EVALUATION OF THE PERIOD OF A FAMILY OF TRIANGLE AND BOX LADDER GRAPHS

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Abstract. We prove that the period of a family of $n$ loop graphs with triangle and box ladders evaluates to $\frac{1}{n} (\frac{2n-2}{n-1}) \zeta(2n-3)$.

1. Introduction

The period of a primitive logarithmically divergent Feynman graph $G$ is the scheme independent residue of the regularized amplitude. It can be written in parametric space as follows. Number the edges of $G$ from 1 to $N = 2h_G$ (where $h_G$ is the number of independent loops in $G$), and to each edge $e$ associate a variable $\alpha_e$. The period of $G$ is given by the convergent (projective) integral [2]:

$$P_G = \int_{\alpha_i > 0} \frac{d\alpha_1 \ldots d\alpha_{N-1}}{\Psi_G(\alpha_1, \ldots, \alpha_{N-1}, 1)^2} \in \mathbb{R}$$

where $\Psi_G \in \mathbb{Z}[\alpha_1, \ldots, \alpha_N]$ is the graph, or Kirchhoff, polynomial of $G$. It is defined by the formula

$$\Psi_G = \sum_{T \subseteq G} \prod_{e \notin T} \alpha_e,$$

where the sum is over all spanning trees $T$ of $G$.

![Figure 1: Completed ($Z_5$) and uncompleted ($Z_6$) zig-zag graphs with 5 and 6 loops.](image)

For the zig-zag graphs $Z_n$ depicted in figure 1 the periods were conjectured by D. Broadhurst and D. Kreimer [3] in 1995 as

$$P_{Z_n} = 4 \left( \frac{2n-2}{n-1} \right) ! \left( 1 - \frac{1 - (-1)^n}{2^{2n-3}} \right) \zeta(2n-3),$$

where $\zeta(z) = \sum_{k \geq 1} k^{-z}$ is the Riemann zeta function. The zig-zag conjecture was recently proved by Francis Brown and the author in [4].
Motivated by a conjecture in \( N = 4 \) Super Yang-Mills theory [1] on the period of the graph \( \tilde{G}_{1,1,m-1} \), we prove that the family \( G_{k,\ell,m} \) and their planar duals \( \tilde{G}_{k,\ell,m} \) have the same period as \( \mathbb{Z}_{2k+2\ell+2m} \). The graph \( G_{k,\ell,m} \) depicted in figure 2 has two vertical ladders of \( 2k-1 \) and \( 2\ell-1 \) triangles which are joined at their longer sides by a ladder of \( m \) boxes. The three-valent vertices in the upper half are connected to a common vertex. Its dual \( \tilde{G}_{k,\ell,m} \) has two roses of \( k \) and \( \ell \) boxes which are joined by a ladder of \( m-1 \) boxes. The two-valent vertices of the roses and the upper vertices of the box ladder are joined to a common vertex. The case \( k = \ell = 1 \) is a ladder of \( m+1 \) boxes whose two-valent vertices together with their upper three-valent vertices are connected to a common vertex.

\textbf{Theorem 1.} Let \( G_{k,\ell,m} \) for \( k,\ell,m \geq 1 \) be the family of graphs depicted in figure 2. Let \( \tilde{G}_{k,\ell,m} \) be their planar duals. Then

\[
P_{G_{k,\ell,m}} = P_{\tilde{G}_{k,\ell,m}} = \frac{4}{n} \binom{2n-2}{n-1} \zeta(2n-3),
\]

where \( n = 2(k+\ell+m) \).

The proof uses the twist-identity [5].

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2. Proof of theorem 1

\begin{align*}
\text{Figure 3: The completed graph } G_{k,\ell,m} \text{ maps under the twist identity with respect to the boxed vertices } a, b, 0, \infty \text{ to } G_{k+1,\ell,m-1}. \text{ The curly lines symbolize propagators of negative weights } m-1 \text{ and } m-2, \text{ respectively.}
\end{align*}

Proof. Firstly, we notice that \( P_{G_{k,\ell,m}} = P_{\hat{G}_{k,\ell,m}} \) because the period is invariant under taking planer duals \([3], [5]\). To prove theorem 1 for \( P_{G_{k,\ell,m}} \) we complete the graph by adding a vertex \( \infty \) and connecting \( \infty \) to all three-valent vertices. To make the graph four-regular (make all vertices four-valent) we need to add an inverse propagator of weight \( m-1 \) that connects \( \infty \) with \( a \) in figure 3. In position space Feynman rules a negative propagator of weight \( w \) from \( x \) to \( y \) gives a factor \( \frac{||x-y||^2}{w} \) in the numerator of the integrand. Here we need negative propagators only in intermediate steps.

We apply the twist identity \([5]\) to the four vertices \( a, b, 0, \infty \) and obtain the graph on the right hand side of figure 3. The twist identity is applied to a four vertex cut by swapping the connections of the left hand side to \( a \) and \( b \) and simultaneously swapping the connections to 0 and \( \infty \). Afterwards we have to move edges to opposite sides of the four-cycle \( ab0\infty \) to render the graph four-regular. In figure 3 we had to move the edge connecting \( b \) and \( 0 \) to an edge connecting \( a \) and \( \infty \). This new edge cancels one of the \( m-1 \) negative propagators leaving a negative weight of \( m-2 \). After the twist we flip the left triangle ladder inside the box with vertices \( b \) and \( 0 \) and we obtain the graph \( G_{k+1,\ell,m-1} \). Upon un-completing by removing \( \infty \) we obtain

\[ P_{G_{k,\ell,m}} = P_{G_{k+1,\ell,m-1}}. \]

By moving every second vertex in figure 1 inside the circle we see that

\[ G_{k,\ell,1} = Z_{2k+2\ell+2}. \]

The theorem follows from (2).

We close this note with the remark that periods that are rational multiples of a single zeta value are rare. The only known periods of this type are the periods of the wheels and the zig-zags. However, with increasing loop order an increasing number of
graphs can be transformed to the wheel or the zig-zag by a sequence of twist identities and taking planar duals (the Fourier identity).

References