i) renormalized parametric:
non-recursive "Zimmermann's"

ii) momentum space via recursion
Bogoliubov-Parasiuk

iii) coordinate space "Epstein Glaser"

iv) Wilson's line
path integral

RENORMALIZED PARAMETRIC FEYNMAN RULES (CONT'D., JAN.20 2021)

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1. Derivation of renormalized Feynman rules in parametric space

We turn to the derivation of Feynman rules. In our Hopf algebra H_Γ, we have graphs Γ with labelled edges $e \in \Gamma^{[1]}$. To a graph Γ, we will assign forms Φ_Γ which depend on the edge labels A_e, the squared masses m_e^2, and the momenta $q(v)$, $v \in \Gamma^{[0]}$. Physicists may wish to consider these external momenta $q(v)$ as external edges, with a splitting as in say $q(v) = q_1 + q_2$ corresponding to two external edges at v, if so desired (for example to achieve homogeneity in the valence of vertices).

We assume that for a product of graphs $\Gamma_1\Gamma_2$, labels are not repeated. The forms Φ_Γ have the structure $\Phi_\Gamma = f_\Gamma(\{A_e\})\Omega_\Gamma$, with $f_\Gamma(\{A_e\})$ a function of all the edge variables and Ω_Γ a standard form, see below. With unrepeated labels, $\Phi_{\Gamma_1\Gamma_2} = f_{\Gamma_1} f_{\Gamma_2} \Omega_{\Gamma_1 \cup \Gamma_2}$.

Renormalized Feynman rules make use of the Hopf algebra H_Γ to construct a linear combination of forms Φ_Γ^R such that it can be integrated against positive real projective $\mathbb{P}^{|\Gamma^{[1]}|-1}$-space. We write $\mathbf{\Phi}^R(\Gamma) \in G = \mathrm{Spec}_{\mathrm{Feyn}}(H)$ for the resulting integral.

$$\phi(h_1 h_2) = \phi(h_1)\phi(h_2) \qquad G^H_{\mathbb{C}}$$

for fixed masses and momenta

1.1. **Schwinger parametrization and the exponential integral.** We first define the two graph polynomials ψ, φ. Both are configuration polynomials. We define them here though using spanning trees and spanning forests. We have (for a connected graph Γ)

$$\psi_\Gamma := \sum_T \prod_{e \notin T} A_e, \tag{1}$$

for spanning trees T and edges e of Γ. Furthermore, we let $q(v)$ be the external momentum entering a vertex $v \in \Gamma$ (it can be zero), and for a subset of vertices $X \subset \Gamma$, we let $Q(X) = \sum_{v \in X} q(v)$. Then,

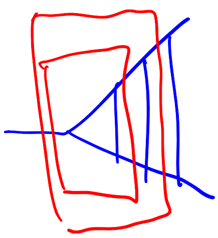
$$\varphi_\Gamma := \sum_{T_1 \cup T_2} Q(T_1) \cdot Q(T_2) \prod_{e \notin T_1 \cup T_2} A_e, \tag{2}$$

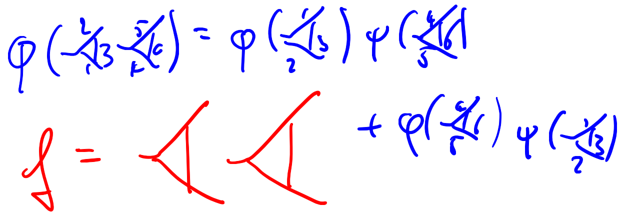
where $T_1 \cup T_2$ is a spanning two-forest. Note: $Q(T_1) = -Q(T_2)$, $Q(T_1)^2 = Q(T_2)^2 = -Q(T_1) \cdot Q(T_2)$.

