1. Derivation of renormalized Feynman rules in parametric space

We turn to the derivation of Feynman rules. In our Hopf algebra $H_\Gamma$, we have graphs $\Gamma$ with labelled edges $e \in \Gamma^{[1]}$. To a graph $\Gamma$, we will assign forms $\Phi_{\Gamma}$ which depend on the edge labels $A_e$, the squared masses $m_e^2$, and the momenta $q(v)$, $v \in \Gamma^{[0]}$. Physicists may wish to consider these external momenta $q(v)$ as external edges, with a splitting as in say $q(v) = q_1 + q_2$ corresponding to two external edges at $v$, if so desired (for example to achieve homogeneity in the valence of vertices).

We assume that for a product of graphs $\Gamma_1 \Gamma_2$, labels are not repeated. The forms $\Phi_\Gamma$ have the structure $\Phi_\Gamma = f_\Gamma(\{A_e\})\Omega_\Gamma$, with $f_\Gamma(\{A_e\})$ a function of all the edge variables and $\Omega_\Gamma$ a standard form, see below. With unrepeated labels, $\Phi_{\Gamma_1 \Gamma_2} = f_{\Gamma_1} f_{\Gamma_2} \Omega_{\Gamma_1 \cup \Gamma_2}$.

Renormalized Feynman rules make use of the Hopf algebra $H_\Gamma$ to construct a linear combination of forms $\Phi_R^\Gamma$ such that it can be integrated against positive real projective $\mathbb{P}^{[1]}-1$-space. We write $\Phi_R(\Gamma) \in G = \text{Spec}_{\text{Feyn}}(H)$ for the resulting integral.
1.1. Schwinger parametrization and the exponential integral. We first define the two graph polynomials $\psi, \varphi$. Both are configuration polynomials. We define them here though using spanning trees and spanning forests. We have (for a connected graph $\Gamma$)

\begin{equation}
\psi_\Gamma := \sum_T \prod_{e \in T} A_e,
\end{equation}

for spanning trees $T$ and edges $e$ of $\Gamma$. Furthermore, we let $q(v)$ be the external momentum entering a vertex $v \in \Gamma$ (it can be zero), and for a subset of vertices $X \subset \Gamma$, we let $Q(X) = \sum_{v \in X} q(v)$. Then,

\begin{equation}
\varphi_\Gamma := \sum_{T_1 \cup T_2} Q(T_1) \cdot Q(T_2) \prod_{e \in T_1 \cup T_2} A_e,
\end{equation}

where $T_1 \cup T_2$ is a spanning two-forest. Note: $Q(T_1) = -Q(T_2)$, $Q(T_1)^2 = Q(T_2)^2 = -Q(T_1) \cdot Q(T_2)$.

We extend these definitions to products of graphs as follows. For $\gamma = \prod_i \gamma_i$,

\begin{equation}
\psi_{\gamma} = \prod_i \psi_{\gamma_i}, \quad \varphi_{\gamma} = \sum_i \left( \varphi_{\gamma_i} \prod_{j \neq i} \psi_{\gamma_j} \right).
\end{equation}
Define \( Q_{vw} := q(v) \cdot q(w) \), let \( S := \sum_{v, w \in \Gamma^{(0)}} c_{vw} Q_{vw} \) a real \((c_{vw} \in \mathbb{R})\) linear combination of scalar products \( Q_{vw} \) which vanishes only when all external momenta \( q(v) \) vanish. We say that \( S \) is in general kinematic position. Let \( \Theta_{vw} := Q_{vw}/S \) and \( \Theta_e := m_e^2/S \).

\[
\phi_{\Gamma}(\Theta) := \frac{\varphi}{S}, \quad \phi_{\Gamma}(S, \Theta) := S\phi_{\Gamma}(\Theta), \quad \phi_{\Gamma}(\Theta) := \varphi_{\Gamma}(\Theta) + \psi_{\Gamma} \left( \sum_{e} A_e \Theta_e \right). 
\]

We usually write \( \phi_{\Gamma} \equiv \phi_{\Gamma}(S, \Theta) \) in the decomposed form (and in slight abuse of notation) as \( \phi_{\Gamma} = S\phi_{\Gamma}(\Theta) \). Extension to products is defined as before.
We have for any $\gamma \subset \Gamma$, with $\gamma = \bigcup \gamma_i$, $\psi_\gamma = \prod_i \psi_{\gamma_i}$.

