\begin{aligned} I^{g,2} &= -(2-2\varepsilon)^2 (q^2)^{D-3} \Big\{ \underbrace{\tilde{\Gamma}^{1,1}_D \tilde{\Gamma}^{1,2-\frac{D}{2}}_D}_{=-\frac{1}{4\varepsilon}+\mathcal{O}(1)} + \underbrace{\tilde{\Gamma}^{1,1}_D \tilde{\Gamma}^{1,3-\frac{D}{2}}_D}_{=\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{5}{2}-\gamma_E\right)+\mathcal{O}(1)} \Big\} \\ &= (q^2)^{1-2\varepsilon}\Big(-4+8\varepsilon+\mathcal{O}(\varepsilon^2)\Big)\left(\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{9}{4}-\gamma_E\right)+\mathcal{O}(1)\right) \\ &= \underline{\underline{(q^2)^{1-2\varepsilon}\left(-\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon}\Big(-5+4\gamma_E\Big)+\mathcal{O}(1)\right)}} \end{aligned} \tag{2.20}$$

Use Eq. (2.16) and Eq. (2.20) to solve Eq. (2.14):

$$
\begin{aligned}
g^{\mu\nu}\Phi^{g} &= -ie^4\left(I^{g,1} + I^{g,2}\right) \\
&= -ie^4 (q^2)^{1-2\varepsilon}\left\{\left(-\frac{2}{\varepsilon}\right) + \left(-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-5 + 4\gamma_E\right)\right) + \mathcal{O}(1)\right\} \\
&= \underline{\underline{-4ie^4 (q^2)^{1-2\varepsilon}\left\{-\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{7}{4} + \gamma_E\right) + \mathcal{O}(1)\right\}}} \qquad (2.21)
\end{aligned}
$$

The double and simple poles in ε have been isolated and everything else is finite when the limit $\varepsilon \to 0$ is taken. Because of the double pole in ε, a simple subtraction at some value $q^2 = \mu^2$ seems insufficient. However, because of the Ward identities in QED, for the sum of all graphs contributing to the two-loop expansion of the photonic amplitude a simple subtraction suffices. We saw about that at the beginning of this chapter, when we learned that the quadratically divergent amplitude can be separated into a factor $g_{\mu\nu}q^2 - q_\mu q_\nu$ and a logarithmically divergent function F, which we have computed here, thus a simple subtraction must be enough. Bear in mind that this accounts for the sum of all graphs alone, not for every single graph. In Section 2.6.2, the subtractions for the graphs in the Landau gauge and for the sum of all two-loop graphs in an arbitrary gauge take place.

2.4.2. The $l^\mu l^\nu$ part

Basically, we are going to repeat the same computation for the gauge-dependent part of the photon propagator. The amplitude is given by Eq. (2.10):

$$
\begin{aligned}
I^{l} &= -ie^4 \frac{\mathrm{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\sigma\gamma_\delta\gamma_\rho\gamma_\eta\right\}(k+q)^\alpha k^\beta (k+l)^\delta k^\eta l^\sigma l^\rho}{(k+q)^2 (k^2)^2 (l^2)^2 (l+k)^2} \\
&=: -ie^4 \frac{\mathrm{num}(I^{l})}{\mathrm{den}(I^{l})} \qquad (2.22)
\end{aligned}
$$

The trace is computed using mathematica (cf. Chapter E) and the contractions, including the contraction with $g^{\mu\nu}$, is also done this way. The result is

$$
\begin{aligned}
g^{\mu\nu}\mathrm{num}(I^{l}) = 2(D-2)(k^2)^2\left[-(k+q)^2 + l^2 - 2l\cdot q + q^2\right] + 8k^2(k+l)^2\left[(k+q)^2 + l\cdot q - q^2\right] + \\
+ 4\left[(k+l)^2\right]^2\left[-(k+q)^2 + q^2\right] + 4(k+l)^2\left[(k+q)^2 l^2 - l^2 q^2\right]
\end{aligned}
$$

Because of the form of the denominator,

$$
\mathrm{den}(I^{l}) = (k+q)^2 (k^2)^2 (l^2)^2 (l+k)^2
$$

and because of the property of the one-loop master integral of dimensional regularization, Eq. (B.2), that the integral will vanish if one of the exponents is zero or a negative integer, not all terms in $g^{\mu\nu}\mathrm{num}(I^{l})$ survive the integration. In fact, since there are three factors with k in $\mathrm{den}(I^{l})$, but only two factors with l, any term involving $(l^2)^2$ or $(l+k)^2$ will lead to a vanishing integral and can therefore be omitted. Moreover, the term $(k^2)^2(k+q)^2$ will also yields a vanishing integral because it will leave only one term with k in the denominator.

The resulting amplitude, with all terms leading to vanishing integrals omitted, is

$$
\begin{aligned}
g^{\mu\nu} I^{l} &= -2(D-2)ie^4 \frac{\left(k^2\right)^2 l^2 + \left(k^2\right)^2 q^2 - 2\left(k^2\right)^2 l \cdot q}{(k+q)^2 \left(k^2\right)^2 \left(l^2\right)^2 (l+k)^2} \\
&= -2(D-2)ie^4 \frac{l^2 + q^2 - 2l \cdot q}{(k+q)^2 \left(l^2\right)^2 (l+k)^2} \\
\Rightarrow \qquad g^{\mu\nu} \Phi^{l} &= -2(D-2)ie^4 \iint \mathrm{d}^D k \, \mathrm{d}^D l \, \frac{l^2 + q^2 - 2l \cdot q}{(k+q)^2 \left(l^2\right)^2 (l+k)^2} \\
&=: -ie^4 \left\{ I^{l,1} + I^{l,2} + I^{l,3} \right\}
\end{aligned} \tag{2.23}
$$

Compute the three integrals:

$$
\begin{aligned}
I^{l,1} &= 2(D-2) \iint \frac{\mathrm{d}^D k \, \mathrm{d}^D l}{(k+q)^2 l^2 (l+k)^2} \\
&= 2(D-2) \int \frac{\mathrm{d}^D k}{(k+q)^2} \mathcal{M}(1,1,D,k^2) \\
&= 2(D-2) \tilde{\Gamma}_D^{1,1} \mathcal{M}\left(1, 2 - \frac{D}{2}, D, q^2\right) \\
&= 2(D-2) \tilde{\Gamma}_D^{1,1} \tilde{\Gamma}_D^{1,2-\frac{D}{2}} \left(q^2\right)^{D-3}
\end{aligned} \tag{2.24}
$$

The result for $\tilde{\Gamma}_D^{1,1} \tilde{\Gamma}_D^{1,2-\frac{D}{2}}$ can be found in Section B.2.5. We get for $I^{l,1}$:

$$
I^{l,1} = 2(D-2)\left(q^2\right)^{D-3} \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{1}{(4-D)} \cdot \frac{1}{(3-D)\left(\frac{3D}{2}-4\right)} \tag{2.25}
$$

$$
\begin{aligned}
&= \underbrace{2(2-2\varepsilon)}_{=4+\mathcal{O}(1)} \left(q^2\right)^{1-2\varepsilon} \left(-\frac{1}{4\varepsilon} + \mathcal{O}(1)\right) \\
&= \left(q^2\right)^{1-2\varepsilon} \left(-\frac{1}{\varepsilon} + \mathcal{O}(1)\right)
\end{aligned} \tag{2.26}
$$

Next is the second integral:

$$
\begin{aligned}
I^{l,2} &= 2(D-2) q^2 \iint \frac{\mathrm{d}^D k \, \mathrm{d}^D l}{(k+q)^2 \left(l^2\right)^2 (l+k)^2} \\
&= 2(D-2) q^2 \iint \frac{\mathrm{d}^D k}{(k+q)^2} \mathcal{M}(1,2,D,k^2) \\
&= 2(D-2) q^2 \tilde{\Gamma}_D^{1,2} \mathcal{M}\left(1, 3 - \frac{D}{2}, D, q^2\right) \\
&= 2(D-2) \left(q^2\right)^{D-3} \tilde{\Gamma}_D^{1,2} \tilde{\Gamma}_D^{1,3-\frac{D}{2}}
\end{aligned}
$$

The result for $\tilde{\Gamma}_D^{1,2}\tilde{\Gamma}_D^{1,3-\frac{D}{2}}$ is written down in Section B.2.11. It yields:

$$\begin{aligned} I^{l,2} &= 2(D-2)(q^2)^{D-3}\frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)}\cdot\frac{1}{\left(\frac{D}{2}-2\right)(4-D)} \qquad (2.27)\\ &= 2(2-2\varepsilon)(q^2)^{1-2\varepsilon}\left(-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{3}{2}+\gamma_E\right)+\mathcal{O}(1)\right)\\ &= (4-8\varepsilon)(q^2)^{1-2\varepsilon}\left(-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{3}{2}+\gamma_E\right)+\mathcal{O}(1)\right)\\ &= (q^2)^{1-2\varepsilon}\left(-\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon}(-4+4\gamma_E)+\mathcal{O}(1)\right) \qquad (2.28) \end{aligned}$$

Last, but not least, compute $I^{l,3}$. In order to avoid an integral similar to the one encountered in Section 2.4.1 with l^μ in the numerator, the k integration will be performed first. In order to do that, there will be a shift $k \to \tilde{k} = k - q$. This is unproblematic because the master integral is translation invariant.

$$\begin{aligned} I^{l,3} &= -4(D-2)\int \mathrm{d}^D l\,\frac{l\cdot q}{(l^2)^2}\int\frac{\mathrm{d}^D k}{(k+q)^2(k+l)^2}\\ &= -4(D-2)\int \mathrm{d}^D l\,\frac{l\cdot q}{(l^2)^2}\int\frac{\mathrm{d}^D \tilde{k}}{(\tilde{k})^2\left(\tilde{k}+(l-q)\right)^2}\\ &= -4(D-2)\int \mathrm{d}^D l\,\frac{l\cdot q}{(l^2)^2}\mathcal{M}\big(1,1,D,(l-q)^2\big)\\ &= -4(D-2)\tilde{\Gamma}_D^{1,1}\int \mathrm{d}^D l\,\frac{l\cdot q}{(l^2)^2[(l-q)^2]^{2-\frac{D}{2}}} \end{aligned}$$

Eq. (A.5) is used to simplify the scalar product in the numerator.

$$\begin{aligned} I^{l,3} &= -4(D-2)\tilde{\Gamma}_D^{1,1}\cdot\frac{1}{2}\int \mathrm{d}^D l\,\frac{l^2+q^2-(l-q)^2}{(l^2)^2(l-q)^{2\,2-\frac{D}{2}}}\\ &= -2(D-2)\tilde{\Gamma}_D^{1,1}\left\{\int\frac{\mathrm{d}^D l}{l^2[(l-q)^2]^{2-\frac{D}{2}}}+q^2\int\frac{\mathrm{d}^D l}{(l^2)^2[(l-q)^2]^{2-\frac{D}{2}}}-\int\frac{\mathrm{d}^D l}{(l^2)^2[(l-q)^2]^{1-\frac{D}{2}}}\right\}\\ &= -2(D-2)\tilde{\Gamma}_D^{1,1}\left\{\mathcal{M}\left(1,2-\frac{D}{2},D,q^2\right)+q^2\mathcal{M}\left(2,2-\frac{D}{2},D,q^2\right)-\mathcal{M}\left(2,1-\frac{D}{2},D,q^2\right)\right\}\\ &= -2(D-2)(q^2)^{D-3}\left\{\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{1,2-\frac{D}{2}}+\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{2,2-\frac{D}{2}}-\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{2,1-\frac{D}{2}}\right\} \qquad (2.29) \end{aligned}$$

The results for $\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{1,2-\frac{D}{2}}$, $\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{2,2-\frac{D}{2}}$, and $\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{2,1-\frac{D}{2}}$ can be found in Section B.2.5, Section B.2.8, and Section B.2.7, respectively. Summed up, the resulting term is

$$\begin{aligned} I^{l,3} &= -2(2-2\varepsilon)(q^2)^{1-2\varepsilon}\left(\left(\frac{-1}{4\varepsilon}\right)+\left(\frac{-1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{3}{2}+\gamma_E\right)\right)-\left(\frac{-1}{2\varepsilon}+\frac{1}{\varepsilon}\left(-\frac{9}{4}+\gamma_E\right)\right)\right)\\ &= (q^2)^{1-2\varepsilon}\left(\frac{-2}{\varepsilon}+\mathcal{O}(1)\right) \qquad (2.30) \end{aligned}$$

Inserting Eq. (2.26), Eq. (2.28), and Eq. (2.30) in Eq. (2.23), we get

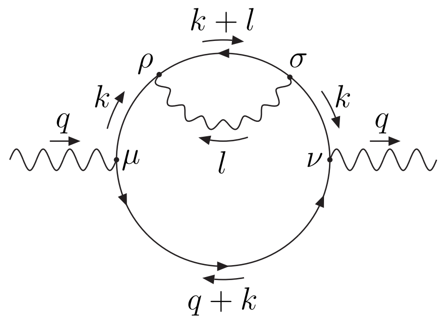
$$\begin{aligned}
g^{\mu\nu} I^{l} &= -ie^4 \Big(I^{l,1} + I^{l,2} + I^{l,3} \Big) \\
&= -ie^4 (q^2)^{1-2\varepsilon} \left(\left(-\frac{1}{\varepsilon} \right) + \left(-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} (-4 + 4\gamma_E) \right) + \left(\frac{-2}{\varepsilon} \right) + \mathcal{O}(1) \right) \\
&= -4ie^4 (q^2)^{1-2\varepsilon} \left(-\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon} \left(-\frac{7}{4} + \gamma_E \right) + \mathcal{O}(1) \right) \qquad (2.31)
\end{aligned}$$

Obviously, Eq. (2.31) looks not only very similar to Eq. (2.21), but is in fact identical. As indicated above, the gauge parameter ξ can be chosen to be $\xi = 1$ (Landau gauge) which abolishes the poles, leaving the amplitude finite. The explicit computation is to be found in Section 2.6.

2.4.3. The Second Graph

As discussed at the very beginning of this section, there is a second graph with a subgraph inside the other fermionic propagator, namely:

$\Gamma' =$

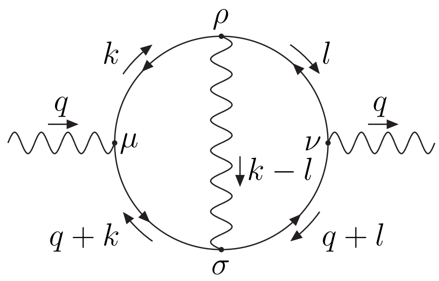


(2.32)

Of course, it is possible to compute the Feynman integrand of and see that it is the same as . On the other hand, this is actually obvious because the direction of fermionic edges is arbitrary as long as it is consistent in every vertex. The upper and the lower leg are not in any way distinguished from each other, or weighted in a certain way, other than the arbitrary labeling done by us, so that it should make no difference in which fermionic line the subgraph appears as long as it is the same subgraph both times. Consequently, when all contributing amplitudes are summed up, it suffices to multiply Eq. (2.21) by two.

2.5. The Graph with a Vertex Subgraph

Compared to , the second graph contributing to the two-loop photonic amplitude, namely the graph

$\Gamma =$  (2.33)

yields some problems. It has two superficially divergent subgraphs, but these two subgraphs are not a forest: Neither are they disjoint, nor does any one contain the other completely. In Section 2.4, one divergent subgraph was nested inside the other, so the problem could be reduced to nested one-loop problems. Now, the divergencies are overlapping, which means that we will not be successfull by referring to one-loop computations alone. A quite helpful formula, the triangle relation, is derived in Chapter C.
As in Section 2.1, the ansatz to follow when computing the Feynman amplitude $\Phi = \int \mathrm{d}^D k \, \mathrm{d}^D l \; I$ is to assume that the propagator can only be a function of the momentum of the external line, q, and on the two Lorentz indices of the two vertices to which external lines are attached, μ and ν. Contraction with $g^{\mu\nu}$ yields this function.

$$F\left(\frac{q^2}{\mu^2}\right) = \frac{1}{3q^2} g^{\mu\nu} \Phi \tag{2.34}$$

We need to determine $g^{\mu\nu}\Phi$ in order to find $F(q^2)$ and, thus, Φ.
The Feynman rules of QED give the Feynman integrand for Eq. (2.33)

$$\begin{aligned} I &= -\operatorname{Tr}\left\{ ie\gamma_\mu i\frac{\not k}{k^2} ie\gamma_\rho i \frac{\not l}{l^2} ie\gamma_\nu i \frac{\not l + \not q}{(l+q)^2} ie\gamma_\sigma i \frac{\not k + q}{(k+q)^2} \right\} i \frac{\left(g^{\rho\sigma} - \xi \frac{(k-l)^\rho (k-l)^\sigma}{(k-l)^2}\right)}{(k-l)^2} \\ &= -ie^4 \operatorname{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta\gamma_\nu\gamma_\delta\gamma_\sigma\gamma_\eta\right\} \frac{k^\alpha l^\beta (l+q)^\delta (k+q)^\eta}{k^2(k+q)^2 l^2 (l+q)^2} \frac{\left(g^{\rho\sigma} - \xi \frac{(k-l)^\rho (k-l)^\sigma}{(k-l)^2}\right)}{(k-l)^2} \\ &=: \left(I^g - \xi I^{kl} \right) \end{aligned} \tag{2.35}$$

2.5.1. The $g^{\mu\nu}$ part

At first, we compute

$$I^g = -ie^4 \frac{\operatorname{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta\gamma_\nu\gamma_\delta\gamma^\rho\gamma_\eta\right\} k^\alpha l^\beta (l+q)^\delta (k+q)^\eta}{k^2(k+q)^2 l^2 (l+q)^2 (k-l)^2} \tag{2.36}$$

The use of Eq. (2.7) in Eq. (2.36) as well as computing and contracting the trace with mathematica (cf. Chapter E), and contracting the result with $g^{\mu\nu}$ (keeping in mind that $g^{\mu\nu}g_{\mu\nu} = D$), yields

$$\begin{aligned}
g^{\mu\nu} I^g &= -\frac{4(2-D)ie^4}{k^2(k+q)^2 l^2 (l+q)^2 (k-l)^2} \cdot \Bigg((D-4)\Big[q^2(k\cdot l) + l^2(k\cdot q) + k^2\big((l\cdot q) + l^2\big)\Big] + \\
&\qquad + 4\big((k\cdot q) + (k\cdot l)\big)\big((k\cdot l) + (l\cdot q)\big)\Bigg) \\
&:= -4ie^4 \frac{\mathrm{num}\left(g^{\mu\nu} I^g\right)}{\mathrm{den}\left(g^{\mu\nu} I^g\right)}
\end{aligned}$$

The numerator is multiplied out and the scalar products are replaced using Eq. (A.4) or, in the case of $k\cdot l$, the similar relation Eq. (A.5) which we need in order to produce $(k-l)^2$ instead of $(k+l)^2$.
The numerator of the result is

$$\begin{aligned}
\mathrm{num}\left(g^{\mu\nu} I^g\right) = &(D-2)(k-l)^2\left(k^2 + l^2 + (k+q)^2 + (l+q)^2\right) + \\
&+ \left(-\frac{D^2}{2} + 3D - 4\right)\left(k^2(l+q)^2 + l^2(k+q)^2\right) + (2-D)\left[(k-l)^2\right]^2 + \\
&+ (2-D)\left(l^2(l+q)^2 + k^2(k+q)^2\right) + (2-D)\left(k^2 l^2 + (k+q)^2(l+q)^2\right) + \\
&+ (D-2)q^2\left(k^2 + l^2 + (k+q)^2 + (l+q)^2\right) + \left(\frac{1}{2}(D-8)(D-2)\right)q^2(k-l)^2 + \\
&+ (2-D)\left(q^2\right)^2
\end{aligned} \tag{2.37}$$

One can easily see that the result is invariant under exchange of $k \leftrightarrow l$ and under simultaneous exchange of $k \leftrightarrow k+q$ and $l \leftrightarrow l+q$. This is obvious from the labeling of the graph: It is completely arbitrary which loop momentum runs through which cycle, and whether the q is carried inside the upper or the lower half circle. Because of this symmetry, Eq. (2.37) can be simplified since several terms will give the same results.

$$\begin{aligned}
\mathrm{num}\left(g^{\mu\nu} I^g\right) = &4(D-2)(k-l)^2(k+q)^2 + 2\left(-\frac{D^2}{2} + 3D - 4\right)k^2(l+q)^2 + \\
&+ (2-D)\left[(k-l)^2\right]^2 + 2(2-D)l^2(l+q)^2 + 2(2-D)(k+q)^2(l+q)^2 + \\
&+ 4(D-2)q^2(k+q)^2 + \frac{1}{2}(D-8)(D-2)q^2(k-l)^2 + (2-D)\left(q^2\right)^2
\end{aligned} \tag{2.38}$$

Taking the denominator into account,

$$\mathrm{den}\left(g^{\mu\nu} I^g\right) = k^2(k+q)^2 l^2 (l+q)^2 (k-l)^2 \quad , \tag{2.39}$$

it is obvious that there are several terms in Eq. (2.38) leading to vanishing integrals. First of all, observe that there are two terms in Eq. (2.39) with k but no l, two terms with l but no k, and one term mixing the two. A $(k-l)^2$ in the numerator deletes this mixing term and the integral falls apart into two one-loop integrals. Therefore, any term in Eq. (2.38) which includes $(k-l)^2$ and one other inverse propagator makes the integral vanish since there are no higher powers than 1 on any of the propagators in Eq. (2.39). This is due to the property of the one-loop master integral $\mathcal{M}(\alpha, \beta, D, q^2)$, which vanishes at $\alpha = 0$ or $\beta = 0$. But even the terms without

$(k-l)^2$ may lead to vanishing integrals if there are more than one term including k (or l, respectively). Thus, the term $l^2(l+q)^2$ in Eq. (2.38) may be omitted. Not as easily spotted, but equally vanishing are the terms $(k+q)^2(l+q)^2$, because they will leave a denominator $k^2l^2(k-l)^2$, which, after the first integration of, say, l, yields $(k^2)^{3-\frac{D}{2}}$, but no $(k+q)^2$, so the property of $\mathcal{M}(\alpha,\beta,D,q^2)$ makes the term negligible since one of the two exponents is zero, regardless of the value of the other. The remaining terms are the ones that might give an actual, non-vanishing contribution to the amplitude.
The numerator leads to the following integrand.

$$
\begin{aligned}
\operatorname{num}\left(g^{\mu\nu} I^{g}_{-\!\oplus\!-}\right) = \Big(& (D-2)(4-D)k^2(l+q)^2 + (2-D)\left[(k-l)^2\right]^2 + \\
& + (4D-8)q^2(k+q)^2 + \frac{1}{2}(D-8)(D-2)q^2(k-l)^2 + (2-D)\left(q^2\right)^2 \Big) \\
\Rightarrow \quad g^{\mu\nu} I^{g}_{-\!\oplus\!-} = -4ie^4 \Bigg(& \frac{(D-2)(4-D)}{(k+q)^2l^2(k-l)^2} + \frac{(2-D)(k-l)^2}{k^2(k+q)^2l^2(l+q)^2} + \frac{(4D-8)q^2}{k^2l^2(l+q)^2(k-l)^2} + \\
& + \frac{1}{2}\frac{q^2(D-8)(D-2)}{k^2(k+q)^2l^2(l+q)^2} + \frac{\left(q^2\right)^2(2-D)}{k^2(k+q)^2l^2(l+q)^2(k-l)^2} \Bigg)
\end{aligned}
$$

Now of course, the amplitude is computed by the integration over the loop momenta, in this case l and k. We will write down the integrals, even though they might not exist (yet). Some of these integrals do diverge, but we will understand the written-down integral as nothing more than something we have written down. We will think about the meaning later, namely in Section 2.6. First, let us examine the amplitude and its poles without thinking about how to get rid of them.

$$
\begin{aligned}
\Rightarrow \quad g^{\mu\nu}\Phi^{g}_{-\!\oplus\!-} &= -4ie^4 \Bigg((D-2)(4-D) \iint \frac{\mathrm{d}^D l \, \mathrm{d}^D k}{(k+q)^2l^2(k-l)^2} + \\
&\quad + (2-D) \iint \mathrm{d}^D l \, \mathrm{d}^D k \frac{(k-l)^2}{k^2(k+q)^2l^2(l+q)^2} + \\
&\quad + (4D-8)q^2 \iint \frac{\mathrm{d}^D l \, \mathrm{d}^D k}{k^2l^2(l+q)^2(k-l)^2} + \\
&\quad + \frac{1}{2}(D-8)(D-2)q^2 \iint \frac{\mathrm{d}^D l \, \mathrm{d}^D k}{k^2(k+q)^2l^2(l+q)^2} + \\
&\quad + (2-D)\left(q^2\right)^2 \iint \frac{\mathrm{d}^D l \, \mathrm{d}^D k}{k^2(k+q)^2l^2(l+q)^2(k-l)^2} \Bigg) \qquad (2.40) \\
&=: -4ie^4 \left(I^{g,1}_{-\!\oplus\!-} + I^{g,2}_{-\!\oplus\!-} + I^{g,3}_{-\!\oplus\!-} + I^{g,4}_{-\!\oplus\!-} + I^{g,5}_{-\!\oplus\!-} \right) \qquad (2.41)
\end{aligned}
$$

There are five different integrals. Of these, only $I^{g,5}_{-\!\oplus\!-}$ cannot be solved using one-loop techniques. Since we are already familiar with the master integral for the one-loop case, we start with the computation of the first four integrals. Then, for $I^{g,5}_{-\!\oplus\!-}$, we will refer to Chapter C.

