ZIMMERMANN FORESTS AND HOPF ALGEBRAS OF GRAPHS AND DECORATED TREES

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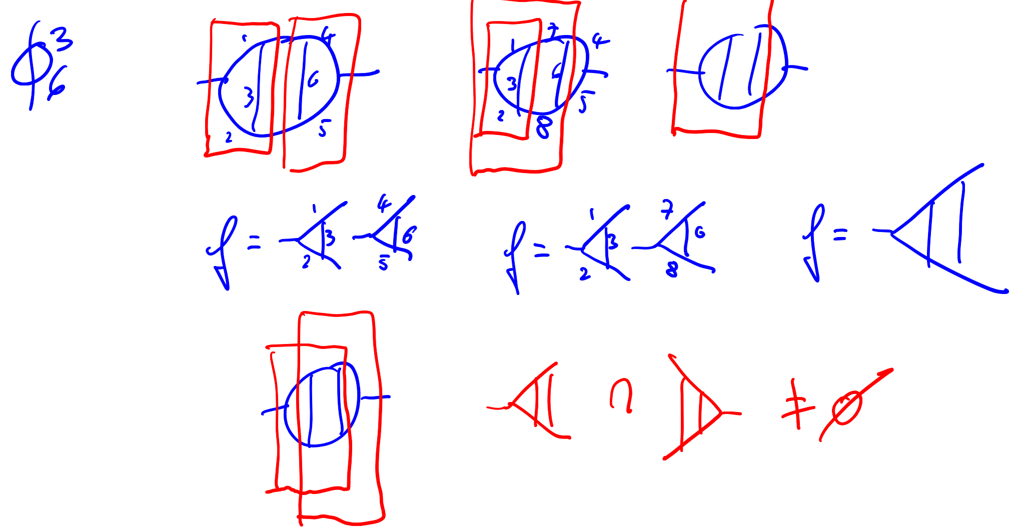
1. Forests

It is useful to collect some notation first.

1.1. **Definitions.** For a 1PI superficially divergent graph Γ, we define a forest f to be a collection of 1PI proper superficially divergent sub-graphs $\Gamma_i \subset \Gamma$, $i \in \mathcal{I}_\Gamma^f$ for some index set $\mathcal{I}_\Gamma^f$, such that either they are disjoint: $\Gamma_i \cap \Gamma_j = \emptyset$, or contained in each other: $\Gamma_i \subset \Gamma_j$ or $\Gamma_j \subset \Gamma_i$. In particular, a forest f is a product of 1PI graphs: $f = \prod_i \gamma_i$. By Γ/f we denote the graph obtained by contracting the graphs γ_i to points in Γ.



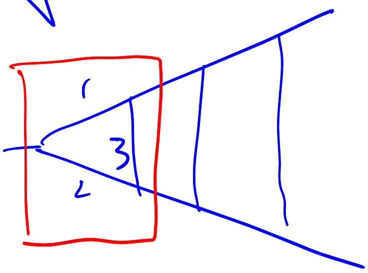
For a 1PI superficially divergent graph Γ, we define a maximal forest to be a forest and furthermore, we demand that $p_\Gamma^f := \Gamma/[\cup_{i\in\mathcal{I}_\Gamma^f}\Gamma_i]$ has no divergent sub-graph. We hence call the index set $\mathcal{I}_\Gamma^f$ maximal for Γ.

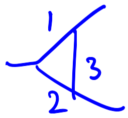
cograph

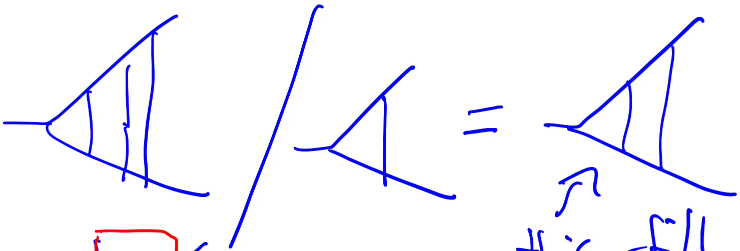
For $f \ni \Gamma_i \subset \Gamma$, each index set $\mathcal{I}_\Gamma^f$ defines an index set $\mathcal{I}_i^f$ of all forests strictly contained in Γ_i, i.e. such that $\Gamma_j \subset \Gamma_i\ \forall j \in \mathcal{I}_i^f$.

