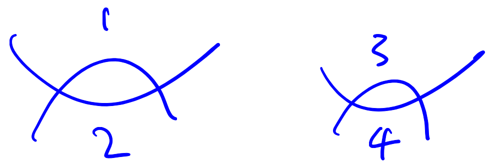
Renormalized Feynman rules in parametric space.

We consider graphs Γ with labeled edges.
To each such graph, we assign a form ϕ_Γ, build on edge variables A_i, $i \in E_\Gamma$, m_i masses and external momenta q_i.
Labels are <u>not repeated</u> for disjoint unions of graphs.



$$\phi_{\Gamma_1 \Gamma_2} = f_{\Gamma_1} f_{\Gamma_2} \Omega_{\Gamma_1 \cup \Gamma_2}.$$

$$\phi_\Gamma = f_\Gamma \Omega_\Gamma$$ ← "canonical form"

← function of A_i, m_i, q_i.

renormalized FR:

$$\Phi_\Gamma \to \Phi_\Gamma^R$$ to be defined as a forest sum

and $$\int_{\mathbb{P}_\Gamma} \Phi_\Gamma^R$$ is the renormalized amplitude:

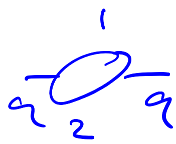
$$\int \Phi_\Gamma^R \,.$$

Warm up:

$$\frac{1}{Q} = \int_0^\infty dA\, e^{-AQ}$$

$$Q_1 = k^2 - m_1^2 + i\eta$$

$$Q_2 = (k+q)^2 - m_2^2 + i\eta$$

$G =$ 

$$\int d^4k \int_0^\infty dA_1\, dA_2\, e^{-A_1 Q_1 - A_2 Q_2}$$

$$\int_0^\infty dA_1\, dA_2\, e^{-A_1(k^2 - m_1^2 + i\eta) - A_2((k+q)^2 - m_2^2 + i\eta)}$$

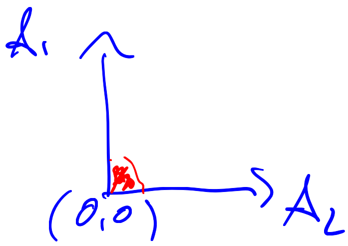
$$= \int_0^\infty dA_1\, dA_2 \int e^{-(A_1+A_2)(k^2+i\eta) - A_2(2k\cdot q + q^2) - A_1 m_1^2 - A_2 m_2^2}\, d^4k$$

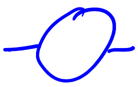
$$= \int_0^\infty dA_1\, dA_2 \int d^4\tilde{k}\; e^{-(A_1+A_2)(\tilde{k}^2 + \ldots) + \frac{A_1 A_2}{(A_1+A_2)^2} q^2 - \frac{(A_1 m_1^2 + A_2 m_2^2)(A_1+A_2)}{(A_1+A_2)}}$$

$$= \int_0^\infty dA_1\, dA_2 \; \frac{e^{\ominus\frac{\overbrace{(q^2 A_1 A_2}^{\varphi_G} - (m_1^2 A_1 + m_2^2 A_2)(A_1+A_2))}{A_1+A_2}}}{\underbrace{(A_1+A_2)^2}_{\psi_G}}$$

log div. for small A_1, A_2

$$\frac{dA_1\, dA_2}{(A_1+A_2)^2}$$



 $= G$

$$\psi_G = (A_1 + A_2)$$

$$\varphi_G = -q^2 (A_1 A_2)$$

$$\phi_G = \varphi_G + (m_1^2 A_1 + m_2^2 A_2)\overbrace{(A_1 + A_2)}^{\psi_G}$$

This generalizes (QFT, ~~textbook~~ I+II Bachelor thesis by Marcel Golz)

$$\psi_\Gamma := \sum_{\substack{T \\ \text{sp. trees}}} \prod_{e \notin T} A_e \qquad 0 \le A_e < \infty \text{ for each internal edge } e$$

fast? S...