Compute the first contribution to the amplitude:

$$
\begin{aligned}
I^{g,1} &= (D-2)(4-D)\int \frac{\mathrm{d}^D k}{(k+q)^2}\mathcal{M}(1,1,D,k^2) \\
&= (D-2)(4-D)\tilde{\Gamma}^{1,1}_D \mathcal{M}\left(1, 2-\frac{D}{2}, D, q^2\right) \\
&= (D-2)(4-D)\tilde{\Gamma}^{1,1}_D \tilde{\Gamma}^{1,2-\frac{D}{2}}_D \left(q^2\right)^{D-3} \\
\textit{Eq. } (B.16) \Rightarrow \quad &= \left(q^2\right)^{D-3}(D-2)(4-D)\frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)}\cdot\frac{1}{(4-D)}\cdot\frac{1}{(3-D)\left(\frac{3D}{2}-4\right)} \\
\textit{Eq. } (B.17) \Rightarrow \quad &= \left(q^2\right)^{1-2\varepsilon}(2-2\varepsilon)(2\varepsilon)\left(-\frac{1}{4\varepsilon}+\mathcal{O}(1)\right) \\
&= \underline{\underline{\mathcal{O}(1)}} \quad \text{at } \varepsilon \to 0
\end{aligned}
$$

Obviously, the integral itself has a pole, but due to the multiplication with $(4-D)$, this pole is cancelled and the term is finite. On account of the subtraction at $q^2 = \mu^2$ which we are going to execute later on, the value of the finite integral is not of our concern because it will cancel.
Continue with the next integral.

$$
\begin{aligned}
I^{g,2} &= (2-D)\iint \mathrm{d}^D l\,\mathrm{d}^D k \frac{k^2+l^2-2(k\cdot l)}{k^2(k+q)^2 l^2(l+q)^2} \\
&= -2(2-D)\underbrace{\int \mathrm{d}^D k \frac{k^\mu}{k^2(k+q)^2}}_{=-\frac{q^\mu}{2}\mathcal{M}(1,1,D,q^2)}\underbrace{\int \mathrm{d}^D l \frac{l_\mu}{l^2(l+q)^2}}_{=-\frac{q_\mu}{2}\mathcal{M}(1,1,D,q^2)} \\
&= \frac{D-2}{2}q^2\mathcal{M}^2(1,1,D,q^2) \\
&= \frac{D-2}{2}\left(q^2\right)^{D-3}\left(\tilde{\Gamma}^{1,1}_D\right)^2 \\
\textit{Eq. } (B.12) \Rightarrow \quad &= \frac{D-2}{2}\left(q^2\right)^{D-3}\frac{\Gamma^4\left(\frac{D}{2}-1\right)\Gamma^2\left(3-\frac{D}{2}\right)}{\Gamma^2(D-2)}\cdot\frac{1}{\left(2-\frac{D}{2}\right)^2} \\
\textit{Eq. } (B.13) \Rightarrow \quad &= \left(q^2\right)^{1-2\varepsilon}\frac{2-2\varepsilon}{2}\left(\frac{1}{\varepsilon^2}+\frac{4-2\gamma_E}{\varepsilon}+\mathcal{O}(1)\right) \\
&= \left(q^2\right)^{1-2\varepsilon}(1-\varepsilon)\left(\frac{1}{\varepsilon^2}+\frac{4-2\gamma_E}{\varepsilon}+\mathcal{O}(1)\right) \\
&= \underline{\underline{\left(q^2\right)^{1-2\varepsilon}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\Big(3-2\gamma_E\Big)+\mathcal{O}(1)\right)}}
\end{aligned}
\tag{2.42}
$$

Carry on with $I^{g,3}$:

$$
\begin{aligned}
I^{g,3} &= (4D-8)q^2 \int \frac{\mathrm{d}^D l}{l^2(l+q)^2}\mathcal{M}(1,1,D,l^2) \\
&= (4D-8)q^2\tilde{\Gamma}_D^{1,1}\mathcal{M}\left(1,3-\frac{D}{2},D,q^2\right) \\
&= \left(q^2\right)^{D-3}(4D-8)\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{1,3-\frac{D}{2}} \\
\mathit{Eq.}\ (B.18) \Rightarrow \quad &= \left(q^2\right)^{D-3}(4D-8)\frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{1}{(4-D)\left(2-\frac{D}{2}\right)} \cdot \frac{1}{(D-3)} \\
\mathit{Eq.}\ (B.19) \Rightarrow \quad &= \left(q^2\right)^{1-2\varepsilon}(8-8\varepsilon)\left(\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{5}{2}-\gamma_E\right)+\mathcal{O}(1)\right) \\
&= \underline{\underline{\left(q^2\right)^{1-2\varepsilon}\left(\frac{4}{\varepsilon^2}+\frac{1}{\varepsilon}\Big(16-8\gamma_E\Big)+\mathcal{O}(1)\right)}}
\end{aligned}
\tag{2.43}
$$

The last integral easily solved with one-loop techniques is $I^{g,4}$:

$$
\begin{aligned}
I^{g,4} &= \frac{1}{2}(D-8)(D-2)q^2\mathcal{M}^2(1,1,D,q^2) \\
\mathit{Eq.}\ (B.12) \Rightarrow \quad &= \frac{1}{2}(D-8)(D-2)\left(q^2\right)^{D-3}\frac{\Gamma^4\left(\frac{D}{2}-1\right)\Gamma^2\left(3-\frac{D}{2}\right)}{\Gamma^2(D-2)} \cdot \frac{1}{\left(2-\frac{D}{2}\right)^2} \\
\mathit{Eq.}\ (B.13) \Rightarrow \quad &= \left(q^2\right)^{1-2\varepsilon}(-2-\varepsilon)(2-2\varepsilon)\left(\frac{1}{\varepsilon^2}+\frac{4-2\gamma_E}{\varepsilon}+\mathcal{O}(1)\right) \\
&= \underline{\underline{\left(q^2\right)^{1-2\varepsilon}\left(\frac{-4}{\varepsilon^2}+\frac{1}{\varepsilon}\Big(-14+8\gamma_E\Big)+\mathcal{O}(1)\right)}}
\end{aligned}
\tag{2.44}
$$

For the last integral, $I^{g,5}$, we need to utilize the triangle relation, Eq. (C.12), derived in Chapter C.

$$
\begin{aligned}
I^{g,5} &= (2-D)\left(q^2\right)^2 \iint \frac{\mathrm{d}^D l\,\mathrm{d}^D k}{k^2(k+q)^2 l^2(l+q)^2(k-l)^2} \\
\mathit{Eq.}\ (C.12) \Rightarrow \quad &= (2-D)\frac{2\left(q^2\right)^2}{D-4}\left\{\iint \frac{\mathrm{d}^D l\,\mathrm{d}^D k}{k^2(k+q)^2\left(l^2\right)^2(l+q)^2} - \iint \frac{\mathrm{d}^D l\,\mathrm{d}^D k}{(k+q)^2\left(l^2\right)^2(l+q)^2(k-l)^2}\right\}
\end{aligned}
$$

Solve the integrals separately:

$$\begin{aligned}
\iint \frac{\mathrm{d}^D l\, \mathrm{d}^D k}{k^2(k+q)^2\left(l^2\right)^2(l+q)^2} &= \mathcal{M}(1,1,D,q^2)\cdot\mathcal{M}(1,2,D,q^2)\\
&= \left(q^2\right)^{D-5}\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{1,2}\\
Eq.\ (B.14) \Rightarrow \qquad &= \left(q^2\right)^{D-5}\frac{\Gamma^4\left(\frac{D}{2}-1\right)\Gamma^2\left(3-\frac{D}{2}\right)}{\Gamma^2(D-2)}\cdot\frac{1}{\left(2-\frac{D}{2}\right)}\cdot\frac{1}{\left(\frac{D}{2}-2\right)}\cdot(D-3)\\
Eq.\ (B.15) \Rightarrow \qquad &= \left(q^2\right)^{-1-2\varepsilon}\left(\frac{-1}{\varepsilon^2}+\frac{1}{\varepsilon}(-2+2\gamma_E)+\mathcal{O}(1)\right)\\
\iint \frac{\mathrm{d}^D l\, \mathrm{d}^D k}{(k+q)^2\left(l^2\right)^2(l+q)^2(k-l)^2} &= \int\frac{\mathrm{d}^D l}{\left(l^2\right)^2(l+q)^2}\mathcal{M}\Big(1,1,D,(l+q)^2\Big)\\
&= \tilde{\Gamma}_D^{1,1}\mathcal{M}\left(2,3-\frac{D}{2},D,q^2\right)\\
&= \left(q^2\right)^{D-5}\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{2,3-\frac{D}{2}}\\
Eq.\ (B.24) \Rightarrow \qquad &= \left(q^2\right)^{D-5}\frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)}\cdot\frac{1}{\left(2-\frac{D}{2}\right)}\frac{1}{\left(\frac{D}{2}-2\right)}\frac{\frac{3D}{2}-5}{(D-3)}\\
Eq.\ (B.25) \Rightarrow \qquad &= \left(q^2\right)^{-1-2\varepsilon}\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}(-2+2\gamma_E)+\mathcal{O}(1)\right)\\
\Rightarrow \qquad I^{g,5} &\propto \frac{1}{D-4}\Big(\mathcal{O}(1)\Big)
\end{aligned}$$

Because the factor $(D-4)$ in the denominator gives another pole once the limit $D\to 4$ is taken, the actual value of the $\mathcal{O}(1)$ term suddenly becomes relevant. So we need to expand the products of Γ functions in Section B.2.4 and Section B.2.9 to $\mathcal{O}(\varepsilon^2)$, not just to $\mathcal{O}(\varepsilon)$.

$$\begin{aligned}
\frac{\Gamma^3(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(2-3\varepsilon)} &= 1+(3-2\gamma_E)\varepsilon+\left(9-6\gamma_E+2\gamma_E^2-\frac{\pi^2}{6}\right)\varepsilon^2+\mathcal{O}(\varepsilon^3)\\
\Rightarrow \qquad \iint \frac{\mathrm{d}^D l\, \mathrm{d}^D k}{k^2(k+q)^2\left(l^2\right)^2(l+q)^2} &= \left(q^2\right)^{-1-2\varepsilon}\left(\frac{-1}{\varepsilon^2}+\frac{1}{\varepsilon}(-2+2\gamma_E)+\right.\\
&\qquad\left.\left(-4+4\gamma_E-2\gamma_E^2+\frac{\pi^2}{6}\right)+\mathcal{O}(\varepsilon)\right)\\
\frac{\Gamma^4(1-\varepsilon)\Gamma^2(1+\varepsilon)}{\Gamma^2(2-2\varepsilon)} &= 1+(4-2\gamma_E)\varepsilon+\left(12-8\gamma_E+2\gamma_e^2-\frac{\pi^2}{6}\right)\varepsilon^2+\mathcal{O}(\varepsilon^3)\\
\Rightarrow \qquad \iint \frac{\mathrm{d}^D l\, \mathrm{d}^D k}{(k+q)^2\left(l^2\right)^2(l+q)^2(k-l)^2} &= \left(q^2\right)^{-1-2\varepsilon}\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}(-2+2\gamma_E)+\right.\\
&\qquad\left.+\left(-4+4\gamma_E-2\gamma_E^2+\frac{\pi^2}{6}\right)+\mathcal{O}(\varepsilon)\right)\\
\Rightarrow \qquad I^{g,5} &\propto \frac{1}{D-4}\Big(\mathcal{O}(\varepsilon)\Big)\propto\mathcal{O}(1)
\end{aligned}$$

Obviously, $I^{g,5}$ is finite and therefore, plays no role for the amplitude.
Of the five integrals necessary for Φ^{g}, two are finite and only three need regular-

ization. Insert the results in Eq. (2.41):

$$
\begin{aligned}
g^{\mu\nu}\Phi^{g}_{\ominus} &= -4ie^4\left(I^{g,2}_{\ominus} + I^{g,3}_{\ominus} + I^{g,4}_{\ominus}\right)\\
&= -4ie^4 (q^2)^{1-2\varepsilon}\Bigg\{\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(3 - 2\gamma_E - 2\ln(\pi)\Big)\right) + \left(\frac{4}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(16 - 8\gamma_E - 8\ln(\pi)\Big)\right)\\
&\qquad + \left(\frac{-4}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(-14 + 8\gamma_E + 8\ln(\pi)\Big)\right) + \mathcal{O}(1)\Bigg\}\\
&= \underline{\underline{-4ie^4 (q^2)^{1-2\varepsilon}\left\{\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(5 - 2\gamma_E - 2\ln(\pi)\Big) + \mathcal{O}(1)\right\}}} \qquad (2.45)
\end{aligned}
$$

2.5.2. The $(k-l)^\mu(k-l)^\nu$ part

Last, but not least, the gauge-dependent part of $\Phi_{\ominus}$ needs to be computed.

$$
\begin{aligned}
I^{kl}_{\ominus} &= -\operatorname{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta\gamma_\nu\gamma_\delta\gamma_\sigma\gamma_\eta\right\}\frac{k^\alpha l^\beta (l+q)^\delta (k+q)^\eta}{k^2(k+q)^2 l^2 (l+q)^2}\frac{(k-l)^\rho (k-l)^\sigma}{\left[(k-l)^2\right]^2}\\
&=: -ie^4\frac{\operatorname{num}(I^{kl}_{\ominus})}{\operatorname{den}(I^{kl}_{\ominus})}
\end{aligned}
$$

As before, the trace of the eight γ matrices and the contraction with the respective momenta as well as with $g^{\mu\nu}$ is done using mathematica (cf. Chapter E). The result is

$$
\begin{aligned}
g^{\mu\nu}\operatorname{num}(I^{kl}_{\ominus}) = 4\Bigg\{&2(D-2)\left((k^2)^2 l^2 + k^2 (l^2)^2 + k^2\Big[(l+q)^2\Big]^2 + l^2\Big[(k+q)^2\Big]^2\right)\\
&+ 2(2-D)\left((k^2)^2(l+q)^2 + (l^2)^2(k+q)^2\right) +\\
&+ 2(2-D)q^2\Big(k^2(l+q)^2 + l^2(k+q)^2\Big)\\
&\\
&+ q^2\left(k^2 - l^2\right)\left((k+q)^2 - (l+q)^2\right) -\\
&- \Big(k^2 l^2\left[(l+q)^2 + (k+q)^2\right] + (k+q)^2(l+q)^2\left[k^2 + l^2\right]\Big) +\\
&+ (k-l)^2\left(2k^2l^2 - k^2(l+q)^2 - l^2(k+q)^2\right)\Bigg\} \qquad (2.46)
\end{aligned}
$$

But which of these terms survive the integration? As we know, integrals vanish as soon as one of the two exponents α or β in $\mathcal{M}(\alpha,\beta,D,q^2)$ is zero or a negative integer. Therefore, if a certain mometum (k or l) appears in only one term in the denominator of an expression, the integral vanishes. Since $(k-l)^2$ appears quadratically in the denominator, but not in Eq. (2.46), we will always keep at least one factor $(k-l)^2$ in the denominator. Therefore, all terms involving k^2 and $(k+q)^2$ as well as all terms involving l^2 and $(l+q)^2$ yield vanishing integrals because they will cancel all terms involving these momenta in the denomintator, except $(k-l)^2$, which is only one factor, but we need two in order for $\mathcal{M}(\alpha,\beta,D,q^2)$ not to vanish. Moreover, some terms may allow a first integration, but vanish during the integration over the second momentum, as is the case with terms involing $(k-l)^2$, $(l+q)^2$ and $(k+q)^2$, and also $(k-l)^2$, l^2 and k^2. Additionally, the term $(k^2)^2 l^2$ vanishes, since

the integration over l produces a new exponent for the $(k+q)^2$ term, and the $(k^2)^2$ leaves k^2 in the denominator with an exponent of -1, which produces a zero due to the properties of the one-loop master integral. Of course, the same is true for $k^2(l^2)^2$.
Because of the symmetry of the graph and the arbitrariness of the labeling of the internal momenta, $g^{\mu\nu}\mathrm{num}(I^{kl})$ is invariant under exchange of $k \leftrightarrow l$ and under simultaneous exchange of $k \leftrightarrow k+q$ and $l \leftrightarrow l+q$, as discussed in Section 2.5.1. With these simplifications, Eq. (2.46) can be reduced to

$$g^{\mu\nu}\mathrm{num}(I^{kl}) = 4ie^4\Big\{2(D-2)(k^2)^2(l+q)^2 + (2-D)q^2k^2(l+q)^2 + (2-D)k^2(l+q)^2(k-l)^2\Big\}$$

$$\begin{aligned}
\Rightarrow \quad g^{\mu\nu}\Phi^{kl} &= -4ie^4\Bigg\{(2D-4)\iint \mathrm{d}^D l\,\mathrm{d}^D k\,\frac{k^2}{(k+q)^2 l^2[(k-l)^2]^2}+\\
&\qquad + (2-D)q^2\iint \frac{\mathrm{d}^D k\,\mathrm{d}^D l}{(k+q)^2 l^2[(k-l)^2]^2}+\\
&\qquad +(2-D)\iint \frac{\mathrm{d}^D k\,\mathrm{d}^D l}{(k+q)^2 l^2 (k-l)^2}\Bigg\}\\
&=: -4ie^4\left\{I^{kl,1} + I^{kl,2} + I^{kl,3}\right\}
\end{aligned} \tag{2.47}$$

Compute the integrals separately:

$$\begin{aligned}
I^{kl,1} &= (2D-4)\int \frac{\mathrm{d}^D k}{(k^2)^{-1}(k+q)^2}\mathcal{M}(1,2,D,k^2)\\
&= (2D-4)\tilde{\Gamma}^{1,2}_D\mathcal{M}\left(1,2-\frac{D}{2},D,q^2\right)\\
&= (q^2)^{D-3}(2D-4)\tilde{\Gamma}^{1,2}_D\tilde{\Gamma}^{1,2-\frac{D}{2}}_D\\
\mathit{Eq.}\ (B.26) \Rightarrow \quad &= (q^2)^{D-3}(2D-4)\frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)}\cdot\frac{1}{(4-D)}\cdot\frac{1}{\left(\frac{3D}{2}-4\right)}\\
\mathit{Eq.}\ (B.27) \Rightarrow \quad &= (q^2)^{1-2\varepsilon}(4-4\varepsilon)\left(\frac{1}{4\varepsilon}+\mathcal{O}(1)\right)\\
&= \underline{\underline{(q^2)^{1-2\varepsilon}\left(\frac{1}{\varepsilon}+\mathcal{O}(1)\right)}}
\end{aligned} \tag{2.48}$$

The next integral is computed quite similarly:

$$\begin{aligned}
I^{kl,2} &= (2-D)q^2\int \frac{\mathrm{d}^D k}{(k+q)^2}\mathcal{M}(1,2,D,k^2)\\
&= q^2(2-D)\tilde{\Gamma}^{1,2}_D\mathcal{M}\left(1,3-\frac{D}{2},D,q^2\right)\\
&= (q^2)^{D-3}(2-D)\tilde{\Gamma}^{1,2}_D\tilde{\Gamma}^{1,3-\frac{D}{2}}_D\\
\mathit{Eq.}\ (B.28) \Rightarrow \quad &= (q^2)^{D-3}(2-D)\frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)}\cdot\frac{1}{\left(\frac{D}{2}-2\right)(4-D)}\\
\mathit{Eq.}\ (B.29) \Rightarrow \quad &= (q^2)^{1-2\varepsilon}(-2+2\varepsilon)\left(-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{3}{2}+\gamma_E\right)+\mathcal{O}(1)\right)\\
&= \underline{\underline{(q^2)^{1-2\varepsilon}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\Big(2-2\gamma_E\Big)+\mathcal{O}(1)\right)}}
\end{aligned} \tag{2.49}$$

At last, we are about to finish the computation of integrals.

$$
\begin{aligned}
I^{kl,3} &= (2-D)\int \frac{\mathrm{d}^D k}{(k+q)^2}\mathcal{M}(1,1,D,k^2) \\
&= (2-D)\tilde{\Gamma}^{1,1}_{D}\mathcal{M}\left(1,2-\frac{D}{2},D,q^2\right) \\
&= \left(q^2\right)^{D-3}(2-D)\tilde{\Gamma}^{1,1}_{D}\tilde{\Gamma}^{1,2-\frac{D}{2}}_{D} \\
Eq.\ (B.16) \Rightarrow \quad &= \left(q^2\right)^{D-3}(2-D)\frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)}\cdot\frac{1}{(4-D)}\cdot\frac{1}{(3-D)\left(\frac{3D}{2}-4\right)} \\
Eq.\ (B.17) \Rightarrow \quad &= \left(q^2\right)^{1-2\varepsilon}(-2+2\varepsilon)\left(-\frac{1}{4\varepsilon}+\mathcal{O}(1)\right) \\
&= \underline{\underline{\left(q^2\right)^{1-2\varepsilon}\left(\frac{1}{2\varepsilon}+\mathcal{O}(1)\right)}}
\end{aligned} \tag{2.50}
$$

Now, using Eq. (2.48), Eq. (2.49), and Eq. (2.50) in Eq. (2.47), we find the solution for $g^{\mu\nu}\Phi$:

$$
\begin{aligned}
g^{\mu\nu}\Phi^{kl} &= -4ie^4\left\{I^{kl,1}+I^{kl,2}+I^{kl,3}\right\} \\
&= -4ie^4\left(q^2\right)^{1-2\varepsilon}\left\{\left(\frac{1}{\varepsilon}\right)+\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\big(2-2\gamma_E-2\ln(\pi)\big)\right)+\left(\frac{1}{2\varepsilon}\right)+\mathcal{O}(1)\right\} \\
&= \underline{\underline{-4ie^4\left(q^2\right)^{1-2\varepsilon}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{7}{2}-2\gamma_E-2\ln(\pi)\right)+\mathcal{O}(1)\right)}}
\end{aligned} \tag{2.51}
$$

2.6. Renormalization of the two-loop photon propagator

As the partial results for the amplitude are here, it is now time to combine the gauge-dependent and gauge-independent parts of the two graphs to get the complete amplitudes of the respective graphs, and then add the two in the way described in Section 2.4.3 in order to get the full amplitude of the photon propagator, up to two loops. Special attention will be paid to the results in the Landau gauge, $\xi = 1$.

2.6.1. The Amplitudes (unrenormalized)

For the Fermionic Subgraph We have successfully computed the gauge-dependent and the gauge-independent contributions to the Feynman amplitude of , which is also the amplitude of .

$$
\begin{aligned}
Eq.\ (2.21) \quad &: \quad g^{\mu\nu}\Phi^{g} = -4ie^4\left(q^2\right)^{1-2\varepsilon}\left\{-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{7}{4}+\gamma_E\right)+\mathcal{O}(1)\right\} \\
Eq.\ (2.31) \quad &: \quad g^{\mu\nu}\Phi^{l} = -4ie^4\left(q^2\right)^{1-2\varepsilon}\left(-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{7}{4}+\gamma_E\right)+\mathcal{O}(1)\right)
\end{aligned}
$$

Together with Eq. (2.8) and Eq. (2.2), let us combine the results to produce the complete amplitude.

$$
\begin{aligned}
&Eq.\ (2.2), Eq.\ (2.8) \quad \Rightarrow \quad F(q^2) = \frac{1}{3q^2}\left(g^{\mu\nu}\Phi^{g} - \xi g^{\mu\nu}\Phi^{l}\right)\\
&\Rightarrow \qquad F(q^2) = -\frac{4}{3} i e^4 (q^2)^{-2\varepsilon}\left((1-\xi)\left(-\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{7}{4} + \gamma_E\right)\right) + \mathcal{O}(1)\right)
\end{aligned}
\tag{2.52}
$$

Obviously, for $\xi = 1$ (Landau gauge), not just the double pole in ε, but also the simple pole in ε, hence the entire (divergent) expression disappears!

$$
F(q^2)\Big|_{\xi=1} = 0 \tag{2.53}
$$

This is obvious once one considers that in the Landau gauge, the graph has an amplitude which is identical to zero. Therefore, anytime this subgraph is inserted into a cograph, the contribution vanishes (but only in the Landau gauge, of course).

