EVALUATION OF THE PERIOD OF A FAMILY OF TRIANGLE AND BOX LADDER GRAPHS

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Abstract. We prove that the period of a family of n loop graphs with triangle and box ladders evaluates to $\frac{4}{n}\binom{2n-2}{n-1}\zeta(2n-3)$.

1. Introduction

The period of a primitive logarithmically divergent Feynman graph G is the scheme independent residue of the regularized amplitude. It can be written in parametric space as follows. Number the edges of G from 1 to $N = 2h_G$ (where h_G is the number of independent loops in G), and to each edge e associate a variable α_e. The period of G is given by the convergent (projective) integral [2]:

$$P_G = \int_{\alpha_i>0} \frac{\mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_{N-1}}{\Psi_G(\alpha_1, \dots, \alpha_{N-1}, 1)^2} \in \mathbb{R} \tag{1}$$

where $\Psi_G \in \mathbb{Z}[\alpha_1, \dots, \alpha_N]$ is the graph, or Kirchhoff, polynomial of G. It is defined by the formula

$$\Psi_G = \sum_{T\subset G} \prod_{e\notin T} \alpha_e,$$

where the sum is over all spanning trees T of G.

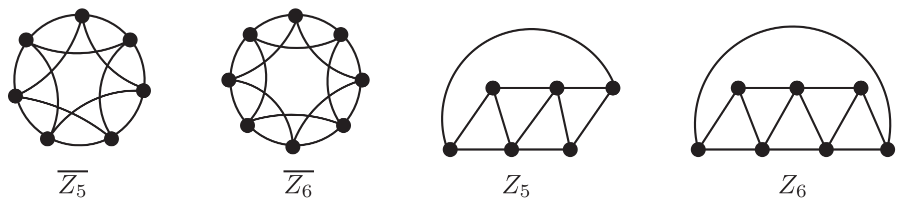

Figure 1: Completed ($\overline{Z_\bullet}$) and uncompleted ($Z_\bullet$) zig-zag graphs with 5 and 6 loops.

For the zig-zag graphs Z_n depicted in figure 1 the periods were conjectured by D. Broadhurst and D. Kreimer [3] in 1995 as

$$P_{Z_n} = 4\frac{(2n-2)!}{n!(n-1)!}\left(1 - \frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3), \tag{2}$$

where $\zeta(z) = \sum_{k\geq 1} k^{-z}$ is the Riemann zeta function. The zig-zag conjecture was recently proved by Francis Brown and the author in [4].

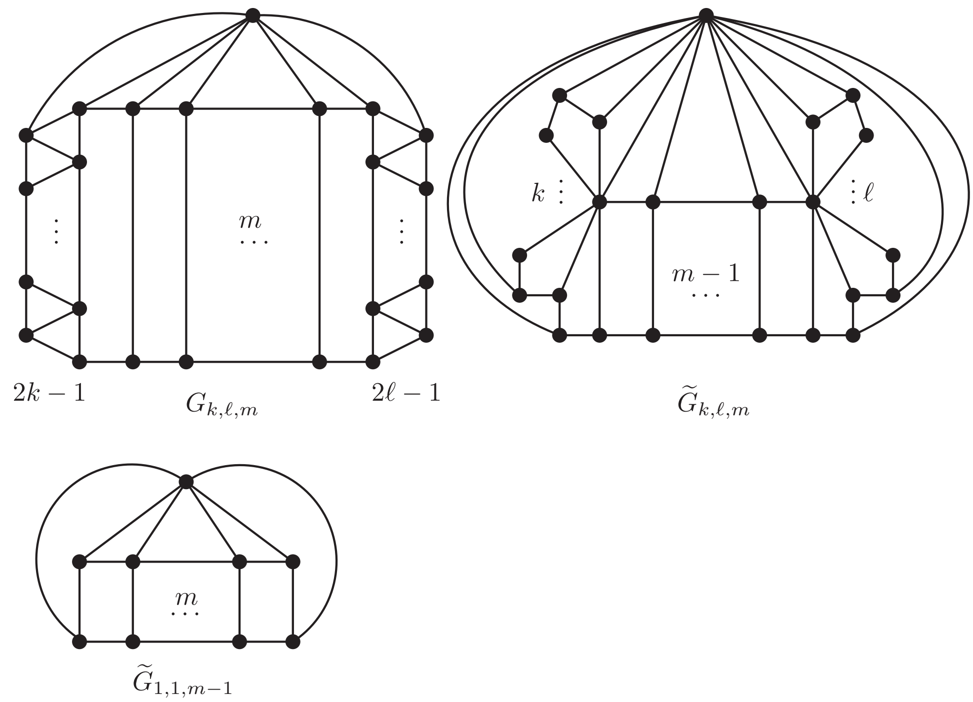

Figure 2: The $G_{k,\ell,m}$ and their planar duals $\widetilde{G}_{k,\ell,m}$ have period $\frac{4}{n}\binom{2n-2}{n-1}\zeta(2n-3)$ where $n = 2(k+\ell+m)$.

Motivated by a conjecture in $N = 4$ Super Yang-Mills theory [1] on the period of the graph $\widehat{G}_{1,1,m-1}$ we prove that the family $G_{k,\ell,m}$ and their planer duals $\widehat{G}_{k,\ell,m}$ have the same period as $Z_{2k+2\ell+2m}$. The graph $G_{k,\ell,m}$ depicted in figure 2 has two vertical ladders of $2k-1$ and $2\ell-1$ triangles which are joined at their longer sides by a ladder of m boxes. The three-valent vertices in the upper half are connected to a common vertex. Its dual $\widehat{G}_{k,\ell,m}$ has two roses of k and ℓ boxes which are joined by a ladder of $m-1$ boxes. The two-valent vertices of the roses and the upper vertices of the box ladder are joined to a common vertex. The case $k = \ell = 1$ is a ladder of $m+1$ boxes whose two-valent vertices together with their upper three-valent vertices are connected to a common vertex.