We call a forest complete, if $\mathcal{I}_\Gamma^f$ is maximal for Γ and $\mathcal{I}_i^f$ maximal for each proper 1PI superficially divergent sub-graph Γ_i of Γ.

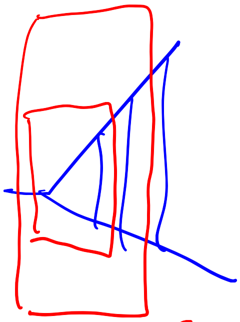
max forest:

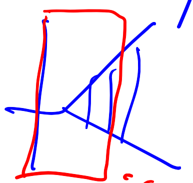


$f =$ 



this still has subgraph





is maximal.

is maximal

cograph must be free of subdivergences!

Need to check superficial degree of divergence: $\omega_D(\Gamma)$.

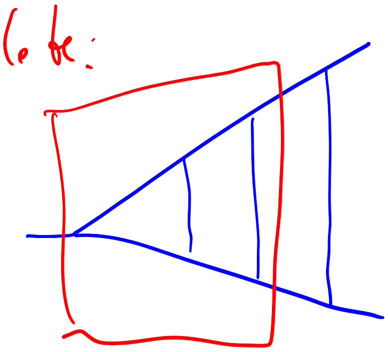
$$\underline{\omega_6}(\Gamma) = 6|\Gamma| - 2|E_6|$$

Each finite graph Γ has a finite number $|C(\Gamma)|$ of complete forests. Here, the set of all such complete forests is denoted by $C(\Gamma)$. Examples are below.

Such complete forests are in one-to-one correspondence with decorated rooted trees where the set of decorations p_v (at vertices v) is given by 1PI superficially divergent graphs free of sub-divergences,

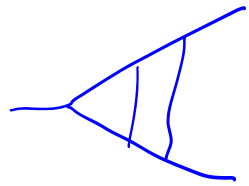
$$p_v := \Gamma_i / \cup_{j \in \mathcal{I}_i^f} \Gamma_j. \tag{1}$$

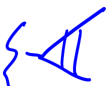
Complete:

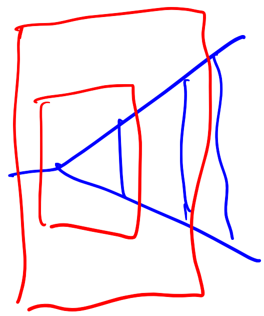


is maximal.

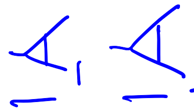
But not complete



$f = \{$  $\}$



is complete,

$f = \{$  $\}$

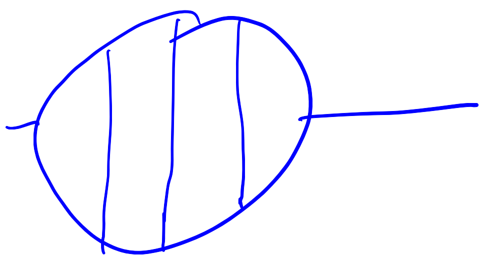
all complete forests

From now on, we write in obvious abuse of notation $T \in C(\Gamma)$ for such a decorated rooted tree.

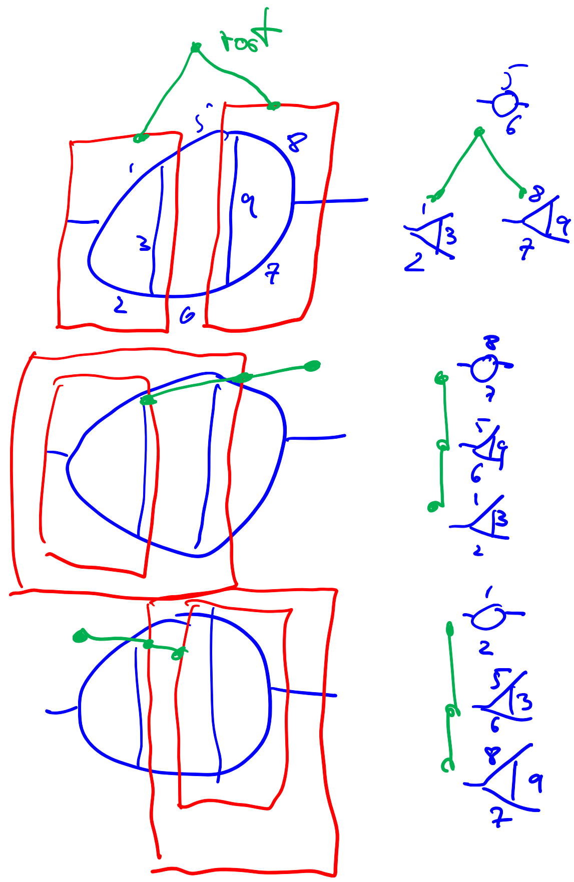
Note that the power set $P_E(T)$ of edges $E(T)$ of such a tree T gives all possible cuts c at the tree T: any $c \in P_E(T)$ defines, for a connected tree T, a union of connected components $T - c$ obtained by removing the edges c, with $R^c(T)$ the unique component containing the root of T, and $P^c(T)$ the union of the remaining components. We have $\cup_{T \in C(\Gamma)} 2^{E(T)}$ as the set of all cuts available altogether, and denote by (c, T) an element of this set.