For the Vertex Subgraph All parts for the amplitude Φ have been computed

$$
\begin{aligned}
Eq.\ (2.45) \quad &: \quad g^{\mu\nu}\Phi^{g} = -4ie^4 (q^2)^{1-2\varepsilon}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left(5 - 2\gamma_E\right) + \mathcal{O}(1)\right)\\
Eq.\ (2.51) \quad &: \quad g^{\mu\nu}\Phi^{l} = -4ie^4 (q^2)^{1-2\varepsilon}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{7}{2} - 2\gamma_E\right) + \mathcal{O}(1)\right)
\end{aligned}
$$

Next, we can put them all together to obtain the scalar function $F(q^2)$ of the entire amplitude of .

$$
\begin{aligned}
F(q^2) &= \frac{1}{3q^2} g^{\mu\nu}\Phi = \frac{1}{3q^2}\left(g^{\mu\nu}\Phi^{g} - \xi g^{\mu\nu}\Phi^{kl}\right)\\
&= -\frac{4}{3} i e^4 (q^2)^{-2\varepsilon}\left\{(1-\xi)\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-2\gamma_E\right)\right) + \frac{1}{\varepsilon}\left(5 - \frac{7}{2}\xi\right) + \mathcal{O}(1)\right\}
\end{aligned}
$$

We find a loss of subdivergencies at $\xi = 1$, as expected!

$$
\begin{aligned}
F(q^2)\Big|_{\xi=1} &= -\frac{4}{3} i e^4 (q^2)^{-2\varepsilon}\left(\frac{3}{2\varepsilon} + \mathcal{O}(1)\right)\\
&= \underline{\underline{-2ie^4 (q^2)^{-2\varepsilon}\left(\frac{1}{\varepsilon} + \mathcal{O}(1)\right)}}
\end{aligned}
$$

2.6.2. The Full Amplitude (renormalized)

In the Landau gauge

In order to control the pole in ε and in order to execute the limit $\varepsilon \to 0$, we apply a single subtraction at some value $q^2 = \mu^2$ (the reference momentum) to the scalar function $F(q^2)$ of the amplitude of we computed in the Landau gauge above.

$$
\begin{aligned}
F^R\left(\frac{q^2}{\mu^2}\right)\Bigg|_{\xi=1} &:= F(q^2)\Big|_{\xi=1} - F(\mu^2)\Big|_{\xi=1} \\
&= -2ie^4\left(\frac{1}{\varepsilon}+\mathcal{O}(1)\right)\left(\left(q^2\right)^{-2\varepsilon}-\left(\mu^2\right)^{-2\varepsilon}\right) \\
&= -2ie^4\left(\frac{1}{\varepsilon}+\mathcal{O}(1)\right)\left(-2\ln\left(\frac{q^2}{\mu^2}\right)\varepsilon+\mathcal{O}(\varepsilon^2)\right) \\
&= 4ie^4\left(\ln\left(\frac{q^2}{\mu^2}\right)+\mathcal{O}(\varepsilon)\right) \\
&\overset{\varepsilon\to 0}{=} 4ie^4\ln\left(\frac{q^2}{\mu^2}\right)
\end{aligned} \tag{2.54}
$$

To get from the second to the third line, the expansion of $\left(q^2\right)^{-2\varepsilon}$ for small ε was used: $\left(q^2\right)^{-2\varepsilon} = 1-2\ln\left(q^2\right)\varepsilon+\mathcal{O}(\varepsilon^2)$.
With this result, we are finally able to write down the two-loop contribution to the vacuum polarization of quantum electrodynamics in the Landau gauge:

$$
\Phi_2^R\Big|_{\xi=1} = 4ie^4(q^2 g_{\mu\nu}-q_\mu q_\nu)\ln\left(\frac{q^2}{\mu^2}\right) \tag{2.55}
$$

The "4" in Eq. (2.55) is the coefficient of the β function of quantum electrodynamics to two loops! [8]

In an arbitrary gauge

In the next step, no gauge will be chosen. Instead, the gauge parameter ξ will be left as it is. In order to regularize, we need the sum of all graphs contributing to this loop order in the amplitude.

$$
\begin{aligned}
F_2(q^2) &= \frac{1}{3q^2}g^{\mu\nu}\Phi_2 \\
&= \frac{1}{3q^2}\left(2g^{\mu\nu}\Phi + g^{\mu\nu}\Phi\right) \\
&= 2F(q^2) + F(q^2) \\
&= -\frac{4}{3}ie^4\left(q^2\right)^{-2\varepsilon}\left((1-\xi)\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{7}{2}+2\gamma_E+2\ln(\pi)\right)\right)+\right. \\
&\qquad\left.+\left((1-\xi)\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\left(-2\gamma_E-2\ln(\pi)\right)\right)+\frac{1}{\varepsilon}\left(5-\frac{7}{2}\xi\right)\right)+\mathcal{O}(1)\right) \\
&= -\frac{4}{3}ie^4\left(q^2\right)^{-2\varepsilon}\left(\frac{1}{\varepsilon}+\mathcal{O}(1)\right)\left((1-\xi)\left(-\frac{7}{2}\right)+5-\frac{7}{2}\xi\right) \\
&= -\frac{4}{3}ie^4\left(q^2\right)^{-2\varepsilon}\left(\frac{1}{\varepsilon}+\mathcal{O}(1)\right)\cdot\left(\frac{3}{2}\right) \\
&= -2ie^4\left(q^2\right)^{-2\varepsilon}\left(\frac{1}{\varepsilon}+\mathcal{O}(1)\right)
\end{aligned} \tag{2.56}
$$

In order to get the renormalized amplitude, the often-discussed subtraction at the reference momenta $q^2 = \mu^2$ must be carried out one last time.

$$\begin{aligned}
F_2^R\left(\frac{q^2}{\mu^2}\right) &= F_2(q^2) - F_2(\mu^2) \\
&= -2ie^4\left(\frac{1}{\varepsilon} + \mathcal{O}(1)\right)\left((q^2)^{-2\varepsilon} - (\mu^2)^{-2\varepsilon}\right) \\
&= -2ie^4\left(\frac{1}{\varepsilon} + \mathcal{O}(1)\right)\left(\left(1 - 2\ln(q^2)\varepsilon + \mathcal{O}(\varepsilon^2)\right) - \left(1 - 2\ln(\mu^2)\varepsilon + \mathcal{O}(\varepsilon^2)\right)\right) \\
&= -2ie^4\left(\frac{1}{\varepsilon} + \mathcal{O}(1)\right)\left(-2\ln\left(\frac{q^2}{\mu^2}\right)\varepsilon + \mathcal{O}(\varepsilon^2)\right) \\
&= 4ie^4\left(\ln\left(\frac{q^2}{\mu^2}\right) + \mathcal{O}(\varepsilon)\right) \\
&\overset{\varepsilon\to 0}{=} 4ie^4\ln\left(\frac{q^2}{\mu^2}\right)
\end{aligned}$$

$$\Rightarrow \qquad \Phi_2^R = \underline{\underline{4ie^4(q^2 g_{\mu\nu} - q_\mu q_\nu)\ln\left(\frac{q^2}{\mu^2}\right)}} \tag{2.57}$$

It is clear that the result in the Landau gauge is the same as in an arbitrary gauge. This is due to the fact that up to two loops, the β function of QED is independent of the renormalization scheme and therefore also of the chosen gauge. Anyway, the coefficient of the two-loop β function is 4.

3. Corolla Approach

In this chapter, we will repeat the same computations, but this time using the Corolla approach, a graph theoretical procedure using a polynomial of half-edges of the scalar equivalent of the graphs. The key component of this thesis will be the computation of the two-loop contribution to the β-function of quantum electrodynamics using not the standard textbook approach, but this graph theoretical approach where one utilizes three graph polynomials: The first and second Symanzik polynomials and the Corolla polynomial. These polynomials appear in scalar field theory, but we will see that it is possible to relate them to gauge theory.

3.1. Combinatorial Properties of Feynmal Graphs

3.1.1. Notation

In order for the use of the corolla polynomial and corolla differentials to be introduced properly, some clarification of notation will be necessary first.
A graph Γ will be perceived as a set of vertices and edges, and each edge connecting two vertices will be regarded as a set of two half-edges.
Let Γ be a connected graph.

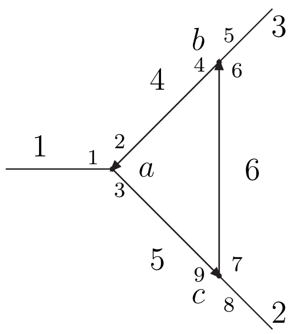
- Denote by V^Γ the set of vertices of Γ.
- Denote by E^Γ the set of edges of Γ.
  - Let E^Γ_{int} be the set of internal edges of Γ.
  - Let E^Γ_{ext} be the set of external edges of Γ.
  - Then, $E^\Gamma_{\text{int}} \cap E^\Gamma_{\text{ext}} = \emptyset$ and $E^\Gamma_{\text{int}} \cup E^\Gamma_{\text{ext}} = E^\Gamma$.

Introduce an orientation on Γ and call $s(e)$ (the source of edge e) the vertex in which e starts, and call $t(e)$ (the target of edge e) the vertex towards which e is directed. Of course, $s(e), t(e) \in V^\Gamma$.
Let us not allow for tadpoles (edges which begin and end in the same vertex). As a consequence, each edge with its orientation can be regarded as an oriented set of two distinct vertices: $s(e)$ and $t(e)$ cannot be identical. However, more than one edge can be adjacent to a vertex v. Call $n(v)$ the set of edges adjacent to $v \in V^\Gamma$, and let its cardinality $|n(v)|$ be the number of edges adjacent to v, called the valence of v. If all vertices of a graph Γ have the same valence m, $|n(v)| = m \ \forall v \in V^\Gamma$, then the graph itself is called m-valent.[1]
Next, half-edges are introduced. Each internal edge will be considered a pair of two half-edges. To be more precise, a half-edge is a pair of a vertex with its adjacent edge.

- Denote by H^Γ the set of half-edges (v, e), with $e \in n(v)$, of Γ.

- Let $\mathrm{cor}\,(v) \equiv \cup_{e \in n(v)}(v, e)$ be the set of all half-edges incident to vertex $v \in V^{\Gamma}$, and call it the corolla at v.
- Let D_v be the sum of all half-edge variables of half-edges in $\mathrm{cor}\,(v)$, $v \in V^{\Gamma}$.
- Let C_{Γ} be the set of all cycles in Γ. Note that we do not restrict this to independent cycles (circuits), but all cycles.
- If Γ is 3-regular (all vertices in Γ have valence 3), and if C is a cycle in C_{Γ}, then there is a unique edge v_C of Γ for every vertex of C such that v_C is not in C. The half-edge (v, v_C) will be denoted by a_{v,v_C}.

Example *Consider the graph*



The bigger numbers indicate decoration of edges, the smaller numbers represent decoration of half-edges, and the letters indicate decoration of vertices. The arrows clarify the orientation of the edges. The vertices and edges are:

$$
\begin{aligned}
V^{-\!\!<} &= \{a, b, c\} \\
E^{-\!\!<} &= \{e_1, e_2, e_3, e_4, e_5, e_6\} \\
E^{-\!\!<}_{ext} &= \{e_1, e_2, e_3\} \\
E^{-\!\!<}_{int} &= \{e_4, e_5, e_6\}
\end{aligned}
$$

Then, for example, $s(e_4) = b$ and $t(e_4) = a$ because e_4 is oriented from b to a. Take vertex a:

$$n(a) = e_1, e_4, e_5 \quad \Rightarrow \quad |n(a)| = 3$$

But not just a, all vertices are 3-valent. Therefore, the entire graph is 3-valent. The half-edges are given by

$$
\begin{aligned}
H^{-\!\!<} &= \Big\{(a, e_1), (a, e_4), (a, e_5), (b, e_4), (b, e_3), (b, e_6), (c, e_6), (c, e_2), (c, e_5)\Big\} \\
&= \{h_1, h_2, h_3, h_4, h_5 h_6, h_7, h_8, h_9\} \\
\Rightarrow \quad \mathrm{cor}\,(a) &= \{h_1, h_2, h_3\} \;\;, \quad \mathrm{cor}\,(b) = \{h_4, h_5, h_6\} \;\;, \quad \mathrm{cor}\,(c) = \{h_7, h_8, h_9\} \\
D_a &= h_1 + h_2 + h_3 \;\;, \quad D_b = h_4 + h_5 + h_6 \;\;, \quad D_c = h_7 + h_8 + h_9
\end{aligned}
$$

There is only one loop in the graph, namely that given by the internal edges and the vertices. Note how the edges e_1, e_2, and e_3 and therefore the half-edges h_1, h_5, and h_8 are not in the cycle, even though their adjacent vertices are.

$$\begin{aligned} C &= \{a, b, c, e_4, e_5, e_6\} \\ (a, a_C) &= h_1 = a_{a,e_1} \\ (b, b_C) &= h_5 = a_{b,e_3} \\ (c, c_C) &= h_8 = a_{c,e_2} \end{aligned}$$

3.1.2. The Symanzik Polynomials

Looking at a graph Γ and using power counting, one can directly obtain the degree of divergence of the Feynman integral associated with Γ by applying the Feynman rules. Moreover, in scalar field theories, such as ϕ_6^3 theory or ϕ_4^4 theory, the Feynman integrand I_Γ can be acquired just by observing the graph, using the Symanzik polynomials. The scalar Feynman integrand is of the form

$$I_\Gamma = \frac{\exp\left(-\frac{\phi_\Gamma}{\psi_\Gamma}\right)}{\psi_\Gamma^{\frac{D}{2}}}, \tag{3.1}$$

where D denotes the dimension of space time, and ψ and ϕ denote the first and second Symanzik polynomials, respectively. These polynomials are of combinatorial nature and, among other things, dependent on the first Betti number of Γ. One way to express them is to assign an edge variable A_e to every edge $e \in E_{\text{int}}^\Gamma$ and define

$$\psi_\Gamma = \sum_T \prod_{e \notin T} A_e \tag{3.2}$$

$$\phi_\Gamma = \sum_{T_1 \cup T_2} Q(T_1) Q(T_2) \prod_{e \notin T_1 \cap T_2} A_e$$

For the first Symanzik polynomial, ψ_Γ, the sum is over all spanning trees T of the graph Γ, meaning $v \in T \ \forall \ v \in \Gamma^{[0]}$ and T is simply-connected. For the second Symanzik polynomial, ϕ_Γ, the sum is over all spanning two-forests $T_1 \cup T_2$, meaning $T_1 \cap T_2 = \emptyset$, $v \in T_1 \cup T_2 \ \forall v \in \Gamma^{[0]}$ and T_i is simply-connected, $i \in \{1, 2\}$. Furthermore, $Q(T_i)$ denotes the momentum flowing through T_i. Since it is not always obvious or can be confusing what momenta flow though the T_i, there is an alternative way to write down the second Symanzik polynomial, using the Pfaffian determinant of a matrix relating to the graph:

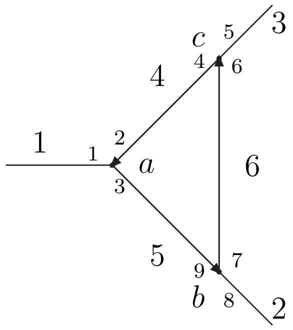
$$\phi_\Gamma = |N|_{\text{Pf}}(\Gamma) = \sum_{T_1 \cup T_2} \left(\sum_{e \notin T_1 \cup T_2} \tau(e) \xi_e \right)^2 \prod_{e \notin T_1 \cup T_2} A_e \tag{3.3}$$

$$\text{with} \qquad \tau(e) = \begin{cases} +1 & \text{if } e \text{ is oriented from } T_1 \to T_2 \\ -1 & \text{if } e \text{ is oriented from } T_2 \to T_1 \\ 0 & \text{else} \end{cases}$$

Here, the ξ_i denote the edge momenta of the edge e_i und $\tau(e)$ refers to the orientation of e.

has one loop, therefore its first Symanzik polynomial will be of first degree in the A_i, and the second Symanzik polynomial on one degree higher, namely second degree. Since and are 1PI two-loop graphs, their first Symanzik polynomial ϕ_Γ will be of second degree in the A_i and the second Symanzik polynomial ϕ_Γ of third degree. The measure of integration includes all two edge variables A_i for the one-loop graph and all five edge variables A_i for the two-loop graphs. Recall that only internal edges get equipped with an edge variable. [2][9]

Example *Take the graph from the other example:*



There are three spanning trees and three spanning two-forests: A spanning tree is two out of the three internal edges, and a spanning two-forest is an internal edge and the vertex it does not touch.

$$
\begin{aligned}
T &\in \left\{ \;,\; , \; \right\} \\
&= \Big\{\{e_5, e_6\}, \{e_4, e_5\}, \{e_4, e_6\}\Big\} \\
T_1 \cup T_2 &\in \left\{ \;,\; , \; \right\} \\
&= \Big\{\{a\} \cup \{e_6\}, \{b\} \cup \{e_4\}, \{c\} \cup \{e_5\}\Big\}
\end{aligned}
$$

Therefore, the first and second Symanzik polynomials are given by

$$
\begin{aligned}
\psi \quad &= \sum_T \prod_{e \notin T} A_e \\
&= A_4 + A_5 + A_6 \\
\phi \quad &= \sum_{T_1 \cup T_2} \left(\sum_{e \notin T_1 \cup T_2} \tau(e) \xi_e^2 \right)^2 \prod_{e \notin T_1 \cup T_2} A_e = A_4 + A_5 + A_6 \\
&= (\xi_5 - \xi_4)^3 A_4 A_5 + (\xi_6 - \xi_5)^2 A_5 A_6 + (\xi_4 - \xi_6)^2 A_4 A_6
\end{aligned}
$$

Sometimes, not the Symanzik polynomials, but their dual, the Kirchhoff polynomials, are used.
Our goal will be to set the dimension of spacetime $D = 4$, even though ϕ^3 theory is renormalizable in $D = 6$ dimensions. We will use $D = 4$ here because we are aiming at quantum electrodynamics, which is renormalizable in $D = 4$ dimensions.

Unfortunately, the Symanzik polynomials only provide the Feynman integrand for scalar field theories. An analogous method for gauge theories is not available at this point. Finding such an easy rule to obtain the Feynman integrand for gauge theory graphs, on the other hand, is of high interest, and recent progress will be discussed in this thesis.

3.1.3. The Corolla Polynomial

It is time to introduce the most important graph polynomial of this thesis, the corolla polynomial. Remember that for a 3-valent graph, for every vertex in any cycle, there is a unique half edge which is not part of the cycle. We define

$$\mathcal{C}^n = \sum_{\substack{C_1,\ldots,C_n \in \mathcal{C}_\Gamma \\ C_j \text{pairwise disjoint}}} \left(\left(\prod_{j=1}^{n} \prod_{v \in C} a_{v,v_C} \right) \prod_{v \notin C_1 \cup \cdots \cup C_n} D_v \right)$$

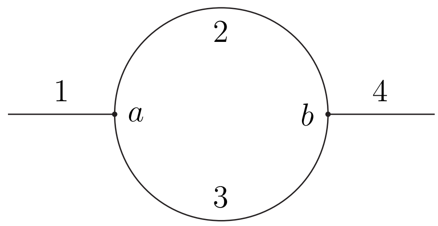
The alternating series

$$\mathcal{C} = \sum_{n \geq 0} (-1)^n \mathcal{C}^n$$

is called the corolla polynomial. Because every finite graph has a finite number of cycles, this series is actually a well-defined sum. It is also strictly positive, despite the alternating sign.[1][2]
For our purposes, only the $n = 0$ and $n = 1$ monomials are of interest:

$$\mathcal{C}^0 = \sum_{v \in V^\Gamma} D_v$$
$$\mathcal{C}^1 = \sum_{C \in \mathcal{C}_\Gamma} \left(\left(\prod_{v \in C} a_{v,v_C} \right) \prod_{v \notin C} D_v \right)$$

Example *Take the graph*

$\Gamma =$ 

- *The set of edges is given by* $E^\Gamma = \{e_1, e_2, e_3, e_4\}$, *where* $E^\Gamma_{ext} = \{e_1, e_4\}$ *and* $E^\Gamma_{int} = \{e_2, e_3\}$.
- *The set of vertices is given by* $V^\Gamma = \{a, b\}$.
- *The set of half-edges is given by* $H^\Gamma = \Big\{(a, e_1), (a, e_2), (a, e_3), (b, e_3), (b, e_2), (b, e_4)\Big\} \equiv \{h_1, h_2, h_3, h_4, h_5, h_6\}$. *Note that there is the same orientation at each vertex.*
- *There is only one cycle in this graph, namely* $C = \{(e_2, e_3)\}$. *For reasons of shortness, only the edges are used in order to identify the cycle, but nevertheless, the vertices* a *and* b *are part of the cycle, too.*

With these information given, the first two Corolla monomials are given by

$$\begin{aligned}\mathcal{C}^0_\Gamma &= \prod_{v\in\{a,b\}} D_v = D_a D_b = (h_1+h_2+h_3)(h_4+h_5+h_6)\\ \mathcal{C}^1_\Gamma &= \prod_{v\in\{a,b\}} a_{v,v_C} = a_{a,e_1}a_{b,e_4} = h_1 h_6\end{aligned}$$

Since there is only one cycle in Γ*, there are no more Corolla monomials. The Corolla polynomial is complete as of* $\mathcal{C}_\Gamma = \mathcal{C}^0_\Gamma - \mathcal{C}^1_\Gamma$.

The Corolla Differential

In order to get from scalar field theory to gauge theories, the Corolla polynomial has the very nice property that it can be transformed into a differential operator, and this differential operator acting on the scalar Feynman integrand creates the integrand for gauge theory.
$\mathcal{C}^0$ creates bosonic self-interactions. It is therefore not used when one wants to create QED amplitudes, because QED is an abelian gauge theory with no self-interactions of the photon. However, the gluon of QCD does interact with itself. Hence, $\mathcal{C}^0$ will be used in order to create some of the necessary QCD graphs. $\mathcal{C}^1$ will create abelian properties, such as fermionic lines and photon-fermion-antifermion vertices, and also ghosts, depending on the differential operator that is being used. Therefore, $\mathcal{C}^1$ is important for both QED and QCD.[2]

The Corolla Differential for Fermionic Lines

In order to create fermions, the Corolla polynomial will be turned into a Corolla differential by carrying out the following substitution for the half-edge variables h:

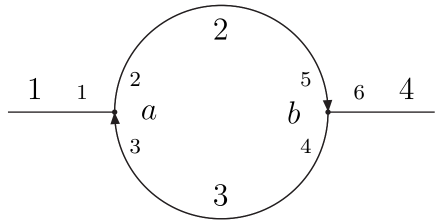
$$h \to D_f(h) = \left(\frac{1}{A_{e(h_+)}}\frac{\partial}{\partial \xi_{e(h_+)_{\mu(h_+)}}}\gamma_{\mu(h_+)}\gamma_{\mu(h)} - \frac{1}{A_{e(h_-)}}\frac{\partial}{\partial \xi_{e(h_-)_{\mu(h_-)}}}\gamma_{\mu(h)}\gamma_{\mu(h_-)}\right) \tag{3.4}$$

In Eq. (3.4), the notations h_+ and h_- refer to the ordering on the oriented graph. In a three-valent vertex v, for any half-edge h adjacent to v, there is one unique half-edge h_- preceding and one unique half-edge h_+ succeeding h, relative to the orientation on the graph and on the vertex. The ξ_i refer to the edge momenta. For example, $\xi_{e(h_+)}$ is the momentum in the edge $e(h_+)$, which is the edge of the half-edge h_+ which follows h. [2]

3.2. The One-Loop Graph

3.2.1. The Corolla Differential

Now that it comes down to an actual computation, we need a decorated graph. We use the same labeling as in the example above, but also denote the half-edges by smaller numbers.

$\Gamma =$ 

In order to transform a scalar graph into a gauge theory graph, one must rewrite the Corolla polynomial using the Corolla differentials, as indicated in Eq. (3.4). In this case, where we regard the photon propagator with one fermion loop, this is quantum electrodynamics, an abelian gauge theory; hence one uses the Corolla monomial of degree one. It is given by

$$\mathcal{C}^1_{-\!\bigcirc\!-} = \prod_{v \in C} a_{v,v_C} = h_1 h_6$$

In order to create fermions, the substitution discussed in Eq. (3.4) for the half-edge variables h must be carried out.

$$\begin{aligned} D_f(h_1) &= \frac{1}{A_2}\frac{\partial}{\partial \xi_{2_{\mu_2}}}\gamma_{\mu_2}\gamma_{\mu_1} - \frac{1}{A_3}\frac{\partial}{\partial \xi_{3_{\mu_3}}}\gamma_{\mu_1}\gamma_{\mu_3} \\ D_f(h_6) &= \frac{1}{A_3}\frac{\partial}{\partial \xi_{3_{\mu_3}}}\gamma_{\mu_3}\gamma_{\mu_4} - \frac{1}{A_2}\frac{\partial}{\partial \xi_{2_{\mu_2}}}\gamma_{\mu_4}\gamma_{\mu_2} \end{aligned}$$

The result for the Corolla differential of degree one is

$$\begin{aligned} \mathcal{C}^1_{-\!\bigcirc\!-}\Big(h \to D_f(h)\Big) &= \left(\frac{1}{A_2}\frac{\partial}{\partial \xi_{2_{\mu_2}}}\gamma_{\mu_2}\gamma_{\mu_1} - \frac{1}{A_3}\frac{\partial}{\partial \xi_{3_{\mu_3}}}\gamma_{\mu_1}\gamma_{\mu_3}\right) \cdot \left(\frac{1}{A_3}\frac{\partial}{\partial \xi_{3_{\mu_3}}}\gamma_{\mu_3}\gamma_{\mu_4} - \frac{1}{A_2}\frac{\partial}{\partial \xi_{2_{\mu_2}}}\gamma_{\mu_4}\gamma_{\mu_2}\right) \\ &= \frac{1}{A_2 A_3}\left(\frac{\partial^2}{\partial \xi_{2_{\mu_2}}\partial \xi_{3_{\mu_3}}}\operatorname{Tr}\gamma_{\mu_2}\gamma_{\mu_1}\gamma_{\mu_3}\gamma_{\mu_4} + \frac{\partial^2}{\partial \xi_{3_{\mu_3}}\partial \xi_{2_{\mu_2}}}\operatorname{Tr}\gamma_{\mu_1}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_2}\right) \\ &\quad + \text{nonlinear terms} \\ &= \frac{1}{A_2 A_3}\operatorname{Tr}\left(\gamma_{\mu_1}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_2}\right)\left(\frac{\partial^2}{\partial \xi_{2_{\mu_2}}\partial \xi_{3_{\mu_3}}} + \frac{\partial^2}{\partial \xi_{3_{\mu_3}}\partial \xi_{2_{\mu_2}}}\right) \qquad (3.5) \\ &= \frac{4\left(g_{\mu_1\mu_3}g_{\mu_4\mu_2} - g_{\mu_1\mu_4}g_{\mu_3\mu_2} + g_{\mu_1\mu_2}g_{\mu_3\mu_4}\right)}{A_2 A_3}\left(\frac{\partial^2}{\partial \xi_{2_{\mu_2}}\partial \xi_{3_{\mu_3}}} + \frac{\partial^2}{\partial \xi_{3_{\mu_3}}\partial \xi_{2_{\mu_2}}}\right) \end{aligned}$$

In the second line of Eq. (3.5), we simply wrote down "nonlinear terms" for all terms with an A_i^2 in the denominator because these terms are neglected and do not have to be given explicitely. In the third line, we have used cyclicity of the trace, and in the forth line the trace identity in four dimensions for the trace of four γ matrices (cf. Eq. (A.2)).