$$\varphi_\Gamma := \sum_{\underbrace{T_1 \sqcup T_2}_{\text{sp. 2-forest}}} Q(T_1) \cdot Q(T_2) \prod_{e \notin T_1 \sqcup T_2} A_e$$

2nd without masses

where $Q(T_i) = \sum_{v \in V_{T_i}} q(v)$

V_{T_i} ← vertices

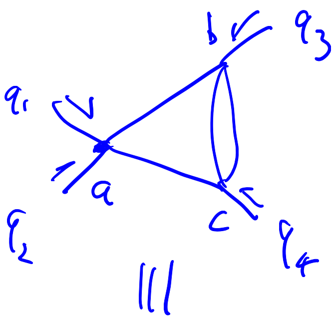
$q(v)$ ← ext. mom. at v

$$Q(T_1) = -Q(T_2)$$

$$Q(T_1)\cdot Q(T_2) = -Q(T_i)^2$$

$$\Phi_\Gamma = \varphi_\Gamma + \underbrace{\left(\sum_{e\in E_\Gamma} A_e m_e^2\right)\varphi_\Gamma}_{A\cdot\Gamma}.$$

} 2nd Sym... pol with masses

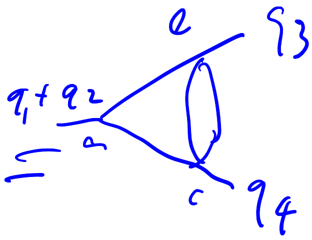


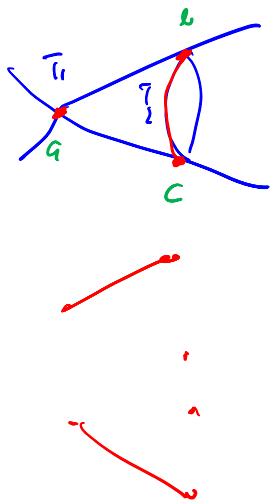
$$q(a) = q_1 + q_2$$

$$q(b) = q_3$$

$$q(c) = q_4$$

|||





Spanning 2-forests

$T_1 =$ · $T_2 =$

$Q(T_1) = q_1 + q_2$

$Q(T_2) = q_3 + q_4$

$Q(T_1)\cdot Q(T_2) = -(q_1+q_2)^2$

These are the two Symanzik polynomials.

This defines them on simple graphs Γ,

for products $\gamma = \prod_i \gamma_i$

we define:

$$\psi_\gamma = \prod_i \psi_{\gamma_i}$$

$$\varphi_\gamma = \sum_i \left(\varphi_{\gamma_i} \prod_{j \neq i} \psi_{\gamma_j} \right).$$

Define $Q_{vw} := q(v) \cdot q(w)$, $v, w \in V_\Gamma$

let $s := \sum_{v,w} c_{vw} Q_{vw}$

$c_{vw} \equiv c_{wv} \in \mathbb{R}$

such that $s = 0 \Rightarrow$ all $q(v) \equiv 0$

"s is in general position".

We call s a scale.

$$\varphi_\Gamma = s\, \varphi_\Gamma(\{\Theta\}), \quad \text{for } \Theta_{vw} = \frac{Q_{vw}}{s}$$

angle

$$\frac{m_e^2}{s} = \Theta_e$$

Properties of ψ_Γ, φ_Γ, ϕ_Γ.

$$\psi_\Gamma = \psi_{\Gamma/\gamma}\,\psi_\gamma + R^\Gamma_\gamma$$

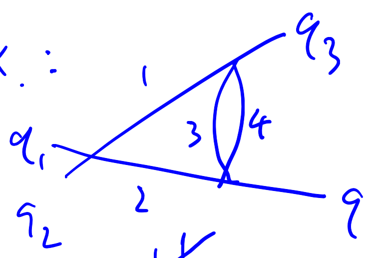
$$\varphi_\Gamma = \varphi_{\Gamma/\gamma}\,\psi_\gamma + \tilde{R}^\Gamma_\gamma,$$

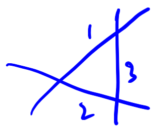
with: $|R^\Gamma_\gamma| = |\psi_\gamma| + 1$

$|\tilde{R}^\Gamma_\gamma| = |\psi_\gamma| + n, \quad n \geq 0$

and $|\ldots|$ means degree in variables A_e, $e \in E_\gamma$.