$P^c(T)$ corresponds to a forest of Γ, with each of its connected components corresponding to a graph γ_i in $f = \prod_i \gamma_i$ [1].

For every complete forest, there is a decorated rooted tree.



3 complete forests.

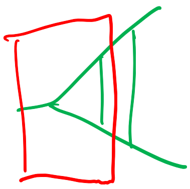


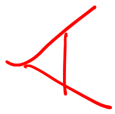
[1] $c \to P^c(T)$ furnishes a surjective map F from $\cup_{T \in C(\Gamma)} 2^{E(T)}$ to the forests f of Γ. The set of pre-images $f_c = F^{-1}(f)$ gives a partition of $\cup_{T \in C(\Gamma)} 2^{E(T)}$ which is a bijection with the forests of Γ.

After having determined the set $C(\Gamma)$, all (non-empty) forests of Γ are in bijection with (non-empty) sets f_c of some cuts (c, T). We describe them as follows.

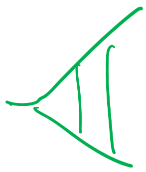
If we let $|T|$ be the number of vertices of a tree T, a tree T allows for $2^{|T|} - 1$ cuts including the empty one. For a graph Γ, this gives us $\sum_{T \in C(\Gamma)} 2^{|T|-1}$ cuts c. By construction, a forest $f = \prod_i \gamma_i$ of a graph Γ assigns to a graph the product $(\Gamma/f) \prod_i \gamma_i$. We have $|C(\Gamma/f)| \prod_i |C(\gamma_i)|$ cuts c_i corresponding to the same forest, and let f_c be the set of cuts (c, T) which correspond to the same forest f.

We often notate a cut (c, T) using T with marked edges, and notate the union f_c then as a sum of such trees. We have $\sum_{T \in C(\Gamma)} 2^{|T|} = \sum_f |f_c|$ by construction[2].



$f =$ 

2^1

$f = \emptyset$ 

[2]The cardinality $|f_c|$ of f_c gives the number of sectors in f and Γ/f.

We can hence label the forests of a graph Γ by subsets of edges on some of the trees $T \in C(\Gamma)$:

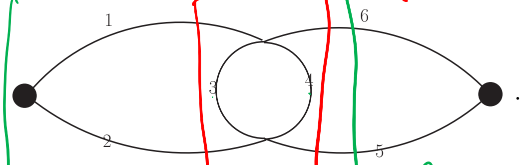
$$\sum_{f} = \sum_{T \in C(\Gamma)} \sum_{c \in f_c} = \sum_{T \in C(\Gamma)} \sum_{c \in P_E(T)} . \tag{2}$$

Furthermore, we identify the empty forest (of Γ) with Γ and write $\sum_f^{\emptyset}$ when we include it in the sum. If we allow forests also to contain Γ itself, we double the sum of forests and write $\sum_{[f]}$ for the corresponding sum.

Example 1. Consider the graph

$\sim f_1$

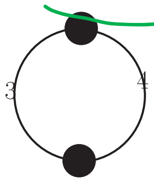
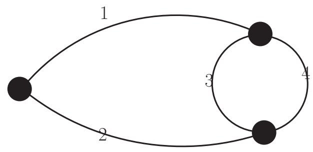
ϕ^4_4

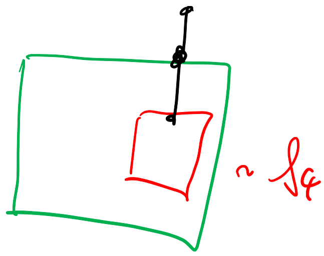
(3) $\Gamma =$  .