3.2.2. The Scalar Integrand

The scalar integrad of the one-loop propagator in ϕ^3_6 theory is formally given by Eq. (3.1).

The First Symanzik Polynomial The first Symanzik polynomial requires the sum over all trees of the graph. A tree is a set of edges that is simply-connected and touches all vertices of the graph. In the case of , the internal edge e_2 is one tree and the internal edge e_3 is another one. There are no more trees.

$$T \in \Big\{ \frown , \smile \Big\} = \Big\{ \{e_2\}, \{e_3\} \Big\}$$

Therefore, using Eq. (3.2), the first Symanzik polynomial of is given by

$$\psi = A_2 + A_3$$

The Second Symanzik Polynomial We use the definition of the second Symanzik polynomial given by Eq. (3.3), the Pfaffian determinant, defined as follows:

$$Eq.\ (3.3) \qquad \Rightarrow \qquad |N|_{\mathrm{Pf}} = \sum_{T_1 \cup T_2} \left(\sum_{e \notin T_1 \cup T_2} \tau(e) \xi_e \right)^2 \prod_{e \notin T_1 \cup T_2} A_e$$

Recall that $\tau(e)$ is $+1$ if e is oriented from T_1 to T_2, -1 if it is oriented from T_2 to T_1, and 0 otherwise. There is only one two-forest, namely the disjoint set of only the two vertices.

$$T_1 \cup T_2 = \Big\{ \cdot \ \ \cdot \Big\} = \Big\{ \{a\} \cup \{b\} \Big\}$$

We have decided to declare T_1 to be vertex a and T_2 to be vertex b. Thus, e_2 is oriented from T_1 to T_2 and e_3 is oriented from T_2 to T_1. The assignment of T_1 and T_2 as well as the orientation of the edges is arbitrary.

$$|N|_{\mathrm{Pf}} = \Big(\tau(2)\xi_2 + \tau(3)\xi_3 \Big)^2 A_2 A_3 = \left(\xi_2 - \xi_3 \right)^2 A_2 A_3$$

The scalar integrand is given by[2]

$$I = \frac{1}{2} \frac{\exp\left(-\frac{(\xi_2 - \xi_3)^2 A_2 A_3}{A_2 + A_3} \right)}{(A_2 + A_3)^2} \tag{3.6}$$

3.2.3. From the Scalar to the Abelian Gauge Amplitude

Applying the Corolla differential Eq. (3.5) to the scalar integrand Eq. (3.6) will produce the integrand for the gauge amplitude.

$$\begin{aligned}
I^{\mathrm{QED}} &= \frac{\mathrm{Tr}\left(\gamma_{\mu_1}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_2}\right)}{2A_2A_3} \left(\frac{\partial^2}{\partial \xi_{2_{\mu_2}} \partial \xi_{3_{\mu_3}}} + \frac{\partial^2}{\partial \xi_{3_{\mu_3}} \partial \xi_{2_{\mu_2}}} \right) \frac{\exp\left(-\frac{(\xi_2 - \xi_3)^2 A_2 A_3}{A_2 + A_3} \right)}{(A_2 + A_3)^2} \\
&= \frac{\mathrm{Tr}\left(\gamma_{\mu_1}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_2}\right)}{A_2A_3(A_2+A_3)^2} \left(\frac{\partial^2}{\partial \xi_{2_{\mu_2}} \partial \xi_{3_{\mu_3}}} \right) \exp\left(-\frac{(\xi_2 - \xi_3)^2 A_2 A_3}{A_2 + A_3} \right) \\
&= \frac{\mathrm{Tr}\left(\gamma_{\mu_1}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_2}\right)}{A_2A_3(A_2+A_3)^2} \left(\frac{\partial}{\partial \xi_{2_{\mu_2}}} \right) \left(+ 2(\xi_2 - \xi_3)^{\mu_3} \frac{A_2A_3}{A_2+A_3} \exp\left(-\frac{(\xi_2 - \xi_3)^2 A_2 A_3}{A_2 + A_3} \right) \right) \\
&= 2\,\mathrm{Tr}\left(\gamma_{\mu_1}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_2}\right) \left(\frac{g^{\mu_2\mu_3}}{(A_2+A_3)^3} \exp\left(-\frac{q^2 A_2 A_3}{A_2 + A_3} \right) - 2q^{\mu_3}q^{\mu_2} \frac{A_2A_3}{(A_2+A_3)^4} \exp\left(-\frac{q^2 A_2 A_3}{A_2 + A_3} \right) \right)
\end{aligned} \tag{3.7}$$

In the last line, we have replaced $\xi_2 - \xi_3$ by q in order to prevent the term from getting too lengthy. The next step would be to contract the trace with the terms on its right, but first, let us inspect the scaling behaviour of the two terms, considering that integration over $\mathrm{d}A_2\,\mathrm{d}A_3$ is intended.
The scaling behaviour of the integrand will tell us a lot about possible divergencies. Let us rescale the A_i by a factor λ, so we replace $A_i \to \lambda A_i$ and $\mathrm{d}A_i \to \lambda\,\mathrm{d}A_i$ and see what happens:

$$\begin{aligned} \frac{g^{\mu_2\mu_3}}{(A_2+A_3)^3}\exp\left(-\frac{q^2A_2A_3}{A_2+A_3}\right)\mathrm{d}A_1\,\mathrm{d}A_2 &\to \frac{g^{\mu_2\mu_3}}{\lambda^3(A_2+A_3)^3}\exp\left(-\frac{q^2\lambda^2A_2A_3}{\lambda(A_2+A_3)}\right)\lambda^2\,\mathrm{d}A_1\,\mathrm{d}A_2 \\ &\sim \frac{1}{\lambda} \end{aligned} \tag{3.8}$$

$$\begin{aligned} q^{\mu_3}q^{\mu_2}\frac{A_2A_3}{(A_2+A_3)^4}\exp\left(-\frac{q^2A_2A_3}{A_2+A_3}\right) &\to q^{\mu_3}q^{\mu_2}\frac{\lambda^2A_2A_3}{\lambda^4(A_2+A_3)^4}\exp\left(-\frac{q^2\lambda^2A_2A_3}{\lambda(A_2+A_3)}\right)\lambda^2\,\mathrm{d}A_1\,\mathrm{d}A_2 \\ &\sim 1 \end{aligned} \tag{3.9}$$

As we see, the metric tensor term scales like $\frac{1}{\lambda}$, whereas the four-momenta term is scale invariant. The scaling invariance shows that the four-momenta term is logarithmically divergent. The metric tensor term, however, does not scale as nicely. The $\frac{1}{\lambda}$ behaviour points towards quadratic divergence. This could be a problem for the integration, so let us try to rewrite the term.

$$G^{\mu_2\mu_3} := g^{\mu_2\mu_3}\iint\limits_{\mathbb{R}_+^2}\frac{e^{-q^2\frac{A_2A_3}{A_2+A_4}}}{(A_2+A_3)^3}\,\mathrm{d}A_2\,\mathrm{d}A_3 \tag{3.10}$$

We substitute $A_2 = a_2A_3,\ \mathrm{d}A_2 = A_3\,\mathrm{d}a_2$. The integral of the metric tensor term then yields:

$$G^{\mu_2\mu_3} = g^{\mu_2\mu_3}\iint\limits_{\mathbb{R}_+^2}\frac{e^{-q^2\frac{a_2A_3}{1+a_2}}}{A_3^2(1+a_2)^3}\,\mathrm{d}a_2\,\mathrm{d}A_3$$

Next, partially integrate with respect to A_3:

$$G^{\mu_2\mu_3} = g^{\mu_2\mu_3}\left\{-\int_0^\infty \mathrm{d}a_2\frac{e^{-q^2\frac{a_2A_3}{1+a_2}}}{A_3(1+a_2)^3}\Bigg|_{A_3=0}^{\infty} - q^2\iint\limits_{\mathbb{R}_+^2}\frac{a_2}{A_3(1+a_2)^4}e^{-q^2\frac{a_2A_3}{1+a_2}}\,\mathrm{d}a_2\,\mathrm{d}A_3\right\}$$

The first term vanishes due to renormalization conditions, so only the second term remains. Now, we resubstitute.

$$G^{\mu_2\mu_3} = -q^2g^{\mu_2\mu_3}\iint\limits_{\mathbb{R}_+^2}\frac{A_2A_3}{(A_2+A_3)^4}e^{-q^2\frac{A_2A_3}{A_2+A_3}}\,\mathrm{d}A_2\,\mathrm{d}A_3$$

This integrand has a similar structure as the four-momenta term in Eq. (3.7). Put the two terms back together:

$$\Phi^{\mathrm{QED}}_{-\!\bigcirc\!-} = -2\,\mathrm{Tr}\left(\gamma_{\mu_1}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_2}\right)\left(q^2g^{\mu_2\mu_3}+2q^{\mu_2}q^{\mu_3}\right)\iint\limits_{\mathbb{R}_+^2}\frac{A_2A_3}{(A_2+A_3)^4}\exp\left(-q^2\frac{A_2A_3}{A_2+A_3}\right)\mathrm{d}A_2\,\mathrm{d}A_3$$

The next step is to compute the contraction of the trace with the factor $(q^2 g^{\mu_2\mu_3} + 2q^{\mu_2}q^{\mu_3})$. As above, the trace of four γ matrices is given in Eq. (A.2). The result of the contraction is:

$$\Phi^{\text{QED}}_{-\!\bigcirc\!-} = -8(q^2 g_{\mu_1\mu_4} - q_{\mu_1}q_{\mu_4}) \iint\limits_{\mathbb{R}^2_+} \frac{A_2 A_3}{(A_2 + A_3)^4} e^{-q^2 \frac{A_2 A_3}{A+2+A_3}} \, \mathrm{d}A_2 \, \mathrm{d}A_3$$

As expected, we see that the result is transversal.
It is finally time to integrate over the Feynman parameters A_2 and A_3. In order to do that, a substitution is carried out:

$$\begin{aligned} A_2 &\to a_2 t \\ A_3 &\to a_3 t \\ \mathrm{d}A_2 \, \mathrm{d}A_3 &\to t \, \mathrm{d}t \, \mathrm{d}\Omega_2(a_2, a_3) \end{aligned}$$

The integral over $\mathrm{d}\Omega_2(a_2, a_3)$ is over the projective space $\mathbb{P}_2$ and t is integrated from c to ∞. In fact, t should actually be integrated from 0 to ∞, but the integral diverges at the lower boundary, so it is replaced by a small c and the limit $c \to 0$ is to be taken after a regularization has taken place.

$$I^{\text{QED}}_{-\!\bigcirc\!-} = -8(q^2 g_{\mu_1\mu_4} - q_{\mu_1}q_{\mu_4}) \int\limits_{\mathbb{P}_2} \mathrm{d}\Omega_2(a_2, a_3) \frac{a_2 a_3}{(a_2 + a_3)^4} \int\limits_c^\infty \mathrm{d}t \frac{e^{-tq^2 \frac{a_2 a_3}{a_2 + a_3}}}{t}$$

As stated above, the integration over t diverges as the lower limit approaches zero. For non-zero c, this integral is known as the (negative of the) exponential integral, $-\mathrm{Ei}(c)$. Let us look at its expansion for small c:

$$-\int\limits_c^\infty \mathrm{d}t \;\; t^{-1} e^{-tq^2 \frac{a_2 a_3}{a_2 + a_3}} = \gamma_E + \ln(c) + \ln\left(q^2 \frac{a_2 a_3}{a_2 + a_3}\right) + \mathcal{O}(c) \tag{3.11}$$

The first term is the Euler-Mascheroni-constant, γ_E. It is independent of c or the momentum, q^2. The second term, the logarithm of c, is the diverging actor in this expression. However, it is independent of q^2 or any other observable! Therefore, a simple subtraction at some momentum $q^2 = \mu^2$ would get rid of the divergence and render the integral finite. After taking the limit $c \to 0$, only the third term would survive, in the form of $-\ln\left(\frac{q^2}{\mu^2}\right)$.
The regularized integral then is:

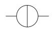
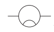
$$I^{\text{QED}}_{-\!\bigcirc\!-\,\text{reg}} = 8(q^2 g_{\mu_1\mu_4} - q_{\mu_1}q_{\mu_4}) \ln\left(\frac{q^2}{\mu^2}\right) \int\limits_{\mathbb{P}_2} \mathrm{d}\Omega_2(a_2, a_3) \frac{a_2 a_3}{(a_2 + a_3)^4}$$

$\mathrm{d}\Omega_2(a_2, a_3)$ can be written as $a_2 \, \mathrm{d}a_3 - a_3 \, \mathrm{d}a_2$. The projective space, over which we integrate, is scaling invariant, so it is possible to set one of the integration variables to a constant and then integrate the other from 0 to ∞. Let us set $a_2 = 1$ and therefore $\mathrm{d}a_2 = 0$ and integrate partially with respect to a_3.

$$
\begin{aligned}
I^{\text{QED}}_{\ \ \text{reg}} &= 8(q^2 g_{\mu_1\mu_4} - q_{\mu_1} q_{\mu_4}) \ln\left(\frac{q^2}{\mu^2}\right) \int_0^\infty \mathrm{d}a_3 \frac{a_3}{(1+a_3)^4} \\
&= 8(q^2 g_{\mu_1\mu_4} - q_{\mu_1} q_{\mu_4}) \ln\left(\frac{q^2}{\mu^2}\right) \Bigg\{ \underbrace{-\frac{1}{3}\frac{a_3}{(1+a_3)^3}\Bigg|_0^\infty}_{=0} + \underbrace{\frac{1}{3}\int_0^\infty \frac{\mathrm{d}a_3}{(1+a_3)^3}}_{=\frac{1}{6}} \Bigg\} \\
&= \underline{\underline{\frac{4}{3}(q^2 g_{\mu_1\mu_4} - q_{\mu_1} q_{\mu_4}) \ln\left(\frac{q^2}{\mu^2}\right)}}
\end{aligned}
\tag{3.12}
$$

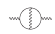
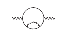
The factor $\frac{4}{3}$ is the coefficient of the one-loop β function.

3.3. The Two-Loop Graphs: From Graph Theory to the Integrand

Before we get into the details of the graphs  and , we will work out graph theoretical properties in order to investigate the integrand and the associated amplitude.

The Scalar Integrand The Feynman integrand I_Γ of a scalar graph Γ in D dimensions is given through the first and second Symanzik polynomials. In contrast to our notation above, we will also include the measure of integration as part of the integrand.

$$
I_\Gamma = \frac{\exp\left(-\frac{\phi_\Gamma}{\psi_\Gamma}\right)}{{\psi_\Gamma}^{\frac{D}{2}}} \bigwedge_i \mathrm{d}A_i \tag{3.13}
$$

The QED Integrand Now comes the interesting part! We will make the transformation from the scalar integrand to the abelian gauge integrand by applying derivatives on I_Γ with repect to the momenta ξ_i of the internal edges e_i. The specific form of the derivatives is given by the corolla polynomial.
The integrand of quantum electrodynamics for the graphs  and  (with the respective ψ_Γ and ϕ_Γ) is obtained by

$$
\begin{aligned}
I^{\text{QED}}_\Gamma &= \frac{1}{A_1}\frac{\partial}{\partial {\xi_1}_{\mu_1}} \frac{1}{A_2}\frac{\partial}{\partial {\xi_2}_{\mu_2}} \frac{1}{A_3}\frac{\partial}{\partial {\xi_3}_{\mu_3}} \frac{1}{A_4}\frac{\partial}{\partial {\xi_4}_{\mu_4}} \, I_\Gamma \underbrace{\operatorname{Tr}\left(\gamma_{\mu_a}\gamma_{\mu_2}\gamma_{\mu_d}\gamma_{\mu_3}\gamma_{\mu_c}\gamma_{\mu_4}\gamma_{\mu_b}\gamma_{\mu_1}\right)}_{=:\operatorname{Tr}_\Gamma} \bigwedge_i \mathrm{d}A_i \\
&= \frac{1}{A_1A_2A_3A_4} \frac{\partial^4}{\partial {\xi_1}_{\mu_1} \partial {\xi_2}_{\mu_2} \partial {\xi_3}_{\mu_3} \partial {\xi_4}_{\mu_4}} \, I_\Gamma \operatorname{Tr}_\Gamma \bigwedge_i \mathrm{d}A_i \\
&=: \frac{\partial_{1234} I_\Gamma}{A_1A_2A_3A_4} \operatorname{Tr}_\Gamma \bigwedge_i \mathrm{d}A_i
\end{aligned}
\tag{3.14}
$$

The trace will be dealt with separately. For now, let us investigate the four-fold derivative. In slight abuse of notation, call Eq. (3.14) with the trace left-out I^{QED}_Γ as well.

There are three types of expressions emerging from it. Introduce the following notation (the first of which has already been used in Eq. (3.14)):

$$\begin{aligned}\partial_{1234} &:= \frac{\partial^4}{\partial\xi_{1_{\mu_1}}\partial\xi_{2_{\mu_2}}\partial\xi_{3_{\mu_3}}\partial\xi_{4_{\mu_4}}} \\ \phi_i^\Gamma &:= \frac{\partial\phi_\Gamma}{\partial\xi_{i_{\mu_i}}} \\ \phi_{ij}^\Gamma &:= \frac{\partial\phi_i^\Gamma}{\partial\xi_{j_{\mu_j}}}\end{aligned}$$

Use this in order to compute the derivative needed for Eq. (3.14):

$$\begin{aligned}\partial_4 I_\Gamma &= -\frac{\phi_4^\Gamma}{\psi_\Gamma^3}\exp\left(-\frac{\phi_\Gamma}{\psi_\Gamma}\right)\mathrm{d}A_1\wedge\cdots\wedge\mathrm{d}A_5 \\ \partial_{34} I_\Gamma &= \left\{-\frac{\phi_{34}^\Gamma}{\psi_\Gamma^3}+\frac{\phi_3^\Gamma\phi_4^\Gamma}{\psi_\Gamma^4}\right\}\exp\left(-\frac{\phi_\Gamma}{\psi_\Gamma}\right)\mathrm{d}A_1\wedge\cdots\wedge\mathrm{d}A_5 \\ \partial_{234} I_\Gamma &= \left\{\frac{\phi_2^\Gamma\phi_{34}^\Gamma\phi_2^\Gamma\phi_{23}^\Gamma+\phi_3^\Gamma\phi_{24}^\Gamma}{\psi_\Gamma^4}-\frac{\phi_2^\Gamma\phi_3^\Gamma\phi_4^\Gamma}{\psi_\Gamma^5}\right\}\exp\left(-\frac{\phi_\Gamma}{\psi_\Gamma}\right)\mathrm{d}A_1\wedge\cdots\wedge\mathrm{d}A_5 \\ \partial_{1234} I_\Gamma &= \left\{\frac{\phi_{12}^\Gamma\phi_{34}^\Gamma+\phi_{23}^\Gamma\phi_{14}^\Gamma+\phi_{13}^\Gamma\phi_{24}^\Gamma}{\psi_\Gamma^4}-\right. \\ &\quad -\frac{\phi_1^\Gamma\phi_2^\Gamma\phi_{34}^\Gamma+\phi_1^\Gamma\phi_3^\Gamma\phi_{24}^\Gamma+\phi_1^\Gamma\phi_4^\Gamma\phi_{23}^\Gamma+\phi_2^\Gamma\phi_3^\Gamma\phi_{14}^\Gamma+\phi_2^\Gamma\phi_4^\Gamma\phi_{13}^\Gamma+\phi_3^\Gamma\phi_4^\Gamma\phi_{12}^\Gamma}{\psi_\Gamma^5}+ \\ &\quad \left.+\frac{\phi_1^\Gamma\phi_2^\Gamma\phi_3^\Gamma\phi_4^\Gamma}{\psi_\Gamma^6}\right\}\exp\left(-\frac{\phi_\Gamma}{\psi_\Gamma}\right)\mathrm{d}A_1\wedge\cdots\wedge\mathrm{d}A_5 \qquad (3.15) \\ &=: \tilde{I}_{\Gamma 1}+\tilde{I}_{\Gamma 2}+\tilde{I}_{\Gamma 3}\end{aligned}$$

The three terms have different degrees of divergence. Firstly, the degrees of divergence will be determined. This is independent of the choice which one of the two graphs is considered. Secondly, the ϕ_i^Γ and ϕ_{ij}^Γ will be computed explicitely in order to see which ones vanish. Of course, the results are different for the two graphs and will we dealt with in the repective sections.

Degrees of Divergence

In order to determine the degrees of divergence in Eq. (3.14), we investigate the scaling behaviour of the three different terms found in Eq. (3.15). We will substutite $A_1 = t$ and $A_i = t\cdot a_i$ for the A_i for which $i \neq 1$. The degree of divergence will be visible in the power of t. Keep in mind that ψ_Γ is a second-degree polynomial in the A_i and ϕ_Γ is a third-degree polynomial.

$$\begin{aligned}\psi_\Gamma &\to t^2\left(a_2a_4+a_3+a_2a_3+a_4+a_3a_5+a_2a_5+a_5+a_4a_5\right) =: t^2\cdot\bar{\psi}_\Gamma \\ \phi_\Gamma &\to t^3\left(q_1^2a_2a_4a_5+q_2^2a_3a_5\right) =: t^3\cdot\bar{\phi}_\Gamma \\ \mathrm{d}A_1\wedge\cdots\wedge\mathrm{d}A_5 &\to t^4\,\mathrm{d}t\wedge\mathrm{d}a_2\wedge\cdots\wedge\mathrm{d}a_5 \qquad (3.16)\end{aligned}$$

The last line is not obvious at first. Keep in mind that the wedge product is antisymmetric, so $\mathrm{d}t\wedge\mathrm{d}t = 0$. Therefore, when the measures of integration for the A_i, $i\neq 1$, are substituted to give $t\,\mathrm{d}a_i + a_i\,\mathrm{d}t$, all $\mathrm{d}t$s can be dropped since every term is multiplied with $\mathrm{d}A_1 = \mathrm{d}t$. The only surviving terms come from the four $t\,\mathrm{d}a_i$, $i\in\{2,3,4,5\}$, hence the factor t^4.

The Quadratically Divergent Part Using the substitutions from above, the first part of Eq. (3.15) becomes (neglecting the trace as we agreed to do)

$$
\begin{aligned}
I_{\Gamma 1} &:= \frac{\tilde{I}_{\Gamma 1}}{A_1A_2A_3A_4} \to \frac{t^6\left(\bar{\phi}^{\Gamma}_{12}\bar{\phi}^{\Gamma}_{34}+\bar{\phi}^{\Gamma}_{23}\bar{\phi}^{\Gamma}_{14}+\bar{\phi}^{\Gamma}_{13}\bar{\phi}^{\Gamma}_{24}\right)}{t^4a_1a_2a_3a_4\cdot t^8\left(\bar{\psi}^4_\Gamma\right)}\exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)t^4\,\mathrm{d}t\wedge\mathrm{d}a_2\wedge\cdots\wedge\mathrm{d}a_5\\
&= \frac{1}{t^2}\cdot\frac{\bar{\phi}^{\Gamma}_{12}\bar{\phi}^{\Gamma}_{34}+\bar{\phi}^{\Gamma}_{23}\bar{\phi}^{\Gamma}_{14}+\bar{\phi}^{\Gamma}_{13}\bar{\phi}^{\Gamma}_{24}}{a_1a_2a_3a_4\bar{\psi}^4_\Gamma}\exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)\mathrm{d}t\wedge\mathrm{d}a_2\wedge\cdots\wedge\mathrm{d}a_5
\end{aligned}
$$

The power -2 of the t determines that I_{Γ_1} is quadratically divergent. Still, only the t^{-2} and the exponential function depend on the integration variable t. The integral from 0 to ∞ diverges at the lower boundary. However, by replacing the lower boundary by some small number c and writing down the limit $c\to\infty$, we can easily perform partial integration with respect to t:

$$
\lim_{c\to 0}\int_c^\infty \frac{\exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)}{t^2}\,\mathrm{d}t = \text{boundary terms} - \frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\lim_{c\to 0}\int_c^\infty \frac{\exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)}{t}\,\mathrm{d}t
$$

The boundary terms vanish due to renormalization conditions. It is true that the integral $\lim_{c\to 0}\int_c^\infty \frac{\exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)}{t}\,\mathrm{d}t$ still diverges, but only logarithmically. It has an expansion for small c (cf. Eq. (3.11)):

$$
\lim_{c\to 0}\int_c^\infty \frac{\exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)}{t}\,\mathrm{d}t = -\gamma_E - \ln(c) - \ln\left(\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right) + \mathcal{O}(c) \tag{3.17}
$$

The pole in c does not depend on any measurable quantity. Therefore, a single subtration at $q^2=\mu^2$ renders Eq. (3.17) finite, if q is the external momentum of the graph and μ is some reference momentum. It can also be shown that

$$
\ln\left(\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right) - \ln\left(\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)\Bigg|_{q^2=\mu^2} = \ln\left(\frac{q^2}{\mu^2}\right) + \text{const.}
$$

and thus

$$
\begin{aligned}
\lim_{c\to 0}\left(\int_c^\infty \frac{\exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)}{t}\,\mathrm{d}t - \int_c^\infty \frac{\exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)}{t}\,\mathrm{d}t\Bigg|_{q^2=\mu^2}\right) &= -\ln\left(\frac{q^2}{\mu^2}\right) \qquad (3.18)\\
\Rightarrow \qquad \lim_{c\to 0}\int_c^\infty \frac{\exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)}{t^2}\,\mathrm{d}t &= \frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\ln\left(\frac{q^2}{\mu^2}\right)
\end{aligned}
$$

To summarize, the quadratically divergent I_{Γ_1} can be transformed into a vanishing boundary term and a logarithmically divergent part which can be rendered finite using a single subtraction at a reference momentum $q^2=\mu^2$. The remaining integrand is

$$
I^*_{\Gamma 1} = \ln\left(\frac{q^2}{\mu^2}\right)\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\frac{\bar{\phi}^{\Gamma}_{12}\bar{\phi}^{\Gamma}_{34}+\bar{\phi}^{\Gamma}_{23}\bar{\phi}^{\Gamma}_{14}+\bar{\phi}^{\Gamma}_{13}\bar{\phi}^{\Gamma}_{24}}{a_1a_2a_3a_4\bar{\psi}^4_\Gamma}\,\mathrm{d}a_2\wedge\cdots\wedge\mathrm{d}a_5
$$

The asterisk indicates that the subtraction at $q^2=\mu^2$ has been carried out.