**Proposition 1.**

\[
\psi_{\Gamma} = \psi_{\Gamma/\gamma} + R_\gamma^\Gamma, |R_\gamma^\Gamma|_\gamma = |\psi(\gamma)|_\gamma + 1, \tag{5}
\]

\[
\phi_{\Gamma}(\Theta) = \phi_{\Gamma/\gamma}(\Theta) \psi_\gamma + \hat{R}_\gamma^\Gamma(\Theta), |\hat{R}_\gamma^\Gamma(\Theta)|_\gamma \geq |\psi(\gamma)|_\gamma + 1, \tag{6}
\]

and $|\phi_{\Gamma}| = |\psi_{\Gamma}| + 1$, and $|U|_V$ is the degree of $U$ in the edge variables of $V$, and $|U| = |U|_U$.

Note that $\phi_{\Gamma/\gamma}(\Theta)$ can be zero, for example when masses are zero and $Q(T_i) = 0$ for all two-forests of $\Gamma/\gamma$.

**Proof.** From the definitions via spanning trees and two-forests. \qed
We now let \( \Box_{\Gamma} \) be the hypercube \( \mathbb{R}^{[\Gamma]} \), and consider the integrand obtained from a Schwinger parametrization of a Feynman graph \( \Gamma \),

\[
\Phi_{\Gamma}(S, \Theta) := \int dA_{\Box_{\Gamma}^{1}} \ldots dA_{\Box_{\Gamma}^{(1)}} e^{\frac{S \phi_{\Gamma}(\Theta)}{\lambda_{T}}} \psi_{\Gamma}^2.
\]

This unrenormalized integrand cannot be integrated yet in the edge variables \( A_{e} \) against \( \Box_{\Gamma} \). Its renormalized counterpart has the form (say for logarithmic divergences, the general case is below and has the same structure)

\[
\Phi_{R,\Gamma}(S, S_{0}, \Theta, \Theta_{0}) = \sum_{f} (-1)^{|f|} \Phi_{f}(S_{0}, \Theta_{0}) \Phi_{\Gamma/f}(S, \Theta).
\]

\[
\Phi_{R,\Gamma}(S, S_{0}, \Theta, \Theta_{0}) = \sum \Phi_{\Gamma'}^{-1}(S_{0}, \Theta_{0}) \Phi_{\Gamma''}(S, \Theta),
\]

where we used Sweedler’s notation \( \Delta_{G}(\Gamma) = \sum \Gamma' \otimes \Gamma'' \) in the second line. This is the traditional forest formula\(^1\).

\(^{1}\)Note that we use \( \psi_{\emptyset} = 1, \phi_{\emptyset}(\Theta) = 0. \)
In the following, we will renormalize this integrand using kinetic renormalization schemes. For that, we let $2s_G \equiv 2sd(\Gamma)$ be the superficial degree of divergence of $\Gamma$ (in the example of a massive scalar field theory with quartic interactions):}

\begin{equation}
2s_G = 4|\Gamma| - 2|\Gamma^{[1]}|.
\end{equation}

Then, all vertex graphs $\Gamma$ have $s_G = 0$ together with $|\Gamma^{[1]}| = 2|\Gamma|$, while for all propagator graphs, $s_G = 1$ with $|\Gamma^{[1]}| = 2|\Gamma| - 1$.

Let us introduce new variables $A_e \rightarrow a_e$, $A_e = ta_e$, and $dA_1 \cdots dA_{|\Gamma^{[1]}|} \rightarrow dt \wedge \Omega_{\Gamma}$, with $\Omega_{\Gamma}$ the usual $(|\Gamma^{[1]}| - 1)$-form $A_1dA_2 \wedge \cdots \wedge dA_{|\Gamma^{[1]}|} - A_2dA_1 \wedge \cdots$. We find

\begin{equation}
\Phi_{\Gamma} := \frac{dt}{t} \wedge \frac{\Omega_{\Gamma} e^{\frac{2s_G - 2s_G^{[1]}}{2s_G^{[1]}}}}{e^{s_G^{[1]}}}.
\end{equation}

\begin{align*}
\int \mathcal{L}(A_1, A_2) \mathcal{L} \mathcal{L} &
\frac{A_1 dA_2}{(A_1 + A_2)^2} \\
\Delta_1 &:= A_1 a_1 \\
\Delta_2 &:= A_1 a_2 e^{-\frac{A_1 a_2 a_2}{(1 + a_2)}}
\end{align*}