We extend these definitions to products of graphs as follows. For $\gamma = \prod_i \gamma_i$,

$$\psi_\gamma = \prod_i \psi_{\gamma_i},\ \varphi_\gamma = \sum_i \left(\varphi_{\gamma_i} \prod_{j \neq i} \psi_{\gamma_j} \right). \tag{3}$$

$\gamma = \prod_i \gamma_i$





$\Phi(\gamma) = \Phi(\prod_i \gamma_i)$

Define $Q_{vw} := q(v) \cdot q(w)$, let $S := \sum_{v,w \in \Gamma^{(0)}} c_{vw} Q_{vw}$ a real ($c_{vw} \in \mathbb{R}$) linear combination of scalar products Q_{vw} which vanishes only when all external momenta $q(v)$ vanish. We say that S is in general kinematic position. Let $\Theta_{vw} := Q_{vw}/S$ and $\Theta_e := m_e^2/S$.

$$\varphi_\Gamma(\Theta) := \frac{\varphi_\Gamma}{S},\ \phi_\Gamma(S,\Theta) := S\phi_\Gamma(\Theta),\ \phi_\Gamma(\Theta) := \varphi_\Gamma(\Theta) + \psi_\Gamma\left(\sum_e A_e \Theta_e\right). \tag{4}$$

We usually write $\phi_\Gamma \equiv \phi_\Gamma(S,\Theta)$ in the decomposed form (and in slight abuse of notation) as $\phi_\Gamma = S\phi_\Gamma(\Theta)$. Extension to products is defined as before.

We have for any $\gamma \subset \Gamma$, with $\gamma = \cup_i \gamma_i$, $\psi_\gamma = \prod_i \psi_{\gamma_i}$,

Proposition 1.

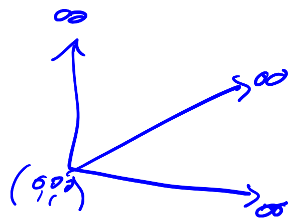
$$\psi_\Gamma = \psi_{\Gamma/\gamma}\psi_\gamma + R^\Gamma_\gamma, |R^\Gamma_\gamma|_\gamma = |\psi(\gamma)|_\gamma + 1, \tag{5}$$

$$\phi_\Gamma(\Theta) = \phi_{\Gamma/\gamma}(\Theta)\psi_\gamma + \bar{R}^\Gamma_\gamma(\Theta), |\bar{R}^\Gamma_\gamma(\Theta)|_\gamma \geq |\psi(\gamma)|_\gamma + 1, \tag{6}$$

and $|\phi_\Gamma| = |\psi_\Gamma| + 1$*, and* $|U|_V$ *is the degree of* U *in the edge variables of* V*, and* $|U| = |U|_U$*.*

Note that $\phi_{\Gamma/\gamma}(\Theta)$ can be zero, for example when masses are zero and $Q(T_i) = 0$ for all two-forests of Γ/γ.

Proof. From the definitions via spanning trees and two-forests. $\square$



We now let $\square_\Gamma$ be the hypercube $\mathbb{R}_+^{|\Gamma^{(1)}|}$, and consider the integrand obtained from a Schwinger parametrization of a Feynman graph Γ,

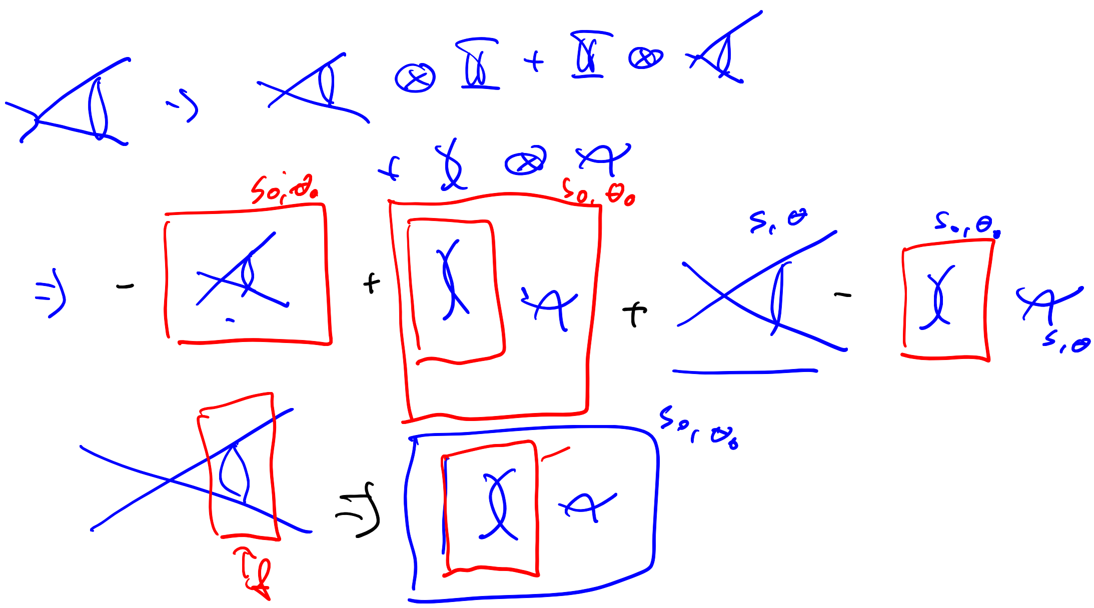
$$\Phi_\Gamma(S,\Theta) := \frac{dA_1\cdots dA_{|\Gamma^{(1)}|}e^{+\frac{S\phi_\Gamma(\Theta)}{\psi_\Gamma}}}{\psi_\Gamma^2}. \tag{7}$$

