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On linear systems of Dyson Schwinger equations

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by
Henry Kißler

Supervised by:

1. *Prof. Dr. Dirk Kreimer*
2. *Prof. Dr. Oliver Schnetz*

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Chapter 1

Introduction

Up to these days quantum field theory remains in a curious situation. On the one hand it is one of the most successful theories for high-precision calculation of physical observables in research fields such as elementary particle or solid state physics, but on the other hand the question about its mathematical foundations is still unanswered. The first aspect in combination with the increasing precision of modern experimental measurements leads to higher demands in loop calculation of Feynman graphs, but perturbative computations result in a blow-up of required arithmetical operations. One possible way to handle these difficulties lies in a non-perturbative formulation of quantum field theory.

One method in non-perturbative quantum field theory are rainbow and ladder approximations of green's functions, leading to differential equations, which can be solved exactly to all orders of perturbation theory. Broadhurst and Kreimer discussed a perturbation series consisting of all graphs obtainable by nestings and chainings of a primitiv self-energy graph in Yukawa and ϕ^3 theory [1].

Another approach connects quantum field theory to Hochschild cohomology. Describing the perturbation series through combinatorial Dyson Schwinger equations using Hochschild one-cocycles [2] allows to determine green's functions via recurrence relations for its Taylor expansion coefficients. The special case of a linear Dyson Schwinger equation is treated in [3]. In this work such a single equation is solved in terms of a scaling solution with an anomalous dimension determined through the Mellin transform of a primitiv self-energy graph in analytic regularization.

It is the purpose of this work to generalise the results of the last reference to systems of such linear Dyson Schwinger equations. The first two chapters provide the background material, starting with a short introduction to the definition and algebraic properties of decorated rooted trees. Then a discussion of Kreimer's toy model follows. We choose this model for its well-studied behaviour in the case of analytic regularization and renormalization in the momentum scheme [4]. After that we state the main results for finding (renomalized) green's functions of a linear system of Dyson Schwinger equations in chapter 4. Chapter 5 illustrates the application of our results in the context of Yukawa theory in dimensional regularization.

Chapter 2

Decorated rooted trees

This chapter intends to define and reveal the Hopf algebra structure of the algebra of decorated rooted trees. As discussed in [5] this structure gives a full description of subdivergence hierarchies in Feynman graphs. This fact motivates the following definitions and the investigation of Feynman rules defined on decorated rooted trees in the next chapters.

2.1 Basic Definitions

In this work we always assume $\mathcal{D}$ to be a nonempty and finite set.[1]

Definition 2.1.1. *A* graph theoretic tree T *consists of finite sets* $V(T)$ *of* nodes *and* edges $E(T) \subseteq \{e \subseteq V(T) : |e| = 2\}$ *such that for any* $v, w \in V(T)$ *there exists an unique* path $(v = v_0, v_1, \ldots, v_n = w)$ *of distinct nodes with* $\{v_i, v_{i+1}\} \in E(T)$ *for* $0 \leq i \leq n$.[2]

As we want to distinguish between certain classes of nodes, we provide a decoration in

Definition 2.1.2. *A* decorated labelled rooted tree *is a tuple* (T, r, d) *consisting of a graph theoretic tree* T*, a distinguished node* $r \in V(T)$ *called* root *and a* decoration $d : V(T) \to \mathcal{D}$*. The set of all decorated labelled rooted trees is denoted by* $\mathcal{T}_{\mathcal{D}}^{labelled}$.

Until now every of the above defined trees incorporates a labelling by set-theoretic distinctness of its nodes. The next step is denoted to refuse this labelling. For two decorated labelled rooted trees $T_1, T_2 \in \mathcal{T}_{\mathcal{D}}^{\text{labelled}}$ we define a relation by $T_1 \sim T_2$ iff there is an isomorphism between T_1 and T_2. That is a bijective mapping $\phi : V(T_1) \to V(T_2)$ which satisfies the following conditions:

1. $\{v, w\} \in E(T_1)$ if and only if $\{\phi(v), \phi(w)\} \in E(T_2)$
2. $\phi(r) = r'$
3. $d_2 \circ \phi = d_1$.

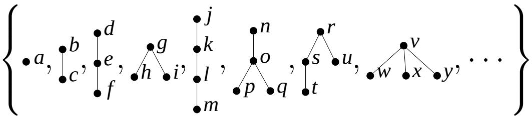
It is easily checked that $\sim$ is an equivalence relation. That allows for

[1]For a proper definition of decorated rooted trees it is not necessary to restrict to finite $\mathcal{D}$. We assume $\mathcal{D}$ to be finite in order to ensure the number of treated Dyson Schwinger Equations to be finite.

[2]In other words a (graph theoretic) tree is a connected graph that contains no cycles or loops

Definition 2.1.3. *A* decorated rooted tree *is an equivalence class arising from* $\sim$ *on* $\mathcal{T}_{\mathcal{D}}^{labelled}$*. The set of all decorated rooted trees is denoted by*

$$\mathcal{T}_{\mathcal{D}} := \mathcal{T}_{\mathcal{D}}^{labelled}/\!\sim \; =$$



Conventionally we print the root on top of the graph and every index attached to a node is an element of $\mathcal{D}$.

After these definitions we like to furnish the set of decorated rooted trees $\mathcal{T}_{\mathcal{D}}$ with a product, or to be more specific with an algebra structure. This can be achieved by the concept of a free commutative unital associative algebra. Nevertheless we state an explicit construction of the algebra of decorated rooted trees. We proceed in two steps. First we give an embedding of the set of decorated rooted trees $\mathcal{T}_{\mathcal{D}}$ into the set $\mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}$, the set of all mappings from $\mathcal{T}_{\mathcal{D}}$ to $\mathbb{N}_0$ with a finite number of non-vanishing values in $\mathbb{N}_0$, such that the obtained monoid structure induces the product of decorated rooted trees. After that the $\mathbb{K}$ vector space structure is added by an embedding into the set of mappings from $\mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}$ to $\mathbb{K}$ with cofinite vanishing values.

The embedding of the set of decorated rooted trees $\mathcal{T}_{\mathcal{D}}$ into the set $\mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}$ is arranged in the following way. An arbitrary decorated rooted tree $t \in \mathcal{T}_{\mathcal{D}}$ is identified with the Kronecker map $\delta_t \in \mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}$ obeying

$$\delta_t(t') = \begin{cases} 1 & \text{if } t' = t \\ 0 & \text{else} \end{cases} \tag{2.1.1}$$

Note that an element $\nu \in \mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}$ is always given by a finite sum of embedded decorated rooted trees. We proceed with our construction in such a way that for a given product of decorated rooted trees $\nu(t)$ is the power of the tree t and hence ν corresponds to a product of decorated rooted trees, which we call a *forest*. Then the neutral element 0, the map sending every decorated tree to zero, of the monoid $\mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}$ represents the *empty forest*.

The next step is the construction of a $\mathbb{K}$ vector space such that the above forests form a basis. Consider the embedding which maps a forest μ to the functional $e_\mu : \mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})} \to \mathbb{K}$ defined by

$$e_\mu(\nu) = \begin{cases} 1 & \text{if } \nu = \mu \\ 0 & \text{else} \end{cases} \quad . \tag{2.1.2}$$

Clearly the set of all functional with cofinite vanishing values $\mathbb{K}[\mathcal{T}_{\mathcal{D}}] := \{(a_\nu)_{\nu \in \mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}} | \; a_\nu \in \mathbb{K}, a_\nu = 0 \text{ for cofinite many } \nu\}$ forms a $\mathbb{K}$ vector space. Moreover we denote the *empty forest* by $\mathbb{1} := \emptyset := e_0$ and the set of all embedded forests by $\mathcal{F}$.[3] From this point on we identify the decorated rooted trees from $\mathcal{T}_{\mathcal{D}}$ and forests from $\mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}$ with their embeddings and simply speak of such trees and forests in $\mathbb{K}[\mathcal{T}_{\mathcal{D}}]$. For two $\mathbb{K}$ vector spaces V and W the Tensor product over $\mathbb{K}$ is denoted by $V \otimes W$. We proceed by defining a product $m : \mathbb{K}[\mathcal{T}_{\mathcal{D}}] \otimes \mathbb{K}[\mathcal{T}_{\mathcal{D}}] \to \mathbb{K}[\mathcal{T}_{\mathcal{D}}]$ satisfying

$$m\left((a_\nu)_{\nu \in \mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}} \otimes (b_\nu)_{\nu \in \mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}}\right) := (c_\nu)_{\nu \in \mathbb{N}_0^{(\mathcal{T}_{\mathcal{D}})}} \text{ with } \qquad c_\nu := \sum_{\mu + \lambda = \nu} a_\mu \cdot b_\lambda .$$

The next proposition is an easy consequence of these definitions.

[3]Note that $\mathcal{F}$ excludes the empty forest, which is not a product of embedded decorated rooted trees.

Proposition 2.1.4. $(\mathbb{K}[\mathcal{T}_\mathcal{D}], m, \mathbb{1})$ *is an unital algebra and a basis of* $\mathbb{K}[\mathcal{T}_\mathcal{D}]$ *is given by* $\mathbb{1}$ *and all finite products of decorated rooted trees or the set all forests* $\mathcal{F}$*, respectively.*

This result is worth to formulate the final definition of the algebra of decorated rooted trees.

Definition 2.1.5. *Let* $\mathbb{K}$ *be a field, then* $H_\mathcal{D} := \mathbb{K}[\mathcal{T}_\mathcal{D}]$ *is called* algebra of decorated rooted trees.

2.2 Graduation and Hopf algebra properties

In this section we explore the structure of the algebra of decorated rooted trees. Doing this we always assume further knowledge about common definitions and statements of the theory of Hopf algebras (for instance see [6, 4, 7]).

2.2.1 Graduation of $H_\mathcal{D}$

First of all we discuss the graduation of the algebra of decorated rooted trees. This structure allows inductive proofs, which we use in the sequel.

For further definitions let us label the elements from $\mathcal{D}$ by $\mathcal{D} = \{1, \dots, N\}$. To every decorated rooted tree $T \in H_\mathcal{D}$ with a given decoration d_T we define a multi-index $\eta_T = (\eta_1, \dots, \eta_N) \in \mathbb{N}_0^N$ by setting $\eta_i := |\{v \in V(T) : d_T(v) = i\}|$ to be the number of nodes decorated with i. To an arbitrary forest $f = T_1 \dots T_M \in H_\mathcal{D}$ we attach a multi-index $\eta_f = \sum_{n=1}^{M} \eta_{T_n}$ by componentwise summation of all multi-indices of its factors.

Let $\eta \in \mathbb{N}_0^N$ we denote the span of all forest with multi-index η by $H^\eta :=$ Span $\{f \in \mathcal{F} : \eta_f = \eta\}$. With these definitions it is easy to recognize the identity

$$H_\mathcal{D} = \bigoplus_{\eta \in \mathbb{N}_0^N} H^\eta.$$

Together with the fact $H^\eta \cdot H^\mu := m(H^\eta \otimes H^\mu) \subseteq H^{\eta+\mu}$ this gives a graduation of the algebra $H_\mathcal{D}$.

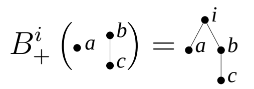
Note that by setting $H^n := \bigoplus_{|\eta|=n} H^\eta$ we rediscover the natural grading simular to the case of undecorated rooted trees which is given by counting the number of nodes of a forest.

In the following we make fertile usage of the graduation of $H_\mathcal{D}$ by observing the canonical filtration and the application of the induction axiom. Because every element of $H_\mathcal{D}$ is contained in a minimal subspace of that filtration and these growing subspaces can be indexed we can provide inductive proofs inducing over the set of elements from $H_\mathcal{D}$.

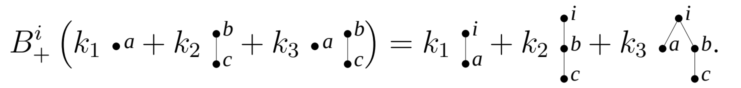
This motivates the definition of a operator which maps elements from one subspace of our filtration to a slightly larger subspace. For this purpose we declare for arbitrary $i \in \mathcal{D}$

Definition 2.2.1. *The* grafting operator $B_+^i : H_\mathcal{D} \to H_\mathcal{D}$ *is defined by linear extension of the map sending every forest* $f \in H_\mathcal{D}$ *to a decorated rooted tree, which is obtained by adding a new root to* f*, decorate this root by* i *and connect it to every root from* f*.*

For example let $k_1, k_2, k_3 \in \mathbb{K}$, then



(2.2.1)



(2.2.2)

2.2.2 Coproduct and the coalgebra $H_{\mathcal{D}}$

Making use of the graduated structure of $H_{\mathcal{D}}$ we define a coproduct on the algebra of decorated rooted trees and show that this turns $H_{\mathcal{D}}$ into a coalgebra.

A definition of the mentioned coproduct requires some previous conventions. Note, the fact that a tree contains no loops allows to establish a partial order on its nodes. Let $T \in H_{\mathcal{D}}$ be a decorated rooted tree. For arbitrary $a, b \in V(T)$ we define:

$$a \leq b \text{ iff } b \text{ is an element of the path } (a = v_0, \ldots, v_n = r)$$

from the node a to the root r. This partial order of nodes allows a formal definition of cutting an edge of a decorated rooted tree. We achieve that by cutting the edge above of a particular node. Therefore for a decorated rooted tree T a *cut* is a subset $C \subseteq V(T)$ which satisfies the condition $a, b \in C \Rightarrow$ neither $a \leq b$ nor $b \leq a$. From now on we will only speak of cuts instead of admissible cuts as all cuts we encounter are of that shape. For the formulation of the next definitions let us assume $T \in H_{\mathcal{D}}$ be an decorated rooted tree, $r \in V(T)$ its root and $C \subseteq V(T)$ a cut.

Definition 2.2.2. *The* pruned part $\mathcal{P}^C(T) \in H_{\mathcal{D}}$ *of a decorated rooted tree $T \in H_{\mathcal{D}}$ and a cut $C \subseteq V(T)$ is defined by setting $\mathcal{P}^{\{r\}}(T) := T$, $\mathcal{P}^{\emptyset}(T) := \mathbb{1}$ and in any other case to be the product $\prod_{v \in C} T_v \in H_{\mathcal{D}}$ of decorated rooted trees $T_v \subseteq T$ with root v, where T_v is the subgraph of T coming from removing the first edge of the path from v to the root of T and restricting the decoration of T to the nodes of T_v.*

The remaining part $\mathcal{R}^C(T) \in H_{\mathcal{D}}$ *is given by the decorated rooted tree coming from removing all those subgraphs from T which are a factor of the pruned part.*

Now we have finished all preparations and continue with the definition of the map Δ which turns out to be a coproduct in proposition 2.2.5.

Definition 2.2.3. *Define $\Delta : H_{\mathcal{D}} \to H_{\mathcal{D}} \otimes H_{\mathcal{D}}$ as algebra morphism and linear extension by setting*

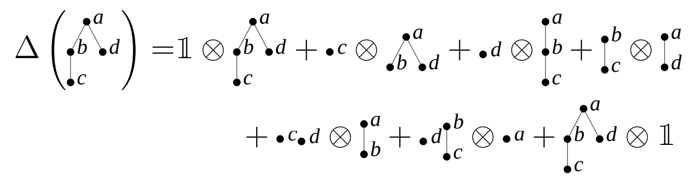
$$\Delta(T) := \sum_{\text{Cuts } C} \mathcal{P}^C(T) \otimes \mathcal{R}^C(T)$$

for any decorated rooted tree T.

The following examples give a demonstration of the map Δ.

$$\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1} \tag{2.2.3}$$

$$\Delta(\bullet_a) = \mathbb{1} \otimes \bullet_a + \bullet_a \otimes \mathbb{1} \tag{2.2.4}$$



(2.2.5)

The next result states that B_+^i is a Hochschild one-cocycle in an abstract Hochschild cocomplex. We only need the identity 2.2.6 which we call *cocycle property*. Furthermore a map satisfying this identity is named *cocycle* in the following. For a further encounter with abstract Hochschild cohomology the reader is referred to [4].