The Logarithmically Divergent Part Carrying out the substitution indicated above in Eq. (3.16) results in

$$\begin{aligned} I_{\Gamma 2} := \frac{\tilde{I}_{\Gamma 2}}{A_1 A_2 A_3 A_4} &\to -\frac{t^9 \left(\bar{\phi}_1^\Gamma \bar{\phi}_2^\Gamma \bar{\phi}_{34}^\Gamma + \cdots + \bar{\phi}_3^\Gamma \bar{\phi}_4^\Gamma \bar{\phi}_{12}^\Gamma\right)}{t^4 a_1 a_2 a_3 a_4 \cdot t^{10} \left(\bar{\psi}_\Gamma^5\right)} \exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right) t^4 \, \mathrm{d}t \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5 \\ &= -\frac{1}{t} \cdot \frac{\bar{\phi}_1^\Gamma \bar{\phi}_2^\Gamma \bar{\phi}_{34}^\Gamma + \cdots + \bar{\phi}_3^\Gamma \bar{\phi}_4^\Gamma \bar{\phi}_{12}^\Gamma}{a_1 a_2 a_3 a_4 \bar{\psi}_\Gamma^5} \exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right) \mathrm{d}t \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5 \\ &= -\frac{\exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right)}{t} \mathrm{d}t \wedge \frac{\bar{\phi}_1^\Gamma \bar{\phi}_2^\Gamma \bar{\phi}_{34}^\Gamma + \cdots + \bar{\phi}_3^\Gamma \bar{\phi}_4^\Gamma \bar{\phi}_{12}^\Gamma}{a_1 a_2 a_3 a_4 \bar{\psi}_\Gamma^5} \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5 \end{aligned}$$

As shown in Eq. (3.18) in the previous paragraph, this logarithmically divergent t-integration yields a logarithm once a single subtration at a reference momentum $q^2 = \mu^2$ is taken.

$$\Rightarrow \qquad I_{\Gamma 2}^* = \ln\left(\frac{q^2}{\mu^2}\right) \frac{\bar{\phi}_1^\Gamma \bar{\phi}_2^\Gamma \bar{\phi}_{34}^\Gamma + \cdots + \bar{\phi}_3^\Gamma \bar{\phi}_4^\Gamma \bar{\phi}_{12}^\Gamma}{a_1 a_2 a_3 a_4 \bar{\psi}_\Gamma^5} \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5$$

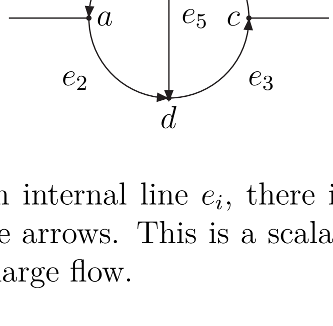
The Convergent Part The last piece of Eq. (3.14) substitutes to

$$\begin{aligned} I_{\Gamma 3} := \frac{\tilde{I}_{\Gamma 3}}{A_1 A_2 A_3 A_4} &\to \frac{t^{12} \bar{\phi}_1^\Gamma \bar{\phi}_2^\Gamma \bar{\phi}_3^\Gamma \bar{\phi}_4^\Gamma}{t^4 a_1 a_2 a_3 a_4 \cdot t^{12} \left(\bar{\psi}_\Gamma^6\right)} \exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right) t^4 \, \mathrm{d}t \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5 \\ &= \frac{\bar{\phi}_1^\Gamma \bar{\phi}_2^\Gamma \bar{\phi}_3^\Gamma \bar{\phi}_4^\Gamma}{a_1 a_2 a_3 a_4 \bar{\psi}_\Gamma^6} \exp\left(-t\frac{\bar{\phi}_\Gamma}{\bar{\psi}_\Gamma}\right) \mathrm{d}t \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5 \end{aligned}$$

There is no power of t here, the t-integration only affects the exponential term. The term is convergent! Therefore, it does not depend on external momenta or other observables, and will cancel once a substitution is used for regularization. $I_{\Gamma 3}$ can be omitted!

3.4. The Two-Loop Graph with Vertex Subgraph

As we have seen in Section 3.3, we definitely need the first and second Symanzik polynomials. Let us write them down. In order to do that, use the following decoration of lines:



(3.19)

Remember that through each internal line e_i, there is a momentum ξ_i running in the direction indicated by the arrows. This is a scalar graph (without charges), so the arrows do not indicate charge flow.

3.4.1. The Spanning Trees and Spanning Two-Forests

As defined above in Section 3.1.2, a spanning tree T of a graph Γ is a set of vertices and edges from Γ such that $V^{\Gamma} = V^{T}$ and T is simply-connected (without loops).
has eight spanning trees.

$$T \in \left\{ , , , , , , , \right\}$$

Since all vertices in V are in V^T, we simplify notation by only giving the edges in the spanning tree, even though spanning trees are technically sets of vertices and edges.

$$T \in \Big\{\{e_1, e_3, e_5\}, \{e_2, e_4, e_5\}, \{e_1, e_4, e_5\}, \{e_2, e_3, e_5\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}, \{e_2, e_3, e_4\}, \{e_1, e_2, e_3\}\Big\}$$

Similar to a spanning tree, which is a simply-connected set of edges on a graph which touches all vertices, a spanning two-forest $T_1 \cup T_2$ of a graph Γ is a disjoint set of two simply-connected sets T_1 and T_2 of edges of Γ such that the union $T_1 \cup T_2$ touches all vertices. Again, the vertices are technically also part of the spanning two-forest, but are not explicitely stated here since they are obviously given by the edges in T_1 and T_2.
has two spanning two-forests.

$$\begin{aligned} T_1 \cup T_2 &\in \left\{ \ , \ \right\} \\ &= \Big\{\{e_1\} \cup \{e_3\}, \{e_2\} \cup \{e_4\}\Big\} \end{aligned}$$

3.4.2. The First and Second Symanzik Polynomial

To obtain the first Symanzik polynomial ψ_Γ of a graph Γ, equip every internal edge e_i with an edge variable A_i. ψ_Γ is given by the sum over all spanning trees T of Γ where all edge variables A_i which are not in T are multiplied.

$$\begin{aligned} \psi_\Gamma &= \sum_T \prod_{e \notin T} A_e \\ \Rightarrow \psi &= A_2A_4 + A_1A_3 + A_2A_3 + A_1A_4 + A_3A_5 + A_2A_5 + A_1A_5 + A_4A_5 \end{aligned}$$

The second Symanzik polynomial ϕ_Γ of a graph Γ has various definitions (cf. Section 3.1.2). Here, we will be using the definition as a Pfaffian determinant. It is a sum over all spanning two-forests, where the edge momenta ξ_i (with their orientations on the graph) which are not in the two-forest are added, squared and multiplied with the edge variables A_i of the edges not in the two-forest.

$$\phi_\Gamma = |N|_{\mathrm{Pf}}(\Gamma) = \sum_{T_1 \cup T_2} \left(\sum_{e \notin T_1 \cup T_2} \tau(e)\xi_e \right)^2 \prod_{e \notin T_1 \cup T_2} A_e$$

$$\text{with} \qquad \tau(e) = \begin{cases} +1 & \text{if } e \text{ is oriented from } T_1 \to T_2 \\ -1 & \text{if } e \text{ is oriented from } T_2 \to T_1 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} \Rightarrow \phi &= (\xi_2 - \xi_4 + \xi_5)^2 A_2A_4A_5 + (-\xi_1 + \xi_3 - \xi_5)^2 A_1A_3A_5 \\ &=: q_1^2 A_2A_4A_5 + q_2^2 A_1A_3A_5 \end{aligned}$$

We have renamed the linear combinations of internal momenta in order to avoid unnecessarily long expressions.

3.4.3. The Derivatives

In Section 3.3, we have introduced ϕ_i^Γ and ϕ_{ij}^Γ as first and second derivatives of the second Symanzik polynomial ϕ_Γ with repect to internal momenta $\xi_{i_{\mu_i}}$ and $\xi_{j_{\mu_j}}$, respectively. It is time to compute these derivatives explicitely.

$$\begin{aligned}
&\phi_{\text{-⦶-}} &&= q_1^2 A_2 A_4 A_5 + q_2^2 A_1 A_3 A_5 \\
\Rightarrow\quad &\phi_1^{\text{-⦶-}} &&= -2{q_2}^{\mu_1} A_1 A_3 A_5 \\
&\phi_2^{\text{-⦶-}} &&= 2{q_1}^{\mu_2} A_2 A_4 A_5 \\
&\phi_3^{\text{-⦶-}} &&= 2{q_2}^{\mu_3} A_1 A_3 A_5 \\
&\phi_4^{\text{-⦶-}} &&= -2{q_1}^{\mu_4} A_2 A_4 A_5 \\
\Rightarrow\quad &\phi_{13}^{\text{-⦶-}} &&= -2g^{\mu_1\mu_3} A_1 A_3 A_5 \\
&\phi_{24}^{\text{-⦶-}} &&= -2g^{\mu_2\mu_4} A_2 A_4 A_5 \\
&\phi_{12}^{\text{-⦶-}} &&= \phi_{14}^{\text{-⦶-}} = \phi_{23}^{\text{-⦶-}} = \phi_{34}^{\text{-⦶-}} = 0
\end{aligned}$$

In the last line, we see that four of the double derivatives vanish! That will shorten the terms in Eq. (3.15) immensely. In fact, the quadratically and logarithmically divergent parts of the integrand are now

$$\begin{aligned}
I_{\text{-⦶-}\,1} &= 4g^{\mu_1\mu_3} g^{\mu_2\mu_4} \frac{\exp\left(-\frac{\phi_{\text{-⦶-}}}{\psi_{\text{-⦶-}}}\right)}{\psi_{\text{-⦶-}}^4} A_5^2 \,\mathrm{d}A_1 \wedge \cdots \wedge \mathrm{d}A_5 \\
I_{\text{-⦶-}\,1}^{\,*} &= 4\ln\left(\frac{q^2}{\mu^2}\right) g^{\mu_1\mu_3} g^{\mu_2\mu_4} a_5^2 \frac{\bar\phi_{\text{-⦶-}}}{\bar\psi_{\text{-⦶-}}^5} \,\mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
I_{\text{-⦶-}\,2} &= -8 \frac{q_2^{\mu_1} q_2^{\mu_3} g^{\mu_2\mu_4} A_1 A_3 A_5^3 + q_1^{\mu_2} q_2^{\mu_4} g^{\mu_1\mu_3} A_2 A_4 A_5^3}{\psi_{\text{-⦶-}}} \frac{\exp\left(-\frac{\phi_{\text{-⦶-}}}{\psi_{\text{-⦶-}}}\right)}{\psi_{\text{-⦶-}}^5} \,\mathrm{d}A_1 \wedge \cdots \wedge \mathrm{d}A_5 \\
I_{\text{-⦶-}\,2}^{\,*} &= 8\ln\left(\frac{q^2}{\mu^2}\right) \frac{q_2^{\mu_1} q_2^{\mu_3} g^{\mu_2\mu_4} a_3 + q_1^{\mu_2} q_2^{\mu_4} g^{\mu_1\mu_3} a_2 a_4}{\bar\psi_{\text{-⦶-}}^5} \,\mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5
\end{aligned} \tag{3.21}$$

3.4.4. Re-Homogenization of the Polynomials

In Eq. (3.20) and Eq. (3.21), there are polynomials in the new edge variables a_i (remember that we substituted $A_1 = t$ and $A_i = t \cdot a_i \ \forall i \neq 1$), but the monomials are not all of the same degree anymore, since the t integration (the A_1 integration) has already been performed. Particularly, the Symanzik polynomials in the new edge variables, $\bar\psi_{\text{-⦶-}}$ and $\bar\phi_{\text{-⦶-}}$, are not homogeneous anymore. This is unfortunate, since some nice properties can be seen in the first Symanzik polynomial if it is written down in a homogeneous form. In order to "re-homogenize" the expressions Eq. (3.20) and Eq. (3.21), we re-substitute:

$$\begin{aligned}
a_i &\to \frac{A_i}{A_1} \qquad \forall i \in \{2,3,4,5\} \\
\mathrm{d}a_i &\to \frac{\mathrm{d}A_i}{A_1} - \frac{A_i\,\mathrm{d}A_1}{A_1^2} \\
&= \frac{1}{A_1^2}\Big(A_1\,\mathrm{d}A_i - A_i\,\mathrm{d}A_1\Big)
\end{aligned} \tag{3.22}$$

Then the substituted first and seconds Symanzik polynomials are resubstituted:

$$\begin{aligned}
\bar{\psi}_{\text{-⦶-}} &= a_3 + a_4 + a_5 + a_2 a_3 + a_2 a_4 + a_2 a_5 + a_3 a_5 + a_4 a_5 \\
&\to \frac{A_3 + A_4 + A_5}{A_1} + \frac{A_2 A_3 + A_2 A_4 + A_2 A_5 + A_3 A_5 + A_4 A_5}{A_1^2} \\
&= \frac{1}{A_1^2}\Big(A_1 A_3 + A_1 A_4 + A_1 A_5 + A_2 A_3 + A_2 A_4 + A_2 A_5 + A_3 A_5 + A_4 A_5\Big) \\
&= \frac{\psi_{\text{-⦶-}}}{A_1^2} \qquad (3.23) \\
\bar{\phi}_{\text{-⦶-}} &= -q_1^2 a_2 a_4 a_5 + q_2^2 a_3 a_5 \\
&\to -q_1^2 \frac{A_2 A_4 A_5}{A_1^3} + q_2^2 \frac{A_3 A_5}{A_1^2} \\
&= \frac{1}{A_1^3}\Big(q_1^2 A_2 A_4 A_5 + q_2^2 A_1 A_3 A_5\Big) \\
&= \frac{\phi_{\text{-⦶-}}}{A_1^3} \qquad (3.24)
\end{aligned}$$

Of course, the $\psi_{\text{-⦶-}}$ and $\phi_{\text{-⦶-}}$ in these expressions are the original first and second Symanzik polynomials. This is obvious because the resubstitution (where $\psi_{\text{-⦶-}}$ was replaced with $t^2\bar{\psi}_{\text{-⦶-}} \equiv A_1^2\bar{\psi}_{\text{-⦶-}}$ and $\phi_{\text{-⦶-}}$ with $t^3\phi_{\text{-⦶-}} = A_1^3\phi_{\text{-⦶-}}$) has just been reversed by setting $a_i \to A_i A_1^{-1}$. Still, it is important for the resubstitution of the entire expressions Eq. (3.20) and Eq. (3.21).
The measure of integration also needs to be resubstituted as indicated above. We find that

$$\mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5 = \frac{1}{A_1^8}\Big(A_1\,\mathrm{d}A_2 - A_2\,\mathrm{d}A_1\Big) \wedge \cdots \wedge \Big(A_1\,\mathrm{d}A_5 - A_5\,\mathrm{d}A_1\Big) \qquad (3.25)$$

Since the wedge product is antisymmetric, $\mathrm{d}x \wedge \mathrm{d}y = -\,\mathrm{d}y \wedge \mathrm{d}x$ and $\mathrm{d}x \wedge \mathrm{d}x = 0$, the wedges can be multiplied out and the vanishing terms can be omitted. The result is

$$\begin{aligned}
&\Big(A_1\,\mathrm{d}A_2 - A_2\,\mathrm{d}A_1\Big) \wedge \cdots \wedge \Big(A_1\,\mathrm{d}A_5 - A_5\,\mathrm{d}A_1\Big) = \\
&\quad = A_1^3\Big(A_1\,\mathrm{d}A_2\,\mathrm{d}A_3\,\mathrm{d}A_4\,\mathrm{d}A_5 - A_2\,\mathrm{d}A_1\,\mathrm{d}A_3\,\mathrm{d}A_4\,\mathrm{d}A_5 + A_3\,\mathrm{d}A_1\,\mathrm{d}A_2\,\mathrm{d}A_4\,\mathrm{d}A_5 - \\
&\qquad\quad A_4\,\mathrm{d}A_1\,\mathrm{d}A_2\,\mathrm{d}A_3\,\mathrm{d}A_5 + A_5\,\mathrm{d}A_1\,\mathrm{d}A_2\,\mathrm{d}A_3\,\mathrm{d}A_4\Big) \\
&\quad =: A_1^3\,\mathrm{d}\Omega_5 \qquad (3.26)
\end{aligned}$$

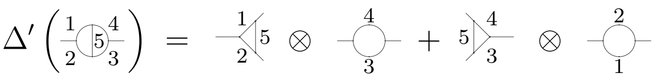
Using Eq. (3.22), Eq. (3.23), Eq. (3.24), and Eq. (3.26), the quadratically and logarithmically divergent parts of the Feynman integrand for the graph $\Gamma = \text{-⦶-}$ are

$$I_{\text{-⦶-}}{}^{*}_{1} = 4\ln\left(\frac{q^2}{\mu^2}\right) g^{\mu_1\mu_3} g^{\mu_2\mu_4} \frac{\phi_{\text{-⦶-}}}{\psi_{\text{-⦶-}}^5} A_5^2\,\mathrm{d}\Omega_5 \qquad (3.27)$$

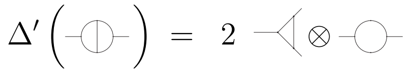
$$I_{\text{-⦶-}}{}^{*}_{2} = 8\ln\left(\frac{\phi_{\text{-⦶-}}}{\psi_{\text{-⦶-}}}\right) \frac{q_2^{\mu_1} q_2^{\mu_3} g^{\mu_2\mu_4} A_1 A_3 + q_1^{\mu_2} q_1^{\mu_4} g^{\mu_1\mu_3} A_2 A_4}{\psi_{\text{-⦶-}}^5} A_5^3\,\mathrm{d}\Omega_5 \qquad (3.28)$$

3.4.5. The Subgraph Structure

It is worthwile to examine the subgraph structure of the graph $\Gamma =$. As we know, its reduced coproduct is not zero, but shows the sub- and co-graph structure.


(3.29)

Since the lines are decorated, the two terms are actually different. Of course, for undecorated graphs, we would get



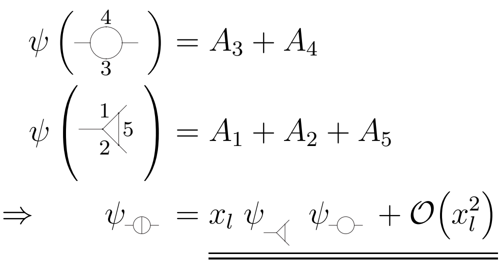
If the first Symanzik polynomial is correct, then the sub- and co-graph structure from Eq. (3.29) should be visible in ψ_Γ. More precisely, if the one-loop subgraph in the left vertex should shrink to zero, that is if the edges e_1, e_2, and e_5 have zero length, then the first Symanzik polynomial should factorize into the two Symanzik polynomials of the sub- and cograph and some other term of a higher order in the shrinking quantity. To show this, substitute

$$A_i \to x_l a_i \quad \forall\, i \in \{1, 2, 5\}$$

in the first Symanzik polynomial:

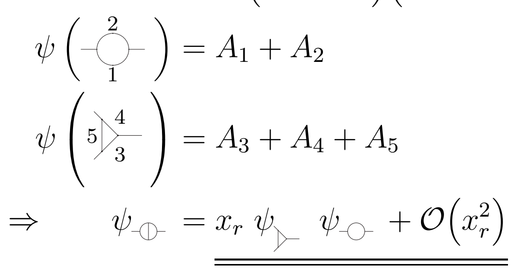
$$\begin{aligned} \psi_\Gamma &\to x_l\Big(a_1A_3 + a_1A_4 + a_2A_3 + a_2A_4 + A_3a_5 + A_4a_5\Big) + x_l^2\Big(a_1a_5 + a_2a_5\Big) \\ &= x_l\Big(a_1 + a_2 + a_3\Big)\Big(A_3 + A_4\Big) + x_l^2(a_1 + a_2)a_5 \end{aligned}$$

This is equal to the product of the Symanzik polynomials of the sub- and co-graph plus a term of higher order in x_l, because



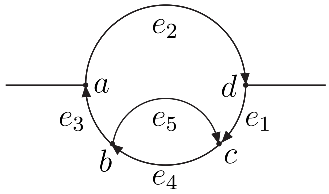
The same is true for the subdivergence in the right vertex:

$$\begin{aligned} & A_i \to x_r a_i \quad \forall\, i \in \{3, 4, 5\} \\ \Rightarrow \qquad & \psi_\Gamma \to x_r\Big(A_1a_3 + A_1a_4 + A_1a_5 + A_2a_3 + A_2a_4 + A_2a_5\Big) + x_r^2\Big(a_3a_5 + a_4a_5\Big) \\ & \qquad = x_r\Big(A_1 + A_2\Big)\Big(a_3 + a_4 + a_5\Big) + x_r^2\Big(a_3 + s_4\Big)a_5 \end{aligned}$$



3.5. The Two-Loop Graph with Fermionic Subgraph

In the same way as in Section 3.4, we are going to compute the Feynman integrand for the gauge theory graph corresponding to the scalar graph . We use the following labeling of lines:



3.5.1. The Spanning Trees and Spanning Two-Forests

The spanning trees of this graph are:

$$
\begin{aligned}
T \in & \Big\{ , , , , , , , \Big\} \\
= & \Big\{ \{e_1, e_3, e_4\}, \{e_1, e_3, e_5\}, \{e_2, e_3, e_4\}, \{e_2, e_3, e_5\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_5\} \Big\}
\end{aligned}
$$

The spanning two-forests are:

$$
\begin{aligned}
T_1 \cup T_2 \in & \Big\{ , , , \Big\} \\
= & \Big\{ \{e_1\} \cup \{e_3\}, \{e_2\} \cup \{e_4\}, \{e_2\} \cup \{e_5\} \Big\}
\end{aligned}
$$

3.5.2. The First and Second Symanzik Polynomial

Consequently, the first and second Symanzik polynomials are given by

$$
\begin{aligned}
\psi &= \sum_T \prod_{e \notin T} A_e \\
&= A_2 A_5 + A_2 A_4 + A_1 A_5 + A_1 A_4 + A_4 A_5 + A_3 A_5 + A_3 A_4 \qquad (3.30) \\
\phi &= |N|_{Pf} \left(\right) = \sum_{T_1 \cup T_2} \left(\sum_{e \notin T_1 \cup T_2} \tau(e) \xi_e \right)^2 \prod_{e \notin T_1 \cup T_2} A_e \\
&= (-\xi_2 + \xi_4 - \xi_5)^2 A_2 A_4 A_5 + (-\xi_1 + \xi_3)^2 A_1 A_3 (A_4 + A_5) \\
&=: q_1^2 A_2 A_4 A_5 + q_2^2 A_1 A_3 (A_4 + A_5) \qquad (3.31)
\end{aligned}
$$

Again, the linear combination of internal momenta ξ_i has been renamed in order to keep the expressions short.