$\omega_\Gamma = 4|\Gamma| - |E_\Gamma| 2$

It has subgraphs

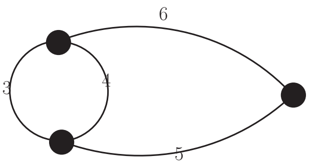
$\sim f_2$

$\gamma_{34} =$  $, \gamma_l =$  ,



$\sim f_4$

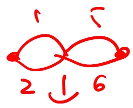
and

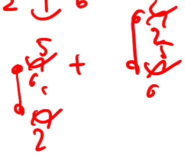
$\gamma_r =$  .

gives rise to two decorated rooted trees

We have $\gamma_{34} \subset \gamma_l$ and $\gamma_{34} \subset \gamma_r$. Its forests are

$$f_0 = \emptyset, |f_{\emptyset c}| = 2, \tag{4}$$
$$f_1 = \gamma_{34}, |f_{1c}| = 2, \tag{5}$$
$$f_2 = \gamma_l, |f_{2c}| = 1, \tag{6}$$
$$f_3 = \gamma_r, |f_{3c}| = 1, \tag{7}$$
$$f_4 = \gamma_{34}, \gamma_l, |f_{4c}| = 1, \tag{8}$$
$$f_5 = \gamma_{34}, \gamma_r, |f_{5c}| = 1. \tag{9}$$

$\Gamma/\gamma_{34} =$ 

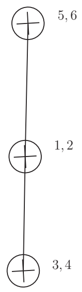


The forest f_1 is neither maximal nor complete. The forests f_2 and f_3 are both maximal, but incomplete. The forests f_4, f_5 are both complete. Hence, $C(\Gamma) = \{f_4, f_5\}$ is a two-element set.

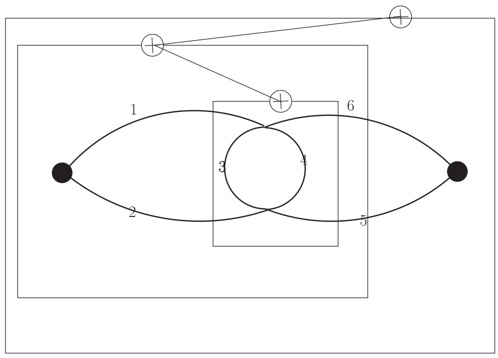
If we add the graph Γ itself to the forests, we double the set, for each f_i, we now have f_i and $f_i \cup \Gamma$.

The decorated trees T_4, T_5 are complete forests. They are given as:

(10)



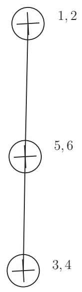
$= T_4 \leftrightarrow$



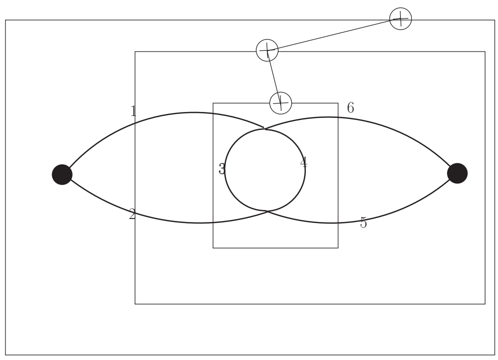
,

and

(11)



$= T_5 \leftrightarrow$



.

We can find the decorations by shrinking all graphs in the subforests of a given forest: we assign to the two maximal complete forests two rooted trees, the root corresponding to the vertex at the outermost box [3].

[3]Also, we can describe those trees as $T_4 := (((3,4),1,2),5,6)$ and $T_5 := (((3,4),5,6),1,2)$, where we indicate the tree structure by bracket configurations and decorations by the edge labels of the corresponding primitive graphs. If we notate forests in trees by square brackets $[\ldots]$ corresponding to cuts, then the correspondences are as follows: $f_0 \leftrightarrow (((3,4),1,2),5,6) + (((3,4),5,6),1,2)$, $f_1 \leftrightarrow (([3,4],1,2),5,6) + (([3,4],5,6),1,2)$, $f_2 \leftrightarrow ([(3,4),1,2],5,6)$, $f_3 \leftrightarrow ([(3,4),5,6],1,2)$, $f_4 \leftrightarrow ([[3,4],1,2],5,6)$, $f_5 \leftrightarrow ([[3,4],5,6],1,2)$. The forests corresponding to $f_i \cup \Gamma$ are then notated by replacing the outermost $(\ldots)$ pair of brackets by $[\ldots]$

1.2. **Hopf structures.** We summarize the relevant Hopf algebra structures as follows.