Proposition 2.2.4. *For any $x \in H_{\mathcal{D}}$ it holds*

$$\Delta \circ B_+^i(x) = \Big[B_+^i \otimes \mathbb{1} + (id \otimes B_+^i) \circ \Delta\Big](x). \tag{2.2.6}$$

Proof. Without loss of generality we can assume x to be a forest f by linearity of Δ and B_+^i.

$$\begin{aligned}
\Delta \circ B_+^i(f) &= \sum_{C \text{ cut of } B_+^i(f)} \mathcal{P}^C(B_+^i(f)) \otimes \mathcal{R}^C(B_+^i(f)) \\
&= \mathcal{P}^{\{r_i\}}(B_+^i(f)) \otimes \mathcal{R}^{\{r_i\}}(B_+^i(f)) + \sum_{C \text{ cut of } f} \mathcal{P}^C(B_+^i(f)) \otimes \mathcal{R}^C(B_+^i(f)) \\
&= B_+^i(f) \otimes \mathbb{1} + \sum_{C \text{ cut of } f} \mathcal{P}^C(f) \otimes B_+^i \circ \mathcal{R}^C(f) \\
&= B_+^i(f) \otimes \mathbb{1} + (id \otimes B_+^i) \circ \Delta f
\end{aligned}$$

Here we used the fact that the set of all cuts in $B_+^i(f)$ decomposes in the cut of the root denoted by $\{r_i\}$ and the set of cuts in f. $\square$

Furthermore define the functional $\epsilon \in \mathrm{Hom}(H_{\mathcal{D}}, \mathbb{K})$ by setting $\epsilon(\mathbb{1}) := 1$ and demanding that $\epsilon(f) = 0$ for any forest $f \neq \mathbb{1}$.

Proposition 2.2.5. *$(H_{\mathcal{D}}, \Delta, \epsilon)$ is a counital coalgebra.*

Proof. First we show the coassociativity of Δ. Because Δ is linear and an algebra morphism it suffices to prove the property for the set of decorated rooted trees. Obviously coassociativity holds for $H^0 \subseteq H_{\mathcal{D}}$. Assuming the statement holds on H^n we show the validity for H^{n+1}. So without loss of generality let $T = B_+^i(f) \in H^{n+1}$ with arbitrary $i \in \mathcal{D}$ and $f \in H^n$.

$$\begin{aligned}
(\Delta \otimes id) \circ \Delta(T) &= (\Delta \otimes id) \circ \Delta \circ B_+^i(f) \\
&= (\Delta \otimes id) \circ \Big[B_+^i \otimes \mathbb{1} + (id \otimes B_+^i) \circ \Delta\Big](f) \\
&= \Big[B_+^i \otimes \mathbb{1} \otimes \mathbb{1} + \Big((id \otimes B_+^i) \circ \Delta\Big) \otimes \mathbb{1} + (id \otimes id \otimes B_+^i) \circ (\Delta \otimes id) \circ \Delta\Big](f) \\
&= \Big[(id \otimes \Delta) \otimes (B_+^i \otimes \mathbb{1}) + \Big(id \otimes (B_+^i \otimes \mathbb{1} + (id \otimes B_+^i) \circ \Delta)\Big) \circ \Delta\Big](f) \\
&= (id \otimes \Delta) \circ \Big[(B_+^i \otimes \mathbb{1}) + (id \otimes B_+^i) \circ \Delta\Big](f) \\
&= (id \otimes \Delta) \circ \Delta(T)
\end{aligned}$$

Which proves coassociativity. Moreover note that

$$\begin{aligned}
(\epsilon \otimes id) \circ \Delta(f) \cong 1 \cdot \mathcal{R}^{\{\emptyset\}}(f) = f = \\
\mathcal{P}^{\{r \in V(f): r \text{ is root}\}}(f) \cdot 1 \cong (id \otimes \epsilon) \circ \Delta(f)
\end{aligned}$$

for any $f \in \mathcal{F}$, i.e. ϵ is the counit. $\square$

2.2.3 Convolution product and Hopf algebra $H_{\mathcal{D}}$

In this paragraph we prove that the algebra of decorated rooted trees is a Hopf algebra. But before we are able to show this, we have to treat one ingredient of a Hopf algebra - the convolution product.

Now let (A, m) be an algebra and (C, Δ) be a Coalgebra.

Definition 2.2.6. *The* convolution product $\star$ *is a map defined by*

$$\star \in \mathrm{Hom}\left(\mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A), \mathrm{Hom}(C, A)\right) \text{ and}$$
$$\star : f \otimes g \mapsto m \circ (f \otimes g) \circ \Delta.$$

It should be remarked that we are constantly suppressing the use of canonical embeddings as $\iota :\ \mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \hookrightarrow \mathrm{Hom}(C \otimes C, A \otimes A)$ in our notation. Additionally we like also to use $\star$ to denote the multiplication map $\star \circ \otimes :\ \mathrm{Hom}(C, A) \times \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, A)$.

Lemma 2.2.7. *The set* $\mathrm{Hom}\,(C, A)_\star := (\mathrm{Hom}_{\mathbb{K}}(C, A), \star)$ *is an associative algebra. If A is unital with unit u and C is counital with counit ϵ, then* $\mathrm{Hom}\,(C, A)_\star$ *is unital with unit $e := u \circ \epsilon$.*

That is exactly Lemma 2.1.7 from [4] where the proof can be found.

Theorem 2.2.8. *H_D is a connected bialgebra, thus $(H_{\mathcal{D}}, m, \mathbb{1}, \Delta, \epsilon, S)$ is a Hopf algebra and the antipode is recursively determined by*

$$S(f) = -f - \sum_f S(f')f'' \tag{2.2.7}$$

for any forest $f \in \mathcal{F}$.[4]

Proof. Firstly note that $H_{\mathcal{D}}$ is a bialgebra by the following argument. By defintion Δ is a morphism of algebras that is $\Delta \circ m = m_{H_{\mathcal{D}} \otimes H_{\mathcal{D}}} \circ (\Delta \otimes \Delta)$ and $\Delta\mathbb{1} = \mathbb{1} \otimes \mathbb{1}$. So we only need to show $\epsilon \otimes \epsilon \cong \epsilon \circ m$. Which is easily seen as both sides of the equation are nonvanishing only at $\mathbb{1} \otimes \mathbb{1}$.

We already showed that the algebra $H_{\mathcal{D}}$ possesses a natural grading by counting the nodes of a forest. In particular it holds $H^0 = \{\mathbb{1}\}$. Moreover the property $\Delta H^n \subseteq \bigoplus_{i+j=n} H^i \otimes H^j = \bigoplus_{i=0}^n H^i \otimes H^{n-i}$ is clearly satisfied by definition of the coproduct Δ. Thus $(H^n)_{n \in \mathbb{N}_0}$ is even a graduation of the bialgebra $H_{\mathcal{D}}$. Defining the filtration $(F^n)_{n \in \mathbb{N}_0}$ through $F^n := \bigoplus_{i=0}^n H^i$ it follows that $H_{\mathcal{D}}$ is connected.

It is a well known result that any connected bialgebra H is a Hopf algebra by explicit construction of a convolution inverse of the identity id_H. The existence of an antipode is a corollary of lemma 2.2.9 below. The antipode is denoted by S and once proven its existence we immediately find

$$\mathbb{1} = e(\mathbb{1}) = (S \star id)\mathbb{1} = S(\mathbb{1}). \tag{2.2.8}$$

Yet the recursion (2.2.7) directly follows from

$$0 = e(f) = (S \star id)(f) = S(f) + f + \sum_f S(f')f''. \tag{2.2.9}$$

This finishes the proof. □

[4] Here we used Sweedler's notation $\Delta f = f \otimes \mathbb{1} + \mathbb{1} \otimes f + \sum_f f' \otimes f''$.

Before we state the next lemma, let us denote

$$G_{\mathcal{A}}^{H} := [\phi \in \mathrm{Hom}(H, \mathcal{A}) : \phi(\mathbb{1}) = \mathbb{1}_{\mathcal{A}}] \quad \text{and} \tag{2.2.10}$$

$$\widetilde{G}_{\mathcal{A}}^{H} := \left\{\phi \in G_{\mathcal{A}}^{H} : \phi \circ m_H = m_{\mathcal{A}} \circ (\phi \otimes \phi)\right\}. \tag{2.2.11}$$

Lemma 2.2.9. *Let H be a connected bialgebra and $\mathcal{A}$ be an algebra, then for any $\phi \in G_{\mathcal{A}}^{H}$ there exists an unique convolution inverse $\phi^{\star -1} \in G_{\mathcal{A}}^{H}$. In other words $(G_{\mathcal{A}}^{H}, \star)$ is a group.*

Proof. Since $(\phi - e)(\mathbb{1}) = 0$ and H is connected for any $x \in H$ there is a $N \in \mathbb{N}$ such that $(\phi - e)^{\otimes n} \circ \Delta^{n-1}(x) = 0$ for any $n \geq N$. Therefore the formal series

$$\phi^{\star -1} = [e - (e - \phi)]^{\star -1} := \sum_{n \in \mathbb{N}_0} (e - \phi)^{\star n} \tag{2.2.12}$$

is locally convergent as finite sum and defines an element from $G_{\mathcal{A}}^{H}$. Recall that the coproduct is a finite sum which we denote in Sweedler's notation

$$\Delta(x) = \sum_{i=1}^{k} x_1^{(i)} \otimes x_2^{(i)} \tag{2.2.13}$$

for some $k \in \mathbb{N}$. Now only dealing with finite sums we can calculate

$$\begin{aligned}
\left[\phi \star \phi^{\star -1}\right](x) &= \sum_{i=1}^{k} \phi\left(x_1^{(i)}\right) \phi^{\star -1}\left(x_2^{(i)}\right) = \sum_{i=1}^{k} \phi\left(x_1^{(i)}\right) \sum_{n=0}^{N} (e - \phi)^{\star}\left(x_2^{(i)}\right) \\
&= \sum_{n=0}^{N} \left[\phi \star (e - \phi)^{\star n}\right](x) = \sum_{n=0}^{N} \left[(e - (e - \phi)) \star (e - \phi)^{\star n}\right](x) \\
&= \left[\sum_{n=0}^{N} (e - \phi)^{\star n} - (e - \phi) \star \sum_{n=0}^{N} (e - \phi)^{\star n}\right](x) \\
&= \left[(e - \phi)^{\star 0} - (e - \phi)^{\star N+1}\right](x) = e(x).
\end{aligned}$$

In the last step we used $(e - \phi)^{\star N+1}(x) = 0$ by definition of N. Analogous one can show the identity $\phi^{\star -1} \star \phi = e$ which proves the existence of an inverse. Uniqueness simply follows from assuming $\psi, \chi \in G_{\mathcal{A}}^{H}$ be a convolution inverse of ϕ:

$$\psi = \psi \star e = \psi \star (\phi \star \chi) = (\psi \star \phi) \star \chi = e \star \chi = \chi \tag{2.2.14}$$

Finally notice that $G_{\mathcal{A}}^{H}$ is closed under the convolution product because for $\phi, \psi \in G_{\mathcal{A}}^{H}$ it holds

$$\phi \star \psi(\mathbb{1}) = \phi(\mathbb{1}) \cdot \psi(\mathbb{1}) = \mathbb{1}_{\mathcal{A}}. \tag{2.2.15}$$

□

2.3 Universal Property of $H_{\mathcal{D}}$

Now the subject of discussion is the universal property of the Hopf algebra of decorated rooted trees. This is the main ingredient for the definition of Feynman rules in the next chapter.

Theorem 2.3.1 (Universal Property)**.** *Let $\mathcal{A}$ be a commutative algebra, $L_i \in \operatorname{End}(\mathcal{A})$ for $i = 1, \cdots N$, then there exists an unique morphism of algebras $\rho : H_{\mathcal{D}} \to \mathcal{A}$ such that for any $i = 1, \cdots N$ it holds*

$$\rho \circ B_+^i = L_i \circ \rho, \quad \textit{or equivalently} \quad \begin{tikzcd} H_{\mathcal{D}} \arrow[r, "\rho"] \arrow[d, "B_+^i"'] & \mathcal{A} \arrow[d, "L_i"] \\ H_{\mathcal{D}} \arrow[r, "\rho"'] & \mathcal{A} \end{tikzcd} \tag{2.3.1}$$

commutes. In addition to that let every L_i be a cocycle and $\mathcal{A}$ be a bi-/Hopf algebra, then ρ is a morphism of bi-/Hopf algebras.

Before we prove this theorem, let us exhibit the following

Lemma 2.3.2. *Let $(H, m, \mathbb{1}, \Delta, \epsilon)$ be a bialgebra and $L \in \operatorname{End}(H)$ be a cocycle, then*

$$L(H) \subseteq \ker \epsilon. \tag{2.3.2}$$

Furthermore in the case of H is a Hopf algebra with antipode S the following identity holds.

$$S \circ L = -S \star L \tag{2.3.3}$$

Proof. The first statement follows from direct calculation:

$$\begin{aligned} \epsilon \circ L &= (\epsilon \otimes \epsilon) \circ \Delta \circ L && (2.3.4)\\ &= (\epsilon \otimes \epsilon) \circ [(id \otimes L) \circ \Delta + L \otimes \mathbb{1}] && (2.3.5)\\ &= \epsilon \circ (L \circ (\epsilon \otimes id) \circ \Delta + L) = 2\epsilon \circ L. && (2.3.6) \end{aligned}$$

Once we have noticed that $L(H) \in \ker \epsilon$ we find

$$\begin{aligned} 0 = u \circ \epsilon \circ L = (S \star id) \circ L &= m \circ (S \otimes id) \circ \Delta \circ L && (2.3.7)\\ = m \circ (S \otimes id) \circ ((id \otimes L) \circ \Delta + L \otimes \mathbb{1}) & && (2.3.8)\\ = S \star L + S \circ L. & && (2.3.9) \end{aligned}$$

□

Now we can prove the above theorem concerning the universal property of $H_{\mathcal{D}}$.

Proof. First notice that ρ is uniquely determined on $F^0 \subseteq H_{\mathcal{D}}$ by $\rho(\mathbb{1}) = \mathbb{1}_{\mathcal{A}}$ due to the assumption to be a morphism of algebras. Now for proving uniqueness of ρ we show that it follows ρ is uniquely determined on F^{n+1} by assuming ρ to be determined on F^n.

As ρ is a morphism of algebras it suffices to show uniqueness on the set of decorated rooted trees. For $T \in F^{n+1}$ there is an $i \in \{1, \cdots, N\}$ such that $T = B_+^i(f)$ with a forest $f \in F^n$. So we have

$$\rho(T) = \rho \circ B_+^i(f) = L_i \circ \rho(f). \tag{2.3.10}$$

That shows uniqueness of ρ on F^{n+1} and hence on $H_{\mathcal{D}}$.

The existence of ρ is provided by taking equation (2.3.1) as inductive definition.

By construction ρ is immediately a morphism of algebras. So it only remains to show that ρ is even a morphism of bi-/Hopf algebras under the assumption that L_i is a cocycle of the bi-/Hopf algebra $\mathcal{A}$.

By the above lemma 2.3.2 we know

$$\rho(\operatorname{im} B_+^i) \subseteq L_i(\operatorname{im} \rho) \subseteq \ker \epsilon_{\mathcal{A}} \tag{2.3.11}$$

for any $i \in \{1, \cdots, N\}$. Therefore it holds

$$\epsilon_{\mathcal{A}} \circ \rho(f) = 0 = \epsilon(f) \quad \forall f \in \mathcal{F}. \tag{2.3.12}$$

Together with

$$\epsilon_{\mathcal{A}} \circ \rho(\mathbb{1}) = \epsilon_{\mathcal{A}}(\mathbb{1}) = 1 = \epsilon(\mathbb{1}) \tag{2.3.13}$$

we obtain $\epsilon_{\mathcal{A}} \circ \rho = \epsilon$ and it remains to show $\Delta_{\mathcal{A}} \circ \rho = \Delta$. Again it suffices to show the identity to be true on the set of decorated rooted trees and it is simply seen that the equality holds on F^0. Assuming the equality to be satisfied on F^n let $t \in F^{n+1}$ be a decorated rooted tree and $t = B_+^i(f)$ for some $f \in F^n$ and $i \in \{1, \cdots N\}$.

$$\begin{aligned}
\Delta_{\mathcal{A}} \circ \rho(t) &= \Delta_{\mathcal{A}} \circ \rho \circ B_+^i(f) = \Delta_{\mathcal{A}} \circ L^i \circ \rho(f) && (2.3.14)\\
&= \Big[(id_{\mathcal{A}} \otimes L^i) \circ \Delta_{\mathcal{A}} + L^i \otimes \mathbb{1}_{\mathcal{A}}\Big] \circ \rho(f) && (2.3.15)\\
&= (id_{\mathcal{A}} \otimes L^i) \circ \Delta_{\mathcal{A}} \circ \rho(f) + L^i \circ \rho(f) \otimes \rho(\mathbb{1}) && (2.3.16)\\
&= (\rho \otimes \rho) \circ \Big[(id_{\mathcal{A}} \otimes B_+^i) \circ \Delta + (B_+^i \otimes \mathbb{1})\Big](f) && (2.3.17)\\
&= (\rho \otimes \rho) \circ \Delta \circ B_+^i(f). && (2.3.18)
\end{aligned}$$

At the 4th line we made use of the induction assumption $\Delta_{\mathcal{A}} \circ \rho(f) = (\rho \otimes \rho) \circ \Delta(f)$. Thus we showed that ρ is a morphism of bialgebras.