3.5.3. The Derivatives

Compute the first and second derivatives of ϕ as defined in Section 3.3. Since only four out of the five internal edges will become fermionic edges, we will only need four derivatives. In our case, the edge e_5 stays bosonic. This choice is arbitrary, we might as well have chosen e_4.

$$
\begin{aligned}
\phi_1 &= -2q_2^{\mu_1} A_1 A_3 (A_4 + A_5) \\
\phi_2 &= -2q_1^{\mu_2} A_2 A_4 A_5 \\
\phi_3 &= 2q_2^{\mu_3} A_1 A_3 (A_4 + A_5) \\
\phi_4 &= 2q_1^{\mu_4} A_2 A_4 A_5 \\
\phi_{13} &= -2g^{\mu_1\mu_3} A_1 A_3 (A_4 + A_5) \\
\phi_{24} &= -2g^{\mu_2\mu_4} A_2 A_4 A_5 \\
\phi_{12} &= \phi_{14} = \phi_{23} = \phi_{34} = 0
\end{aligned}
$$

In the same way as for , four out of the six double derivatives vanish. This leaves Eq. (3.15) as short as

$$
\begin{aligned}
I_{1} &= 4g^{\mu_1\mu_3} g^{\mu_2\mu_4} \frac{\exp\left(-\frac{\phi}{\psi}\right)}{\psi^4} A_5 (A_4 + A_5)\, \mathrm{d}A_1 \wedge \cdots \wedge A_5 \\
I^{*}_{1} &= -4g^{\mu_1\mu_3} g^{\mu_2\mu_4} \frac{\bar{\phi}}{\bar{\psi}} \ln\left(\frac{q^2}{\mu^2}\right) a_5 (a_4 + a_5)\, \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5 \\
I_{2} &= -8 \frac{\exp\left(-\frac{\phi}{\psi}\right)}{\psi^5} \Big(q_2^{\mu_1} q_2^{\mu_3} g^{\mu_2\mu_4} A_1 A_3 + q_1^{\mu_2} q_1^{\mu_4} g^{\mu_1\mu_3} A_2 A_4 \Big) A_5 (A_4 + A_5)\, \mathrm{d}A_1 \wedge \cdots \wedge A_5 \\
I^{*}_{2} &= -8 \ln\left(\frac{q^2}{\mu^2}\right) \frac{q_2^{\mu_1} q_2^{\mu_3} g^{\mu_2\mu_4} a_3 (a_4 + a_5) + q_1^{\mu_2} q_1^{\mu_4} g^{\mu_1\mu_3} a_2 a_4 a_5}{\bar{\psi}^5} a_5 (a_4 + a_5)\, \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_5
\end{aligned}
$$

3.5.4. Re-Homogenization of the Polynomials

Again, in the same way as in Eq. (3.22), it is helpful to transform the first and second Symanzik polynomials back into a homogeneous form. This also works out as easily as above: The measure of integration is the same as seen in Eq. (3.25) and Eq. (3.26), and the Symanzik polynomials become the old polynomials from before:

$$
\begin{aligned}
\bar{\psi} &= a_2 a_5 + a_2 a_4 + a_4 a_5 + a_3 a_5 + a_3 a_4 + a_4 + a_5 \\
&\to \frac{A_2 A_5 + A_2 A_4 + A_4 A_5 + A_3 A_5 + A_3 A_4}{A_1^2} + \frac{A_4 + A_5}{A_1} \\
&= \frac{A_2 A_5 + A_2 A_4 + A_4 A_5 + A_3 A_5 + A_3 A_4 + A_1 A_4 + A_1 A_5}{A_1^2} \\
&= \frac{\psi}{A_1^2}
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
\bar{\phi} &= q_1^2 a_2 a_4 a_5 + q_2^2 a_3 (a_4 + a_5) \\
&\to \frac{q_1^2 A_2 A_4 A_5}{A_1^3} + \frac{q_2^2 A_3 (A_4 + A_5)}{A_1^2} \\
&= \frac{q_1^2 A_2 A_4 A_5 + q_2^2 A_1 A_3 (A_4 + A_5)}{A_1^3} \\
&= \frac{\phi}{A_1^3}
\end{aligned} \tag{3.33}
$$

This gives the re-homogenized expressions:

$$I^{*}_{1} = -4g^{\mu_1\mu_3}g^{\mu_2\mu_4}\frac{\bar{\phi}}{\psi}\ln\left(\frac{q^2}{\mu^2}\right)A_5(A_4+A_5)\,\mathrm{d}\Omega_5 \tag{3.34}$$

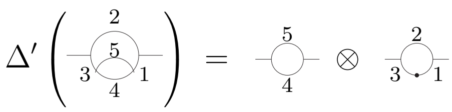
$$I^{*}_{2} = -8\ln\left(\frac{q^2}{\mu^2}\right)\frac{q_2^{\mu_1}q_2^{\mu_3}g^{\mu_2\mu_4}A_1A_3(A_4+A_5)+q_1^{\mu_2}q_1^{\mu_4}g^{\mu_1\mu_3}A_2A_4A_5}{\psi^5}A_5(A_4+A_5)\,\mathrm{d}\Omega_5 \tag{3.35}$$

3.5.5. The Subgraph Structure

The graph is not primitive, but has a subgraph, namely . Therefore, the reduced co-product is

$$\Delta'\left(\ \right) = \ \otimes\ \tag{3.36}$$

However, since the residue of the subgraph is not a vertex, but a two-point function, the labeling of the edges becomes tricky when decorating the cograph. Since the subgraph is shrunk to a point, it becomes a two-point vertex, in a manner of speaking. Therefore, Eq. (3.36) should be decorated in the following way:



(3.37)

What is the first Symanzik polynomial of the cograph? Well, its three spanning trees are each two of the three edges, respectively. It has three internal edges, and since ψ is given by the product over all edges not in a spanning tree, summing over all spanning trees, the first Symanzik polynomial is given by

$$\psi = A_1 + A_2 + A_3$$

If the subgraph shrinks to a point, the first Symanzik polynomial of the original graph should resemble that of the cograph, plus something of a higher order in the parameter that is shrinking. In order to verify this, we substitute the edge variables A_i of the subgraph by xa_i and then let x approach zero.

$$\begin{aligned}
& A_4 \to xa_4 \\
& A_5 \to xa_5 \\
\Rightarrow \quad & \psi \to xA_2a_5 + xA_2a_4 + x^2a_4a_5 + xA_3a_5 + xA_3a_4 + xA_1a_4 + xa_1A_5 \\
& \quad = x(a_4+a_5)(A_1+A_2+A_3) + x^2a_4a_5 \\
& \quad = \underline{\underline{x\psi\ \psi + \mathcal{O}\left(x^2\right)}}
\end{aligned} \tag{3.38}$$

4. Conclusion

For the one-loop case, both computational methods give the same result for the coefficient of the β function, $\frac{4}{3}$. The methods are completely different: In the momentum space calculations, the graph is drawn with wiggly lines for the photons and oriented lines for the electrons, and the Feynman rules for QED (see Chapter D) are used to transition from the drawing to an integrand. Then, the loop momenta are stubbornly integrated out using dimensional regularization. In the end, a simple subtracting suffices to control the logarithmic divergence. This was only possible because a transversal factor $q^2 g_{\mu\nu} - q_\mu q_\nu$ could be extracted from the quadratically divergent integral which reduced the level of divergence by two. So one could argue, it was necessary to know about the properties of external photons to be transversal. On the other hand, during the Corolla computation, transversality was never presupposed. Transversality is a *result* of the computation. This shows how much deeper transversality of the photons is deep-seated in the theory. It is not an assumption made by us, but a property of the theory.
For the two-loop case, the result is the well-known 4 (cf. [8]), which could be confirmed in this thesis. However, it was not possible during the time-span of working on this thesis to compute the amplitude with the Corolla approach, this still needs to be done. However, we have derived the integrand and analyzed its structure. We have seen the expected properties, namely the factorization of the first Symanzik polynomial into co- and subgraph.
As already implied in the introduction, dimensional regularization is not the most mathematically rigorous method. It centers around the poles at space-time dimension $D = 4$, as its main purpose is to distort the space-time dimension just a bit by making it $D = 4 - 2\varepsilon$, where ε has a small positive real part. However, a non-integer space-time dimension has no physical interpretation. Moreover, the goal of quantum field theory is to find non-perturbative methods because the asymptotic behaviour of the β function is not determined yet. The Corolla polynomial and differential help understand the underlying structure a lot better and they show the connection between scalar field theory and gauge theory.
It would be helpful to develop a good, clever implementation of the Corolla computations needed for higher orders than just two loops. It would pose a further test to the method and maybe it could even offer better methods than the ones already at use. The lengthiness of the computations has created some problems during the development of this thesis and the integrals, however nicely derived, are not more-easily solved than the ones when doing dimensional regularization in momentum space. However, due to the analysis of the structure of the Symanzik polynomials, it can be assumed that the computations, however not complete, are correct, and that the Corolla approach portrays a very promising method for the computation of gauge theory amplitudes.

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A. γ traces and scalar products

A.1. γ traces

Throughout this work, we need to evaluate traces involving γ matrices. These are representations of a Clifford algebra, therefore they obey the anticommutator relation $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. This is all we need to know about γ matrices in order to compute their traces. We also make use of the fact that the trace is cyclic, $\mathrm{Tr}\{ABC\} = \mathrm{Tr}\{BCA\} = \mathrm{Tr}\{CAB\}$, and linear.

Trace of two γ matrices This computation is rather simple and short. We use the anticommutator once and cyclicity of the trace.

$$\begin{aligned}
\mathrm{Tr}\left\{\gamma_\mu\gamma_\nu\right\} &= \mathrm{Tr}\left\{2g_{\mu\nu} - \gamma_\nu\gamma_\mu\right\} \\
&= \mathrm{Tr}\left\{2g_{\mu\nu}\right\} - \mathrm{Tr}\left\{\gamma_\nu\gamma_\mu\right\} \\
&= \mathrm{Tr}\left\{2g_{\mu\nu}\right\} - \mathrm{Tr}\left\{\gamma_\mu\gamma_\nu\right\} \\
\Rightarrow \quad \mathrm{Tr}\left\{\gamma_\mu\gamma_\nu\right\} &= \mathrm{Tr}\left\{g_{\mu\nu}\right\} \\
&= g_{\mu\nu}\,\mathrm{Tr}\left\{\mathbb{I}\right\} = 4g_{\mu\nu}
\end{aligned} \tag{A.1}$$

Note that we have used $\mathrm{Tr}\left\{\mathbb{I}\right\} = 4$. This is true even in $D \neq 4$ dimensions because the trace of unity may be any smooth function which equals 4 at $D = 4$.

Trace of four γ matrices This computation is a bit longer than Eq. (A.1), but it uses the same basic steps. One permutes the γ matrix on the very left to the very right and then uses cyclicity of the trace to bring it back to the left.

$$\begin{aligned}
\mathrm{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\right\} &= 2g_{\mu\alpha}\underbrace{\mathrm{Tr}\left\{\gamma_\nu\gamma_\beta\right\}}_{=4g_{\nu\beta}} - \mathrm{Tr}\left\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\beta\right\} \\
&= 8g_{\mu\alpha}g_{\nu\beta} - \left(2g_{\mu\nu}\,\mathrm{Tr}\left\{\gamma_\alpha\gamma_\beta\right\} - \mathrm{Tr}\left\{\gamma_\alpha\gamma_\nu\gamma_\mu\gamma_\beta\right\}\right) \\
&= 8g_{\mu\alpha}g_{\nu\beta} - 8g_{\mu\nu}g_{\alpha\beta} + \left(2g_{\mu\beta}\,\mathrm{Tr}\left\{\gamma_\alpha\gamma_\nu\right\} - \mathrm{Tr}\left\{\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\right\}\right) \\
&= 8\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\alpha\nu}\right) - \underbrace{\mathrm{Tr}\left\{\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\right\}}_{=\mathrm{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\right\}} \\
\Rightarrow \quad \mathrm{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\right\} &= 4\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\alpha\nu}\right)
\end{aligned} \tag{A.2}$$

Trace of six γ matrices One works out the trace of six γ matrices in the exact same way, by permuting the one on the far left to the very right, applying the anti-commutation relation of the Clifford algebra, and known traces of four γ matrices.

Since the computation is comparably lengthy, we will only present the result here.

$$\begin{aligned}\mathrm{Tr}\left\{\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\delta\gamma_\eta\right\} =4\Big\{& g_{\mu\alpha}\left(g_{\nu\beta}g_{\delta\eta} - g_{\nu\delta}g_{\beta\eta} + g_{\nu\eta}g_{\beta\delta}\right) - \\ &- g_{\mu\nu}\left(g_{\alpha\beta}g_{\delta\eta} - g_{\alpha\delta}g_{\beta\eta} + g_{\alpha\eta}g_{\beta\delta}\right) + \\ &+ g_{\mu\beta}\left(g_{\alpha\nu}g_{\delta\eta} - g_{\alpha\delta}g_{\nu\eta} + g_{\alpha\eta}g_{\nu\delta}\right) - \\ &- g_{\mu\delta}\left(g_{\alpha\nu}g_{\beta\eta} - g_{\alpha\beta}g_{\nu\eta} + g_{\alpha\eta}g_{\nu\beta}\right) + \\ &+ g_{\mu\eta}\left(g_{\alpha\nu}g_{\beta\delta} - g_{\alpha\beta}g_{\nu\delta} + g_{\alpha\delta}g_{\nu\beta}\right)\Big\}\end{aligned} \tag{A.3}$$

A.2. Scalar products

When computing the Feynman amplitude of a graph Γ from the Feynman rules, there will be terms encountered that have the form $a \cdot b$, where a and b are four-momenta. Since the denominator of such expressions only contains squares of such four-momenta, it is helpful and necessary to rewrite scalar products in the two following manners, depending on whether one prefers the expression $(a+b)^2$ or $(a-b)^2$ to show up in the rewritten form:

$$a \cdot b = \frac{1}{2}\Big((a+b)^2 - a^2 - b^2\Big) \tag{A.4}$$

$$a \cdot b = \frac{1}{2}\Big(a^2 + b^2 - (a-b)^2\Big) \tag{A.5}$$

B. The Master Integral

The one-loop master integral for dimensional regularization is given by

$$\mathcal{M}\left(\alpha, \beta, D, q^2\right) := \int \frac{\mathrm{d}^D k}{(k^2)^\alpha \left[(k+q)^2\right]^\beta} \tag{B.1}$$

$$= (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{\Gamma\left(\frac{D}{2}-\alpha\right)\Gamma\left(\frac{D}{2}-\beta\right)\Gamma\left(\alpha+\beta-\frac{D}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D-\alpha-\beta)} \tag{B.2}$$

$$=: (q^2)^{\frac{D}{2}-\alpha-\beta}\Gamma_D^{\alpha,\beta} \tag{B.3}$$

The abbriviation $\Gamma_D^{\alpha,\beta}$ is just for short-hand notation. $\Gamma(x)$ denotes the Gamma function with the usual properties, $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(1) = 1$. We note that because of the poles of $\Gamma(x)$ at $x = 0$, $M(0, \beta, D, q^2) = M(\alpha, 0, D, q^2) \equiv 0$.

B.1. Derivation

In order to derive the result for $\mathcal{M}$, we need to introduce a few identities. The first one is a rather complicated-looking expression for some $u^{-\rho}$. Take the integral representation for the Γ function:

$$\Gamma(\rho) = \int_0^\infty e^{-A} A^\rho \frac{\mathrm{d}A}{A}$$

The measure of integration, $\frac{\mathrm{d}A}{A}$, is invariant under rescaling, so we set $A = ua$ and get

$$\begin{aligned}\Gamma(\rho) &= \int_0^\infty e^{-ua}(ua)^\rho \frac{\mathrm{d}a}{a} \\ &= u^\rho \int_0^\infty e^{-ua} a^\rho \frac{\mathrm{d}a}{a}\end{aligned}$$

$$\Rightarrow \qquad u^{-\rho} = \frac{1}{\Gamma(\rho)} \int_0^\infty e^{-ua} a^\rho \frac{\mathrm{d}a}{a} \tag{B.4}$$

Next, we take the D dimensional Gaussian integral,

$$\int \mathrm{d}^D k \, e^{-Xk^2} = \left(\frac{\pi}{X}\right)^{\frac{D}{2}} \tag{B.5}$$

but the space-time dimension only makes sense in real life, so to speak, if it is a positive integer. However, when we integrate in a complex-dimensional spacetime,

we use Eq. (B.5) as a *definition* for the complex measure.
Now, the integral we are looking to compute is the following:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^{\frac{D}{2}}} \frac{1}{[k^2]^\alpha [(k+q)^2]^\beta} \tag{B.6}$$

This integral, in which α, β, and D can be complex numbers, is crucial for QFT. We use Eq. (B.4) in order to rewrite the integrand of Eq. (B.6).

$$\begin{aligned}
\int \frac{\mathrm{d}^D k}{(2\pi)^{\frac{D}{2}}} \frac{1}{[k^2]^\alpha [(k+q)^2]^\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int \frac{\mathrm{d}^D k}{(2\pi)^{\frac{D}{2}}} \iint\limits_0^\infty \frac{\mathrm{d}A\,\mathrm{d}B}{AB} e^{-k^2 A - (k+q)^2 B} A^\alpha B^\beta \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int \frac{\mathrm{d}^D k}{(2\pi)^{\frac{D}{2}}} \iint\limits_0^\infty \mathrm{d}A\,\mathrm{d}B A^{\alpha-1} B^{\beta-1} e^{-(A+B)\left\{k^2 + 2kq\frac{B}{A+B} + q^2 \frac{B}{A+B}\right\}}
\end{aligned}$$

Using translation invariance of the k integration and completing the square in the exponent, and in the next step performing the k integration, the expression becomes

$$\begin{aligned}
\int \frac{\mathrm{d}^D k}{(2\pi)^{\frac{D}{2}}} \frac{1}{[k^2]^\alpha [(k+q)^2]^\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int \mathrm{d}^D \bar{k} \iint\limits_0^\infty \mathrm{d}A\,\mathrm{d}B A^{\alpha-1} B^{\beta-1} e^{-(A+B)\left\{\bar{k}^2 + q^2 \frac{AB}{(A+B)^2}\right\}} \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint\limits_0^\infty \mathrm{d}A\,\mathrm{d}B A^{\alpha-1} B^{\beta-1} \frac{1}{(A+B)^{\frac{D}{2}}} e^{-q^2 \frac{AB}{A+B}} \\
(\text{subst. } B = Ab) \quad \rightarrow \quad &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint\limits_0^\infty \mathrm{d}A\,\mathrm{d}b A^{\alpha+\beta-1} \frac{b^{\beta-1}}{A^{\frac{D}{2}}(1+b)^{\frac{D}{2}}} e^{-q^2 a \frac{B}{1+B}} \\
(\text{subst. } A = a\frac{1+b}{bq^2}) \quad \rightarrow \quad &= (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint\limits_0^\infty \mathrm{d}a\,\mathrm{d}b \frac{1+b}{b} \left(a\frac{1+b}{b}\right)^{\alpha+\beta-\frac{D}{2}-1} \frac{b^{\beta-1}}{(1+b)^{\frac{D}{2}}} e^{-a}
\end{aligned}$$

The a integration yields another Γ function:

$$\begin{aligned}
\int \mathrm{d}^D k \frac{1}{[k^2]^\alpha [(k+q)^2]^\beta} &= (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{1\Gamma\left(\alpha+\beta-\frac{D}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)} \int\limits_0^\infty \mathrm{d}b \left(\frac{1+b}{b}\right)^{\alpha+\beta-\frac{D}{2}} \frac{b^{\beta-1}}{(1+b)^{\frac{D}{2}}} \\
(\text{subst. } b' = \frac{b}{1+b}) \quad \rightarrow \quad &= (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{\Gamma\left(\alpha+\beta-\frac{D}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)} \int\limits_0^1 \mathrm{d}b' \, (1-b')^{\frac{D}{2}-\beta-1} (b')^{\frac{D}{2}-\alpha-1}
\end{aligned} \tag{B.7}$$

Eq. (B.7) is the integral representation of the B (beta) function. (Not the β function of some QFT, however, but the Euler integral of first kind.)

$$\begin{aligned}
B(x,y) &= \int\limits_0^1 \mathrm{d}t \, (t)^{x-1}(1-t)^{y-1} \\
&= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad \int \mathrm{d}^D k \frac{1}{[k^2]^\alpha [(k+q)^2]^\beta} &= (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{\Gamma\left(\alpha+\beta-\frac{D}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)} B\left(\frac{D}{2}-\alpha, \frac{D}{2}-\beta\right) \\
&= (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{\Gamma\left(\frac{D}{2}-\alpha\right)\Gamma\left(\frac{D}{2}-\beta\right)\Gamma\left(\alpha+\beta-\frac{D}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D-\alpha-\beta)}
\end{aligned}$$

This is the result Eq. (B.2). [10]

B.2. Some important expicit results

In this master thesis, we will encounter several explicit values for α and β which yield different results for $\Gamma_D^{\alpha,\beta}$. In this section, we demonstrate the calculations in order to keep computations to a minimum throughout the actual thesis.

B.2.1. $\alpha = 1, \beta = 1$

$$\begin{aligned}
\Gamma_D^{1,1} &= \frac{\Gamma^2\left(\frac{D}{2}-1\right)\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(1)\Gamma(1)\Gamma(D-2)} \\
&= \frac{\Gamma^2\left(\frac{D}{2}-1\right)\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(D-2)} \\
&= \frac{\Gamma^2\left(\frac{D}{2}-1\right)\Gamma\left(3-\frac{D}{2}\right)}{\Gamma(D-2)} \cdot \frac{1}{2-\frac{D}{2}} \qquad \text{(B.8)} \\
&= \frac{\Gamma^2\left(1-\varepsilon\right)\Gamma\left(1+\varepsilon\right)}{\Gamma(2-2\varepsilon)} \cdot \frac{1}{\varepsilon} \\
&= \frac{1}{\varepsilon} + \mathcal{O}(1) \qquad \text{(B.9)}
\end{aligned}$$

B.2.2. $\alpha = 1, \beta = 2$

$$\begin{aligned}
\Gamma_D^{1,2} &= \frac{\Gamma\left(\frac{D}{2}-1\right)\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(3-\frac{D}{2}\right)}{\Gamma(1)\Gamma(2)\Gamma(D-3)} \\
&= \frac{\Gamma^2\left(\frac{D}{2}-1\right)\Gamma\left(3-\frac{D}{2}\right)}{\Gamma(D-3)} \cdot \frac{1}{\frac{D}{2}-2} \qquad \text{(B.10)} \\
&= \frac{\Gamma^2\left(1-\varepsilon\right)\Gamma\left(1+\varepsilon\right)}{\Gamma(1-2\varepsilon)} \cdot \frac{1}{-\varepsilon} \\
&= -\frac{1}{\varepsilon} + \mathcal{O}(1) \qquad \text{(B.11)}
\end{aligned}$$

Since this thesis deals with two-loop computations, there will be two terms $\Gamma_D^{\alpha,\beta}$ and $\Gamma_D^{\alpha',\beta'}$. Some of the Γ functions cancel due to this multiplication. Therefore, it is a lot more sensible to look at products of $\Gamma_D^{\alpha,\beta}$.

B.2.3. $\alpha = 1, \beta = 1, \alpha' = 1, \beta' = 1$

$$
\begin{aligned}
\left(\Gamma_D^{1,1}\right)^2 &= \frac{\Gamma^4\left(\frac{D}{2}-1\right)\Gamma^2\left(2-\frac{D}{2}\right)}{\Gamma^2(D-2)} \\
&= \frac{\Gamma^4\left(\frac{D}{2}-1\right)\Gamma^2\left(2-\frac{D}{2}\right)}{\Gamma^2(D-2)} \\
&= \frac{\Gamma^4\left(\frac{D}{2}-1\right)\Gamma^2\left(3-\frac{D}{2}\right)}{\Gamma^2(D-2)} \cdot \frac{1}{\left(2-\frac{D}{2}\right)^2} \qquad \text{(B.12)} \\
&= \frac{\Gamma^4(1-\varepsilon)\Gamma^2(1+\varepsilon)}{\Gamma^2(2-2\varepsilon)} \cdot \frac{1}{\varepsilon^2} \\
&= \left(1+(4-2\gamma_E)\varepsilon + \mathcal{O}(\varepsilon^2)\right)\frac{1}{\varepsilon^2} \\
&= \frac{1}{\varepsilon^2} + \frac{4-2\gamma_E}{\varepsilon} + \mathcal{O}(1) \qquad \text{(B.13)}
\end{aligned}
$$

B.2.4. $\alpha = 1, \beta = 1, \alpha' = 1, \beta' = 2$

$$
\begin{aligned}
\tilde{\Gamma}_D^{1,1}\tilde{\Gamma}_D^{1,2} &= \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(3-\frac{D}{2}\right)}{\Gamma(D-2)\Gamma(D-3)} \\
&= \frac{\Gamma^4\left(\frac{D}{2}-1\right)\Gamma^2\left(3-\frac{D}{2}\right)}{\Gamma^2(D-2)} \cdot \frac{1}{\left(2-\frac{D}{2}\right)} \cdot \frac{1}{\left(\frac{D}{2}-2\right)} \cdot (D-3) \qquad \text{(B.14)} \\
&= \frac{\Gamma^4(1-\varepsilon)\Gamma^2(1+\varepsilon)}{\Gamma^2(2-2\varepsilon)} \cdot \frac{-1}{\varepsilon^2} \cdot (1-2\varepsilon) \\
&= \left(1+(4-2\gamma_E)\varepsilon + \mathcal{O}(\varepsilon^2)\right)\frac{-1}{\varepsilon^2}(1-2\varepsilon) \\
&= \frac{-1}{\varepsilon^2} + \frac{1}{\varepsilon}(-2+2\gamma_E) + \mathcal{O}(1) \qquad \text{(B.15)}
\end{aligned}
$$

B.2.5. $\alpha = 1, \beta = 1, \alpha' = 1, \beta' = 2 - \frac{D}{2}$

$$
\begin{aligned}
\Gamma_D^{1,1}\Gamma_D^{1,2-\frac{D}{2}} &= \frac{\Gamma^2\left(\frac{D}{2}-1\right)\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(D-2)} \cdot \frac{\Gamma\left(\frac{D}{2}-1\right)\Gamma(D-2)\Gamma(3-D)}{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{3D}{2}-3\right)} \\
&= \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{1}{(4-D)} \cdot \frac{1}{(3-D)\left(\frac{3D}{2}-4\right)} \qquad \text{(B.16)} \\
&= \frac{\Gamma^3(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(2-3\varepsilon)} \cdot \frac{1}{2\varepsilon} \cdot \frac{1}{(-1+2\varepsilon)(2-3\varepsilon)} \\
&= \left(1+\mathcal{O}(\varepsilon)\right) \cdot \frac{1}{2\varepsilon} \cdot \left(-\frac{1}{2} + \mathcal{O}(\varepsilon)\right) \\
&= -\frac{1}{4\varepsilon} + \mathcal{O}(1) \qquad \text{(B.17)}
\end{aligned}
$$

B.2.6. $\alpha = 1, \beta = 1, \alpha' = 1, \beta' = 3 - \frac{D}{2}$

$$\begin{aligned}
\Gamma_D^{1,1}\Gamma_D^{1,3-\frac{D}{2}} &= \frac{\Gamma^2\left(\frac{D}{2}-1\right)\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(D-2)} \cdot \frac{\Gamma\left(\frac{D}{2}-1\right)\Gamma\left(D-3\right)\Gamma\left(4-D\right)}{\Gamma\left(3-\frac{D}{2}\right)\Gamma\left(\frac{3D}{2}-4\right)} \\
&= \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{1}{(4-D)\left(2-\frac{D}{2}\right)} \cdot \frac{1}{(D-3)} \qquad \text{(B.18)} \\
&= \frac{\Gamma^3\left(1-\varepsilon\right)\Gamma\left(1+2\varepsilon\right)}{\Gamma(2-3\varepsilon)} \cdot \frac{1}{2\varepsilon^2} \cdot \frac{1}{(1-2\varepsilon)} \\
&= \left(1+(3-2\gamma_E)\varepsilon + \mathcal{O}(\varepsilon^2)\right) \cdot \frac{1}{2\varepsilon^2} \cdot \left(1+2\varepsilon+\mathcal{O}(\varepsilon^2)\right) \\
&= \frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{5}{2}-\gamma_E\right) + \mathcal{O}(1) \qquad \text{(B.19)}
\end{aligned}$$

