$\downarrow$

Graphs, Zimmermann forests, parametric renormalization

What is a graph?

G is a triple

(H_G, V_G, E_G) where

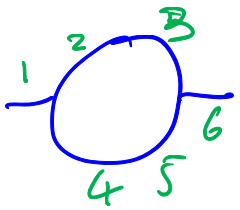
H_G is the set of half-edges

V_G is a partitions into parts of cardinality ≥ 3,

E_G is a partition of h-e into parts of cardinality < 3,

cardinality 1 ~ external edge

" 2 ~ internal "



g

$$H_g = \{1, 2, 3, 4, 5, 6\}$$

$$V_g = \{\underbrace{\{1, 2, 4\}}_{a}, \underbrace{\{3, 5, 6\}}_{b}\}$$

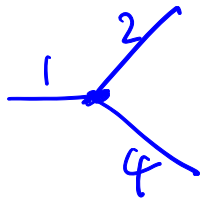
2 vertices

$$\mathcal{E}_g = \{\underbrace{\{2, 3\}, \{4, 5\}}_{E_G}, \{1\}, \{6\}\}$$

C_v is the corolla at v.

$C_a = \{1, 2, 4\}$ $\quad C_b = \{3, 5, 6\}$

graphically:



E_g is the set of parts of card. 2 in $\mathcal{E}_G$

$$|E_G| = e_G$$

$$|V_G| = v_G.$$

Typically, we consider bridge-free & 2-connected & 1PI graphs.

There is a valuation of 1PI graphs,

$$\omega_D(G) = D \cdot \underset{\substack{\uparrow \\ \text{loop number}}}{|G|} - \sum_{e \in E_G} \omega_e - \sum_{v \in V_G} \omega_v$$

For a renormalizable theory, one has

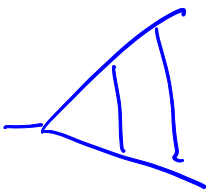
$$\omega_D(G_1) = \omega_D(G_2) \quad \forall\, G_1, G_2$$

such that $\ell_{G_1} = \ell_{G_2}$, regardless of $|G_1|$, $|G_2|$.

ℓ_G is number of parts of cardinality $\underline{1}$ in $\mathcal{E}_G$

$= \#$ of ext. edges.

$$|G| = E_G - V_G + 1 \quad \forall \text{ connected graphs.}$$



$\to 6 - 5 + 1 = 2$ loops

$$\omega_6\Big(\Big) = 6 \cdot 2 - 2 \cdot 6 + 0 \cdot 5 = 0$$

$$\omega_6\Big(\Big) = 6 \cdot 1 - 2 \cdot 3 + 0 \cdot 3 = 0$$

Zimmermann forests.

For a 1PI $\underbrace{\text{superficially divergent}}_{\omega_D(\Gamma) \geq 0}$ graph Γ, we define a forest $\mathcal{F}$ to be a collection of 1PI proper superficially div. subgraphs $\Gamma_i \subsetneq \Gamma$, $i \in I^{\mathcal{F}}_{\Gamma}$ an index set, s.t.

either $\Gamma_i \cap \Gamma_j = \emptyset$ $\forall\, i,j \in I^{\mathcal{F}}_{\Gamma}$, or

$\Gamma_i \subsetneq \Gamma_j$ or $\Gamma_j \subsetneq \Gamma_i$.

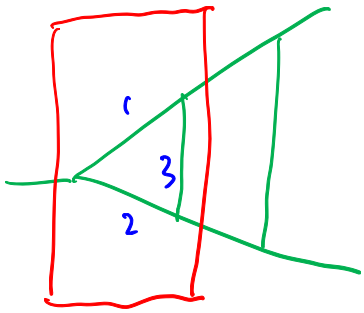
In particular, a forest is a product, hence a disjoint union, of graphs.