Finally we consider the Hopf algebra case. We have to show $S_{\mathcal{A}} \circ \rho = \rho \circ S$. Obviously this equality holds on F^0. Once again we make use of the induction axiom and assume the equality to be hold on F^n. Now we show the equation to be satisfied for any decorated rooted tree $t = B_+^i(f) \in F^{n+1}$.

$$\begin{aligned}
S_{\mathcal{A}} \circ \rho(t) &= S_{\mathcal{A}} \circ \rho \circ B_+^i(f) = S \circ L^i \circ \rho(f) && (2.3.19)\\
&= -(S_{\mathcal{A}} \star L^i) \circ \rho(f) = -(S_{\mathcal{A}} \circ \rho) \star (L^i \circ \rho)(f) && (2.3.20)\\
&= -\rho \circ (S \star B_+^i)(f) = \rho \circ S \circ B_+^i(f) && (2.3.21)
\end{aligned}$$

Besides of the induction assumption we made use of lemma 2.3.2 and the cocycle property of B_+^i[5]. Now the validity of the equality for any element from F^{n+1} is proven by the property that any antipode is an antimorphism of algebras. For any $t_1, t_2 \in \mathcal{T}_{\mathcal{D}}$

$$\begin{aligned}
S_{\mathcal{A}} \circ \rho(t_1 t_2) &= S_{\mathcal{A}} \circ m\left(\rho(t_1) \otimes \rho(t_2)\right) && (2.3.22)\\
&= m \circ \tau\left(S_{\mathcal{A}} \circ \rho(t_1) \otimes S_{\mathcal{A}} \circ \rho(t_2)\right) && (2.3.23)\\
&= m \circ \tau\left(\rho \circ S(t_1) \otimes \rho \circ S(t_2)\right) && (2.3.24)\\
&= \rho \circ m \circ \tau\left(S(t_1) \otimes S(t_2)\right) && (2.3.25)\\
&= \rho \circ S(t_1 t_2). && (2.3.26)
\end{aligned}$$

Here τ denotes a permutation of the components in the tensor product, i.e. $\tau(t_1 \otimes t_2) = t_2 \otimes t_1$. We do not prove the antipode to be an antimorphism at this place but the interested reader may consult to [6]. □

[5] see proposition 2.2.4

Chapter 3

Kreimer's toy model

After the discussion of the Hopf algebra of decorated rooted trees, we proceed with a definion of Feynman rules on this algebra. The combination of both, the set of decorated rooted trees and these Feynman rules constitute Kreimer's toy model of a quantum field theory. The treatment of regularization and renormalization in this chapter ensures physical results and allow a comparison between the perturbation series and the non-perturbative results of the next chapter. We mainly review the results of [4].

3.1 Analytically regularized Feynman rules

Before we set up the common definiton of Kreimer's toy model we like to study one special example in order to motivate this toy model and to show the analogy between the well known methods of regularization and analytic continuation of a Mellin transform that we use in the sequel.

Define $f : \mathbb{R}_+^2 \to \mathbb{C}, (\zeta, s) \mapsto \frac{1}{\zeta+s}$ and regard its Mellin transform

$$F(z,s) = \int_0^\infty f(\zeta,s)\zeta^{-z} d\zeta = s^{-z} \int_0^\infty \frac{\zeta^{-z}}{\zeta+1} d\zeta = s^{-z}\beta(1-z,z). \tag{3.1.1}$$

In the last step we used the integral representation of Euler's beta function

$$\beta(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\, dt \quad \text{for } \Re(x), \Re(y) > 0. \tag{3.1.2}$$

Therefore we find $0 < \Re(z) < 1$ as domain of convergence of the Mellin transform F. However the beta function can be understood as meromorphic continuation of the Mellin integral. This is the central concept of analytic regularization. In order to obtain finite integrals we introduced the artificial parameter z. But we are interested in the physical limit of a vanishing regularizer $z \to 0$. Therefore it is necessary to use further methods of renormalization to obtain holomorphic expressions. That will be done in the sequel. However before we do that, let us continue by using the gamma function representation of the beta function and end up with

$$F(z,s) = s^{-z}\frac{1}{z}\Gamma(1+z)\Gamma(1-z). \tag{3.1.3}$$

This expression gives qualitatively the same pole-structure as the computation of a massless one loop graph in dimensional regularization:

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k-q)^2} = \frac{1}{(4\pi)^2}\left(\frac{q^2}{4\pi}\right)^{-\epsilon} \frac{1}{\epsilon}\frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} \tag{3.1.4}$$

Where the dimension is set to $D = 4 - 2\epsilon$.

This motivates the following definition of our toy model. Henceforth for $i = 1, \cdots N$ we assume $f_i : \mathbb{R}_+^2 \mapsto \mathbb{R}$ to be a homogeneous function of degree -1. That is $f(\zeta, s) = s^{-1} f(\zeta/s, 1) =: s^{-1} f(\zeta/s)$ and we keep on suppressing the second variable in our notation whenever it is equal to 1. Moreover $f(\zeta)$ is assumed to be an analytic function at $\zeta = 0$ and $\mathcal{O}(\zeta^{-1})$ as $\zeta \to \infty$. As above these assumptions ensure the existence of a meromorphic continuation of the Mellin transform $F_i(z, s)$ for at least $-1 < \Re(z) < 1$. Furthermore the assumed asymptotic behaviour implies the occurrence of a single pole of order one in the denoted domain of z. So F_i is given as Laurent series[1]

$$F_i(z, s) = \frac{c_{i,-1}}{z} + \sum_{n \in \mathbb{N}_0} c_{i,n}(s)\, z^n. \tag{3.1.5}$$

Note, at this point we restricted our discussion to *logarithmically divergent graphs*. The process of renormalization discussed in the following can be generalised to higher degrees of divergence using suitable projective characters in accordance with the BPHZ scheme. Nevertheless we like to restrict ourselves by studying Dyson Schwinger equations which are renormalized by a simple subtraction and continue with

Definition 3.1.1. *For any pair $(z, s) \in \mathbb{R}_+^2$ the* regularized Feynman rules $\phi_{z,s}$ *are defined as algebra morphism through the universal property of $H_{\mathcal{D}}$ where $\mathcal{A} = \mathbb{C}[\ln s][z^{-1}, z]]$ and*

$$\phi_{z,s} \circ B_+^i(\cdot) = \int_0^\infty f_i(\zeta, s) \zeta^{-z} \phi_{z,\zeta}(\cdot) d\zeta. \tag{3.1.6}$$

The next proposition provides a characterisation of the multiplicative structure of Kreimer's toy model and we reveal this structure by studying a class of Feynman graphs in Yukawa theory in section 5.

Proposition 3.1.2. *Let $f \in \mathcal{F}$ be a forest, then it holds*

$$\phi_{z,s}(f) = s^{-z|f|} \prod_{v \in V(f)} F_{d(v)}\left(z \left|\mathcal{P}^{\{v\}} f\right|\right). \tag{3.1.7}$$

Were we abbreviated the number of nodes of the forest f by $|f|$.

Proof. The inductive proof use the graduated structure of $H_{\mathcal{D}}$. Due to definition (3.1.6) the induction basis on H_0 becomes trivial. Moreover it suffices to prove the identity for the set of decorated rooted trees by multiplicativity of both sides of equation (3.1.7). Now let $t = B_+^i(f)$ with a forest f satisfying the condition (3.1.7), then it follows

$$\begin{aligned}
\phi_{z,s}(B_+^i(f)) &= \int_0^\infty f_i(\zeta)(s\zeta)^{-z} \phi_{z,s\zeta}(f)\, d\zeta \\
&= \int_0^\infty f_i(\zeta)(s\zeta)^{-z}(s\zeta)^{-z|f|} \prod_{v \in V(f)} F_{d(v)}\left(z \left|\mathcal{P}^{\{v\}}(f)\right|\right) \\
&= s^{-z(1+|f|)} \prod_{v \in V(f)} F_{d(v)}\left(z \left|\mathcal{P}^{\{v\}}(f)\right|\right) \int_0^\infty f_i(\zeta)\zeta^{-z(1+|f|)}\, d\zeta \\
&= s^{-z|t|} \prod_{v \in V(t)} F_{d(v)}\left(z \left|\mathcal{P}^{\{v\}}(t)\right|\right).
\end{aligned}$$

Which finishes the proof. □

[1] It should be remarked that the residue is not s dependent due to the homogeneity of f. This assumption ensures local counter terms.

3.2 The Birkhoff decomposition

Now we give a common statement of the Birkhoff decomposition in order to apply it to the regularized Feynman rules in the next paragraph. During this paragraph we suppose H to be a conncted bialgebra, $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ an algebra decomposed into the direct sum of vector spaces $\mathcal{A}_\pm$.

Definition 3.2.1. *Let $\phi \in G_\mathcal{A}^H$ then a pair of mappings $\phi_\pm \in G_\mathcal{A}^H$ satisfying*

$$\phi = \phi_-^{\star -1} \star \phi_+ \quad \textit{and} \quad \phi_\pm(\ker \epsilon) \subseteq \mathcal{A}_\pm \tag{3.2.1}$$

is called Birkhoff decomposition *of ϕ.*

Proposition 3.2.2. *For $\phi \in G_\mathcal{A}^H$ and a projection $R : \mathcal{A} \twoheadrightarrow \mathcal{A}_-$ there exists a unique Birkhoff decomposition which can be recursively computed for $x \in \ker \epsilon$ by*

$$\phi_-(x) = -R\left(\bar{\phi}(x)\right) \quad \textit{and} \quad \phi_+(x) = (id - R)\left(\bar{\phi}(x)\right) \tag{3.2.2}$$

$$\bar{\phi} = \phi + m \circ (\phi_- \otimes \phi) \circ \widetilde{\Delta} \quad \textit{i.e.} \quad \bar{\phi}(x) = \phi(x) + \sum \phi_-(x')\phi(x''). \tag{3.2.3}$$

Moreover if $\mathcal{A}$ is commutative, $\phi \in \widetilde{G}_\mathcal{A}^H$ is a morphism of unital algebras and $R \in \widetilde{G}_\mathcal{A}^\mathcal{A}$ is a projective character, then $\phi_\pm$ are algebra morphisms themselves.

Proof. The uniqueness of an existing Birkhoff decomposition $\phi_\pm$ comes from

$$\phi_+ - \phi_- = \phi + \phi_- \star \phi - \phi - \phi_- = \bar{\phi}. \tag{3.2.4}$$

Which implies the identity (3.2.2). Since $\phi_\pm(\mathbb{1}) = \mathbb{1}_\mathcal{A}$ the Birkhoff decomposition is uniquely determined on the start $H^0 \subseteq H$ of the filtration and recursively defined on H^{n+1} through (3.2.2) and $\widetilde{\Delta}(H^{n+1}) \subseteq \sum_{i=1}^n H^i \otimes H^{n+1-i}$ in an unique manner. For existence we define ϕ_- by (3.2.2) and set $\phi_+ := \phi_- \star \phi$. For $x \in \ker \epsilon$ it simply follows

$$\phi_+(x) = [\phi_- \star \phi](x) = \left[\phi_- + \bar{\phi}\right](x) = \left[(id - R) \circ \bar{\phi}\right](x). \tag{3.2.5}$$

Which proves the second identity (3.2.2) for ϕ_+ and $\phi_+(\ker \epsilon) \subseteq \mathcal{A}_+ = \ker R$.

For showing the algebra morphism property firstly notice that a projective character fulfils the Rota-Baxter equation

$$R \circ [(Rx)y + x(Ry) - xy] = R(x)R(y). \tag{3.2.6}$$

Due to the equation $\phi_+ = \phi_- + \bar{\phi}$ and the recursive formula for $\bar{\phi}$ it suffices to show ϕ_- is a morphism of algebras. Note that $\phi_-(\mathbb{1}) = \mathbb{1}_\mathcal{A}$ implies $\phi_-(\mathbb{1} \cdot x) = \phi_-(\mathbb{1}) \cdot \phi_-(x)$ for any $x \in H$. So the statement is clearly satisfied on H^0 giving the start of the induction. Now assume $x \cdot y \in H^{n+1}$ with $x \in H^{n_1}$, $y \in H^{n_2}$ and without loss of generality $0 < n_1, n_2 < n$ and the induction assumption ϕ_- to be a morphism of algebras on H^n in particular $\phi_-(xy') = \phi_-(x)\phi_-(y')$ and $\phi_-(x'y) = \phi_-(x')\phi_-(y)$, then in follows:

$$
\begin{aligned}
\phi_-(xy) \underset{(3.2.2)}{=} & -R\left[\phi(xy) + \sum_{xy} \phi_-\left(\{xy\}'\right)\phi\left(\{xy\}''\right)\right] \\
= & -R\Bigg[\phi(x)\phi(y) + \sum_x \phi_-(x')\phi(x'') \sum_y \phi_-(y')\phi(y'') + \phi(x)\phi_-(y) + \phi_-(x)\phi(y) \\
& + \{\phi(x) + \phi_-(x)\} \sum_y \phi_-(y')\phi(y'') + \{\phi(y) + \phi_-(y)\} \sum_x \phi_-(x')\phi(x'')\Bigg] \\
= & -R\Bigg[\left\{\phi(x) + \sum_x \phi_-(x')\phi(x'')\right\} \cdot \left\{\phi(y) + \sum_y \phi_-(y')\phi(y'')\right\} \\
& + \phi_-(x) \cdot \left\{\phi(y) + \sum_y \phi_-(y')\phi(y'')\right\} + \left\{\phi(x) + \sum_x \phi_-(x')\phi(x'')\right\} \cdot \phi_-(y)\Bigg] \\
= & \; R\left[\left\{R\bar{\phi}(x)\right\}\bar{\phi}(y) + \bar{\phi}(x)\left\{R\bar{\phi}(y)\right\} - \bar{\phi}(x)\bar{\phi}(y)\right] \\
\underset{(3.2.6)}{=} & \left[R\bar{\phi}(x)\right] \cdot \left[R\bar{\phi}(y)\right] \\
\underset{(3.2.2)}{=} & \left[-\phi_-(x)\right] \cdot \left[-\phi_-(y)\right] \quad = \quad \phi_-(x) \cdot \phi_-(y)
\end{aligned}
$$

In the 2nd equation we rewrote

$$
\begin{aligned}
\widetilde{\Delta}(xy) &= \Delta(xy) - \mathbb{1} \otimes xy - xy \otimes \mathbb{1} = \Delta(x)\Delta(y) - \mathbb{1} \otimes xy - xy \otimes \mathbb{1} \\
&= (\widetilde{\Delta}(x) + \mathbb{1} \otimes x + x \otimes \mathbb{1})(\widetilde{\Delta}(y) + \mathbb{1} \otimes y + y \otimes \mathbb{1}) - \mathbb{1} \otimes xy - xy \otimes \mathbb{1}
\end{aligned}
$$

and used the commutativity of $\mathcal{A}$.

□

3.3 Feynman rules on $H_\mathcal{D}$ and their regularity

By considering regularized Feynman rules we found results as meromorphic continuation in z. Therefore the physical limit of a vanishing regulator $z \to 0$ is not convergent yet. Before taking this limit we have to construct renormalized Feynman rules depending holomorphic on z. This task is provided by the Birkhoff decomposition as we see in the sequel.

From this point on we specify the notation through

$$\mathcal{A} := \mathbb{C}[\ln s][z^{-1}, z]] \tag{3.3.1}$$

$$\mathcal{A}_- := \mathbb{C}[z^{-1}, z]] \tag{3.3.2}$$

$$R_\mu F(s, z) := F(\mu, z) \in \mathbb{C}[z^{-1}, z]] \tag{3.3.3}$$

Here $\mathcal{A}$ denotes the set of Laurent series with finite negative order with coefficents in $\mathbb{C}[\ln s]$ and R_μ maps a polynomial from $\mathbb{C}[\ln s]$ to $\mathbb{C}$ by evaluation at $s = \mu$.