Theorem 1. *Let $G_{k,\ell,m}$ for $k, \ell, m \geq 1$ be the family of graphs depicted in figure 2. Let $\widehat{G}_{k,\ell,m}$ be their planar duals. Then*

$$P_{G_{k,\ell,m}} = P_{\widehat{G}_{k,\ell,m}} = \frac{4}{n}\binom{2n-2}{n-1}\zeta(2n-3), \tag{3}$$

where $n = 2(k+\ell+m)$.

The proof uses the twist-identity [5].

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2. Proof of theorem 1

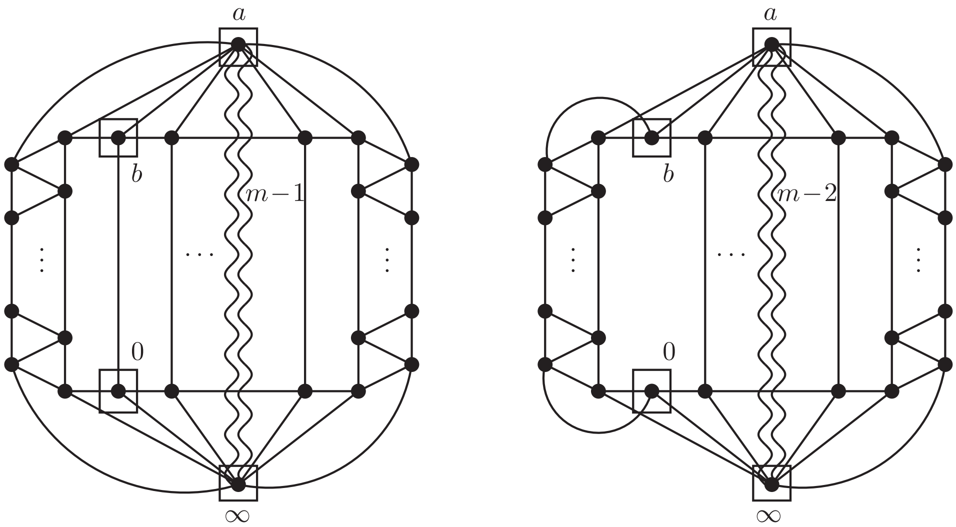

Figure 3: The completed graph $\overline{G}_{k,\ell,m}$ maps under the twist identity with respect to the boxed vertices $a, b, 0, \infty$ to $\overline{G}_{k+1,\ell,m-1}$. The curly lines symbolize propagators of negative weights $m-1$ and $m-2$, respectively.

Proof. Firstly, we notice that $P_{G_{k,\ell,m}} = P_{\widehat{G}_{k,\ell,m}}$ because the period is invariant under taking planer duals [3], [5]. To prove theorem 1 for $P_{G_{k,\ell,m}}$ we complete the graph by adding a vertex ∞ and connecting ∞ to all three-valent vertices. To make the graph four-regular (make all vertices four-valent) we need to add an inverse propagator of weight $m-1$ that connects ∞ with a in figure 3. In position space Feynman rules a negative propagator of weight w from x to y gives a factor $||x-y||^{2w}$ in the numerator of the integrand. Here we need negative propagators only in intermediate steps.

We apply the twist identity [5] to the four vertices a, b, 0, ∞ and obtain the graph on the right hand side of figure 3. The twist identity is applied to a four vertex cut by swapping the connections of the left hand side to a and b and simultaneously swapping the connections to 0 and ∞. Afterwards we have to move edges to opposite sides of the four-cycle $a0b\infty$ to render the graph four-regular. In figure 3 we had to move the edge connecting b and 0 to an edge connecting a and ∞. This new edge cancels one of the $m-1$ negative propagators leaving a negative weight of $m-2$. After the twist we flip the left triangle ladder inside the box with vertices b and 0 and we obtain the graph $\overline{G}_{k+1,\ell,m-1}$. Upon un-completing by removing ∞ we obtain

$$P_{G_{k,\ell,m}} = P_{G_{k+1,\ell,m-1}}.$$

By moving every second vertex in figure 1 inside the circle we see that

$$G_{k,\ell,1} = Z_{2k+2\ell+2}.$$

The theorem follows from (2). $\square$

We close this note with the remark that periods that are rational multiples of a singe zeta value are rare. The only known periods of this type are the periods of the wheels and the zig-zags. However, with increasing loop order an increasing number of graphs can be transformed to the wheel or the zig-zag by a sequence of twist identities and taking planar duals (the Fourier identity).

References

[1] **Z. Bajnok, N. Drukker, A. Hegedus, R. Nepomechie, L. Palla, C. Sieg, R. Suzuki**, in preparation.
[2] **S. Bloch, H. Esnault, D. Kreimer**, *On Motives Associated to Graph Polynomials*, Comm. Math. Phys. 267, 181-225 (2006).
[3] **D. Broadhurst, D. Kreimer**, *Knots and numbers in ϕ^4 theory to 7 loops and beyond*, Int. J. Mod. Phys. C6, 519-524 (1995).
[4] **F. Brown, O. Schnetz**, *Proof of the zig-zag conjecture*, arXiv:1208.1890 [math.NT] (2012).
[5] **O. Schnetz**, *Quantum periods: A census of ϕ^4 transcendentals*, Jour. Numb. Theory and Phys. 4 no. 1, 1-48 (2010).