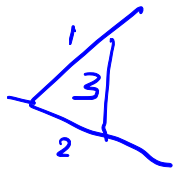
$$\mathcal{F} = \coprod_i \gamma_i$$

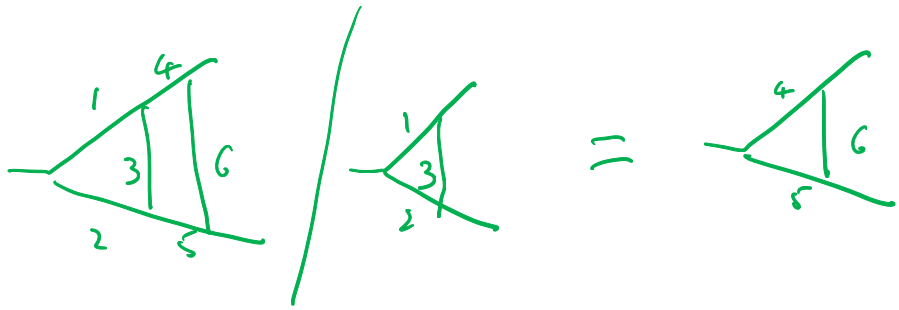
By $\Gamma/\mathcal{F}$ we denote the graph obtained by contracting all γ_i to points (shrink internal edges of γ_i's)

For a 1PI superficially divergent graph Γ we define a <u>maximal forest</u> to be a forest, and furthermore, we demand that $P^{\mathcal{F}}_{\Gamma} := \Gamma \Big/ \left(\bigcup_{i \in I^{\mathcal{F}}_{\Gamma}} \Gamma_i \right)$ has no divergent subgraph.

ϕ_6^3



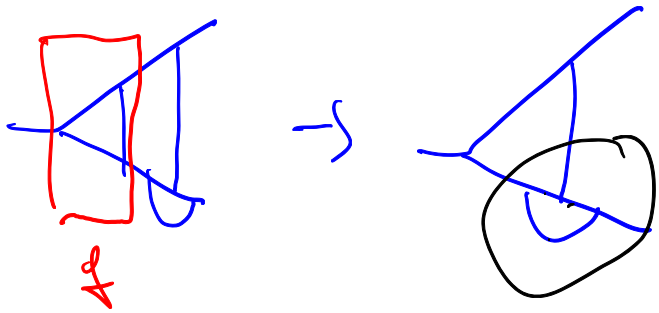
$f =$ 



← has no div. subgraph,

So f is maximal.

(Also see Ex. 1 in paper).



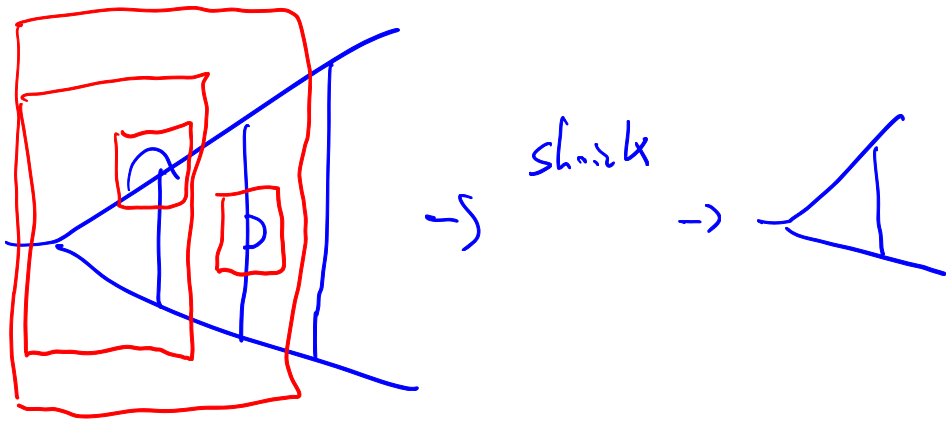
If you have a maximal forest f, we call I_Γ^f maximal for Γ.

For $f \ni \Gamma_i \subset \Gamma$, each index set I_Γ^f defines a dedicated index set I_i^f for all forests strictly contained in Γ_i, s.t.

