Biinjecting of $H$, Hochschild cohomology, and universality.

We know $H^{(j)}$, set of $j$-roots with $j$ vertices.

Now $B_+$: $H^{(j)} \to H^{(j+1)}$.

Is this all, or is there a new grading associated to $B_+$?

As we have a co-radical filtration and this gives a grading.

How often can I apply $\Delta$ until I get primitives?

Let us consider $\Delta \cong \Delta_0 \cap (\partial \ominus \Theta \oplus \Theta)$. Let $n$ such that $\imath \cdot n = 0$?
\[ \Delta (\cdot) = \cdot \otimes \overline{\Delta} + \overline{\Delta} \otimes \cdot \]
\[ \Delta (\cdot) = 0 \]
\[ \Delta (\cdot) = \cdot \otimes \cdot \implies \Delta^2 = (\cdot \otimes \cdot) \overline{\otimes} (\cdot \otimes \cdot) \cdot \overline{\Delta} (1) = 0 \]
\[ \Delta^2 (1) = 0 \implies \Delta^n (1) = 0 \quad \forall n \geq 2 \]

For \( h \in \mathfrak{h} \) and \( \ell \) large enough such that
\[ \Delta^{\ell h, h} (h) \neq 0, \quad h \text{ co-radical degree} \]
and because of
\[ \Delta B = B \otimes B + (\partial \otimes B) \Delta, \]

\[ \ell \| h \| = \| h \|^2 \quad \text{A priori:} \quad \text{no.} \]

\[ 2 \| h \| = \| h \| \quad \text{no.} \]

\[ \ell \| h \| = \| h \| \quad \text{no.} \]

\[ \Delta (2 \| h \| - \cdot \cdot) = (2 \cdot \otimes \cdot - 2 \cdot \otimes \cdot) = 0 \]

Remark: \( \gamma : H \to H \)
\[ \gamma (h) = \gamma \cdot h \]
\[ \gamma (g) = \gamma g \]
Consider $S \ast Y : \mathfrak{m}_H(\mathfrak{S} \otimes Y) \xrightarrow{\Delta} \mathfrak{m}_H(\mathfrak{S} \otimes Y)(\mathfrak{T} \otimes \mathfrak{T} \otimes \mathfrak{S} + \cdots) = \left(-\mathfrak{T} \otimes \mathfrak{T} \otimes \mathfrak{S} + 2\mathfrak{T} \otimes \mathfrak{S} + \cdots\right)$

$B_{+}^n(\mathcal{E}) = \mathfrak{T}_n$

Prop. \[ \| S \ast Y(t_n) \| = 1 \]
\[ |t_n| = n. \]

$B_{+}$ is a Hochschild 1-cocycle.

$\Leftrightarrow \Delta B_{+} = B_{+} \otimes 1 + (1 \otimes B_{+}) \Delta \subseteq$

This implies universality for the pair $(H, B_{+})$. \[ \Delta \Delta = 1 = 1 \cdot 1 \]

Thus, $(H, B_{+})$ is unique and universal:

for any other commutative Hopf algebra $(H', L_1)$, $H = (\mathfrak{T}, \mathfrak{m}_1, \mathfrak{m}_1, \mathfrak{S}_1, \mathfrak{S}_1)$
\[ \Delta \circ l_1 = l_1 \otimes I_1 + (\text{id} \otimes l_1) l_1. \]

(\text{So } l_1 \text{ is a 1-cocycle}).

There exists a unique Hopf algebra morphism \( g : H \to H_1 \) s.t.

\[ l_1 \circ g = s \circ B_+ \cdot \]

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**What means \( B_+ \) is Hochschild closed?**

**Given any map** \( \mathbf{Z} : H \to H \otimes \cdots \otimes H \)

\[ \text{\( b_2 : H \to H \otimes \cdots \otimes H \) \( \underset{(n+1) \text{-times}}{\underset{n \text{-times}}{\text{\( n \text{-times} \)}}} \)} \]

\[ b_2(h) = (\text{id} \otimes 2)^n \Delta(h) - \sum_{j=1}^{n} (-1)^{j+1} A_{s_j} \Xi(c_j) \]

\[ + (-1)^{n+1} \Xi(h) \otimes 1 \]

(*)

**Easy to check:**

\[ b \circ b = 0. \]

The cohomology of this is "Hochschild cohomology."
Check (x) $d u = 1$. $\exists H: H \to H$

\[ \Delta z = z \otimes 1 + (id \otimes 2) \cdot z \]

$\implies b R_+ = 0$

\[ b R_+ = 3 \cdot b + (id \otimes b) \Delta - \Delta b \]

So, $R_+$ is $b$-closed.

Is $R_+$ $b$-exact?

\[ \exists R_+ \implies 3 \otimes 2 \implies 3: H \to H^\otimes \cong \mathbb{Q} \]

$R_+$ is not exact, as

\[ b (2) = -1 \implies T = b 2 \]

\[ T (2) = \Delta (2) I - (id \otimes 2) \Delta I \]

\[ = 0 \]

So, $R_+$ is 1-cocycle.