From proposition 3.2.2 we get the

Corollary 3.3.1. *For a projective character $R = R^2 \in \widetilde{G}^{\mathcal{A}}_{\mathcal{A}}$ the Birkhoff decomposition of regularized Feynman rules reads*

$$\phi^R_{z,s} := (\phi_{z,s})_+ = (R \circ \phi_{z,s} \circ S) \star \phi_{z,s} \tag{3.3.4}$$

$$Z_{z,s} := (\phi_{z,s})_- = R \circ \phi_{z,s} \circ S \tag{3.3.5}$$

with algebra morphisms $\phi^R_{z,s}$ called regularized and renormalized Feynman rules *and $Z_{z,s}$ the* counter term.

Proof. First notice that the convolution inverse of $\phi_{z,s}$ is given by composition with the antipode $\phi_{z,s}^{\star -1} = \phi_{z,s} \circ S$. Indeed for any $f \in \mathcal{F}$ we find

$$\begin{aligned}(\phi_{z,s} \circ S) \star \phi_{z,s}(f) &= m_{\mathcal{A}} \circ (\phi_{z,s} \circ S \otimes \phi_{z,s}) \circ \Delta(f) = \phi_{z,s} \circ m(S \otimes id) \circ \Delta(f) \\ &= \phi_{z,s} \circ u \circ \epsilon(f) = u_{\mathcal{A}} \circ \epsilon(f) = e(f).\end{aligned}$$

Now we define $(\phi_{z,s})_- := R \circ \phi_{z,s} \circ S = R \circ \phi_{z,s}^{\star -1}$ and $(\phi_{z,s})_+ := (\phi_{z,s})_- \star \phi_{z,s}$ and determine their images in $\mathcal{A}$. Because $R \circ \phi_{z,s}^{\star -1}(\ker \epsilon) \subseteq \operatorname{im} R \subseteq \mathcal{A}_-$ and

$$\begin{aligned}R \circ \left(R \circ \phi_{z,s}^{\star -1} \star \phi_{z,s}\right) &= \left(R^2 \circ \phi_{z,s}^{\star -1}\right) \star (R \circ \phi_{z,s}) \\ &= R \circ \left(\phi_{z,s}^{\star -1} \star \phi_{z,s}\right) = R \circ e = e\end{aligned}$$

we find that $(\phi_{z,s})_\pm (\ker \epsilon) \subseteq \mathcal{A}_\pm$. By construction of $(\phi_{z,s})_\pm$ it follows that

$$\phi_{z,s} = (\phi_{z,s})_-^{\star -1} \star (\phi_{z,s})_+ \,. \tag{3.3.6}$$

Therefore the pair $(\phi_{z,s})_\pm$ is a Birkhoff decomposition of $\phi_{z,s}$ and by proposition 3.2.2 it is unique. ☐

Finally we go to the physical limit $z \to 0$ of the regularized and renormalized Feynman rules. This is the subject of

Definition 3.3.2. *The strong limit of the regularized and renormalized algebra morphisms* $\lim\limits_{z\to 0} \phi_{z,s}^R =: \phi_s$ *is called* Feynman rules.

In preparation of the existence theorem we prove the following

Lemma 3.3.3. *Let H be a connected bialgebra, $\mathcal{A}$ a commutative algebra with endomorphisms $L_i \in \operatorname{End}(\mathcal{A})$ for $i \in \mathcal{D}$ and $\rho : H \to \mathcal{A}$ the universal morphism through* (2.3.1) *and $R \in \operatorname{End}(\mathcal{A})$ a projection such that*

$$L \circ m_{\mathcal{A}}(\phi_- \otimes id) = m_{\mathcal{A}} \circ (\phi_- \otimes L). \tag{3.3.7}$$

Then it holds

$$\bar{\phi} \circ B_+^i = L_i \circ \phi_+. \tag{3.3.8}$$

Here $\phi_\pm$ denotes the Birkhoff decomposition of ϕ with the projection R.

Proof. The proof is given by an explicit calculation:

$$\begin{aligned}\bar{\phi} \circ B_+^i &= (\phi_+ - \phi_-) \circ B_+^i = (\phi_- \star \phi - \phi_-) \circ B_+^i \\ &= m_{\mathcal{A}} \circ (\phi_- \otimes \phi) \circ \Delta \circ B_+^i - \phi_- \circ B_+^i \\ &= m_{\mathcal{A}} \circ (\phi_- \otimes \phi) \circ \left[(id \otimes B_+^i) \circ \Delta + B_+^i \otimes \mathbb{1}\right] - \phi_- \circ B_+^i \\ &= m_{\mathcal{A}} \circ \left[\phi_- \otimes (\phi \circ B_+^i)\right] \circ \Delta = m_{\mathcal{A}} \circ [\phi_- \otimes (L \circ \phi)] \circ \Delta \\ &= L \circ m_{\mathcal{A}}(\phi_- \otimes \phi) \circ \Delta = L \circ (\phi_- \star \phi) = L \circ \phi_+\end{aligned}$$

☐

Indeed in the context of the regularized Feynman rules $\rho = \phi_{z,s}$ equation (3.3.7) is satisfied because $(\phi_{z,s})_-$ maps to a constant which can be pulled out the integral of (3.1.6). Therefore we may apply the above lemma in

Proposition 3.3.4 (Existence of Feynman rules)**.** *The limit* $\lim_{z\to 0}\phi_{z,s}$ *exists and maps* $H_{\mathcal{D}}$ *to* $\mathbb{C}[\ln\frac{s}{\mu}]$.

Proof. The inductive proof has a trivial start on H^0 given by the Mellin transform (3.1.5). Now assuming the statement to hold on H^n we show that it also holds for every decorated rooted tree $t \in H^{n+1}$. Using the lemma 3.3.3 we obtain:

$$\begin{aligned}\phi_s(t) &= \lim_{z\to 0}(id - R_\mu)\,\phi^{-}_{z,s}\circ B^i_+(f) = \lim_{z\to 0}(id-R_\mu)\,L_i\circ\phi^R_{z,s}\\ &= \lim_{z\to 0}\int_0^\infty \underbrace{\left[\frac{f_i(\zeta/s)}{s} - \frac{f_i(\zeta/\mu)}{\mu}\right]}_{\mathcal{O}(\zeta^{-2})}\zeta^{-z}\phi^R_{z,\zeta}(f)\,d\zeta \end{aligned} \tag{3.3.9}$$

By the induction assumption $\phi^R_{z,\zeta}(f)\in\mathcal{O}(\ln^N\zeta)$ for a $N\in\mathbb{N}$. As integrals of type $\int_0^\infty {}^1\!/\!{}_{\zeta^2+1}\ln^N(\zeta)\,d\zeta$ are finite for any $N\in\mathbb{N}$ (integration by parts and induction over N), the integral in (3.3.9) converges for a suitable majorant and thus by the dominated convergence theorem it holds

$$\phi_s(t) = \int_0^\infty \left[\frac{f_i(\zeta/s)}{s} - \frac{f_i(\zeta/\mu)}{\mu}\right]\phi_\zeta(f)\,d\zeta \tag{3.3.10}$$

$$= \int_0^\infty \left[\frac{f_i(\zeta\frac{\mu}{s})}{\frac{s}{\mu}} - f_i(\zeta)\right]\underbrace{\phi_{\mu\zeta}(f)}_{\in\mathbb{C}[\ln\zeta]}\,d\zeta \tag{3.3.11}$$

Which implies $\phi_s(t)\in\mathbb{C}[\ln\frac{s}{\mu}]$. This proves the statement to be satisfied on H^{n+1} because, as limit of algebra morphisms, ϕ_s is a morphism of algebras as well. □

Now we are in a very well-behaved situation. We found that ϕ_s maps $H_{\mathcal{D}}$ to $\mathbb{C}[\ln s/\mu]$. Therefore we may assume $\mu = 1$ without loss of generality or substitute $\ln {}^s\!/\!{}_\mu =: x$. Moreover we remind the reader that the set of polynomials $\mathbb{C}[x]$ is also a graded Hopf algebra by the requirement $\Delta x = \mathbb{1}\otimes x + x\otimes\mathbb{1}$.

The next step in our discussion comes from the fact that we defined ϕ_s as a limit of universal morphisms. Now we explore if the Feynman rules arise from the universal property.

For $i\in\mathcal{D}$ we define $\eta_i\in\mathbb{C}[x]'$ and $L_i\in\mathrm{End}(\mathbb{C}[x])$ through

$$\eta_i(x^N) := (-1)^N N!\,c_{i,N} \quad \forall N\in\mathbb{N}_0 \tag{3.3.12}$$

$$L_i(\phi_s) := (-c_{i,-1}\int_0 + \partial\eta_i)(\phi_s) \tag{3.3.13}$$

with a *coboundary* ∂ in an abstract Hochschild cohomology, whereas it suffices to know the property $\partial\eta_i(x^N) = \sum_{j=1}^N\binom{N}{j}\eta_i(x^{N-j})x^j$ at this place.

Proposition 3.3.5. *The universal algebra morphism due to theorem 2.3.1 and* L_i *is given by the Feynman rules* ϕ_s.

$$\begin{tikzcd} H_{\mathcal{D}} \arrow[r, "\phi_s"] \arrow[d, "B^i_+"'] & \mathbb{C}[\ln {}^s\!/\!{}_\mu] \arrow[d, "L_i"] \\ H_{\mathcal{D}} \arrow[r, "\phi_s"'] & \mathbb{C}[\ln {}^s\!/\!{}_\mu] \end{tikzcd} \quad \textit{or} \quad \phi_s\circ B^i_+ = L_i\circ\phi_s \quad \forall i\in\mathcal{D} \tag{3.3.14}$$

Proof. The proof mainly consists of calculating the possible integrals with integrands in $\mathbb{C}[\ln s]$ (set $\mu = 1$). For arbitrary $N \in \mathbb{N}_0$ and $i \in \mathcal{D}$ it holds:

$$\begin{aligned}
&\lim_{z\to 0}(id - R_1)\left[\int_0^\infty f_i(\zeta)(s\zeta)^{-z}\ln^N(s\zeta)\,d\zeta\right] \\
&= \left(-\frac{\partial}{\partial z}\right)^N_{z=0} (id - R_1)\int_0^\infty f_i(\zeta)(s\zeta)^{-z}\,d\zeta \\
&= \left(-\frac{\partial}{\partial z}\right)^N_{z=0} (s^{-z} - 1)\int_0^\infty f_i(\zeta)\zeta^{-z}\,d\zeta \\
&= \left(-\frac{\partial}{\partial z}\right)^N_{z=0} \left[\frac{s^{-z}-1}{z} zF_i(z)\right] \\
&= (-1)^N \sum_{k=0}^N \binom{N}{k} \left[\left(\frac{\partial}{\partial z}\right)^k_{z=0} \frac{s^{-z}-1}{z}\right]\left[\left(\frac{\partial}{\partial z}\right)^{N-k}_{z=0}(zF_i(z))\right] \\
&= (-1)^N \sum_{k=0}^N \binom{N}{k} k!\frac{(-\ln s)^{k+1}}{(k+1)!}(N-k)!\,c_{i,N-k-1} \\
&= \left[\sum_{k=0}^N \frac{N!}{(k+1)!}x^{k+1}(-1)^{N-k-1}c_{i,N-k-1}\right]_{x=\ln s} \\
&= \left[-c_{i,-1}\frac{x^{N+1}}{N+1} + \sum_{k=1}^N \binom{N}{k} x^k(-1)^{N-k}(N-k)!\,c_{i,N-k}\right]_{x=\ln s} \\
&= L_i(x^N)\Big|_{x=\ln s}
\end{aligned}$$

Now let $f \in H_{\mathcal{D}}$. We know that $\phi_s(f)$ is a polynomial. Therefore we can apply the above identity and use linearity of ϕ_s and L_i.

$$\begin{aligned}
\phi_s \circ B_+^i(f) &= \int_0^\infty \left[\frac{f_i(\zeta/s)}{s} - \frac{f_i(\zeta)}{1}\right]\phi_\zeta(f)\,d\zeta \\
&= \lim_{z\to 0}\int_0^\infty \left[\frac{f_i(\zeta/s)}{s} - \frac{f_i(\zeta)}{1}\right]\zeta^{-z}\phi_\zeta(f)\,d\zeta \\
&= \lim_{z\to 0}(id - R_1)\int_0^\infty f_i(\zeta)(s\zeta)^{-z}\phi_{s\zeta}(f)\,d\zeta \\
&= L_i \circ \phi_s(f)
\end{aligned}$$

$\square$

Proposition 3.3.5 implies the following

Corollary 3.3.6. *The Feynman rules ϕ_s are a morphism of Hopf algebras.*

Proof. Due to theorem 2.3.1 it remains to show that L is a cocycle. Firstly notice that $\int_0$ is a cocycle on $\mathbb{C}[x]$:

$$\begin{aligned}
\Delta\left(\int_0 x^N\right) &= \frac{1}{N+1}\Delta x^{N+1} = \frac{1}{N+1}\sum_{k=0}^{N+1}\binom{N+1}{k}x^k \otimes x^{N+1-k} \\
&= \frac{x^{N+1}}{N+1}\otimes 1 + N!\sum_{k=0}^{N}\frac{x^k}{k!}\otimes\frac{x^{N+1-k}}{(N+1-k)!} \\
&= \left(\int_0 x^N\right)\otimes 1 + \left(id\otimes\int_0\right)\circ\left(\sum_{k=0}^{N}\binom{N}{k}\frac{x^k}{k!}\otimes\frac{x^{N-k}}{(N-k)!}\right) \\
&= \left[\int_0 \otimes 1 + \left(id\otimes\int_0\right)\circ\Delta\right](x^N)
\end{aligned}$$

The fact that $\partial\eta_i$ can be understood as coboundary in a cochain complex trivially implies the cocycle property to be satisfied for the endomorphism $\int_0 + \partial\eta_i$. Nevertheless we want to give an explicit calculation.

$$\begin{aligned}
\Delta\left(\partial\eta_i(x^N)\right) &= \sum_{j=1}^{N}\binom{N}{j}\eta_i(x^{N-j})\Delta(x^j) \\
&= \sum_{j=1}^{N}\sum_{k=0}^{j}\binom{N}{j}\binom{j}{k}\eta_i(x^{N-j})x^{j-k}\otimes x^k \\
&= \sum_{j=1}^{N}\binom{N}{j}\eta_i(x^{N-j})x^j\otimes 1 + \sum_{j=1}^{N}\sum_{k=1}^{j}\binom{N}{j}\binom{j}{k}\eta_i(x^{N-j})x^{j-k}\otimes x^k \\
&= \left(\Delta x^N\right)\otimes 1 + \underbrace{\sum_{k=0}^{N}\sum_{j=k}^{N}\binom{N}{j}\binom{j}{k}\eta_i(x^{N-j})x^{j-k}\otimes x^k}_{=:(*)}
\end{aligned}$$

We finish the proof by observing

$$\begin{aligned}
(*) &= \sum_{k=0}^{N}\sum_{j=0}^{N-k}\binom{N}{j+k}\binom{j+k}{k}\eta_i(x^{N-j-k})x^j\otimes x^k \\
&= \sum_{j=0}^{N-1}\binom{N}{j}x^j\otimes\left(\sum_{k=1}^{N-j}\binom{N-j}{k}\eta_i(x^{N-j-k})x^k\right) \\
&= (id\otimes\partial\eta_i)\circ\left(\sum_{j=0}^{N-1}\binom{N}{j}x^j\otimes x^{N-j}\right) \\
&= (id\otimes\partial\eta_i)\circ\left(\Delta x^N - x^N\otimes 1\right) = (id\otimes\partial\eta_i)\circ\Delta x^N.
\end{aligned}$$

□

Our final task is to investigate the behaviour of Feynman rules under the convolution product. As the renormalization group equation the next lemma implies consequences for the shape of green's functions as we see in the following chapter.