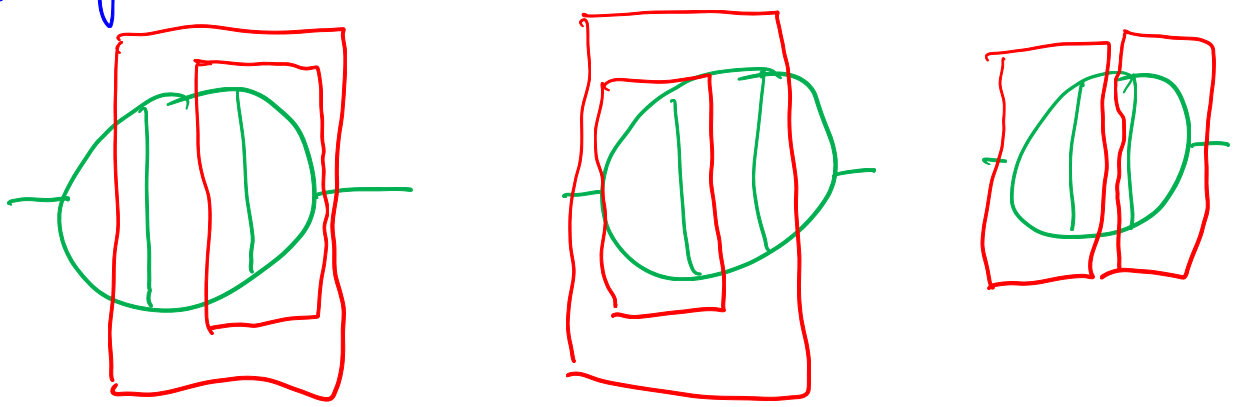
$\Gamma_j \subset \Gamma_i \quad \forall\ j \in \underset{\sim}{I}_i^f$.

We call a forest complete, if $\underset{\sim}{I}_\Gamma^f$ is maximal for Γ and $\underset{\sim}{I}_i^f$ maximal for each proper subgraph Γ_i of Γ.

Ex. Complete maximal forests.



Each graph Γ has only a finite number of complete maximal forests.

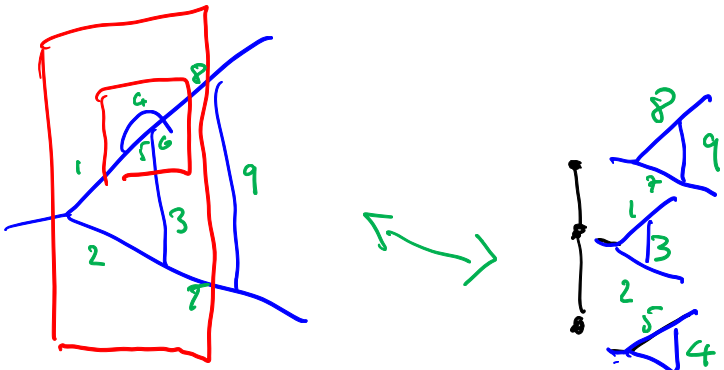


} complete max. forests.

Complete max. forests are in 1:1 correspondence with decorated rooted trees. The set of decorations p_v at vertices v of a tree is given by

superficially disjoint PI graphs free of scd-div's.

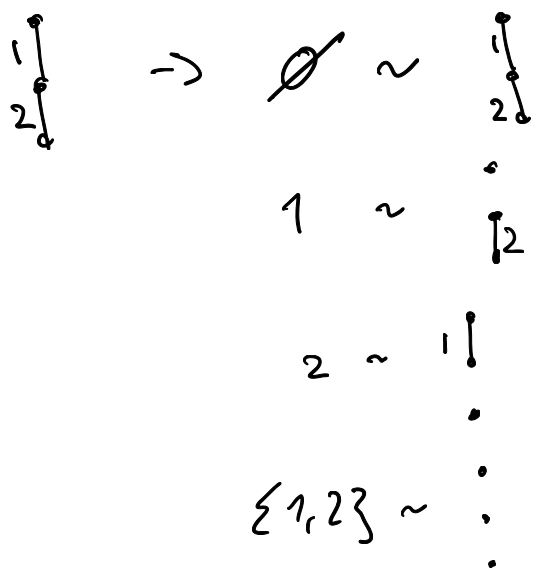
$$p_v = \Gamma_i \Big/ \bigcup_{j \in I_i} \Gamma_j .$$



max. compl. forest.