Thus, $R_+$ is the only generator

of Hochschild cohomology

for commutative $H$ of $F$ algebras.
(Foissy) using results in Harrison cohomology.

This makes \((H, B_t)\) interesting.

Consider the pair \((H, L_1)\),

\[ L_1 \circ S = S \circ B_t \] that shall uniquely determine a \(H_1\) algebra homomorphism \(S: H \to H_1\).

\[
S \left( \prod T_i \right) = \prod_i S \left( T_i \right)
\]

\[
S(\cdot) = L_1 \left( \prod T_i \right)
\]

\[
S B_t \left( \prod T_i \right) = L_1 \left( S \left( \prod T_i \right) \right).
\]

Non-trivial is needed to check

\[
(S \otimes S) \circ \Delta (h) = \Delta_1 \circ S (h).
\]
Check \( \Delta \).

\[ T = \mathcal{B}_+ (X) \]

\[ \Delta_i L_1 (S (\bar{\tau} \bar{\tau}_i)) = L_1 (S (\bar{\tau} \bar{\tau}_i)) \otimes \bar{\tau}, \]

\[ + (\text{id} \otimes L_1) \Delta_i (S (\bar{\tau} \bar{\tau}_i)), \text{ (**) \}

Use induction:

\[ (\text{id} \otimes L_1) \Delta_i S (\bar{\tau} \bar{\tau}_i) = (S \otimes S) (\text{id} \otimes \mathcal{B}_+) \Delta \bar{\tau} \bar{\tau}_i. \]

To (**)\]

\[ = \Delta_i L_1 (S (\bar{\tau} \bar{\tau}_i)) = L_1 o S (\bar{\tau} \bar{\tau}_i) \otimes \bar{\tau}, \]

\[ + (S \otimes S) (\text{id} \otimes \mathcal{B}_+) \Delta \bar{\tau} \bar{\tau}_i. \]
\[(S \otimes \mathcal{B}_t) \Delta (\prod_i T_i)\]
\[= S \circ \mathcal{B}_t (\prod_i T_i) \otimes S (\mathcal{B}_t) + \sum_i \mathcal{T}_i\]
\[+ (S \otimes S) (\text{id} \otimes \mathcal{B}_t) \Delta (\prod_i T_i)\]
\[= \bigcup_i S (\prod_i T_i) \otimes \mathcal{T}_i + (S \otimes S) . (\text{id} \otimes \mathcal{B}_t) \Delta (\prod_i T_i) . \square\]

When you have a comm. Hopf algebra \(H_1\), with a \(1\)-cocycle \(\lambda_1\) for \((H_1, \mathcal{B}_t)\) is a \(\mathbb{R}\) model.
The most beautiful simple count.

Hold algebra.

Polynomials in one variable.
P(x) = \sum_j f_j x^j

\[ \Delta x = x \otimes 1 + \otimes x \]
\[ \Delta \sum_j f_j x^j = \sum_j \Delta(f_j) x^j \]
\[ = \sum_j f_j \Delta(x)^j \]

\[ \Delta x^2 = x^2 \otimes 1 + \otimes x^2 + 2x \otimes x \]
\[ \Delta x^3 = x^3 \otimes 1 + 3x^2 \otimes x + 3x \otimes x^2 + \otimes x^3 \]

What is its flat seed? I- cococho?

\[ \Delta x^2 = \Delta x = (x \otimes 1 + \otimes x)^2 \]
\[ = (x \otimes 1 + \otimes x) \cdot (x \otimes 1 + \otimes x) \]
\[ = xx \otimes 1 + (1 \cdot x \otimes x) + x \cdot (1 \otimes x) + (1 \otimes x) \cdot x \]

But so this is

\[ \Delta \int x^n = \int x^n \otimes 1 + (\text{id} \otimes f) \Delta x^n \]
\[
\int x^n = \frac{1}{n+1} x^{n+1}
\]

\[
\begin{align*}
\Delta \int x & = \int \frac{1}{2} x^2 = \frac{1}{2} \left( x^2 \otimes 1 + (\otimes x^2 \otimes 2 \otimes x) \right) \\
& = \frac{1}{2} x^2 \otimes 1 + (\otimes \frac{1}{2} x^2 + x \otimes x) \\
& = \int x \otimes 1 + (\otimes \frac{1}{2} x^2 + x \otimes x) \\
& = \int x \otimes 1 + (\otimes \frac{1}{2} x^2 + x \otimes x) \\
& = \int x \otimes x + (\otimes \frac{1}{2} x^2)
\end{align*}
\]

\[
\begin{align*}
\int & \rightarrow \frac{1}{2} L^2 \\
\Lambda & \rightarrow \frac{1}{3} L^3 \\
\rho & = \frac{1}{6} L^3 \quad \int \frac{\rho n^2 x}{x} \, dx
\end{align*}
\]

\[
a, k, \in \Xi \ldots \Xi
\]
$\neq \neq \neq$

Skeletion expansion