1.2.1. *For trees.* For the free commutative algebra of decorated rooted trees H_{Dec} (typically, decorations are provided by either the graphs p_v or their set of edge labels) we have a co-product Δ_T defined by

$$\Delta_T \circ B_+^p(\cdot) = B_+^p(\cdot) \otimes 1 + (\mathrm{id} \otimes B_+^p)\Delta_T, \tag{12}$$

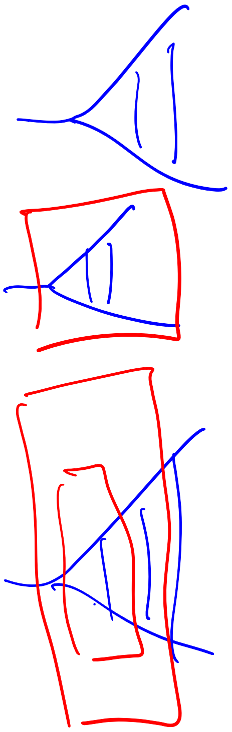
and an antipode given by

$$S(T) = -T - \sum_{c \in P_E(T)} (-1)^{|c|} P^c(T) R^c(T), \tag{13}$$

where $R^c(T)$ contains the root with decoration p and $P^c(T)$ are the other trees in $T - c$. B_+^p are Hochschild 1-cocycles.

We let $\mathrm{shad} : H_{\mathrm{Dec}} \to H_\emptyset$ be the map which forgets decorations.

$\gamma \subsetneq \Gamma$



1.2.2. *For graphs.* For graphs we have a Hopf algebra of graphs H_Γ with co-product

$$\Delta_G(\Gamma)=\Gamma\otimes 1+1\otimes\Gamma+\sum_{\gamma\subseteq\Gamma}\gamma\otimes\Gamma/\gamma, \tag{14}$$

← internal edges of γ shrink to zero length

where γ is a disjoint union $\gamma=\cup_i\gamma_i$ of 1PI graphs which are superficially divergent.