Ex.:





$= \gamma$

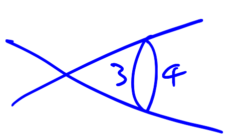
$$\psi_\Gamma = \underline{(A_1+A_2)(A_3+A_4) + A_3 A_4}$$

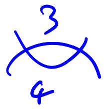
$$\psi_\gamma = A_1 + A_2 + A_3$$

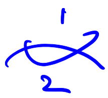
$$\psi_{\Gamma/\gamma} = \psi_{\cancel{\circ}} = A_4$$

$$\psi_{\Gamma/\gamma}\,\psi_\gamma = \underline{A_4(A_1+A_2+A_3)}$$

$$R^\Gamma_\gamma = \underline{A_2 A_3 + A_1 A_3} \quad \checkmark$$



$\gamma =$ 

$\Gamma/\gamma =$ 

$\gamma = 3 \cancel{\ } 4$

$\underline{\Gamma/\gamma =}$ 

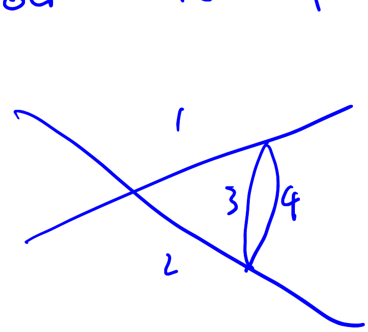
$(A_1+A_2)(\underline{A_3+A_4}) + \underline{\underline{A_3 A_4}}$

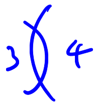
$$\varphi_\Gamma = (q_1+q_2)^2 (A_3+A_4)(A_1A_2) + q_3^2 A_1A_3A_4 + q_4^2 A_2A_3A_4 .$$

$$\varphi_{\Gamma/\gamma} = (q_1+q_2)^2 A_1A_2$$

$$\psi_\gamma = (A_3+A_4) .$$

How to prove statement on degrees?

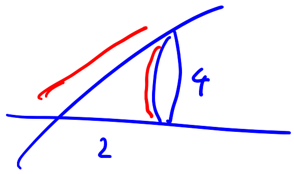


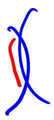
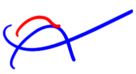
$\gamma =$ 

$\Gamma/\gamma =$ 

remind yourself: $\psi_\Gamma = \sum_{\substack{\text{sp.t.}\\ T}} \prod_{e\notin T} A_e$

choose sp. tree say 1,3:



γ  $\Gamma/\gamma =$ 

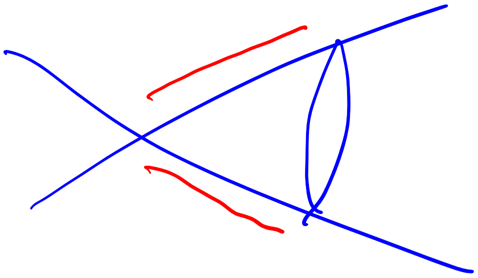
$A_2\, A_4$ comes from A_2 not in sp. tree of Γ/γ, A_4 not in sp. tree of γ.

$\varphi_{\Gamma/\gamma}\ \psi_\gamma$

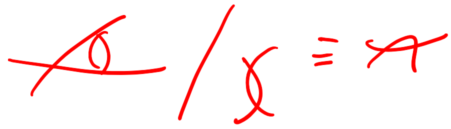
every edge which I add to the spanning tree in γ gives a loop in γ.