Lemma 3.3.7. *For any $a, b \in \mathbb{K}$ the Feynman rules satisfy*

$$\phi_a \star \phi_b = \phi_{ab/\mu}. \tag{3.3.15}$$

Proof. By corollary 3.3.6 we know that $\phi_s : H_{\mathcal{D}} \to \mathbb{K}[\ln {}^{s}\!/\!{}_{\mu}]$ is a morphism of Hopf algebras. Now for every $a \in \mathbb{K}$ we define the map $\mathrm{ev}_a : \mathbb{K}[\ln {}^{s}\!/\!{}_{\mu}] \to \mathbb{K}$ as evaluation of s at a. It is easily checked that these evaluations are characters of the Hopf algeba of polynomials $\mathbb{K}[\ln {}^{s}\!/\!{}_{\mu}]$, i.e. $\mathrm{ev}_a \in \widetilde{G}_{\mathbb{K}}^{\mathbb{K}[\ln {}^{s}\!/\!{}_{\mu}]}$. Moreover for any $a, b \in \mathbb{K}$ it holds

$$\begin{aligned}
[\mathrm{ev}_a \star \mathrm{ev}_b]\,(\ln {}^{s}\!/\!{}_{\mu})^n &= [(\mathrm{ev}_a \otimes \mathrm{ev}_b) \circ \Delta(\ln {}^{s}\!/\!{}_{\mu})]^n \\
&= [\mathrm{ev}_a(\ln {}^{s}\!/\!{}_{\mu})\,\mathrm{ev}_b(1) + \mathrm{ev}_a(1)\,\mathrm{ev}_b(\ln {}^{s}\!/\!{}_{\mu})]^n \\
&= (\ln {}^{a}\!/\!{}_{\mu} + \ln {}^{b}\!/\!{}_{\mu})^n = [\mathrm{ev}_{ab/\mu}(\ln {}^{s}\!/\!{}_{\mu})]^n \\
&= \mathrm{ev}_{ab/\mu}((\ln {}^{s}\!/\!{}_{\mu})^n)
\end{aligned}$$

for any $n \in \mathbb{N}$ and therefore $\mathrm{ev}_a \star \mathrm{ev}_b = \mathrm{ev}_{ab/\mu}$. Now we find

$$\begin{aligned}
\phi_a \star \phi_b &= (\mathrm{ev}_a \circ \phi_s) \star (\mathrm{ev}_b \circ \phi_s) = m \circ [(\mathrm{ev}_a \circ \phi_s) \otimes (\mathrm{ev}_b \circ \phi_s)] \circ \Delta \\
&= m \circ [\mathrm{ev}_a \otimes \mathrm{ev}_b] \circ \Delta \circ \phi_s = [\mathrm{ev}_a \star \mathrm{ev}_b] \circ \phi_s = \mathrm{ev}_{ab/\mu} \circ \phi_s \\
&= \phi_{ab/\mu}
\end{aligned}$$

which proves the lemma. $\square$

Chapter 4

Systems of linear Dyson Schwinger equations

In this chapter we investigate linear systems of combinatorial Dyson Schwinger equations in the Hopf algebra of decorated rooted trees. Furthermore adding Feynman rules from Kreimer's toy model we study the relationship between the perturbation series in $H_{\mathcal{D}}$ and Green's functions as non-perturbative solutions of a system of corresponding integral Dyson Schwinger equations.

4.1 Basic notation

The general case of a *system* with a number *of N linear Dyson Schwinger equations* is given by

$$X^k(\alpha) = \mathbb{1} + \alpha \sum_{i=1}^{N} B_+^i \left(\sum_{j=1}^{N} a_{ijk} X^j(\alpha) \right) \tag{4.1.1}$$

with $a_{ijk} \in \mathbb{C}$ for all $i, j, k \in \mathcal{D} = \{1, \cdots, N\}$ and a *coupling parameter* $\alpha \in \mathbb{C}$. The solution $X^k(\alpha) \in H_{\mathcal{D}}[[\alpha]]$ of this system is a formal power series in α with coefficients in $H_{\mathcal{D}}$. We call it *perturbation series* in $H_{\mathcal{D}}$. As the equation (4.1.1) gives a recursive definition of every coefficient in the formal series starting with $\mathbb{1}$, the solution exists and is unique. Note that in this type of equations every grafting operator is multiplied with the same coupling parameter α. In the next chapter we discuss examples of linear systems and assign to every index $i \in \mathcal{D}$ a primitive *skeleton graphs* in massless Yukawa theory. In this context the coupling parameter corresponds to the loop number and we have to restrict to skeletons with the same loop number. Skeleton graphs of a different loop order would require terms with different powers of the coupling parameter α.

We can simplify this system by defining cocycles $B_j^k : H_{\mathcal{D}} \to H_{\mathcal{D}}$ by

$$B_j^k := \sum_{i=1}^{N} B_+^i a_{ijk}. \tag{4.1.2}$$

Moreover we rewrite our decoration by attaching a pair of labels from $\{1, \cdots, N\}$ to any node such that

$$\bullet_{k,j} := B_j^k(\mathbb{1}) = \sum_{i=1}^{N} a_{ijk} B_+^i(\mathbb{1}) = \sum_{i=1}^{N} a_{ijk} \bullet_i \tag{4.1.3}$$

and so on. In this notation the linear system (4.1.1) is reduced to

$$X^k(\alpha) = \mathbb{1} + \alpha \sum_{j=1}^{N} B_j^k \left(X^j(\alpha) \right). \tag{4.1.4}$$

Assuming the set up of paragraph 3.1 we also rewrite the corresponding Mellin transforms.

$$f_{kj} := \sum_{i=1}^{N} a_{ijk} f_i \quad \text{and} \quad F_{kj}(z) := \sum_{i=1}^{N} a_{ijk} F_i(z) = \sum_{n=-1}^{\infty} c_{kj,n} z^n \tag{4.1.5}$$

The Feynman rules ϕ_s constructed in the previous chapter now obey

$$\phi_s(B_j^k(f)) = \int_0^\infty \left[\frac{f_{kj}(\zeta/s)}{s} - \frac{f_{kj}(\zeta/\mu)}{\mu} \right] \phi_\zeta(f)\, d\zeta \tag{4.1.6}$$

and by proposition 3.3.5 the Feynman rules satisfy $\phi_s \circ B_j^k(\cdot) = L_{kj} \circ \phi_s(\cdot)$ with endomorphisms $L_{kj} := \sum_{i=1}^{N} a_{ikj} L_i$.

Now the application of Feynman rules to the solution (4.1.4) of the combinatorial Dyson Schwinger equations $\phi_s(X^k(\alpha))$ are considered as the actual *perturbation series*. If we define $G^k(\ln {}^s\!/\!_\mu, \alpha) := \phi_s(X^k(\alpha))$, then by use of (4.1.6) the Dyson Schwinger equation turns into an integral equation

$$G^k(\ln {}^s\!/\!_\mu, \alpha) = 1 + \alpha \sum_{j=1}^{N} (id - R_\mu) \int_0^\infty f_{kj}(\zeta, s) G^j(\ln {}^\zeta\!/\!_\mu, \alpha)\, d\zeta. \tag{4.1.7}$$

In the following we reproduce the perturbation series by solving this system of *integral Dyson Schwinger equations*. A solution of this non-perturbative problem is called a set of *Green's functions*.

4.2 Solutions of Dyson Schwinger integral equations

Here we construct solutions of the Dyson Schwinger integral equation and analyse their relation to the perturbation series.

We start our discussion with a general observation. While there is an unique solution of the linear system (4.1.4), namely the perturbation series, we can not expect uniqueness for a solution of the integral equation (4.1.7). Indeed in the case of a single Dyson Schwinger equation the associated integral equation possesses a pair of scaling solutions which differ in their anomalous dimensions [3]. But among the solutions of the integral equation there are *non-physical* solutions. They do not correspond to the perturbation series which comes from the application of Feynman rules to the solution of the Dyson Schwinger equation (4.1.4) in the Hopf algebra $H_{\mathcal{D}}$. In order to make clear what is meant by a physical solution we establish

Definition 4.2.1. *A solution of the integral equation* (4.1.7) *is called* physical *iff its coefficients of a Taylor series expansion in the coupling parameter at $\alpha = 0$ are equal to the perturbation series coming from the linear system* (4.1.4).

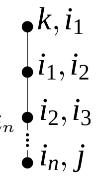
4.2.1 Algebraic study of the perturbation series

In this paragraph we explore the shape of a physical Green's functions by analysing the coefficients of its perturbation series.

Since we are interested in the coefficients of the solution let us define the following useful expressions. For $n \in \mathbb{N}_0$ and $j, k = 1, \cdots, N$ define $x_n^{kj}, x_n^k \in H_{\mathcal{D}}$ by

$$x_0^{kj} := \delta_j^k \mathbb{1} \tag{4.2.1}$$

$$x_{n+1}^{kj} := \sum_{i_1,\cdots,i_n=1}^{N} B_{i_1}^k \circ B_{i_2}^{i_1} \circ B_{i_3}^{i_2} \circ \cdots \circ B_j^{i_n}(\mathbb{1}) = \sum_{i_1,\cdots,i_n} \tag{4.2.2}$$



$$x_n^k := \sum_{j=0}^{N} x_n^{kj} \tag{4.2.3}$$

Now the solution of (4.1.4) is evidently given by

$$X^k(\alpha) = \sum_{n=0}^{\infty} x_n^k \alpha^n. \tag{4.2.4}$$

Knowing the explicit shape of the perturbation series we are able to find a characterisation of the physical Green's functions.

Proposition 4.2.2. *Assume* $\mathbf{G}(\ln {}^s\!/\!_\mu, \alpha) = \phi_s(\mathbf{X}(\alpha))$[1] *to be differentiable with respect to* $\ln {}^s\!/\!_\mu$*, then it holds*

$$\mathbf{G}(\ln {}^s\!/\!_\mu, \alpha) = \exp\left(\ln {}^s\!/\!_\mu \, \boldsymbol{\gamma}(\alpha)\right) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{4.2.5}$$

Here $\boldsymbol{\gamma}$ denotes a $N \times N$ matrix with coefficients in $\mathbb{C}$ which may depend on α. We will see that the Green's functions critically depend on the eigenvalues of this matrix. Therefore we like to name the eigenvalues. For a Green's function satisfying (4.2.5) we declare

Definition 4.2.3. *An eigenvalue* $\gamma_i(\alpha), i = 1, \cdots, N$ *of* $\boldsymbol{\gamma}(\alpha)$ *is called* anomalous dimension*.*

In general this definition only corresponds to the physical anomalous dimension (from the renormalization group equation) in the case of $N = 1$. In the canonical case (4.2.15) treated below, the actual physical anomalous dimension is given by

$$\sum_{j,k} A_{ij}\left(\gamma_j(\alpha)\right) \gamma_j(\alpha) A_{jk}^{-1}\left(\gamma_j(\alpha)\right). \tag{4.2.6}$$

It follows the proof of proposition 4.2.2.

[1]To shorten notation we suppress indices whenever a statement is satisfied for all indices. Additionally we use a matrix notation as in equation 4.2.5 and print such matrix expressions bold.

Proof. The final equation (4.2.5) is a consequence of the group property of the Feynman rules under convolution. But before we can connect lemma 3.3.7 to the above perturbation series (4.2.4), it is necessary to figure out the coproduct of this perturbation series. Therefore we derive:

$$\begin{aligned}\Delta x_{n+1}^{kj} &= \mathbb{1} \otimes x_{n+1}^{kj} + \sum_{i_n=1}^{N} x_1^{i_n j} \otimes x_n^{k i_n} + \sum_{i_{n-1}=1}^{N} x_2^{i_{n-1} j} \otimes x_{n-1}^{k i_{n-1}} + \\ &\quad + \cdots + \sum_{i_1=1}^{N} x_n^{i_1 j} \otimes x_1^{k i_1} + x_{n+1}^{kj} \otimes \mathbb{1} \\ &= \sum_{m=0}^{n+1} \sum_{i=1}^{N} x_m^{ij} \otimes x_{n+1-m}^{ki} \end{aligned} \tag{4.2.7}$$

In this calculation we used the fact that every decorated rooted tree x_{n+1}^{kj} of the perturbation series is a planar tree. Because application of the coproduct to a planar tree with node number $n+1$ gives a sum of all tensor products of planar trees with an overall node number of $n+1$. Taking the decoration of the decorated rooted tree (4.2.2) into account we find the above results. By linearity it immediately follows for the $(n+1)$th coefficient of the solution (4.2.4).

$$\Delta(\alpha^{n+1} x_{n+1}^{k}) = \sum_{m=0}^{n+1} \sum_{i=1}^{N} \alpha^m x_m^i \otimes \alpha^{n+1-m} x_{n+1-m}^{ki} \tag{4.2.8}$$

Collecting all terms with a specific power in α on the left component of the tensor product, we end up with

$$\begin{aligned}\Delta X^k(\alpha) &= \sum_{j=1}^{N} \left[X^j(\alpha) \otimes \left(\alpha x_1^{kj} + \alpha^2 x_2^{kj} + \alpha^3 x_3^{kj} + \cdots \right) \right] + X^k(\alpha) \otimes \mathbb{1} \\ &= \sum_{j=1}^{N} \left(X^j(\alpha) \otimes X^{kj}(\alpha) \right). \end{aligned} \tag{4.2.9}$$

Where we introduced the formal series

$$X^{kj}(\alpha) := \sum_{n=0}^{\infty} \alpha^n x_n^{kj}. \tag{4.2.10}$$

Now we make use of lemma 3.3.7 and find

$$\begin{aligned} G^k(\ln {}^s\!/\!_\mu + \ln {}^t\!/\!_\mu, \alpha) &= G^k(\ln {}^{st}\!/\!_{\mu^2}, \alpha) := \phi_{({}^{st}\!/\!_\mu)}(X^k(\alpha)) \\ &= (\phi_s \star \phi_t)\left(X^k(\alpha)\right) = \sum_{j=1}^{N} \phi_s(X^j(\alpha)) \phi_t(X^{kj}(\alpha)) \\ &= \sum_{j=1}^{N} G^{kj}(\ln {}^t\!/\!_\mu, \alpha) G^j(\ln {}^s\!/\!_\mu, \alpha). \end{aligned}$$

Differentiation with respect to $\ln {}^t\!/\!_\mu$ at $\ln {}^t\!/\!_\mu = 0$ leads to the following differential equation.

$$G'^k(\ln {}^s\!/\!_\mu, \alpha) = \sum_{j=1}^{N} \underbrace{G'^{kj}(0, \alpha)}_{=:\gamma^{kj}(\alpha)} G^j(\ln {}^s\!/\!_\mu, \alpha) = \sum_{j=1}^{N} \gamma^{kj}(\alpha) G^j(\ln {}^s\!/\!_\mu, \alpha) \tag{4.2.11}$$

This differential equation together with the initial condition

$$G^k(0, \alpha) = \phi_\mu(X^k(\alpha)) = 1 \tag{4.2.12}$$

has an unique solution which is given by (4.2.4). $\square$

4.2.2 The canonical case

In the last paragraph we determined the form of the Green's functions as solution of the linear system of Dyson Schwinger equations. With this knowledge we can start constructing physical solutions for different kinds of Mellin transform matrices.

Firstly let us assume that the matrix of Mellin transforms $(F(z)_{kj})_{k,j=1}^{N}$ is diagonalizable and therefore given by

$$\mathbf{F}(z) = \mathbf{A}(z) \begin{bmatrix} \lambda_1(z) & & 0 \\ & \ddots & \\ 0 & & \lambda_N(z) \end{bmatrix} \mathbf{A^{-1}}(z) \tag{4.2.13}$$

By rescaling with a proper diagonal matrix one achieves that $\mathbf{A}(z)$ is holomorphic and therefore analytic at $z = 0$. Further we assume that $\det[\mathbf{A}(0)] \neq 0$. Then Cramer's rule implies that also $\mathbf{A}^{-1}$ is analytic at $z = 0$. Moreover the assumption that every Mellin transform is a Laurent series with pole order one at $z = 0$ also restricts the eigenvalues λ_i, $i = 1, \cdots, N$ to be this type of Laurent series. This can be seen by a simple argument: Rescaling the Mellin matrix with a factor of z gives an holomorphic matrix $\widetilde{\mathbf{F}}(z) := z\mathbf{F}(z)$. Let $\widetilde{\lambda}(z)$ be an eigenvalue of $\widetilde{\mathbf{F}}(z)$ corresponding to an eigenvector v. Then the absolute value of the eigenvalue λ is bounded by $\|\widetilde{\mathbf{F}}\|$:

$$\left|\widetilde{\lambda}(z)\right| = \frac{\|\widetilde{\lambda}(z)v\|}{\|v\|} = \frac{\|\widetilde{\mathbf{F}}(z)v\|}{\|v\|} \leq \|\widetilde{\mathbf{F}}(z)\| \tag{4.2.14}$$

Therefore $\lim_{z\to 0} \widetilde{\lambda}(z)$ exists and $\widetilde{\lambda}$ has no pole at $z = 0$. Now the statement is proven since every eigenvalue $\lambda(z)$ of the Mellin matrix $\mathbf{F}(z)$ corresponds to an eigenvalue $\widetilde{\lambda}(z) := z\lambda(z)$ of the matrix $\widetilde{\mathbf{F}}(z)$. With these preparations we are able to state a solution of the Dyson Schwinger integral equation.