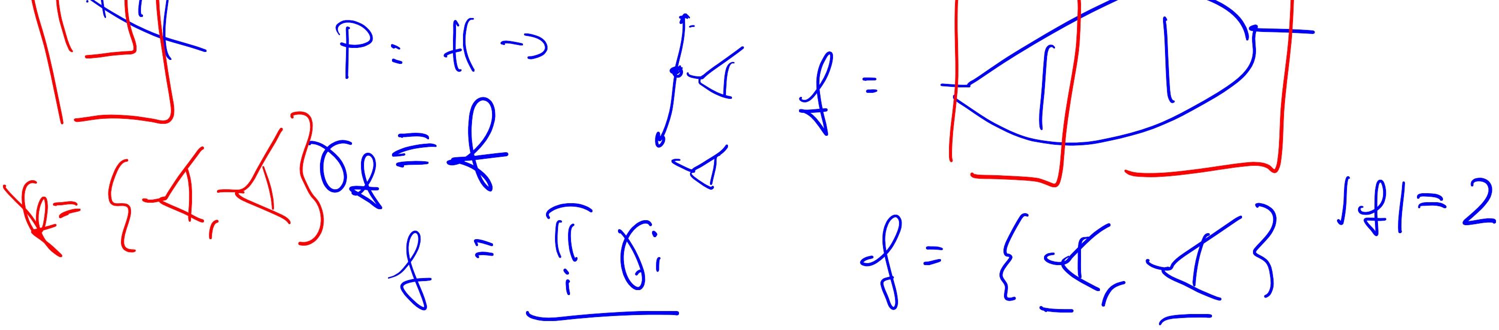
The antipode is given by

$$S(\Gamma)=-\Gamma-\sum_f(-1)^{|f|}\gamma_f\,\Gamma/\gamma_f. \tag{15}$$

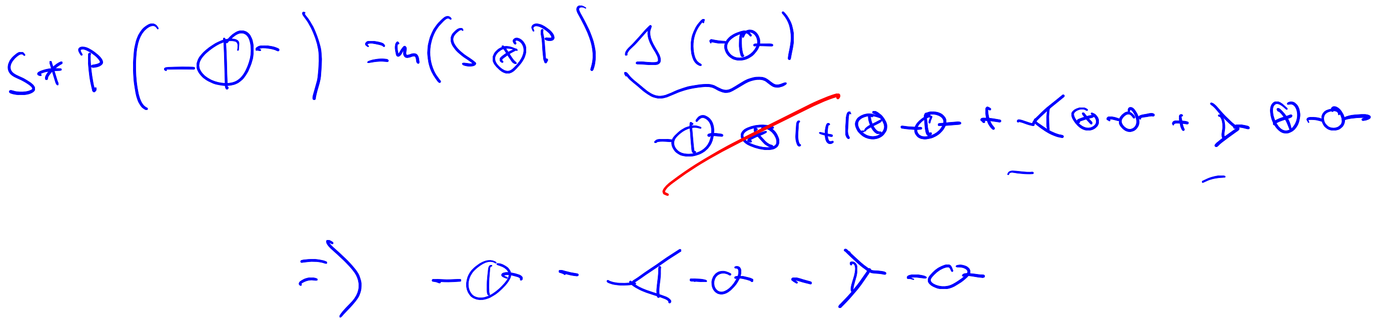
We have a Hopf algebra homomorphism $\rho : H_\Gamma\to H_{\mathrm{Dec}}$ given by $\rho(\Gamma)=\sum_{T\in C(\Gamma)}T$ and with

$$[\rho\otimes\rho]\Delta_G=\Delta_T\rho. \tag{16}$$

For any Hopf algebra $H\in(H_{\mathrm{Dec}},H_\Gamma)$, we let P be the projection into the augmentation ideal. We set $\sigma:=S\star P\equiv m_H(S\otimes P)\Delta$, which vanishes on scalars $\mathbb{Q}\mathbb{I}$. For the Hopf algebra of graphs, one has $\sigma(\Gamma)=\sum_f^{\emptyset}(-1)^{|f|}f(\Gamma/f)$.



$|f|=2$



We need a well-known lemma:

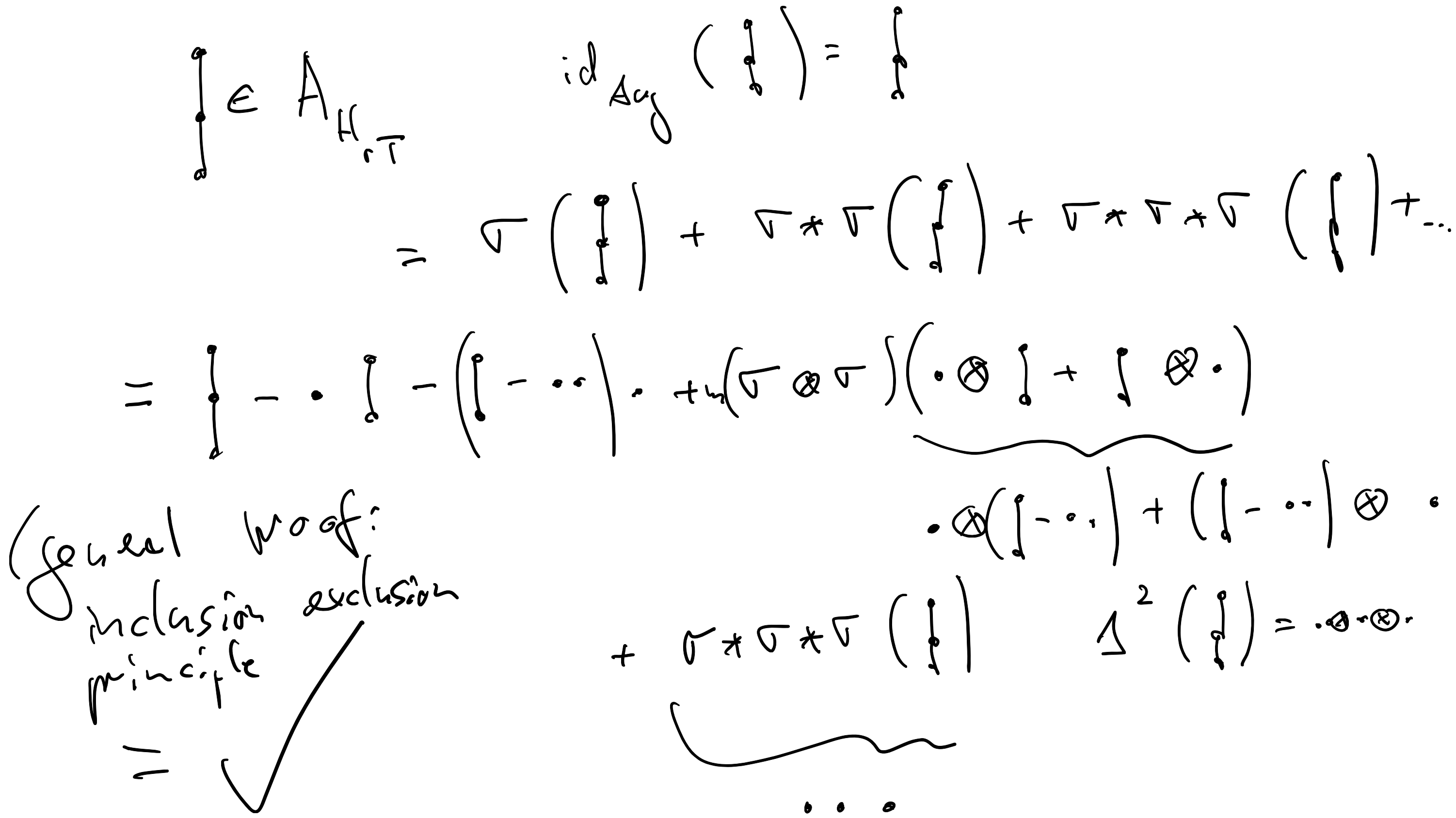
Lemma 2. *Let* $\mathrm{id}_{\mathrm{Aug}}$ *be the identity map* $\mathrm{Aug} \to \mathrm{Aug}$ *in the augmentation ideal. We have*

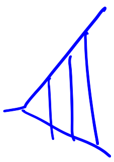
$$\mathrm{id}_{\mathrm{Aug}} = \sum_{j=1}^{\infty} \sigma^{\star j} =: \sum_{j=1}^{\infty} \sigma_j. \tag{17}$$

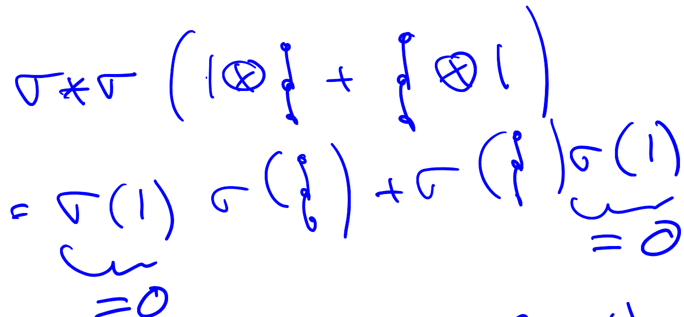
Note that the sums terminate when applied to any element of finite degree in the Hopf algebra. Hopf algebras H allow for a co-radical filtration

$$\mathbb{QI} = H^{(0)} \subset H^{(1)} \cdots \subset H^{(n)} \subset \cdots \subset H. \tag{18}$$

The maps σ_j vanish on elements in the Hopf algebra which are in $H^{(k)}$, $k < j$, and the coradical filtration is defined by the kernels of σ_j: elements in $H^{(k)}$ vanish when acted upon by $\sigma_j, \forall j > k$.







$\sigma(1) = m(S \otimes P)(1 \otimes 1)$
$= S(1) \cdot \underbrace{P(1)}_{=0}$

$\sigma_j = \underbrace{\sigma * \cdots * \sigma}_{j} \sim J^{*j}$

$\sigma_2(\Gamma) = m(\sigma \otimes \sigma)(\ \otimes\ ) = \ \in \mathrm{gr}^2$

Γ is a Feynman graph $c_1 L + c_2 L^2$ $\quad \sum_j \frac{1}{j!}\sigma_j$ gives L^j part

Now any map σ_j above corresponds to a finite sum over forests $\bar{\sigma}_j$. As the empty forest corresponds to the identity map of a graph Γ, we can write for forests