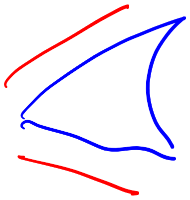
But I can add an edge to the sp. tr. of Γ/γ and remove one from γ, and still have a sp. tr. for Γ.

Proof:



this is not a sp. tree for

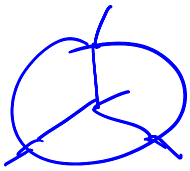




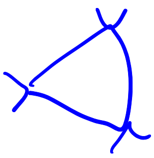
$|R^\Gamma_\gamma| > |\varphi_\gamma|$.

The proof for φ_Γ is similar, by working with 2-forests. Known since 1870's.

Hint: work out say

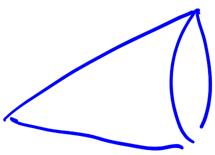


for any

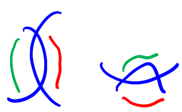


subgraph.

$(A_3+A_4)\ (A_1+A_2)$

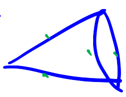


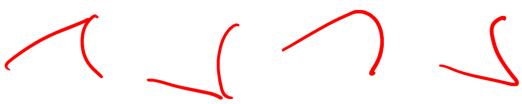
$\to$



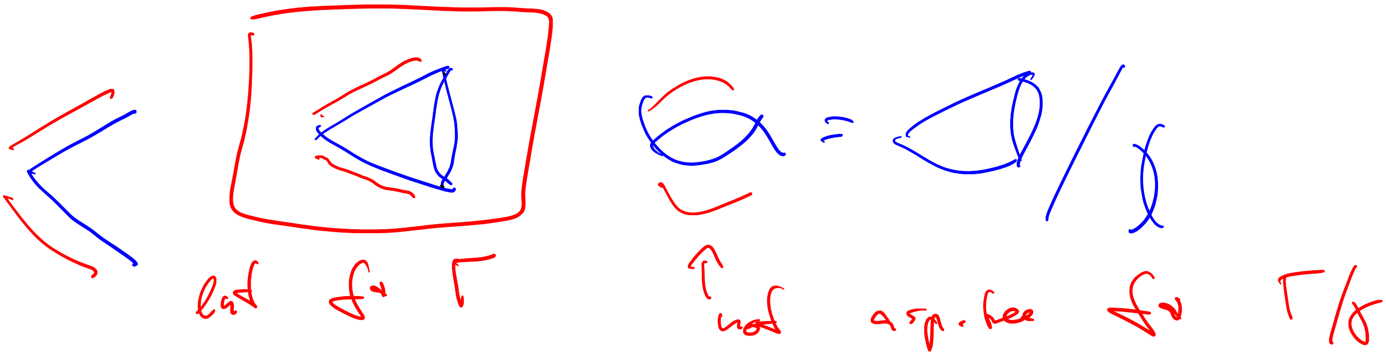
$\gamma \quad \Gamma/\gamma$

R^Γ_γ

$\varphi_\Gamma($  $)$ comes from all five sp. trees.

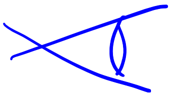


$\to (A_1+A_2)(A_3+A_4)$



All sp. trees which are not sp. tree for Γ/γ and for γ must have one more edge in $\Gamma\setminus\gamma$.

How to use this to get renormalized Feynman graphs.