Theorem 4.2.4. *Let $\alpha\lambda_i(\gamma_i(\alpha)) = 1$ for $i = 1, \cdots, N$, then the ansatz*

$$G^i(\ln {}^{s}/_{\mu}, \alpha) = \sum_{j,k=1}^{N} A_{ij}(\gamma_j(\alpha)) \left(\frac{s}{\mu}\right)^{-\gamma_j(\alpha)} A^{-1}_{jk}(\gamma_j(\alpha)) \tag{4.2.15}$$

is a solution of the integral equation (4.1.7).

Before we start the proof it should be remarked that in the following proposition 4.2.7 and the corollary 4.2.6 it turns out that this solution is *physical.* Now let us declare the following abbreviation.

$$\widetilde{A}_{ij} := \sum_{k=1}^{N} A_{ij}(\gamma_j(\alpha)) A^{-1}_{jk}(\gamma_j(\alpha)) \tag{4.2.16}$$

Proof. For a proof we just put in the ansatz (4.2.15) in the equation (4.1.7):

$$
\begin{aligned}
G^i(\ln{}^s\!/\!_\mu,\alpha) - 1 &= \alpha \sum_{j=1}^{N} \int_0^\infty f_{ij}(\zeta) \left[G^j(\ln{}^{s\zeta}\!/\!_\mu,\alpha) - G^j(\ln\zeta,\alpha) \right] d\zeta \\
&= \alpha \sum_{j,k=1}^{N} \int_0^\infty f_{ij}(\zeta) \widetilde{A}_{jk} \zeta^{-\gamma_k(\alpha)} \left[\left(\frac{s}{\mu}\right)^{-\gamma_k(\alpha)} - 1 \right] d\zeta \\
&= \sum_{j,k=1}^{N} \alpha F_{ij}(\gamma_k(\alpha)) \widetilde{A}_{jk} \left[\left(\frac{s}{\mu}\right)^{-\gamma_k(\alpha)} - 1 \right] \\
&= \sum_{j,k=1}^{N} A_{ij}(\gamma_j(\alpha)) \underbrace{\alpha\lambda_j(\gamma_j(\alpha))}_{=1} A^{-1}_{jk} \left[\left(\frac{s}{\mu}\right)^{-\gamma_j(\alpha)} - 1 \right] \\
&= \sum_{j,k=1}^{N} A_{ij}(\gamma_j(\alpha)) \left(\frac{s}{\mu}\right)^{-\gamma_j(\alpha)} A^{-1}_{jk}(\gamma_j(\alpha)) - 1
\end{aligned}
$$

□

Notice the good convergence behaviour of the Green's functions under Mellin transformations. While defining the Mellin transform for a single element of the perturbation series needed the introduction of a regularizer z, for the non-perturbative Green's functions the integral converges without an artificial regularizer.

The next step is to explore the properties of the anomalous dimensions $\gamma_i(\alpha)$.

Lemma 4.2.5. *Let $\gamma(\alpha)$ be an anomalous dimension of a* physical *Green's function fulfilling*

$$\alpha\lambda(\gamma(\alpha)) = 1 \quad \text{for} \quad \lambda(z) = \sum_{i=-1}^{\infty} r_i\, z^i, \tag{4.2.17}$$

then $\gamma(\alpha)$ is analytic at $\alpha = 0$ and uniquely determined by

$$\gamma(\alpha) = \sum_{i=1}^{\infty} \gamma_i\, \alpha^i, \quad \text{with} \quad \gamma_1 = r_{-1} \tag{4.2.18}$$

$$\text{and for } n \geq 2 \quad \gamma_n = \sum_{i=0}^{n-2} r_i \sum_{j_1+\cdots+j_{i+1}=n-1} \gamma_{j_1}\cdots\gamma_{j_{i+1}}. \tag{4.2.19}$$

Proof. First notice a physical Green's function has to satisfy

$$\mathbf{G}(\ln{}^s\!/\!_\mu, 0) = \phi_s(\mathbf{X}(0)) = \phi_s(\mathbb{1}) = 1. \tag{4.2.20}$$

By proposition 4.2.2 there is an $N \times N$ matrix $\widetilde{A}(\ln{}^s\!/\!_\mu,\alpha)$ with coefficients given by polynomials in $\ln{}^s\!/\!_\mu$ such that

$$G^i({}^s\!/\!_\mu, 0) = \sum_{j=1}^{N} \widetilde{A}_{ij} \left(\frac{s}{\mu}\right)^{-\gamma_j(0)}. \tag{4.2.21}$$

Since this expression has to be constant with respect to ${}^s\!/\!_\mu$ every anomalous dimension must vanish at $\alpha = 0$,

$$\gamma(0) = 0. \tag{4.2.22}$$

Now we can derive a recursive formula for γ_n.

$$\gamma(\alpha) = \alpha\left(r_{-1} + \sum_{i=0}^{\infty} r_i \gamma^{i+1}(\alpha)\right) \tag{4.2.23}$$

Since γ is at least $\mathcal{O}(\alpha)$ we find that $\gamma_1 = r_{-1}$. For $n \geq 2$ it holds

$$\gamma_n = \frac{1}{n!}\left[\frac{d}{d\alpha}\right]^n_{\alpha=0} \gamma(\alpha) = \frac{1}{(n-1)!}\sum_{i=0}^{n-2} r_i \left[\frac{d}{d\alpha}\right]^{n-1}_{\alpha=0} \gamma^{i+1}(\alpha) \tag{4.2.24}$$

$$= \frac{1}{(n-1)!}\sum_{i=0}^{n-2} r_i \left[\frac{d}{d\alpha}\right]^{n-1}_{\alpha=0} \left(\sum_{j=1}^{n-1} \gamma_j \alpha^j\right)^{i+1} \tag{4.2.25}$$

$$= \frac{1}{(n-1)!}\sum_{i=0}^{n-2} r_i \left[\frac{d}{d\alpha}\right]^{n-1}_{\alpha=0} \left(\sum_{j_1,\cdots,j_{i+1}=1}^{n-1} \gamma_{j_1}\cdots\gamma_{j_{i+1}} \alpha^{j_1+\cdots+j_{i+1}}\right) \tag{4.2.26}$$

$$= \sum_{i=0}^{n-2} r_i \sum_{j_1+\cdots+j_{i+1}=n-1} \gamma_{j_1}\cdots\gamma_{j_{i+1}}. \tag{4.2.27}$$

This proves uniqueness of γ. Finally recall as implicit function of an analytic function the anomalous dimension $\gamma(\alpha)$ is also analytic at $\alpha = 0$. □

This result directly implies

Corollary 4.2.6. *The solution* (4.2.15) *is analytic at* $\alpha = 0$.

The final task in the discussion of systems with a diagonalizable Mellin transform matrix is a comparison to the perturbation series. This is the subject of the next proposition which ensures that (4.2.15) is a physical solution.

Proposition 4.2.7. *Let* $\mathbf{G}(\ln {}^{s}\!/\!{}_{\mu}, \alpha)$ *be a Green's function of type* (4.2.5)*, analytic at* $\alpha = 0$ *and* $\mathbf{G}(\ln {}^{s}\!/\!{}_{\mu}, 0) = \mathbf{1}$*, then it is a physical solution.*

Proof. Notice that every coefficient of the perturbation series from $H_{\mathcal{D}}$ is uniquely determined by the property

$$\phi_s(x^i_{n+1}\alpha^{n+1}) = \alpha \sum_{j=1}^{N} \int_0^\infty (id - R_\mu) f_{ij}(\zeta, s)\phi_\zeta(x^j_n \alpha^n)\, d\zeta \tag{4.2.28}$$

and it starts at $\phi_s(\mathbf{X}(0)) = 1$. To prove that an analytic Green's function

$$G^i(\ln {}^{s}\!/\!{}_{\mu}, \alpha) = 1 + \sum_{j=1}^{\infty} g^i_j(\ln {}^{s}\!/\!{}_{\mu})\, \alpha^j. \tag{4.2.29}$$

is physical it suffices to show

$$g^i_{n+1}\alpha^{n+1} = \alpha \sum_{j=1}^{N} \int_0^\infty (id - R_\mu) f_{ij}(\zeta, s) g^j_n \alpha^n\, d\zeta \tag{4.2.30}$$

for any $n \in \mathbb{N}_0$ and all $i = 1, \cdots, N$. We prove this equation by showing a contradiction:

Let m be the minimum such that

$$C^i_{m+1}\alpha^{m+1} := \alpha \sum_{j=1}^{N} \int_0^\infty (id - R_\mu) f_{ij}(\zeta, s) g^j_m \alpha^m\, d\zeta \neq g^i_{m+1}\alpha^{m+1} \tag{4.2.31}$$

i.e. the equality (4.2.30) does not hold. By proposition 4.2.2 any coefficient $g_j^i(\ln {}^s\!/\!_\mu)$ is a polynomial in $\ln {}^s\!/\!_\mu$ and hence the renormalized Mellin integral converges for any $g_j^i(\ln {}^s\!/\!_\mu)$. Now we take a look at the associated integral equation, which should be satisfied by the Green's function. For a reasonable notation we assume $m \geq 1$ here. The remaining case $m = 0$ can be treated in the same way.

$$\alpha \sum_{j=1}^{N} \int_0^\infty (id - R_\mu) f_{ij}(\zeta, s) \left(G^j(\ln {}^\zeta\!/\!_\mu, \alpha) - 1 - \sum_{k=1}^{m} g_k^j \alpha^k \right) d\zeta \tag{4.2.32}$$

$$= G^i(\ln {}^s\!/\!_\mu, \alpha) - 1 - \sum_{k=1}^{m} g_k^i \alpha^k - C_{m+1}^i \alpha^{m+1} \tag{4.2.33}$$

While the upper series starts with a term in α^{m+2} the lower series has by assumption of m a non vanishing term in α^{m+1}. This is a contradiction. □

4.2.3 The degenerated case

Now the remaining issue to discuss are systems with a Mellin transform matrix which is not diagonalizable. A general *degenerated* system decouples in separate systems of Jordan blocks. First we construct a physical solution of one elementary Jordan block. After that we treat a generalisation of the Jordan block result.

So let us assume the matrix of Mellin transforms to be

$$\mathbf{F}(z) = \begin{bmatrix} \lambda(z) & & & & 0 \\ 1 & \lambda(z) & & & \\ & 1 & \lambda(z) & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \lambda(z) \end{bmatrix} \in \mathcal{M}(\mathbb{C}[z^{-1}, z]], M) \quad \text{with} \quad M \leq N. \tag{4.2.34}$$

In this set up the first integral equation decouples and we immediately know a physical solution of the first Green's function

$$G^1(\ln {}^s\!/\!_\mu, \alpha) = \left(\frac{s}{\mu} \right)^{-\gamma(\alpha)} \tag{4.2.35}$$

with an anomalous dimension $\gamma(\alpha)$ obeying $\alpha\lambda(\gamma(\alpha)) = 1$ and lemma 4.2.5. The other components are obtained by

Theorem 4.2.8. *The ansatz*

$$G^n(\ln {}^s\!/\!_\mu, \alpha) = \left(1 + \sum_{j=1}^{n-1} A_j^n(\alpha)(\ln {}^s\!/\!_\mu)^j \right) \left(\frac{s}{\mu} \right)^{-\gamma(\alpha)} \qquad n = 1, \cdots, M \tag{4.2.36}$$

with $A_0^n := 1$, $A_n^n := 0$ *and recursive defined amplitudes*

$$A_m^n = \alpha \left(A_m^{n-1} - \frac{\alpha\gamma'(\alpha)}{m} A_{m-1}^{n-1} \right) \qquad m = 1, \cdots, n-1 \tag{4.2.37}$$

provides a physical solution of the integral equation with the Mellin matrix (4.2.34). *Moreover for* $m = 1$ *and* $m = n - 1$ *the amplitudes are explicitly given by*

$$A_1^n = - \left(\sum_{i=2}^{n} \alpha^i \right) \gamma'(\alpha) \quad \textit{and} \tag{4.2.38}$$

$$A_{n-1}^n = - \frac{(-1)^n}{(n-1)!} \left(\alpha^2 \gamma'(\alpha) \right)^{n-1}. \tag{4.2.39}$$

Proof. Since we already know the solution in the case $n = 1$ we may assume $n \geq 2$ without loss of generality. We put the ansatz (4.2.36) into the Dyson Schwinger integral equation and calculate conditions for the amplitudes.

$$
\begin{aligned}
G^n(\ln {}^s/_\mu, \alpha) - 1 &= \sum_{i=n-1}^{n} \alpha \int_0^\infty (id - R_\mu) f_{ni}(\zeta, s) G^i(\ln {}^\zeta/_\mu, \alpha)\, d\zeta \\
&= \sum_{i=n-1}^{n} \alpha \int_0^\infty (id - R_\mu) f_{ni}(\zeta, s) \left[1 + \sum_{j=1}^{i-1} A^i_j \left(\frac{-1}{\gamma'(\alpha)} \frac{\partial}{\partial \alpha}\right)^j\right] \left(\frac{\zeta}{\mu}\right)^{-\gamma(\alpha)} d\zeta \\
&= \sum_{i=n-1}^{n} \alpha \left[1 + \sum_{j=1}^{i-1} A^i_j \left(\frac{-1}{\gamma'(\alpha)} \frac{\partial}{\partial \alpha}\right)^j\right] \underbrace{F_{ni}(\gamma(\alpha))}_{\delta_{n\,i}/\alpha + \delta_{n-1\,i}} \left[\left(\frac{s}{\mu}\right)^{-\gamma(\alpha)} - 1\right] \\
&= \left[1 + \sum_{j=1}^{n-1} A^n_j \left(\frac{-1}{\gamma'(\alpha))} \frac{\partial}{\partial \alpha}\right)^j\right] \left(\frac{s}{\mu}\right)^{-\gamma(\alpha)} \\
&\quad + \alpha \left[\sum_{j=1}^{n-1} \sum_{k=1}^{j-1} A^n_j \binom{j}{k} \frac{k!}{\alpha(\alpha\gamma'(\alpha))^k} (\ln {}^s/_\mu)^{j-k} + \sum_{j=1}^{n-2} A^{n-1}_j (\ln {}^s/_\mu)^j\right] \left(\frac{s}{\mu}\right)^{-\gamma(\alpha)} \\
&\quad + \alpha \left[1 + \sum_{j=1}^{n-1} A^n_j \frac{j!}{\alpha} \left(\frac{1}{\alpha\gamma'(\alpha)}\right)^j\right] \left(\frac{s}{\mu}\right)^{-\gamma(\alpha)} \\
&\quad - \alpha \left[\frac{1}{\alpha} + \sum_{j=1}^{n-1} A^n_j \frac{j!}{\alpha} \left(\frac{1}{\alpha\gamma'(\alpha)}\right)^j + 1\right]
\end{aligned}
$$

To satisfy the integral equation (4.1.7) the second and third line have to vanish. This requirement leads to two conditions.

$$
A^n_1 = -\alpha^2 \gamma'(\alpha) - \alpha\gamma'(\alpha) \sum_{j=2}^{n-1} A^n_j \frac{j!}{(\alpha\gamma'(\alpha))^j} \tag{4.2.40}
$$

$$
A^{n-1}_m = - \sum_{j=m+1}^{n-1} A^n_j \binom{j}{j-m} \frac{(j-m)!}{\alpha(\alpha\gamma'(\alpha))^{j-m}} \tag{4.2.41}
$$

Let us arrange the abbreviation $\widetilde{A}^i_m := \frac{m!}{(\alpha\gamma'(\alpha))^m} A^i_m$ for $i = n-1, n$ and $m = 1, \cdots n$. Then the condition (4.2.41) is equivalent to

$$
-\alpha \begin{bmatrix} \widetilde{A}^{n-1}_{n-2} \\ \widetilde{A}^{n-1}_{n-3} \\ \vdots \\ \widetilde{A}^{n-1}_1 \end{bmatrix} = \begin{bmatrix} 1 & & & \text{\huge 0} \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \widetilde{A}^n_{n-1} \\ \widetilde{A}^n_{n-2} \\ \vdots \\ \widetilde{A}^n_2 \end{bmatrix} \tag{4.2.42}
$$