\begin{align*}
\mathcal{A}_1 &:= \int dA_1 a_1 \frac{a_1 a_2 a_2}{(a_1 + a_2)^2} \\
\mathcal{A}_2 &:= \int dA_2 a_2 \frac{a_1 a_2 a_2}{(a_1 + a_2)^2}
\end{align*}
We want to study the overall $t$-integration as a function of the superficial degree of divergence $s^\Gamma$ first. Concretely, we are interested to define and find the limit in the $t$-integration

$$\lim_{c \to 0} \int_c^\infty \Phi_G, \tag{12}$$

where $c \in \mathbb{R}_+$. We use renormalization conditions on $\Phi_G \equiv \Phi_G(S, \Theta)$. 

$$\int_c^\infty \Phi_G \quad \text{exists} \quad \forall \ c > 0$$
Kinetic renormalization conditions imply that we choose values \( S_0, \Theta_0 \) for the scale and for the angles, such that the renormalized amplitudes of a graph \( \Gamma \), together with their first \( s_F \) derivatives in an expansion around that point, vanish. For \( s_F = 0 \), we can simply subtract at a chosen \( S_0, \Theta_0 \):

\[
\Phi_{\Gamma}(S, \Theta) \rightarrow \Phi_{\Gamma}(S, \Theta) - \Phi_{\Gamma}(S_0, \Theta_0)
\]

which takes care of the overall divergence in the graph \( \Gamma \). For \( s_F = 1 \), we are dealing with a quadratically divergent propagator function. We will subtract at \( q^2 = m^2 \). Note that there are no angles \( \Theta_{vw} \) for a two-point function, the \( \Theta_e \) remain though. Kinetic renormalization conditions are determined by the requirement that the renormalized amplitude vanishes at \( q^2 = m^2 \), together with its first derivative \( \partial_{q^2} \), so that the pole in the propagator has a on-shell unit residue.\(^2\)

\(^2\)For a massless propagator, vanishing of \( \Phi_R^{\mu} \) at \( q^2 = 0 \) and of \( \Phi_R^{\mu}/q^2 \) at \( q^2 = \mu^2 \) are also convenient renormalization conditions.
1.2. $s_T = 0$. Let us start with the case $s_T = 0$. The limit is

$$\lim_{c \to 0} \int_c^\infty \frac{e^{-tX}}{t} dt = -\ln c + \ln X + \gamma_E + O(c).$$

Here, $\gamma_E$ is the Euler–Mascheroni constant. Note that we can decompose the logarithm as

$$\ln \frac{S}{S_0} \frac{\phi_T(\Theta)}{\phi_T(\Theta_0)} = \ln\left(\frac{S}{S_0}\right) + \ln\left(\frac{\phi_T(\Theta)}{\phi_T(\Theta_0)}\right),$$

(we assume $S/S_0 > 0$). We assume also that the angles $\Theta, \Theta_0$ are chosen such that we are off Landau singularities. Approaching such singularities means studying the corresponding variation of the logarithm above.
Let us now look at logarithmic sub-divergences. A typical term in the forest formula provides an integrand of the form

\begin{equation}
\begin{aligned}
e^{\frac{S_0}{f}(\theta_0)} & e^{\frac{S_0}{f}(\theta_0)} - e^{\frac{S_0}{f}(\theta_0)} e^{\frac{S_0}{f}(\theta_0)} + e^{\frac{S_0}{f}(\theta_0)} e^{\frac{S_0}{f}(\theta_0)} \\
\psi^2 & \psi^2 \\
\psi^2 & \psi^2
\end{aligned}
\end{equation}
Combining each of the two products of exponentials into a single exponential and using the exponential integral as above delivers

\[ M_f^\Gamma := \ln \frac{S\delta \Gamma f(\Theta) \psi_f + S_0\delta \Gamma(\Theta_0) \psi_f}{\psi_f^2}. \]

Summing over all forests including the empty one delivers the renormalized integrand as the homogeneous of degree zero form

\[ \Phi^\Gamma := \sum_f (-1)^{|f|} M_f^\Gamma. \]

\( \Phi^\Gamma \) is an integrand which can, this is just a rewriting of the forest formula, be integrated against \( P^{[\Gamma]} \). An explicit proof from scratch is given below though, after we decomposed Feynman rules suitably.