B.2.7. $\alpha = 1, \beta = 1, \alpha' = 2, \beta' = 1 - \frac{D}{2}$

$$\begin{aligned}
\Gamma_D^{1,1}\Gamma_D^{2,1-\frac{D}{2}} &= \frac{\Gamma^2\left(\frac{D}{2}-1\right)\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(D-2)} \cdot \frac{\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(D-1\right)\Gamma\left(3-D\right)}{\Gamma\left(1-\frac{D}{2}\right)\Gamma\left(\frac{3D}{2}-3\right)} \\
&= \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{1}{\left(\frac{D}{2}-2\right)(4-D)} \cdot \frac{(D-2)\left(1-\frac{D}{2}\right)}{(3-D)\left(\frac{3D}{2}-4\right)} \qquad \text{(B.20)} \\
&= \frac{\Gamma^3\left(1-\varepsilon\right)\Gamma\left(1+2\varepsilon\right)}{\Gamma(2-3\varepsilon)} \cdot \frac{-1}{2\varepsilon^2} \cdot \frac{(2-2\varepsilon)(-1+\varepsilon)}{(-1+2\varepsilon)(2-3\varepsilon)} \\
&= \left(1+(3-2\gamma_E)\varepsilon + \mathcal{O}(\varepsilon^2)\right) \cdot \frac{-1}{2\varepsilon^2} \cdot \left(1+\frac{3}{2}\varepsilon+\mathcal{O}(\varepsilon^2)\right) \\
&= \left(1+(3-2\gamma_E)\varepsilon + \mathcal{O}(\varepsilon^2)\right) \cdot \left(-\frac{1}{2\varepsilon^2} - \frac{3}{4\varepsilon} + \mathcal{O}(1)\right) \\
&= -\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{9}{4}+\gamma_E\right) + \mathcal{O}(1) \qquad \text{(B.21)}
\end{aligned}$$

B.2.8. $\alpha = 1, \beta = 1, \alpha' = 2, \beta' = 2 - \frac{D}{2}$

$$\begin{aligned}
\Gamma_D^{1,1}\Gamma_D^{2,2-\frac{D}{2}} &= \frac{\Gamma^2\left(\frac{D}{2}-1\right)\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(D-2)} \cdot \frac{\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(D-2\right)\Gamma\left(4-D\right)}{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{3D}{2}-4\right)} \\
&= \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{1}{\left(\frac{D}{2}-2\right)(4-D)} \qquad \text{(B.22)} \\
&= \frac{\Gamma^3\left(1-\varepsilon\right)\Gamma\left(1+2\varepsilon\right)}{\Gamma(2-3\varepsilon)} \cdot \frac{-1}{2\varepsilon^2} \\
&= \left(1+(3-2\gamma_E)\varepsilon + \mathcal{O}(\varepsilon^2)\right) \cdot \frac{-1}{2\varepsilon^2} \\
&= -\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{3}{2}+\gamma_E\right) + \mathcal{O}(1) \qquad \text{(B.23)}
\end{aligned}$$

B.2.9. $\alpha = 1, \beta = 1, \alpha' = 2, \beta' = 3 - \frac{D}{2}$

$$
\begin{aligned}
\Gamma_D^{1,1}\Gamma_D^{2,3-\frac{D}{2}} &= \frac{\Gamma^2\left(\frac{D}{2}-1\right)\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(D-2)} \cdot \frac{\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(D-3\right)\Gamma\left(5-D\right)}{\Gamma\left(3-\frac{D}{2}\right)\Gamma\left(\frac{3D}{2}-5\right)} \\
&= \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{1}{\left(2-\frac{D}{2}\right)} \frac{1}{\left(\frac{D}{2}-2\right)} \frac{\frac{3D}{2}-5}{(D-3)} \qquad \text{(B.24)} \\
&= \frac{\Gamma^3\left(1-\varepsilon\right)\Gamma\left(1+2\varepsilon\right)}{\Gamma(2-3\varepsilon)} \cdot \frac{-1}{\varepsilon^2} \cdot \frac{1-2\varepsilon}{1-3\varepsilon} \\
&= \left(1+(3-2\gamma_E)\varepsilon + \mathcal{O}(\varepsilon^2)\right) \cdot \frac{-1}{\varepsilon^2} \cdot \left(1-\varepsilon+\mathcal{O}(\varepsilon^2\right) \\
&= -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-2+2\gamma_E\right) + \mathcal{O}(1) \qquad \text{(B.25)}
\end{aligned}
$$

B.2.10. $\alpha = 1, \beta = 2, \alpha' = 1, \beta' = 2 - \frac{D}{2}$

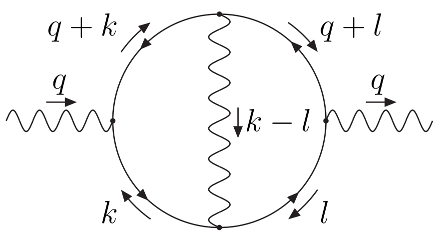
$$
\begin{aligned}
\Gamma_D^{1,2}\Gamma_D^{1,2-\frac{D}{2}} &= \frac{\Gamma\left(\frac{D}{2}-1\right)\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(3-\frac{D}{2}\right)}{\Gamma(D-3)} \cdot \frac{\Gamma\left(\frac{D}{2}-1\right)\Gamma\left(D-2\right)\Gamma\left(3-D\right)}{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{3D}{2}-3\right)} \\
&= \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{\left(2-\frac{D}{2}\right)}{\left(\frac{D}{2}-2\right)(4-D)} \cdot \frac{(D-3)}{(3-D)\left(\frac{3D}{2}-4\right)} \\
&= \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{1}{(4-D)} \cdot \frac{1}{\left(\frac{3D}{2}-4\right)} \qquad \text{(B.26)} \\
&= \frac{\Gamma^3\left(1-\varepsilon\right)\Gamma\left(1+2\varepsilon\right)}{\Gamma(2-3\varepsilon)} \cdot \frac{1}{2\varepsilon} \cdot \frac{1}{(2-3\varepsilon)} \\
&= \left(1+\mathcal{O}(\varepsilon)\right) \cdot \frac{1}{2\varepsilon} \cdot \left(\frac{1}{2}+\mathcal{O}(\varepsilon)\right) \\
&= \frac{1}{4\varepsilon} + \mathcal{O}(1) \qquad \text{(B.27)}
\end{aligned}
$$

B.2.11. $\alpha = 1, \beta = 2, \alpha' = 1, \beta' = 3 - \frac{D}{2}$

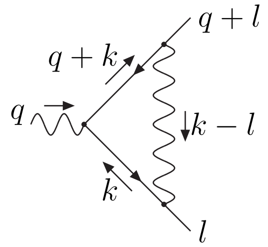
$$
\begin{aligned}
\Gamma_D^{1,2}\Gamma_D^{1,3-\frac{D}{2}} &= \frac{\Gamma\left(\frac{D}{2}-1\right)\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(3-\frac{D}{2}\right)}{\Gamma(D-3)} \cdot \frac{\Gamma\left(\frac{D}{2}-1\right)\Gamma\left(D-3\right)\Gamma\left(4-D\right)}{\Gamma\left(3-\frac{D}{2}\right)\Gamma\left(\frac{3D}{2}-4\right)} \\
&= \frac{\Gamma^3\left(\frac{D}{2}-1\right)\Gamma(5-D)}{\Gamma\left(\frac{3D}{2}-4\right)} \cdot \frac{1}{\left(\frac{D}{2}-2\right)(4-D)} \qquad \text{(B.28)} \\
&= \frac{\Gamma^3\left(1-\varepsilon\right)\Gamma\left(1+2\varepsilon\right)}{\Gamma(2-3\varepsilon)} \cdot \frac{-1}{2\varepsilon^2} \\
&= \left(1+(3-2\gamma_E)\varepsilon + \mathcal{O}(\varepsilon^2)\right) \cdot \frac{-1}{2\varepsilon^2} \\
&= -\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{3}{2}+\gamma_E\right) + \mathcal{O}(1) \qquad \text{(B.29)}
\end{aligned}
$$

C. The triangle relation

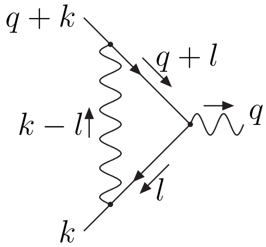
Whenever a subgraph with three external momenta appears in a graph, it is no longer possible to relate the graph to the one-loop master formula because this only works for propagator-like subgraphs. In Section 2.5, we encounter the graph (see Eq. (2.33)):



which has two subgraphs,



and

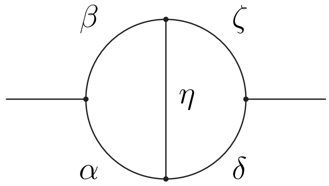


, (C.1)

but neither does one contain the other completely, nor are they disjoint. Therefore one cannot rely on the forest formula, which would effectively make it possible to reduce the problem to one-loop computations.
In general, a graph like the one above in Eq. (2.33), will give an integral of the form

$$I_{\mathrm{t}}(\alpha,\beta,\delta,\zeta,\eta,D,q^2) = \iint \frac{\mathrm{d}^D k\,\mathrm{d}^D l}{(k^2)^\alpha\left[(k+q)^2\right]^\beta (l^2)^\delta \left[(l+q)^2\right]^\zeta \left[(l-k)^2\right]^\eta} \;. \tag{C.2}$$

This would correspond to the graph



where the labels denote the exponents of the propagators on the respective edge. One can easily see that if $\eta = 0$, I_{t} would simply be the product of two one-loop master formulae,

$$I_{\mathrm{t}}(\alpha,\beta,\delta,\zeta,\eta,D,q^2)\Big|_{\eta=0} = M(\alpha,\beta,D,q^2)\cdot M(\delta,\zeta,D,q^2)\;. \tag{C.3}$$

Also, if one of the other exponents is zero, for example α, I_{t} would give two one-loop master integrals where one is nested in the other one.
The cases where one of the exponents vanishes are feasible without further knowledge

other than that of the one-loop master formula. It would be useful to relate the general case, in which all exponents are positive, to cases where one of the exponents is reduced.

In order to derive such a formula using integration by parts, one utilizes the fact that the integral of a total derivative vanishes if the function vanishes at the endpoints. Take, for example, the integration in l.

$$\int \mathrm{d}^D l \, \frac{\partial}{\partial l_\mu} F(l, \dots) = 0$$

In this case, the derivative acts on the μ^{th} component in l, erasing the integral. More generally, one could sum over all μ which would still result in zero.

$$\int \mathrm{d}^D l \, \sum_\mu \frac{\partial}{\partial l_\mu} F(l, \dots) = 0 \tag{C.4}$$

Since all functions we are dealing with are scalar functions, we contract with the momentum of the internal photon in order to get a scalar result. The integral still vanished because the integrand is still a total derivative and all three exponents in $F(l, \dots)$ are at least quadratic in l, so the product of $F(l, \dots)$ and $k - l$ will still fall fast enough at the endpoints.

For reasons of clarity, ignore the k-part of the integration for the moment. This also makes α and β obsolete, so we define the function $\tilde{I}_\mathrm{t}$ to be the l-integration part of the entire integral, I_t, and with contraction, derivatio and summation as in Eq. (C.4) understood.

$$\tilde{I}_\mathrm{t}(\delta, \zeta, \eta, D, (q+k)^2) := \int \mathrm{d}^D l \, \frac{\partial}{\partial l_\mu} \Big((k-l)_\mu F(l, \dots) \Big) = 0 \tag{C.5}$$

$$\text{with} \qquad F(l, \dots) = \frac{1}{(l^2)^\delta \, [(l+q)^2]^\zeta \, [(l-k)^2]^\eta}$$

We execute the derivatives:

$$\frac{\partial}{\partial l_\mu} (l-k)_\mu = D \tag{C.6}$$

$$\begin{aligned} (l-k)_\mu \frac{\partial}{\partial l_\mu} \frac{1}{(l^2)^\delta} &= (l-k)_\mu \frac{-\delta}{(l^2)^{\delta+1}} \, (2l^\mu) \\ &= -\delta \left(\frac{(l-k)^2 - k^2}{(l^2)^{\delta+1}} + \frac{1}{(l^2)^\delta} \right) \end{aligned} \tag{C.7}$$

$$\begin{aligned} (l-k)_\mu \frac{\partial}{\partial l_\mu} \frac{1}{[(l+q)^2]^\zeta} &= (l-k)_\mu \frac{-\zeta}{[(l+q)^2]^{\zeta+1}} \, (2(l+q)^\mu) \\ &= -\zeta \left(\frac{(l-k)^2 - (k+q)^2}{[(l+q)^2]^{\zeta+1}} + \frac{1}{[(l+q)^2]^\zeta} \right) \end{aligned} \tag{C.8}$$

$$\begin{aligned} (l-k)_\mu \frac{\partial}{\partial l_\mu} \frac{1}{[(l-k)^2]^\eta} &= (l-k)_\mu \frac{-\eta}{[(l-k)^2]^{\eta+1}} \, (2(l-k)^\mu) \\ &= -\frac{2\eta}{[(l-k)^2]^\eta} \end{aligned} \tag{C.9}$$

Eq. (C.2) then results in

$$
\begin{aligned}
0 = \iint \mathrm{d}^D l\, \mathrm{d}^D k \Big\{ &(D - \delta - \zeta - 2\eta) F(\alpha, \beta, \delta, \zeta, \eta) \\
&+ \delta \Big(F(\alpha - 1, \beta, \delta + 1, \zeta, \eta) - F(\alpha, \beta, \delta + 1, \zeta, \eta - 1) \Big) \\
&+ \zeta \Big(F(\alpha, \beta - 1, \delta, \zeta + 1, \eta) - F(\alpha, \beta, \delta, \zeta + 1, \eta - 1) \Big) \Big\}
\end{aligned}
\tag{C.10}
$$

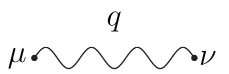
$$
\begin{aligned}
\Rightarrow \quad & \iint \mathrm{d}^D l\, \mathrm{d}^D k\, F(\alpha, \beta, \delta, \zeta, \eta) = \\
\frac{1}{D - \delta - \zeta - 2\eta} \iint \mathrm{d}^D l\, \mathrm{d}^D k \Big\{ &\delta \Big(F(\alpha, \beta, \delta + 1, \zeta, \eta - 1) - (F(\alpha - 1, \beta, \delta + 1, \zeta, \eta) \Big) \\
&+ \zeta \Big(F(\alpha, \beta, \delta, \zeta + 1, \eta - 1) - (F(\alpha, \beta - 1, \delta, \zeta + 1, \eta) \Big) \Big\}
\end{aligned}
\tag{C.11}
$$

where $F(\alpha, \beta, \delta, \zeta, \eta)$ denotes the integrand of Eq. (C.2). Due to the interchangeability of k and l in the integration (Eq. (C.2)), I_t is invariant under $(\alpha \leftrightarrow \beta) \wedge (\delta \leftrightarrow \zeta)$ and $(\alpha \leftrightarrow \delta) \wedge (\beta \leftrightarrow \zeta)$. So for the case $\alpha = \beta = \delta = \zeta = \eta = 1$, as needed in Section 2.5.1, the triangle relation Eq. (C.11) gives

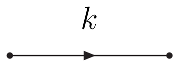
$$
\begin{aligned}
F(1,1,1,1,1) &= \frac{1}{D-4} \Big\{ F(1,1,2,1,0) - F(0,1,2,1,1) + \underbrace{F(1,1,1,2,0)}_{=F(1,1,2,1,0)} - \underbrace{F(1,0,1,2,1)}_{=F(0,1,2,1,1)} \Big\} \\
&= \frac{2}{D-4} \Big\{ F(1,1,2,1,0) - F(0,1,2,1,1) \Big\} \; .
\end{aligned}
\tag{C.12}
$$

D. Feynman Rules of QED

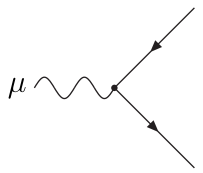
Quantum electrodynamics is an abelian gauge theory. Therefore, there are no boson-boson interactions, and there is only one fermion if the theory is massless. In a Feynman diagram, the fermion propagator is denoted by a straight line with an arrow which indicates charge flow. The boson of QED is the photon, a spin-1 boson, denoted by a wavy line. The only possible interaction is a photon-fermion-antifermion vertex. To translate from the diagram to the (unrenormalized) Feynman integrand, one starts at a vertex or fermionic line of choice and starts writing down the terms corresponding to the vertices or lines one encounters, while proceding against the direction of the arrows. The corresponding expressions are:



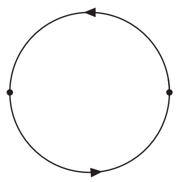
$$\longleftrightarrow \qquad i\frac{g^{\mu\nu} - \xi\frac{q^\mu q^\nu}{q^2}}{q^2}$$



$$\longleftrightarrow \qquad i\frac{1}{\not{k}} = i\frac{\not{k}}{k^2}$$



$$\longleftrightarrow \qquad ie\gamma_\mu$$



$$\longleftrightarrow \qquad -1$$

The Dirac-slash notation $\not{q}$ is short for $\not{q} = \gamma_\mu q^\mu$. In the tree-level vertex expression, e is the electric charge of the fermion. The internal photonic propagator is dependent on the gauge as ξ can be chosen freely. The Feynman gauge is $\xi = 0$, where the analytic expression for the photonic progator gets rather short. The Landau gauge is $\xi = 1$, with a transversal photon. Because of the Ward identities, in QED all subdivergencies of higher-loop graphs vanish if the Landau gauge is being used. The last lines indicates that for every closed fermion loop, the expression needs to be multiplied with -1.[10]

E. Mathematica Code

```
In[1]:= << HighEnergyPhysics`FeynCalc`

Loading FeynCalc from /u/grauel/.Mathematica/Applications/HighEnergyPhysics
FeynCalc 8.2.0 For help, type ?FeynCalc, open FeynCalcRef8.nb or visit www.feyncalc.org
Loading FeynArts, see www.feynarts.de for documentation
Loop::shdw :
  Symbol Loop appears in multiple contexts {FeynArts`, HighEnergyPhysics`FeynCalc`Loop`}; definitions
     in context FeynArts` may shadow or be shadowed by other definitions. >>
FeynAmp::shdw :
  Symbol FeynAmp appears in multiple contexts {FeynArts`, HighEnergyPhysics`FeynCalc`FeynAmp`};
     definitions in context FeynArts` may shadow or be shadowed by other definitions. >>
FeynAmpList::shdw :
  Symbol FeynAmpList appears in multiple contexts {FeynArts`, HighEnergyPhysics`FeynCalc`FeynAmpList`};
     definitions in context FeynArts` may shadow or be shadowed by other definitions. >>
PropagatorDenominator::shdw : Symbol PropagatorDenominator appears in
     multiple contexts {FeynArts`, HighEnergyPhysics`FeynCalc`PropagatorDenominator`};
     definitions in context FeynArts` may shadow or be shadowed by other definitions. >>
FeynAmpDenominator::shdw : Symbol FeynAmpDenominator appears in
     multiple contexts {FeynArts`, HighEnergyPhysics`FeynCalc`FeynAmpDenominator`};
     definitions in context FeynArts` may shadow or be shadowed by other definitions. >>
GaugeXi::shdw :
  Symbol GaugeXi appears in multiple contexts {FeynArts`, HighEnergyPhysics`FeynCalc`GaugeXi`};
     definitions in context FeynArts` may shadow or be shadowed by other definitions. >>
NonCommutative::shdw : Symbol NonCommutative appears in
     multiple contexts {FeynArts`, HighEnergyPhysics`FeynCalc`NonCommutative`};
     definitions in context FeynArts` may shadow or be shadowed by other definitions. >>
Optional::opdef : The default value for the optional argument a : f s_. : F | S | V | T | U | SV contains a pattern. >>
Optional::opdef : The default value for the optional argument a : f s_. : F | S | V | T | U | SV contains a pattern. >>
Global`PolarizationVector::shdw : Symbol PolarizationVector appears
     in multiple contexts {Global`, HighEnergyPhysics`FeynCalc`PolarizationVector`};
     definitions in context Global` may shadow or be shadowed by other definitions. >>
Global`DiracSpinor::shdw :
  Symbol DiracSpinor appears in multiple contexts {Global`, HighEnergyPhysics`FeynCalc`DiracSpinor`};
     definitions in context Global` may shadow or be shadowed by other definitions. >>
Global`DiracTrace::shdw :
  Symbol DiracTrace appears in multiple contexts {Global`, HighEnergyPhysics`fctools`DiracTrace`};
     definitions in context Global` may shadow or be shadowed by other definitions. >>
FeynArts 3.7 patched for use with FeynCalc
FeynArts 3.7 patched for use with FeynCalc

In[2]:= (* Computation of the gμν-part of the graph with one-loop subgraph in a fermionic leg *)

In[3]:= argumentFG = (2 - D) *
          DiracGamma[LorentzIndex[μ]].DiracGamma[LorentzIndex[α]].DiracGamma[LorentzIndex[ν]].
           DiracGamma[LorentzIndex[β]].DiracGamma[LorentzIndex[δ]].DiracGamma[LorentzIndex[η]]
```

Out[3]= $(2-D)\,\gamma^{\mu}.\gamma^{\alpha}.\gamma^{\nu}.\gamma^{\beta}.\gamma^{\delta}.\gamma^{\eta}$

```
In[4]:= traceFG = DiracTrace[argumentFG, DiracTraceEvaluate → True];
```

```
In[5]:= term = Contract[traceFG, FourVector[k + q, LorentzIndex[α]]];

In[6]:= term = Contract[term, FourVector[k, LorentzIndex[β]]];

In[7]:= term = Contract[term, FourVector[l + k, LorentzIndex[δ]]];

In[8]:= term = Contract[term, FourVector[k, LorentzIndex[η]]];

In[9]:= term = term /. MetricTensor[LorentzIndex[μ], LorentzIndex[ν]] →
          D / 4 * MetricTensor[LorentzIndex[μ], LorentzIndex[ν]];

In[10]:= gterm = Contract[term, MetricTensor[LorentzIndex[μ], LorentzIndex[ν]]];

In[11]:= Simplify[gterm]
```

Out[11]= $4\,(D-2)^2\left(k^2\,(k\cdot l+k\cdot q-l\cdot q)+2\,k\cdot l\,k\cdot q+k^{2^2}\right)$

```
In[12]:= gterm = gterm /. Contract[FourVector[k, LorentzIndex[μ]],
            FourVector[l, LorentzIndex[μ]]] → 1 / 2 * (KL^2 - K^2 - L^2);

In[13]:= gterm = gterm /. Contract[FourVector[k, LorentzIndex[μ]],
            FourVector[q, LorentzIndex[μ]]] → 1 / 2 * (KQ^2 - K^2 - Q^2);

In[14]:= gterm =
          gterm /. Contract[FourVector[k, LorentzIndex[μ]], FourVector[k, LorentzIndex[μ]]] → K^2;

In[15]:= gterm =
          gterm /. Contract[FourVector[l, LorentzIndex[μ]], FourVector[l, LorentzIndex[μ]]] → L^2;

In[16]:= gterm =
          gterm /. Contract[FourVector[q, LorentzIndex[μ]], FourVector[q, LorentzIndex[μ]]] → Q^2;

In[17]:= gterm = Expand[gterm]
```

Out[17]= $2\,D^2\,K^4-4\,D^2\,K^2\,l\cdot q+2\,D^2\,\mathrm{KL}^2\,\mathrm{KQ}^2-2\,D^2\,\mathrm{KL}^2\,Q^2-2\,D^2\,\mathrm{KQ}^2\,L^2+2\,D^2\,L^2\,Q^2-8\,D\,K^4+16\,D\,K^2\,l\cdot q-8\,D\,\mathrm{KL}^2\,\mathrm{KQ}^2+$
$8\,D\,\mathrm{KL}^2\,Q^2+8\,D\,\mathrm{KQ}^2\,L^2-8\,D\,L^2\,Q^2+8\,K^4-16\,K^2\,l\cdot q+8\,\mathrm{KL}^2\,\mathrm{KQ}^2-8\,\mathrm{KL}^2\,Q^2-8\,\mathrm{KQ}^2\,L^2+8\,L^2\,Q^2$

```
In[18]:= Simplify[gterm]
```

Out[18]= $2\,(D-2)^2\left(K^4-2\,K^2\,l\cdot q+\left(\mathrm{KL}^2-L^2\right)\left(\mathrm{KQ}^2-Q^2\right)\right)$

```
In[19]:= (* Computation of the lμlν-
          part of the graph with one-loop subgraph in a fermionic leg *)

In[20]:= argumentFL = DiracGamma[LorentzIndex[μ]].DiracGamma[LorentzIndex[α]].
          DiracGamma[LorentzIndex[ν]].DiracGamma[LorentzIndex[β]].DiracGamma[LorentzIndex[σ]].
          DiracGamma[LorentzIndex[δ]].DiracGamma[LorentzIndex[ρ]].DiracGamma[LorentzIndex[η]]
```

Out[20]= $\gamma^{\mu}.\gamma^{\alpha}.\gamma^{\nu}.\gamma^{\beta}.\gamma^{\sigma}.\gamma^{\delta}.\gamma^{\rho}.\gamma^{\eta}$

```
In[21]:= traceFL = DiracTrace[argumentFL, DiracTraceEvaluate → True];

In[22]:= term = Contract[traceFL, FourVector[k + q, LorentzIndex[α]]];

In[23]:= term = Contract[term, FourVector[k, LorentzIndex[β]]];

In[24]:= term = Contract[term, FourVector[l + k, LorentzIndex[δ]]];

In[25]:= term = Contract[term, FourVector[k, LorentzIndex[η]]];

In[26]:= term = Contract[term, FourVector[l, LorentzIndex[σ]]];

In[27]:= term = Contract[term, FourVector[l, LorentzIndex[ρ]]];

In[28]:= term = term /. MetricTensor[LorentzIndex[μ], LorentzIndex[ν]] →
          D / 4 * MetricTensor[LorentzIndex[μ], LorentzIndex[ν]];

In[29]:= gterm = Contract[term, MetricTensor[LorentzIndex[μ], LorentzIndex[ν]]];
```