For a recursive description we have to invert the matrix on the right hand side. This results in

$$
\begin{bmatrix} \widetilde{A}^n_{n-1} \\ \widetilde{A}^n_{n-2} \\ \vdots \\ \widetilde{A}^n_2 \end{bmatrix} = -\alpha \begin{bmatrix} 1 & & & \text{\huge 0} \\ -1 & 1 & & \\ & \ddots & \ddots & \\ \text{\huge 0} & & -1 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{A}^{n-1}_{n-2} \\ \widetilde{A}^{n-1}_{n-3} \\ \vdots \\ \widetilde{A}^{n-1}_1 \end{bmatrix}. \tag{4.2.43}
$$

Now this is easily identified to be equivalent to the recursion condition (4.2.37) for $m = 2, \cdots, n-1$. In the case of $m = 1$ we have to analyse the condition (4.2.40).

$$\begin{aligned} A_1^n &= -\alpha^2\gamma'(\alpha) - \alpha\gamma'(\alpha)\sum_{j=2}^{n-1}\widetilde{A}_j^n \underset{(4.2.43)}{=} -\alpha^2\gamma'(\alpha) - \alpha\gamma'(\alpha)\left(-\alpha\widetilde{A}_1^{n-1}\right) \\ &= \alpha\left(A_1^{n-1} - \frac{\alpha\gamma'(\alpha)}{1}A_0^{n-1}\right). \end{aligned} \tag{4.2.44}$$

This proves the validity of (4.2.37). Notice that this recursion condition already proves the inductive step of (4.2.38) and (4.2.39). But with use of (4.2.37) we found in particular

$$A_1^2 = -\alpha^2\gamma'(\alpha) \quad \text{and} \quad A_2^3 = \frac{1}{2}\left(\alpha^2\gamma'(\alpha)\right)^2. \tag{4.2.45}$$

The basis for an inductive proof of the stated explicit formulas (4.2.38) and (4.2.39). Note due to lemma 4.2.5 the Green's function is analytic at $\alpha = 0$ and by proposition 4.2.7 it is physical. □

Unfortunately we can not generalise the solution of theorem 4.2.8 to a degenerated Mellin matrix in an arbitrary basis. In general a degenerated system is given by

$$\widetilde{\mathbf{F}}(z) := \mathbf{S}(z)\mathbf{F}(z)\mathbf{S}^{-1}(z) \tag{4.2.46}$$

with a Jordan block $\mathbf{F}$ as in (4.2.34) and an invertible transformation $\mathbf{S}(z) \in \mathcal{M}(\mathbb{C}[z^{-1}, z]], M)$ which may depend on z. The combination of this z dependence and the appearance of non-diagonal terms in the Mellin matrix causes additional terms which forbid a transformed solution as in the canonical case (4.2.15). However we may expect a well-behaved result in the case of a transformation on the level of the Hopf algebra $H_{\mathcal{D}}$ which conserves the start of the perturbation series given by $\mathbb{1}$.

Still denoting the Jordon block of (4.2.34) by $\mathbf{F}(z)$ and the physical solution from (4.2.36) by $\mathbf{G}(\ln {}^s/_\mu, \alpha)$ we formulate

Lemma 4.2.9. *Let* $\mathbf{S} \in \mathcal{M}(\mathbb{C}, M)$ *an invertible transformation satisfying* $\sum_{j=1}^M S_{ij} = 1$ *or equivalently* $\sum_{j=1}^M S_{ij}^{-1} = 1$ *for all* $i = 1, \cdots, M$*, then a physical solution of the system*

$$\widetilde{\mathbf{F}}(z) = \mathbf{S}\mathbf{F}(z)\mathbf{S}^{-1} \tag{4.2.47}$$

is provided by

$$\widetilde{\mathbf{G}}(\ln {}^s/_\mu, \alpha) := \mathbf{S}\mathbf{G}(\ln {}^s/_\mu, \alpha). \tag{4.2.48}$$

Proof. The proof follows by a simple calculation and the use of theorem 4.2.8. First of all we set

$$f_{ij}(\zeta, s) := \sum_{k,l=1}^M S_{ik}^{-1}\widetilde{f}_{kl}(\zeta, s)S_{lj} \tag{4.2.49}$$

such that f_{ij} has the Mellin transform F_{ij}. Now we can put the ansatz in the integral Dyson Schwinger equations.

$$
\begin{aligned}
\widetilde{G}^i(\ln {}^s\!/\!{}_\mu, \alpha) - 1 &= \alpha \sum_{j=1}^{M} \int_0^\infty (id - R_\mu) \widetilde{f}_{ij}(\zeta, s) \widetilde{G}^j(\ln {}^\zeta\!/\!{}_\mu, \alpha)\, d\zeta \\
&= \alpha \sum_{j,k=1}^{M} \int_0^\infty (id - R_\mu) S_{ij} f_{jk}(\zeta, s) G^k(\ln {}^\zeta\!/\!{}_\mu, \alpha)\, d\zeta \\
&= \sum_{j=1}^{M} S_{ij} \Big(G^j(\ln {}^s\!/\!{}_\mu, \alpha) - 1 \Big) \\
&= \widetilde{G}^i(\ln {}^s\!/\!{}_\mu, \alpha) - \sum_{j=1}^{M} S_{ij}
\end{aligned}
$$

We inverted (4.2.49) in the second line and used the assumption that $G(\ln {}^s\!/\!{}_\mu, \alpha)$ is a solution of the Jordan block in the third line. The last line shows the requirement of the above assumptions on $\mathbf{S}$ in order to leave the first Taylor coefficient of the Green's function unaltered. Finally notice that this solution gives the right perturbation series, since our transformation of the Mellin matrix corresponds to a transformation of the combinatorial Dyson Schwinger equation (4.1.4). We are considering the transformed system

$$
\widetilde{X}^i(\alpha) = \mathbb{1} + \alpha \sum_{j,k,l=1}^{M} S_{ij} B_k^j(\widetilde{X}^l(\alpha)) S_{kl}^{-1} \tag{4.2.50}
$$

which is solved in terms of the original perturbation series $\mathbf{X}(\alpha)$ by

$$
\widetilde{\mathbf{X}}(\alpha) = \mathbf{S}\mathbf{X}(\alpha). \tag{4.2.51}
$$

$\square$

Chapter 5

Applications in massless Yukawa theory

In the following we discuss applications of the previous results in two examples. This is arranged in the context of a massless Yukawa theory and dimensional regularization in $D = 4 - 2z$ dimensions. The first example treats the perturbation series of fermionic and hadronic self-energy graphs and their linear insertions into each other. Thereby we show how such a model can be related to the toy model from chapter 3. Another example is studied to explore contributions to the rainbow approximation from the introduction of a flavour for the fermionic and hadronic fields.

In the context of Yukawa theory we speak of Feynman graphs without giving an introduction to their combinatorial definition. For details of this topic the reader refers to the second chapter of [8]

5.1 Linear insertions of selfenergy graphs

In our first example we like to consider massless Yukawa theory described by the lagrangian

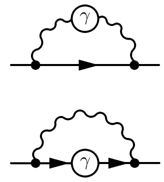
$$\mathscr{L} = i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - g\bar{\psi}\phi\psi, \tag{5.1.1}$$

such that all Feynman graphs in this physical theory consisting of edges and vertices in the set

$$\mathcal{R} = \left\{ \quad , \quad , \quad \right\}. \tag{5.1.2}$$

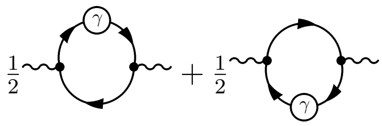
As usual we denote the set of edges by $\mathcal{R}_E$. Now we calculate the hadronic and fermionic two point functions in an extended rainbow approximation. That is we restrict ourselves to selfenergy graphs which can be obtained by inserting the primitives and into eachother but ignore other graph contributions. In this approximation the perturbation series is described by a linear system of Dyson Schwinger equations with cocycles B_+ and B_+ defined on $\mathcal{H}_{\mathrm{Yuk}}$, the set consisting of all products of Yukawa graphs with residue in $\mathcal{R}_{\mathrm{E}}$. By linearity of the considered Dyson Schwinger equations we only insert connected graphs. For such a connected Feynman graph γ an application of the insertion operators gives

$$B_+(\gamma) = \begin{cases} \quad & \text{for} \quad \operatorname{res}\gamma = \sim \\ \quad & \text{for} \quad \operatorname{res}\gamma = \rightarrow \end{cases} \tag{5.1.3}$$



and

$$B_+(\gamma) = \begin{cases} 0 & \text{for} \quad \operatorname{res}\gamma = \sim \\ \tfrac{1}{2} \quad + \tfrac{1}{2} & \text{for} \quad \operatorname{res}\gamma = \rightarrow \end{cases} \tag{5.1.4}$$



Notice that these insertion laws indeed fulfil the cocycle equation from (2.2.6). Under consideration of some combinatorial factors those insertion operators can be defined on $\mathcal{H}_{\text{Yuk}}$ in such a way that the cocycle property is satisfied - for a detailed discussion see [9]. Now we are able to formulate a system of Dyson Schwinger equations for the sought after perturbation series.

$$\begin{aligned} X^{\rightarrow}(\alpha) &= \mathbb{1} + \alpha B_+ \left(X^{\rightarrow}(\alpha)\right) + \alpha B_+ \left(X^{\sim}(\alpha)\right) \\ X^{\sim}(\alpha) &= \mathbb{1} + \alpha B_+ \left(X^{\rightarrow}(\alpha)\right) \end{aligned} \tag{5.1.5}$$

Here we introduce the coupling parameter $\alpha := \sqrt{g}$.

Before we can determine a commutation law for the cocycles B_+, B_+ and Feynman rules $\phi(\gamma, p)$, where p denotes the external momentum of the Feynman graph γ, we briefly sum up some combinatorial facts of connected graphs in $\mathcal{H}_{\text{Yuk}}$. For such a graph γ we denote the number of its interal and external edges by $I(\gamma)$ and $E(\gamma)$, respectively. Only counting edges of a special type we put the corresponding subindex ϕ or ψ to these symbols. In addition to that we denote the number of independent loops of γ by $|\gamma|$. Using simple power counting we determine the superficial degree of divergence to be

$$\operatorname{sdd}(\gamma) = (4 - 2z)|\gamma| - I_\psi(\gamma) - 2I_\phi(\gamma). \tag{5.1.6}$$

Defining $V(\gamma)$ the number of vertices of γ, the vertex adjacency relations are given by

$$V(\gamma) = 2I_\phi(\gamma) + E_\phi(\gamma) \tag{5.1.7}$$

$$2V(\gamma) = 2I_\psi(\gamma) + E_\psi(\gamma). \tag{5.1.8}$$

Moreover a Feynman graph appearing in the perturbation series (5.1.5) has to satisfy the loop condition

$$|\gamma| - 1 = I_\phi(\gamma) + I_\psi(\gamma) - V(\gamma). \tag{5.1.9}$$

Using these combinatorial identities and the restriction $E(\gamma) = 2$ we find the loop number dependency of the superficial degree of divergence.

$$\operatorname{sdd}(\gamma) = 2 - 2z\,|\gamma| - \frac{1}{2}E_\psi = \begin{cases} 1 - 2z|\gamma| & \text{for} \quad \operatorname{res}\gamma = \rightarrow \\ 2 - 2z|\gamma| & \text{for} \quad \operatorname{res}\gamma = \sim \end{cases} \tag{5.1.10}$$

This result motivates the definition of form factor reduced Feynman rules - the algebra morphism $\widetilde{\phi}$ given by

$$\phi(\gamma, p) = \begin{cases} \not{p}\,\widetilde{\phi}(\gamma, p) & \text{for} \quad \operatorname{res}\gamma = \\ p^2\,\widetilde{\phi}(\gamma, p) & \text{for} \quad \operatorname{res}\gamma = \end{cases} \tag{5.1.11}$$

Now using scaling invariance of dimensional regularization and Lorentz invariance the expression

$$(p^2)^{+z|\gamma|}\widetilde{\phi}(\gamma, p) \tag{5.1.12}$$

is independent of p and thus can be evaluated at any external momentum. We use this consideration for handling the subdivergences in the calculation of the commutation relation between $\widetilde{\phi}$ and the cocycles. In the case of fermionic graphs, that is $\operatorname{res}\gamma =$, it follows

$$\begin{aligned}
\widetilde{\phi}\left(B_+(\gamma), p\right) &= \operatorname{Tr}\frac{1}{4\not{p}}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2}\frac{1}{\not{k}+\not{p}}\phi(\gamma, k+p)\frac{1}{\not{k}+\not{p}} \\
&= \frac{1}{2p^2}\int\frac{d^Dk}{(2\pi)^D}\frac{(k+p)^2+p^2-k^2}{k^2(k+p)^2}\left(\frac{1}{(k+p)^2}\right)^{z|\gamma|}\left((k+p)^2\right)^{z|\gamma|}\widetilde{\phi}(\gamma, k+p) \\
&= (p^2)^{z|\gamma|}\widetilde{\phi}(\gamma, p)\frac{1}{2p^2}\int\frac{d^Dk}{(2\pi)^D}\frac{(k+p)^2+p^2}{k^2\left[(k+p)^2\right]^{1+z|\gamma|}} \\
&= (p^2)^{-z}\widetilde{\phi}(\gamma, p)\,\frac{1}{2}\left(\frac{1}{4\pi}\right)^{2-z}\left[G\left(z(1+|\gamma|), z\right) + G\left(1+z(1+|\gamma|), z\right)\right]
\end{aligned}$$

In the last line we used the master integral of dimensional regularization and abbreviated

$$G(w, z) := \frac{\Gamma(1-z)\Gamma(2-w)\Gamma(w-1)}{\Gamma(w-z)\Gamma(3-w-z)}. \tag{5.1.13}$$

Now it is seen that

$$\widetilde{G}(w, z) := \frac{1}{2}\left(\frac{1}{4\pi}\right)^{2-z} G(w, z) = \sum_{n=-1}^{\infty} w^n c_n(z) \tag{5.1.14}$$

with $c_n(z)$ holomorphic at $z = 0$. Since we used the momentum scheme the renormalized value of a Feynman graph is a convergent integral and thus independent of the regulator z. Dimensional regulation gives a z dependent result, which converges to the physical result. But the holomorphic coefficients $c_n(z)$ also have an existing limit as $z \to 0$, therefore it is possible to specify the regularized results by setting $c_n(0)$ and demanding

$$\widetilde{\phi}\left(B_+(\gamma), p\right) = (p^2)^{-z}\left(\widetilde{G}\left(z(1+|\gamma|), 0\right) + \widetilde{G}\left(1+z(1+|\gamma|), 0\right)\right)\widetilde{\phi}(\gamma, p) \tag{5.1.15}$$

without changing the renormalized value. Repeating the same steps for $\operatorname{res}\gamma =$ results in the same values for

$$\begin{aligned}
\widetilde{\phi}\left(B_+(\gamma), p\right) &= \widetilde{\phi}\left(B_+(\gamma), p\right) \\
&= (p^2)^{-z}\left(\widetilde{G}\left(1+z(1+|\gamma|), 0\right) - \widetilde{G}\left(z(1+|\gamma|), 0\right)\right)\widetilde{\phi}(\gamma, p).
\end{aligned} \tag{5.1.16}$$

Denoting $F_n\left(z(1+|\gamma|)\right) := \widetilde{G}\left(2-n+z(1+|\gamma|),0\right)$ for $n=1,2$ we have

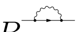
$$F_1(z) = \frac{1}{2}\left(\frac{1}{4\pi}\right)^2 \frac{1}{z(1-z)} \qquad F_2(z) = -\frac{1}{2}\left(\frac{1}{4\pi}\right)^2 \frac{1}{(1-z)(2-z)}. \tag{5.1.17}$$

All this evaluates to a Mellin matrix of type

$$\mathbf{F}(z) = F_1(z)\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} + F_2(z)\begin{bmatrix}1 & -1\\ -1 & 0\end{bmatrix}. \tag{5.1.18}$$