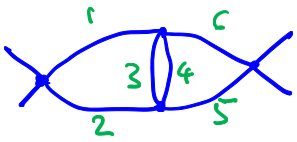
decorated rooted tree

Next: the powerset $P_E(T)$ of edges $E(T)$ of such a tree T gives all possible cuts of T.



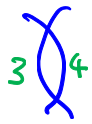
R^c, P^c as before, and $c \in P_E(T)$.

One more example. ϕ^4_4

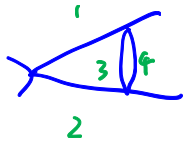


Γ

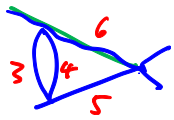
$\omega_4(\Gamma) = 0$



$= \gamma_{34}$



$\sim \gamma_\ell$



$\sim \gamma_r$

$\gamma_{34} \subsetneq \gamma_\ell \subsetneq \Gamma$

$\gamma_{34} \subsetneq \gamma_r \subsetneq \Gamma$

Forests:

$\mathcal{F}_0 = \emptyset$

$\mathcal{F}_1 = \gamma_{34}$ neither max, nor complete

$\mathcal{F}_2 = \gamma_\ell$

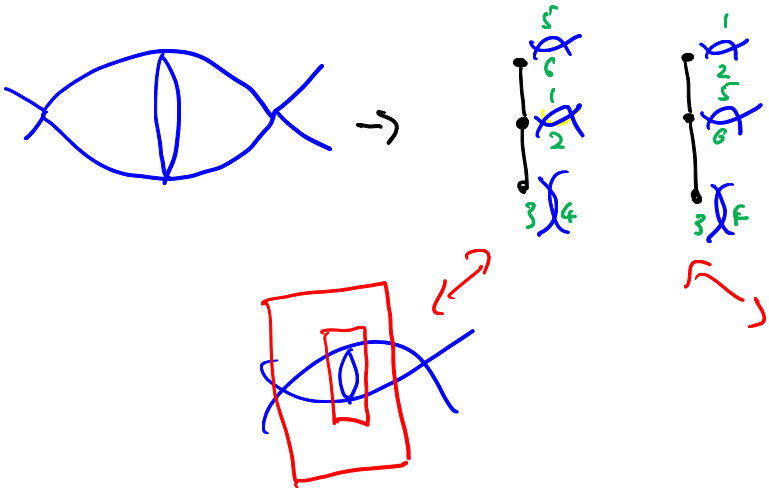
$\mathcal{F}_3 = \gamma_r$

both maximal not complete

$\mathcal{F}_4 = \{\gamma_{34}, \gamma_\ell\}$

$\mathcal{F}_5 = \{\gamma_{34}, \gamma_r\}$

maximal and complete



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On Wednesday, we map back and forth between the Hopf alg. of decorated rooted trees and H_Γ for graphs.

$$\rho : H_\Gamma \to H_{Dec}$$

$$(\rho \otimes \rho)\Delta_G \equiv \Delta_T \circ \rho .$$

And then we introduce the map

$$\sigma = S * P$$

antipode

$$P : H_\Gamma \to Aug_{H_\Gamma} \equiv A_{H_\Gamma}$$

and then there is an identity

$$id_{Aug} : A_{H_\Gamma} \to A_{H_\Gamma}$$

$$id_{Aug} = \sum_{j=1}^{\infty} \sigma^{*j} =: \sum_{j=1}^{\infty} \sigma_j .$$

this is the co-radical filtration alluded to last time

so that σ_1 lives $\subset$ gr^1.

This will give us the RGE.