This unrenormalized integrand cannot be integrated yet in the edge variables A_e against $\square_\Gamma$. Its renormalized counterpart has the form (say for logarithmic divergences, the general case is below and has the same structure)

$$\Phi_\Gamma^R(S,S_0,\Theta,\Theta_0) = \sum_{[f]}^{\emptyset}(-1)^{|f|}\Phi_f(S_0,\Theta_0)\Phi_{\Gamma/f}(S,\Theta). \tag{8}$$

$$= \sum \Phi_{\Gamma'}^{-1}(S_0,\Theta_0)\Phi_{\Gamma''}(S,\Theta), \tag{9}$$

where we used Sweedler's notation $\Delta_G(\Gamma)=\sum\Gamma'\otimes\Gamma''$ in the second line. This is the traditional forest formula[1].



[1] Note that we use $\psi_\emptyset = 1$, $\phi_\emptyset(\Theta)=0$.

In the following, we will renormalize this integrand using kinetic renormalization schemes. For that, we let $2s_\Gamma \equiv 2sd(\Gamma)$ be the superficial degree of divergence of Γ (in the example of a massive scalar field theory with quartic interactions):

$$2s_\Gamma = 4|\Gamma| - 2|\Gamma^{[1]}|. \tag{10}$$

Then, all vertex graphs Γ have $s_\Gamma = 0$ together with $|\Gamma^{[1]}| = 2|\Gamma|$, while for all propagator graphs, $s_\Gamma = 1$ with $|\Gamma^{[1]}| = 2|\Gamma| - 1$.

Let us introduce new variables $A_e \to a_e$, $A_e = ta_e$, and $dA_1 \cdots dA_{|\Gamma^{[1]}|} \to dt \wedge \Omega_\Gamma$, with Ω_Γ the usual $(|\Gamma^{[1]}|-1)$-form $A_1 dA_2 \wedge \cdots \wedge dA_{|\Gamma^{[1]}|} - A_2 dA_1 \cdots \pm \cdots$. We find

$$\Phi_\Gamma := \frac{dt}{t} \wedge \frac{\Omega_\Gamma e^{t\frac{S\phi_\Gamma(\Theta)}{\psi_\Gamma}}}{t^{s_\Gamma}\psi_\Gamma^2}. \tag{11}$$

$A_1 > A_2$

$A_2 > A_1$

 $\to$ $f(A_1, A_2)\,\underbrace{\Omega_\gamma}_{A_1 dA_2 - A_2 dA_1}$

$$\frac{e^{-\frac{A_1 A_2 q^2}{(A_1+A_2)}}}{(A_1+A_2)^2}$$

$A_2 = A_1 a_2$ $\Rightarrow$ $\dfrac{e^{-\frac{A_1 a_2 q^2}{(1+a_2)}}}{A_1 (1+a_2)^2}\, dA_1 da_2$

$A_1 = A_2 a_1$

$$\frac{e^{-t\frac{a_1 a_2 q^2}{(a_1+a_2)}}}{(a_1+a_2)^2}\,(a_1 da_2 - a_2 da_1)$$

$b_2 = A_1 a_2$ $\qquad \cancel{dA_1}$ $\qquad dA_1 a_2$

$a_1 da_2 + a_2 da_1$

$$\downarrow \int_0^\infty \frac{dt}{t}\, \frac{e^{-t\frac{q^2 a_1 a_2}{(a_1+a_2)}}}{(a_1+a_2)^2}\,\Omega_\gamma$$

We want to study the overall t-integration as a function of the superficial degree of divergence s_Γ first. Concretely, we are interested to define and find the limit in the t-integration

$$\lim_{c\to 0}\int_c^\infty \Phi_\Gamma, \tag{12}$$

where $c\in\mathbb{R}_+$. We use renormalization conditions on $\Phi_\Gamma \equiv \Phi_\Gamma(S,\Theta)$.