Now realising that any factor in front of the subdivergence $\widetilde{\phi}(\gamma,p)$ in (5.1.15) and (5.1.16) depends only on $z(1+|\gamma|)$ as in the case of Kreimer's toy model (3.1.7), we rediscover the toy model in the following way. Set $\mathcal{D} := \{1,2\}$ and make use of the universal property 2.3.1 to define an algebra morphism $\rho : H_{\mathcal{D}} \to \mathcal{H}_{\text{Yuk}}$ such that

$$\rho\circ B_+^1 = B_+^{\;\cdot} \circ \rho \quad \text{and} \quad \rho\circ B_+^2 = B_+^{\;\cdot} \circ \rho. \tag{5.1.19}$$

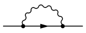
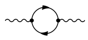
 

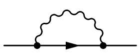
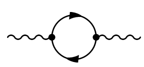
With further definitions $s := p^2$ and $\phi_{z,s}(\cdot) := \widetilde{\phi}(\rho(\cdot),p)$ we see that the system of linear Dyson Schwinger equations (5.1.5) evaluated under the Feynman rules $\widetilde{\phi}$ gives exactly the same results as the system

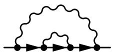
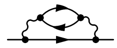
$$\begin{aligned} X^1(\alpha) &= \mathbb{1} + \alpha B_+^1\left(X^1(\alpha)\right) + \alpha B_+^1\left(X^2(\alpha)\right)\\ X^2(\alpha) &= \mathbb{1} + \alpha B_+^2\left(X^1(\alpha)\right) + \alpha B_+^2\left(X^2(\alpha)\right)\end{aligned} \tag{5.1.20}$$

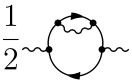
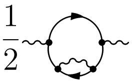
with coefficients in $H_{\mathcal{D}}$ under evaluation of the above defined toy model rules $\phi_{z,s}$. Moreover for any forest $f\in\mathcal{F}$ and $i,j\in\mathcal{D}$ it holds

$$\begin{aligned}\phi_{z,s}\circ B_j^i(f) &= \widetilde{\phi}(\rho\circ B_+^i(f),p) = \widetilde{\phi}(B_+^\Gamma\circ\rho(f),p)\\ &= \left(p^2\right)^{-z} F_{ij}\left(z|B_+^\Gamma\circ\rho(f)|\right)\widetilde{\phi}\left(\rho(f),p\right)\\ &= s^{-z}F_{ij}\left(z|B_j^i(f)|\right)\phi_{z,s}(f).\end{aligned}$$

Here F_{ij} are coefficients of the Mellin matrix (5.1.18) and Γ corresponds to the primitive element  if $i=1$ and $\Gamma =$  in the case of $i=2$. This reproduces the formula (3.1.7) from section 3. Therefore we can represent the Yukawa graphs of (5.1.5) by means of decorated rooted trees and Kreimer's toy model. For particular graphs and suppressing the notation of the second index, we have

$\rho\left(\bullet^1\right) =$  $\qquad \rho\left(\bullet^2\right) =$  (5.1.21)

$\rho\left(\begin{smallmatrix}\bullet^1\\ \bullet_1\end{smallmatrix}\right) =$  $\qquad \rho\left(\begin{smallmatrix}\bullet^1\\ \bullet_2\end{smallmatrix}\right) =$  (5.1.22)

$\rho\left(\begin{smallmatrix}\bullet^2\\ \bullet_1\end{smallmatrix}\right) = \frac{1}{2}$  $+\ \frac{1}{2}$  (5.1.23)

and so on.

Finally we want to find the Green's function corresponding to the linear system of Dyson Schwinger equations (5.1.5). As said in section 4.2.2 we have to calculate the eigenvalues of the Mellin matrix (5.1.18) and diagonalize it. The eigenvalues are easily computed

$$\lambda_{1,2}(z) = \frac{1}{2}\left(F_1(z)+F_2(z)\mp\sqrt{5F_1(z)^2-6F_1(z)F_2(z)+5F_2(z)^2}\right) \tag{5.1.24}$$

such that we have indeed the canonical case. Say $\mathbf{F}$ is diagonalized by the linear transformation $\mathbf{S}$.

$$\mathbf{F}(z) = \mathbf{S}(z) \begin{bmatrix} \lambda_1(z) & \\ & \lambda_2(z) \end{bmatrix} \mathbf{S}^{-1}(z) \tag{5.1.25}$$

Then we find by a computation of the eigenvectors of $\mathbf{F}$ the transformation

$$\mathbf{S}(z) = \begin{bmatrix} \frac{\lambda_1(z)}{F_1(z)-F_2(z)} & \frac{\lambda_2(z)}{F_1(z)-F_2(z)} \\ 1 & 1 \end{bmatrix} \tag{5.1.26}$$

and its inverse

$$\mathbf{S}^{-1}(z) = \frac{1}{\lambda_2(z) - \lambda_1(z)} \begin{bmatrix} -(F_1(z) - F_2(z)) & \lambda_2(z) \\ F_1(z) - F_2(z) & -\lambda_1(z) \end{bmatrix}. \tag{5.1.27}$$

Due to lemma 4.2.5 there are anomalous dimensions implicitly defined by

$$\alpha \lambda_n(\gamma_n(\alpha)) = 1 \quad \text{for} \quad n = 1, 2 \tag{5.1.28}$$

only depending on the coefficients of the Mellin transformations (5.1.18). With these facts and theorem 4.2.4 a physical Green's function is given by

$$\begin{aligned} \mathbf{G}\left(\ln {}^{p^2}\!/_{\mu^2}, \alpha\right) = & \frac{2\alpha F_2 - 1}{\alpha(F_1 + F_2) - 2} \begin{bmatrix} \frac{1}{\alpha(F_1 - F_2)} \\ 1 \end{bmatrix} \exp\left(-\gamma_1(\alpha) \ln {}^{p^2}\!/_{\mu^2}\right) \\ & - \frac{2\alpha \widetilde{F}_2 - 1}{\alpha\left(\widetilde{F}_1 + \widetilde{F}_2\right) - 2} \begin{bmatrix} \frac{1}{\alpha\left(\widetilde{F}_1 - \widetilde{F}_2\right)} \\ 1 \end{bmatrix} \exp\left(-\gamma_2(\alpha) \ln {}^{p^2}\!/_{\mu^2}\right) \end{aligned} \tag{5.1.29}$$

Where the Mellin transform in front of the γ_1 exponential are evaluated at $\gamma_1(\alpha)$ that is $F_n = F_n(\gamma_1(\alpha))$ and the Mellin transforms with a tilde are evaluated at $\gamma_2(\alpha)$ such that $\widetilde{F}_n = \widetilde{F}_n(\gamma_2(\alpha))$.

5.2 Rainbow approximation with flavoured fields

In this section we study the rainbow approximation in an extended Yukawa theory, which involves different kinds of fermionic and hadronic flavours but is obeying conservation of flavour.

In this sense we start with the lagrangian

$$\mathscr{L} = i \sum_{j=1}^{N} \bar{\psi}_j \partial\!\!\!/ \psi_j + \sum_{j=1}^{2N-1} \frac{1}{2} \partial_\mu \phi_j \partial^\mu \phi_j - \sum_{j,k=1}^{N} \sum_{l=1}^{2N-1} g_{jkl} \bar{\psi}_j \psi_k \phi_l. \tag{5.2.1}$$

with the flavour indices j, k and l. We have further options to specify the occurring interaction vertices by making different choices of g_{jkl}. In the following we treat two cases:

1. The flavour is conserved at any interaction vertex, every exchange process may occur and none of these processes is distinguished.

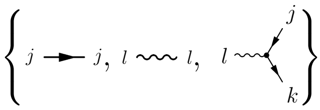
$$g_{jkl} := \begin{cases} g & \text{if } j + k - l = 1 \\ 0 & \text{else} \end{cases} \tag{5.2.2}$$

   With a coupling parameter g.

2. The flavour exchange behaves almost as in the first case but fermionic flavour has to change.

$$\widetilde{g}_{jkl} := \begin{cases} g_{jkl} & \text{if } j \neq k \\ 0 & \text{else} \end{cases} \tag{5.2.3}$$

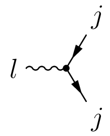
The lagrangian corresponds to the set of residue

$\mathcal{R} =$  (5.2.4)

with indices $j, k = 1, \cdots, N$ and $l = 1, \cdots, 2N-1$. Notice that the condition

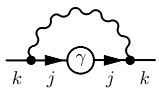
$$j + k - l = 1 \tag{5.2.5}$$

in the definition of the coupling tensor (5.2.2) fixes the bosonic flavour by choosing a pair of fermionic flavour $j, k = 1, \cdots, N$. Therefore we constantly suppress the notation of bosonic flavours in the sequel. Furthermore the condition (5.2.5) corresponds to flavour conservation - the bosonic flavour should be regarded as $l-1$. The same argument holds in the second case but the g_{jjl} term in the definition of $\widetilde{g}_{jkl}$ forbids all vertices

 $j = 1, \cdots, N \quad l = 1, \cdots, 2N-1$ (5.2.6)

with non-changing fermionic flavour.

Now define $\mathcal{H}_{\text{Yuk}}$ to be the set of all products of Yukawa graphs with fermionic residue $j \rightarrow j$. Further define insertion operators $B_+^k : \mathcal{H}_{\text{Yuk}} \to \mathcal{H}_{\text{Yuk}}$ such that for res $\gamma = j \rightarrow j$ it holds

$B_+^k(\gamma) =$  $, \quad \forall j, k = 1, \cdots, N.$ (5.2.7)

Again, note that the insertion operators B_+^k fulfil the cocycle property on $\mathcal{H}_{\text{Yuk}}$ - for a detailed discussion of Hochschild one cocycles see [9]. Now rearranging the coupling parameter $\alpha := \sqrt{g}$, the perturbation series of rainbow graphs with different fermionic fields can be written as

$$X^k(\alpha) = \mathbb{1} + \alpha B_+^k \left(\sum_{j=1}^{N} X^j(\alpha) \right) \quad \forall k = 1, \cdots, N \tag{5.2.8}$$

in the first case and in the second case we find

$$\widetilde{X}^k(\alpha) = \mathbb{1} + \alpha B_+^k \left(\sum_{j \neq k} \widetilde{X}^j(\alpha) \right) \quad \forall k = 1, \cdots, N. \tag{5.2.9}$$

Also note that the combinatorics of section 5.1 also applies to the flavoured Yukawa graphs and we define the Feynman rules $\phi(\gamma, p)$ in a flavour independent way. Thus it is possible to reproduces the results of the first example. Defining form factor reduced Feynman rules $\phi(\gamma,p) = \not{p}\widetilde{\phi}(\gamma,p)$ and specifying the regularization prescription it follows

$$\widetilde{\phi}\left(B_+^k(\gamma),p\right) = (p^2)^{-z} F\left(z \left| B_+^k(\gamma) \right|\right) \widetilde{\phi}(\gamma,p), \tag{5.2.10}$$

such that renormalization is provided as in Kreimer's toy model in chapter 3. Here $F(z) := F_1(z) + F_2(z)$ with $F_{1/2}(z)$ from the first example is $k-$independent as the Feynman rules do not distinguish between fermionic flavours. So we end up with Mellin matrices

$$\mathbf{F}_{jk}(z) = F(z) \qquad \text{and} \qquad \widetilde{\mathbf{F}}_{jk}(z) = F(z)\left(1 - \delta_{jk}\right). \tag{5.2.11}$$

In the first case we have to diagonalize the matrix which entries are identically 1. It is easy to see that a basis of eigenvectors in given by

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{5.2.12}$$

with the eigenvalues $\lambda = 0$ (multiplicity $N-1$) and $\lambda = N$. Therefore we are in the canonical case as treated in section 4.2.2 and we can define the matrix $\mathbf{S}$ by the sequence of eigenvectors from (5.2.12). It is also easy to figure out that the inverse of that matrix is given by

$$S^{-1}_{jk} = \frac{1}{N} - \delta_{jk+1}. \tag{5.2.13}$$

This gives a diagonalization of the first Mellin matrix.

$$\mathbf{F}(z) = F(z)\mathbf{S} \begin{bmatrix} 0 & & \text{\huge 0} \\ & \ddots & \\ & & 0 \\ \text{\huge 0} & & & N \end{bmatrix} \mathbf{S}^{-1} \tag{5.2.14}$$

Following the theorem 4.2.4 we calculate

$$\sum_{k=1}^{N} S^{-1}_{jk} = 1 - \sum_{k=1}^{N} \delta_{jk+1} = \delta_{jN} \tag{5.2.15}$$

and see that only the last eigenvalue of (5.2.12) contributes to the Green's function. Now by lemma 4.2.5 the anomalous dimension is determined via the condition

$$\alpha N F\left(\gamma(\alpha)\right) = 1. \tag{5.2.16}$$

Finally in the first case a physical Green's function is provided by the simple scaling solution

$$G^j\left({}^{s}\!/_{\mu}, \alpha\right) = \left({}^{s}\!/_{\mu}\right)^{-\gamma(\alpha)} \quad \forall j = 1, \cdots, N. \tag{5.2.17}$$

Returning to the second case we find that the vectors of the basis (5.2.12) are also eigenvectors of the Mellin matrix $\widetilde{\mathbf{F}}(z)$. This results in the diagonalization

$$\widetilde{\mathbf{F}}(z) = F(z)\mathbf{S}\begin{bmatrix} -1 & & & \Large 0 \\ & \ddots & & \\ & & -1 & \\ \Large 0 & & & N-1 \end{bmatrix}\mathbf{S}^{-1}. \tag{5.2.18}$$

Again, due to equation (5.2.15) only the last eigenvector contributes to the Green's function and

$$\alpha(N-1)F\left(\tilde{\gamma}(\alpha)\right) = 1. \tag{5.2.19}$$

In combination with lemma 4.2.5 this determines the anomalous dimension $\tilde{\gamma}(\alpha)$. As above we find a physical solution

$$\widetilde{G}^j\left({}^s\!/\!_\mu, \alpha\right) = \left({}^s\!/\!_\mu\right)^{-\tilde{\gamma}(\alpha)} \quad \forall j = 1, \cdots, N. \tag{5.2.20}$$

The found results for the Green's functions (5.2.17), (5.2.20) and the anomalous dimensions (5.2.16) and (5.2.19) come as no surprise as the coefficients of the Mellin transform are multiplied with the number of fermionic flavours participating in the interaction. This behaviour may be expected from the Dyson Schwinger equations (5.2.8) and (5.2.9) and flavour independence of the Feynman rules.

Chapter 6

Conclusion

In the first chapter we introduced the Hopf algebra of decorated rooted trees $H_{\mathcal{D}}$ and studied some of its basic properties. Many proofs made exhaustive use of the graduated structure - a concept well-fitting to the appearance of different orders in perturbation theory.

The review of Kreimer's toy model in the second chapter gave a rigorous definition of regularized and renormalized Feynman rules. For renormalization we made use of the Hopf algebraic properties of $H_{\mathcal{D}}$. In particular it is the antipode which determines the renormalized Feynman rules and the counter terms (see corollary 3.3.1). This toy model formed the setup for our studies of systems of linear Dyson Schwinger equations in the subsequent chapters.

The third chapter contained the main results of our investigation of linear Dyson Schwinger equations. In order to find non-perturbative solutions we turned these equations in a system of integral Dyson Schwinger equations. As such systems provide Green's functions which do not correspond to the actual perturbation series in $H_{\mathcal{D}}$, we proved a characterisation of physical Green's functions - see proposition 4.2.2. With theorem 4.2.4 and 4.2.8 we found physical Green's functions for canonical and degenerated Mellin matrices. Thereby the analytic dependency of the coupling parameter α ensured the accordance with the perturbation series.

It chapter 4 our results were applied in the context of massless Yukawa theory in dimensional regularization. We studied perturbation series of linear insertions of self-energy graphs and connected this approximation to Kreimer's toy model. Furthermore we looked for an answer to the following question: What happens to the rainbow approximation if different kinds of flavours are involved?

However, up to this point the question about a general form of the Green's function in the degenerated case remains unanswered. The attempt to generalise lemma 4.2.9 leads to recurrence relations which also depend on the derivatives of the amplitudes. We have not solved these relations yet.

Furthermore it would be interesting to study the treatment of perturbation series with higher order skeleton graphs - as it has been done in the case of a single equation [10].

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Certification

Hereby I declare that the present masterthesis

On linear systems of Dyson Schwinger equations

is my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution and I have noticed the examination regulations in their current version.

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