In[30]:=
```
gterm
```

Out[30]= $4 D k^2 l^2 k\cdot q + 8 D k^2 k\cdot l\, l\cdot q + 4 D k^2 l^2 l\cdot q - 16 D k\cdot l^2 k\cdot q - 8 D l^2 k\cdot l\, k\cdot q + 4 D k^{2^2} l^2 - 8 D k^2 k\cdot l^2 - 4 D k^2 l^2 k\cdot l - 16 k^2 l^2 k\cdot q - 16 k^2 k\cdot l\, l\cdot q - 16 k^2 l^2 l\cdot q + 8 k^2 l^2 (k\cdot q + l\cdot q) + 32 k\cdot l^2 k\cdot q + 16 l^2 k\cdot l\, k\cdot q - 16 k^{2^2} l^2 + 16 k^2 k\cdot l^2 + 8 k^2 l^2 \left(k\cdot l + k^2\right)$

In[31]:=
```
gterm = gterm /. Contract[FourVector[k, LorentzIndex[μ]],
      FourVector[l, LorentzIndex[μ]]] → 1 / 2 * (KL^2 - K^2 - L^2);
```

In[32]:=
```
gterm = gterm /. Contract[FourVector[k, LorentzIndex[μ]],
      FourVector[q, LorentzIndex[μ]]] → 1 / 2 * (KQ^2 - K^2 - Q^2);
```

In[33]:=
```
gterm =
   gterm /. Contract[FourVector[k, LorentzIndex[μ]], FourVector[k, LorentzIndex[μ]]] → K^2;
```

In[34]:=
```
gterm =
   gterm /. Contract[FourVector[l, LorentzIndex[μ]], FourVector[l, LorentzIndex[μ]]] → L^2;
```

In[35]:=
```
gterm =
   gterm /. Contract[FourVector[q, LorentzIndex[μ]], FourVector[q, LorentzIndex[μ]]] → Q^2;
```

In[36]:=
```
gterm = Expand[gterm]
```

Out[36]= $-2 D K^4 \mathrm{KQ}^2 - 4 D K^4 l\cdot q + 2 D K^4 L^2 + 2 D K^4 Q^2 + 4 D K^2 \mathrm{KL}^2 \mathrm{KQ}^2 + 4 D K^2 \mathrm{KL}^2 l\cdot q - 4 D K^2 \mathrm{KL}^2 Q^2 - 2 D \mathrm{KL}^4 \mathrm{KQ}^2 + 2 D \mathrm{KL}^4 Q^2 + 2 D \mathrm{KL}^2 \mathrm{KQ}^2 L^2 - 2 D \mathrm{KL}^2 L^2 Q^2 + 4 K^4 \mathrm{KQ}^2 + 8 K^4 l\cdot q - 4 K^4 L^2 - 4 K^4 Q^2 - 8 K^2 \mathrm{KL}^2 \mathrm{KQ}^2 - 8 K^2 \mathrm{KL}^2 l\cdot q + 8 K^2 \mathrm{KL}^2 Q^2 + 4 \mathrm{KL}^4 \mathrm{KQ}^2 - 4 \mathrm{KL}^4 Q^2 - 4 \mathrm{KL}^2 \mathrm{KQ}^2 L^2 + 4 \mathrm{KL}^2 L^2 Q^2$

In[37]:=
```
Simplify[gterm]
```

Out[37]= $-2 (D-2)\left(K^4 \left(\mathrm{KQ}^2 - L^2 - Q^2\right) + 2 K^2 \mathrm{KL}^2 \left(Q^2 - \mathrm{KQ}^2\right) + 2 \left(K^4 - K^2 \mathrm{KL}^2\right) l\cdot q + \mathrm{KL}^2 \left(\mathrm{KL}^2 - L^2\right)\left(\mathrm{KQ}^2 - Q^2\right)\right)$

In[38]:=
```
(* Computation of the gμν-part of the graph with one photon running across *)
```

In[39]:=
```
argumentBG = DiracGamma[LorentzIndex[μ]].
   DiracGamma[LorentzIndex[α]].((D - 6) * DiracGamma[LorentzIndex[δ]].
        DiracGamma[LorentzIndex[ν]].DiracGamma[LorentzIndex[β]] + 2 * (4 - D) *
       (DiracGamma[LorentzIndex[β]] * MetricTensor[LorentzIndex[ν], LorentzIndex[δ]] -
         DiracGamma[LorentzIndex[ν]] * MetricTensor[LorentzIndex[β], LorentzIndex[δ]] +
         DiracGamma[LorentzIndex[δ]] *
          MetricTensor[LorentzIndex[ν], LorentzIndex[β]])).DiracGamma[LorentzIndex[η]]
```

Out[39]= $\gamma^\mu.\gamma^\alpha.\left((D-6)\,\gamma^\delta.\gamma^\nu.\gamma^\beta + 2(4-D)\left(-\gamma^\nu g^{\beta\delta} + \gamma^\delta g^{\beta\nu} + \gamma^\beta g^{\delta\nu}\right)\right).\gamma^\eta$

In[40]:=
```
traceBG = DiracTrace[argumentBG, DiracTraceEvaluate → True];
```

In[41]:=
```
term = Contract[traceBG, FourVector[k, LorentzIndex[α]]];
```

In[42]:=
```
term = Contract[term, FourVector[l, LorentzIndex[β]]];
```

In[43]:=
```
term = Contract[term, FourVector[l + q, LorentzIndex[δ]]];
```

In[44]:=
```
term = Contract[term, FourVector[k + q, LorentzIndex[η]]];
```

In[45]:=
```
term = term /. MetricTensor[LorentzIndex[μ], LorentzIndex[ν]] →
     D / 4 * MetricTensor[LorentzIndex[μ], LorentzIndex[ν]];
```

In[46]:=
```
gterm = Contract[term, MetricTensor[LorentzIndex[μ], LorentzIndex[ν]]];
```

```
In[47]:= gterm
```

Out[47]= $-4D^2 l^2 k\cdot q - 4D^2 k^2 l\cdot q - 4D^2 q^2 k\cdot l - 4D^2 k^2 l^2 - 8Dk\cdot l(k\cdot l + k\cdot q) + 8Dl^2 k\cdot q + 8Dl^2(k\cdot q + k^2) + 8Dk^2 l\cdot q - 16Dk\cdot q\, l\cdot q + 8D(k\cdot q + k^2)\, l\cdot q - 8Dl\cdot q(k\cdot l + k\cdot q) - 8Dk\cdot l(k\cdot l + l\cdot q) - 8Dk\cdot q(k\cdot l + l\cdot q) + 8Dk^2(l\cdot q + l^2) + 8Dk\cdot q(l\cdot q + l^2) + 24Dq^2 k\cdot l + 8Dk^2 l^2 - 16k\cdot l\, k\cdot q + 32k\cdot l(k\cdot l + k\cdot q) - 16l^2(k\cdot q + k^2) - 16k\cdot l\, l\cdot q - 16(k\cdot q + k^2)\, l\cdot q + 32l\cdot q(k\cdot l + k\cdot q) + 32k\cdot l(k\cdot l + l\cdot q) + 32k\cdot q(k\cdot l + l\cdot q) - 16k^2(l\cdot q + l^2) - 16k\cdot q(l\cdot q + l^2) - 16q^2 k\cdot l - 16k\cdot l(k\cdot l + k\cdot q + l\cdot q + q^2) - 16k\cdot l^2$

```
In[48]:= gtermsim = Simplify[gterm]
```

Out[48]= $-4(D-2)\left(k\cdot l\left((D-4)q^2 + 4k\cdot q + 4l\cdot q\right) + (D-4)k^2(l\cdot q + l^2) + k\cdot q\left((D-4)l^2 + 4l\cdot q\right) + 4k\cdot l^2\right)$

```
In[49]:= gterm = gterm /. Contract[FourVector[k, LorentzIndex[μ]],
               FourVector[q, LorentzIndex[μ]]] → 1 / 2 * (- K^2 - Q^2 + KQ^2);

In[50]:= gterm = gterm /. Contract[FourVector[l, LorentzIndex[μ]],
               FourVector[q, LorentzIndex[μ]]] → 1 / 2 * (- L^2 - Q^2 + LQ^2);

In[51]:= gterm = gterm /. Contract[FourVector[k, LorentzIndex[μ]],
               FourVector[l, LorentzIndex[μ]]] → 1 / 2 * (L^2 + K^2 - KL^2);

In[52]:= gterm =
           gterm /. Contract[FourVector[k, LorentzIndex[μ]], FourVector[k, LorentzIndex[μ]]] → K^2;

In[53]:= gterm =
           gterm /. Contract[FourVector[l, LorentzIndex[μ]], FourVector[l, LorentzIndex[μ]]] → L^2;

In[54]:= gterm =
           gterm /. Contract[FourVector[q, LorentzIndex[μ]], FourVector[q, LorentzIndex[μ]]] → Q^2;

In[55]:= gterm = Expand[gterm]
```

Out[55]= $-2D^2K^2\,\mathrm{LQ}^2 + 2D^2\,\mathrm{KL}^2 Q^2 - 2D^2\,\mathrm{KQ}^2 L^2 + 4DK^2\,\mathrm{KL}^2 - 4DK^2\,\mathrm{KQ}^2 - 4DK^2L^2 + 12DK^2\,\mathrm{LQ}^2 + 4DK^2Q^2 - 4D\,\mathrm{KL}^4 + 4D\,\mathrm{KL}^2\,\mathrm{KQ}^2 + 4D\,\mathrm{KL}^2L^2 + 4D\,\mathrm{KL}^2\,\mathrm{LQ}^2 - 20D\,\mathrm{KL}^2Q^2 + 12D\,\mathrm{KQ}^2L^2 - 4D\,\mathrm{KQ}^2\,\mathrm{LQ}^2 + 4D\,\mathrm{KQ}^2Q^2 - 4DL^2\,\mathrm{LQ}^2 + 4DL^2Q^2 + 4D\,\mathrm{LQ}^2Q^2 - 4DQ^4 - 8K^2\,\mathrm{KL}^2 + 8K^2\,\mathrm{KQ}^2 + 8K^2L^2 - 16K^2\,\mathrm{LQ}^2 - 8K^2Q^2 + 8\,\mathrm{KL}^4 - 8\,\mathrm{KL}^2\,\mathrm{KQ}^2 - 8\,\mathrm{KL}^2L^2 - 8\,\mathrm{KL}^2\,\mathrm{LQ}^2 + 32\,\mathrm{KL}^2Q^2 - 16\,\mathrm{KQ}^2L^2 + 8\,\mathrm{KQ}^2\,\mathrm{LQ}^2 - 8\,\mathrm{KQ}^2Q^2 + 8L^2\,\mathrm{LQ}^2 - 8L^2Q^2 - 8\,\mathrm{LQ}^2Q^2 + 8Q^4$

```
In[56]:= gterm = gterm /. {KL^2 → 0, l^2 * LQ^2 → 0, k^2 * KQ^2 → 0, k^2 * l^2 → 0, KQ^2 * LQ^2 → 0}
```

Out[56]= $-2D^2K^2\,\mathrm{LQ}^2 - 2D^2\,\mathrm{KQ}^2L^2 - 4DK^2\,\mathrm{KQ}^2 - 4DK^2L^2 + 12DK^2\,\mathrm{LQ}^2 + 4DK^2Q^2 - 4D\,\mathrm{KL}^4 + 12D\,\mathrm{KQ}^2L^2 + 4D\,\mathrm{KQ}^2Q^2 - 4DL^2\,\mathrm{LQ}^2 + 4DL^2Q^2 + 4D\,\mathrm{LQ}^2Q^2 - 4DQ^4 + 8K^2\,\mathrm{KQ}^2 + 8K^2L^2 - 16K^2\,\mathrm{LQ}^2 - 8K^2Q^2 + 8\,\mathrm{KL}^4 - 16\,\mathrm{KQ}^2L^2 - 8\,\mathrm{KQ}^2Q^2 + 8L^2\,\mathrm{LQ}^2 - 8L^2Q^2 - 8\,\mathrm{LQ}^2Q^2 + 8Q^4$

```
In[57]:= (* Computation of the (k-l)μ(k-l)ν-part of the graph with one photon running across *)

In[58]:= argumentBKL = DiracGamma[LorentzIndex[μ]].DiracGamma[LorentzIndex[α]].
           DiracGamma[LorentzIndex[ρ]].DiracGamma[LorentzIndex[β]].DiracGamma[LorentzIndex[ν]].
           DiracGamma[LorentzIndex[δ]].DiracGamma[LorentzIndex[σ]].DiracGamma[LorentzIndex[η]]
```

Out[58]= $\gamma^\mu.\gamma^\alpha.\gamma^\rho.\gamma^\beta.\gamma^\nu.\gamma^\delta.\gamma^\sigma.\gamma^\eta$

```
In[59]:= traceBKL = DiracTrace[argumentBKL, DiracTraceEvaluate → True];

In[60]:= term = Contract[traceBKL, FourVector[k, LorentzIndex[α]]];

In[61]:= term = Contract[term, FourVector[l, LorentzIndex[β]]];

In[62]:= term = Contract[term, FourVector[l + q, LorentzIndex[δ]]];

In[63]:= term = Contract[term, FourVector[k + q, LorentzIndex[η]]];

In[64]:= term = Contract[term, FourVector[k - l, LorentzIndex[ρ]]];
```

In[65]:=
```
term = Contract[term, FourVector[k - l, LorentzIndex[σ]]];
```

In[66]:=
```
term = term /. MetricTensor[LorentzIndex[μ], LorentzIndex[ν]] →
    D / 4 * MetricTensor[LorentzIndex[μ], LorentzIndex[ν]];
```

In[67]:=
```
gterm = Contract[term, MetricTensor[LorentzIndex[μ], LorentzIndex[ν]]];
```

In[68]:=
```
gterm = Expand[gterm]
```

Out[68]=
$$-8\,D\,k^2\,l^2\,q^2 - 4\,D\,k^{2^2}\,l\cdot q + 8\,D\,k^2\,l\cdot q^2 - 4\,D\,k^2\,l^2\,k\cdot q + 8\,D\,k^2\,k\cdot l\,l\cdot q - 8\,D\,k^2\,k\cdot q\,l\cdot q - 4\,D\,k^2\,l^2\,l\cdot q + 4\,D\,k^2\,q^2\,k\cdot l - 4\,D\,l^{2^2}\,k\cdot q + 8\,D\,l^2\,k\cdot q^2 + 8\,D\,l^2\,k\cdot l\,k\cdot q - 8\,D\,l^2\,k\cdot q\,l\cdot q + 4\,D\,l^2\,q^2\,k\cdot l - 4\,D\,k^{2^2}\,l^2 - 4\,D\,k^2\,l^{2^2} + 8\,D\,k^2\,l^2\,k\cdot l + 16\,k^2\,l^2\,q^2 + 8\,k^{2^2}\,l\cdot q - 16\,k^2\,l\cdot q^2 + 8\,k^2\,l^2\,k\cdot q - 16\,k^2\,k\cdot l\,l\cdot q + 16\,k^2\,k\cdot q\,l\cdot q + 8\,k^2\,l^2\,l\cdot q - 8\,k^2\,q^2\,k\cdot l + 8\,l^{2^2}\,k\cdot q - 16\,l^2\,k\cdot q^2 - 16\,l^2\,k\cdot l\,k\cdot q + 16\,l^2\,k\cdot q\,l\cdot q - 8\,l^2\,q^2\,k\cdot l + 8\,k^{2^2}\,l^2 + 8\,k^2\,l^{2^2} - 16\,k^2\,l^2\,k\cdot l$$

In[69]:=
```
gterm = gterm /. Contract[FourVector[k, LorentzIndex[μ]],
     FourVector[l, LorentzIndex[μ]]] → 1 / 2 * (l^2 + k^2 - KL^2);
```

In[70]:=
```
gterm = gterm /. Contract[FourVector[k, LorentzIndex[μ]],
     FourVector[q, LorentzIndex[μ]]] → 1 / 2 * (KQ^2 - k^2 - q^2);
```

In[71]:=
```
gterm = gterm /. Contract[FourVector[l, LorentzIndex[μ]],
     FourVector[q, LorentzIndex[μ]]] → 1 / 2 * (LQ^2 - l^2 - q^2);
```

In[72]:=
```
gterm =
   gterm /. Contract[FourVector[k, LorentzIndex[μ]], FourVector[k, LorentzIndex[μ]]] → k^2;
```

In[73]:=
```
gterm =
   gterm /. Contract[FourVector[l, LorentzIndex[μ]], FourVector[l, LorentzIndex[μ]]] → l^2;
```

In[74]:=
```
gterm = Expand[gterm]
```

Out[74]=
$$2\,D\,k^4\,\mathrm{LQ}^2 - 2\,D\,k^4\,q^2 + 2\,D\,k^4\,q^2 - 2\,D\,k^2\,\mathrm{KL}^2\,\mathrm{LQ}^2 + 2\,D\,k^2\,\mathrm{KL}^2\,q^2 - 2\,D\,k^2\,\mathrm{KL}^2\,q^2 - 2\,D\,k^2\,\mathrm{KQ}^2\,l^2 - 2\,D\,k^2\,\mathrm{KQ}^2\,\mathrm{LQ}^2 + 2\,D\,k^2\,\mathrm{KQ}^2\,q^2 - 2\,D\,k^2\,l^2\,\mathrm{LQ}^2 + 4\,D\,k^2\,l^2\,q^2 - 4\,D\,k^2\,l^2\,q^2 + 2\,D\,k^2\,\mathrm{LQ}^4 - 2\,D\,k^2\,\mathrm{LQ}^2\,q^2 - 2\,D\,\mathrm{KL}^2\,\mathrm{KQ}^2\,l^2 + 2\,D\,\mathrm{KL}^2\,l^2\,q^2 - 2\,D\,\mathrm{KL}^2\,l^2\,q^2 + 2\,D\,\mathrm{KQ}^4\,l^2 + 2\,D\,\mathrm{KQ}^2\,l^4 - 2\,D\,\mathrm{KQ}^2\,l^2\,\mathrm{LQ}^2 - 2\,D\,\mathrm{KQ}^2\,l^2\,q^2 - 2\,D\,l^4\,q^2 + 2\,D\,l^4\,q^2 + 2\,D\,l^2\,\mathrm{LQ}^2\,q^2 - 4\,k^4\,\mathrm{LQ}^2 + 4\,k^4\,q^2 - 4\,k^4\,q^2 + 4\,k^2\,\mathrm{KL}^2\,\mathrm{LQ}^2 - 4\,k^2\,\mathrm{KL}^2\,q^2 + 4\,k^2\,\mathrm{KL}^2\,q^2 + 4\,k^2\,\mathrm{KQ}^2\,l^2 + 4\,k^2\,\mathrm{KQ}^2\,\mathrm{LQ}^2 - 4\,k^2\,\mathrm{KQ}^2\,q^2 + 4\,k^2\,l^2\,\mathrm{LQ}^2 - 8\,k^2\,l^2\,q^2 + 8\,k^2\,l^2\,q^2 - 4\,k^2\,\mathrm{LQ}^4 + 4\,k^2\,\mathrm{LQ}^2\,q^2 + 4\,\mathrm{KL}^2\,\mathrm{KQ}^2\,l^2 - 4\,\mathrm{KL}^2\,l^2\,q^2 + 4\,\mathrm{KL}^2\,l^2\,q^2 - 4\,\mathrm{KQ}^4\,l^2 - 4\,\mathrm{KQ}^2\,l^4 + 4\,\mathrm{KQ}^2\,l^2\,\mathrm{LQ}^2 + 4\,\mathrm{KQ}^2\,l^2\,q^2 + 4\,l^4\,q^2 - 4\,l^4\,q^2 - 4\,l^2\,\mathrm{LQ}^2\,q^2$$

In[75]:=
```
Expand[%]
```

Out[75]=
$$2\,D\,k^4\,\mathrm{LQ}^2 - 2\,D\,k^4\,q^2 + 2\,D\,k^4\,q^2 - 2\,D\,k^2\,\mathrm{KL}^2\,\mathrm{LQ}^2 + 2\,D\,k^2\,\mathrm{KL}^2\,q^2 - 2\,D\,k^2\,\mathrm{KL}^2\,q^2 - 2\,D\,k^2\,\mathrm{KQ}^2\,l^2 - 2\,D\,k^2\,\mathrm{KQ}^2\,\mathrm{LQ}^2 + 2\,D\,k^2\,\mathrm{KQ}^2\,q^2 - 2\,D\,k^2\,l^2\,\mathrm{LQ}^2 + 4\,D\,k^2\,l^2\,q^2 - 4\,D\,k^2\,l^2\,q^2 + 2\,D\,k^2\,\mathrm{LQ}^4 - 2\,D\,k^2\,\mathrm{LQ}^2\,q^2 - 2\,D\,\mathrm{KL}^2\,\mathrm{KQ}^2\,l^2 + 2\,D\,\mathrm{KL}^2\,l^2\,q^2 - 2\,D\,\mathrm{KL}^2\,l^2\,q^2 + 2\,D\,\mathrm{KQ}^4\,l^2 + 2\,D\,\mathrm{KQ}^2\,l^4 - 2\,D\,\mathrm{KQ}^2\,l^2\,\mathrm{LQ}^2 - 2\,D\,\mathrm{KQ}^2\,l^2\,q^2 - 2\,D\,l^4\,q^2 + 2\,D\,l^4\,q^2 + 2\,D\,l^2\,\mathrm{LQ}^2\,q^2 - 4\,k^4\,\mathrm{LQ}^2 + 4\,k^4\,q^2 - 4\,k^4\,q^2 + 4\,k^2\,\mathrm{KL}^2\,\mathrm{LQ}^2 - 4\,k^2\,\mathrm{KL}^2\,q^2 + 4\,k^2\,\mathrm{KL}^2\,q^2 + 4\,k^2\,\mathrm{KQ}^2\,l^2 + 4\,k^2\,\mathrm{KQ}^2\,\mathrm{LQ}^2 - 4\,k^2\,\mathrm{KQ}^2\,q^2 + 4\,k^2\,l^2\,\mathrm{LQ}^2 - 8\,k^2\,l^2\,q^2 + 8\,k^2\,l^2\,q^2 - 4\,k^2\,\mathrm{LQ}^4 + 4\,k^2\,\mathrm{LQ}^2\,q^2 + 4\,\mathrm{KL}^2\,\mathrm{KQ}^2\,l^2 - 4\,\mathrm{KL}^2\,l^2\,q^2 + 4\,\mathrm{KL}^2\,l^2\,q^2 - 4\,\mathrm{KQ}^4\,l^2 - 4\,\mathrm{KQ}^2\,l^4 + 4\,\mathrm{KQ}^2\,l^2\,\mathrm{LQ}^2 + 4\,\mathrm{KQ}^2\,l^2\,q^2 + 4\,l^4\,q^2 - 4\,l^4\,q^2 - 4\,l^2\,\mathrm{LQ}^2\,q^2$$

In[76]:=
```
gterm = gterm /. {k^2 * KQ^2 → 0, l^2 * LQ^2 → 0, k^2 * l^2 * KL^2 → 0, KQ^2 * LQ^2 * KL^2 → 0};
```

In[77]:=
```
gterm
```

Out[77]=
$$2\,D\,k^4\,\mathrm{LQ}^2 - 2\,D\,k^4\,q^2 + 2\,D\,k^4\,q^2 - 2\,D\,k^2\,\mathrm{KL}^2\,\mathrm{LQ}^2 + 2\,D\,k^2\,\mathrm{KL}^2\,q^2 - 2\,D\,k^2\,\mathrm{KL}^2\,q^2 + 4\,D\,k^2\,l^2\,q^2 - 4\,D\,k^2\,l^2\,q^2 + 2\,D\,k^2\,\mathrm{LQ}^4 - 2\,D\,k^2\,\mathrm{LQ}^2\,q^2 - 2\,D\,\mathrm{KL}^2\,\mathrm{KQ}^2\,l^2 + 2\,D\,\mathrm{KL}^2\,l^2\,q^2 - 2\,D\,\mathrm{KL}^2\,l^2\,q^2 + 2\,D\,\mathrm{KQ}^4\,l^2 + 2\,D\,\mathrm{KQ}^2\,l^4 - 2\,D\,\mathrm{KQ}^2\,l^2\,q^2 - 2\,D\,l^4\,q^2 + 2\,D\,l^4\,q^2 - 4\,k^4\,\mathrm{LQ}^2 + 4\,k^4\,q^2 - 4\,k^4\,q^2 + 4\,k^2\,\mathrm{KL}^2\,\mathrm{LQ}^2 - 4\,k^2\,\mathrm{KL}^2\,q^2 + 4\,k^2\,\mathrm{KL}^2\,q^2 - 8\,k^2\,l^2\,q^2 + 8\,k^2\,l^2\,q^2 - 4\,k^2\,\mathrm{LQ}^4 + 4\,k^2\,\mathrm{LQ}^2\,q^2 + 4\,\mathrm{KL}^2\,\mathrm{KQ}^2\,l^2 - 4\,\mathrm{KL}^2\,l^2\,q^2 + 4\,\mathrm{KL}^2\,l^2\,q^2 - 4\,\mathrm{KQ}^4\,l^2 - 4\,\mathrm{KQ}^2\,l^4 + 4\,\mathrm{KQ}^2\,l^2\,q^2 + 4\,l^4\,q^2 - 4\,l^4\,q^2$$

Selbständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Berlin, den