$\int_c^\infty \Phi_\Gamma$ exists $\forall\, c>0$

Kinetic renormalization conditions imply that we choose values S_0, Θ_0 for the scale and for the angles, such that the renormalized amplitudes of a graph Γ, together with their first s_Γ derivatives in an expansion around that point, vanish. For $s_\Gamma = 0$, we can simply subtract at a chosen S_0, Θ_0:

$$\Phi_\Gamma(S, \Theta) \to \Phi_\Gamma(S, \Theta) - \Phi_\Gamma(S_0, \Theta_0) \tag{13}$$

which takes care of the overall divergence in the graph Γ.
For $s_\Gamma = 1$, we are dealing with a quadratically divergent propagator function. We will subtract at $q^2 = m^2$. Note that there are no angles Θ_{vw} for a two-point function, the Θ_e remain though. Kinetic renormalization conditions are determined by the requirement that the renormalized amplitude vanishes at $q^2 = m^2$, together with its first derivative ∂_{q^2}, so that the pole in the propagator has a on-shell unit residue[2].

[2]For a massless propagator, vanishing of Φ_Γ^R at $q^2 = 0$ and of Φ_Γ^R/q^2 at $q^2 = \mu^2$ are also convenient renormalization conditions.

1.2. $s_\Gamma = 0$**.** Let us start with the case $s_\Gamma = 0$. The limit is

$$\lim_{c\to 0}\int_c^\infty [\Phi_\Gamma(S,\Theta) - \Phi_\Gamma(S_0,\Theta_0)] = \frac{\Omega_\Gamma \ln \frac{S\phi_\Gamma(\Theta)}{S_0\phi_\Gamma(\Theta_0)}}{\psi_\Gamma^2}, \tag{14}$$

using that for small $c > 0$,

$$\int_c^\infty \frac{e^{-tX}dt}{t} = -\cancel{\ln c} + \ln X + \cancel{\gamma_E} + \cancel{\mathcal{O}(c)}. \tag{15}$$

Here, γ_E is the Euler–Mascheroni constant. Note that we can decompose the logarithm as

$$\ln \frac{\frac{S}{S_0}\phi_\Gamma(\Theta)}{\phi_\Gamma(\Theta_0)} = \ln(S/S_0) + \ln(\phi_\Gamma(\Theta)/\phi_\Gamma(\Theta_0)),$$

(we assume $S/S_0 > 0$). We assume also that the angles Θ, Θ_0 are chosen such that we are off Landau singularities. Approaching such singularities means studying the corresponding variation of the logarithm above.

$$\ln\frac{S}{S_0}\underbrace{\int \frac{1}{(a_1+a_2)^2}\,\Omega_{\alpha}}_{\int_0^\infty d\alpha_1 \frac{1}{(1+\alpha_1)^2} d\alpha_1 \equiv 1}$$