For now 

all $m_i = 0$

$q_1 + q_2 = q \qquad q_3 = 0 \qquad q_4 = -q_1 - q_2$

just a function of q^2:

renormalize at $q^2 = \mu^2$:

On Wednesday, we see that the overall convergent (but no subdiv. renormalized) gives:

$$\left(\ln \frac{q^2}{\mu^2}\right)\left\{\frac{1}{\underbrace{\big((A_1+A_2)(A_3+A_4)+A_3A_4\big)}_{\Psi_{\triangleleft}}{}^2}\right\}\Omega_{\triangleleft}$$

Keep A_1, A_2 fixed, send $A_3, A_4 \to 0$

$$\to \frac{1}{(A_1+A_2)^2(A_3+A_4)^2} \qquad \Omega \to \Omega_\gamma \times \dots$$

$$\Omega_\gamma = dA_3\, dA_4$$

$$\frac{dA_3\, dA_4}{(A_3+A_4)^2} \quad \lightning$$

But: Hopf alg. tells you to consider

co-prd ← c.f.

$$\phi_\Gamma - \phi_{\Gamma/\gamma}\,\phi_\gamma$$

$$\frac{1}{(\psi_{\triangleleft})^2} - \frac{1}{\psi_\alpha^2\,\psi_\gamma^2}$$

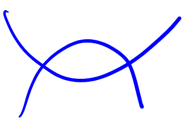
$$\left(\frac{1}{(\underline{\psi_{\Gamma/\gamma}\psi_\gamma} + R^\Gamma_\gamma)^2} - \frac{1}{\underline{\psi_\alpha^2\,\psi_\gamma^2}}\right) = X$$

but for $A_3 A_4 \to 0$

R^Γ_γ vanishes faster than ψ_γ.

Therefore $X \to 0$.

This is the mechanism for renormalization.



$$\to \int_{\mathbb{P}^1} \frac{\ln\left(\frac{\phi_\alpha}{\phi_\alpha^0}\right)}{(A_1+A_2)^2} \underline{(A_1\,dA_2 - A_2\,dA_1)}$$

$$\phi_\alpha = -q^2 A_1 A_2 + (m_1^2 A_1 + m_2^2 A_2)(A_1 + A_2)$$

$$\phi_\alpha^0 = \mu^2 A_1 A_2 + (\ldots\ldots)$$

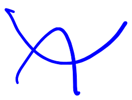
$$-\frac{q^2}{\mu^2} \quad \frac{\phi_\alpha(\Theta)}{\phi_\alpha^0(\Theta_0)}$$

$$\Rightarrow \int_{\mathbb{P}^1} \frac{\ln\left(\frac{-q^2}{\mu^2}\right) + \ln\left(\frac{\phi_\alpha(\theta)}{\phi^0_\alpha(\theta_0)}\right)}{(A_1+A_2)^2}\,\Omega_\alpha$$

Scale

$$\to \ln\frac{s}{s_0}\int_{\mathbb{P}^1}\frac{1}{(A_1+A_2)^2}\,\Omega_\alpha$$

q_1^2



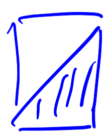
$$\int dA_1\, dA_2 \; \frac{e^{-\frac{A_1 A_2 q^2 + \dots}{(A_1+A_2)}}}{(A_1+A_2)^2}$$

$A_i = t\, a_i$

$$dA_1\, dA_2 \to dt \wedge \underbrace{\left(\overset{a_1}{A_1}\, d\overset{a_2}{\cancel{A_2}} - \overset{a_2}{\cancel{A_2}}\, d\overset{a_1}{\cancel{A_1}}\right)}_{\Omega_\alpha}$$

$$\mathrm{Exp}(c,\theta) = \int_{\mathbb{P}^1}\int_{c}^{\infty} \frac{e^{-t(a_i \dots)}}{t}\,\frac{dt\wedge\Omega}{(a_1+a_2)^2}$$

$$\int_0^1\int_0^x$$



$$= \int_{\mathbb{P}^1} \frac{\left(\cancel{\ln c} \pm \cancel{\gamma_E} + \ln\frac{(\dots)}{\theta_0}\right)}{(a_1+a_2)^2}\,\Omega$$

$\mathbb{P}(\mathbb{R}_+) \sim$ simplex in top-C

Schwinger variables — Feynman par...