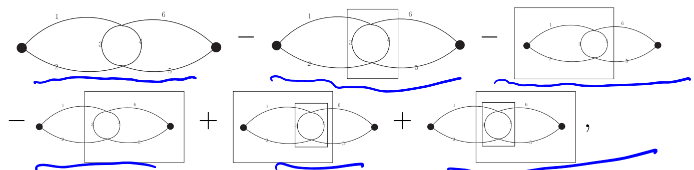
(19)
$$\emptyset = \sum_{j=1}^{\infty} \bar{\sigma}_j.$$

The following gives an example for the maps $\bar{\sigma}_j$, acting on the graph Γ of Example 1.

Example 3.

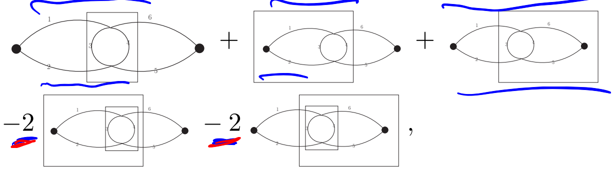
$\bar{\sigma}_1$:

(20)

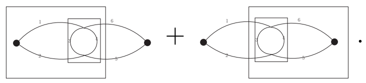


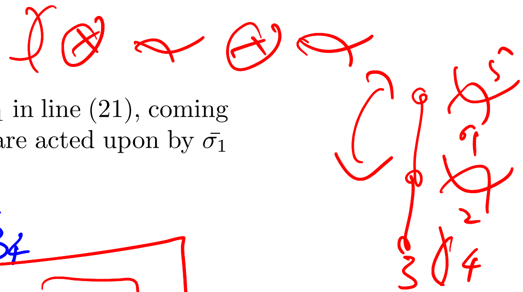
$\bar{\sigma}_2$:

(21)

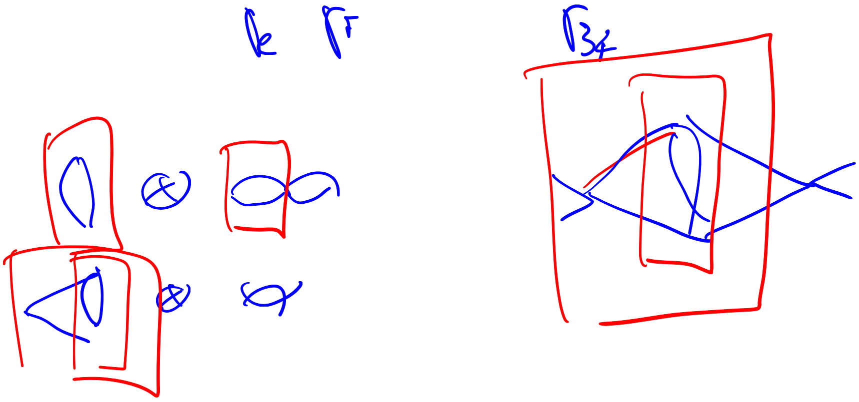


(22) $\bar{\sigma}_3$:





Note the multiplicity two generated in two terms in $\bar{\sigma}_2 = \bar{\sigma}_1 \star \bar{\sigma}_1$ in line (21), coming from the fact that the subgraphs γ_2, γ_3 and the cograph Γ/γ_1 are acted upon by $\bar{\sigma}_1$ with the same results.



On Monday: Feynman rules.

Eg. in ϕ^4_4, or ϕ^3_6 scalar field th.

i) parametrically ←

ii) " in momentum space

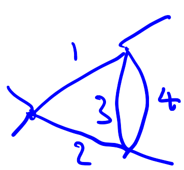
i) we need two polynomials.

ψ_Γ first Symanzik pol. for Γ

Φ_Γ second " "
(with masses)

φ_Γ without masses.

How do you prove renormalizability using properties of ψ_Γ, φ_Γ ?

Ex.  $\rightarrow \quad \psi_{\Gamma} = (A_1+A_2)(A_3+A_4) + A_3A_4$

$$\rightarrow \quad \cdots \quad \int_{\mathbb{P}_{\Gamma}} \frac{\Omega_{\Gamma}}{[(A_1+A_2)(A_3+A_4)+A_3A_4]^2}$$

↑ has a problem when $A_3, A_4 \to \infty$