co-forest

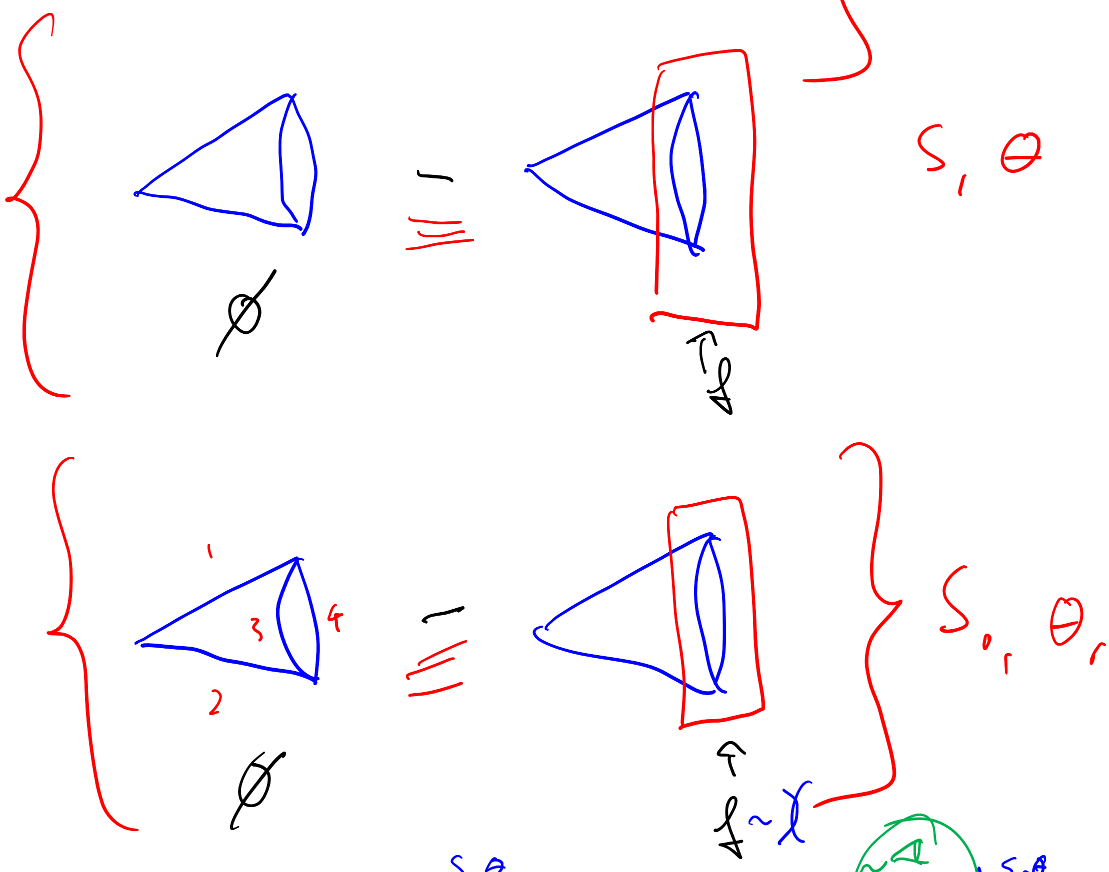
Let us now look at logarithmic sub-divergences. A typical term in the forest formula provides an integrand of the form

$$\frac{e^{+\frac{S\phi_{\Gamma/f}(\Theta)}{\psi_{\Gamma/f}}}}{\psi^2_{\Gamma/f}}\,\frac{e^{+\frac{S_0\phi_f(\Theta_0)}{\psi_f}}}{\psi_f^2} - \frac{e^{+\frac{S_0\phi_{\Gamma/f}(\Theta_0)}{\psi_{\Gamma/f}}}}{\psi^2_{\Gamma/f}}\,\frac{e^{+\frac{S_0\phi_f(\Theta_0)}{\psi_f}}}{\psi_f^2}. \tag{16}$$

co

sub-forest

Now: apply exponential integral



$$\emptyset \quad \frac{\ln \dfrac{\phi^{S,\Theta}_{\triangleleft}}{\phi^{S_0,\Theta_0}_{\triangleleft}}}{\psi^2_{\triangleleft}} - \frac{\ln\left(\dfrac{\phi^{S,\Theta}_{\alpha}\psi_\gamma + \cancel{\phi^{S_0,\Theta_0}_\gamma \psi_\alpha}}{\phi^{S_0,\Theta_0}_\alpha\psi_\gamma+\cancel{\phi^{S_0,\Theta_0}_\gamma\psi_\alpha}}\right)}{\psi_\alpha^2\,\psi_\gamma^2} \qquad f = \gamma$$

$\phi_{\triangleleft} = \phi_\alpha\psi_\gamma + \tilde{R}^{\triangleleft}_\gamma$

$\psi_{\triangleleft} = \psi_\alpha\psi_\gamma + R^{\triangleleft}_\gamma$

Ω $\frac{dA_2\, dA_3\, dA_4}{[(A_3+A_4)(1+A_2)+\cancel{A_3A_4}]^2}$

$\ln a - \ln b = \ln\frac{a}{b}$

Combining each of the two products of exponentials into a single exponential and using the exponential integral as above delivers

$$M_f^\Gamma := \frac{\ln \frac{S\phi_{\Gamma/f}(\Theta)\psi_f + S_0\phi_f(\Theta_0)\psi_{\Gamma/f}}{S_0\phi_{\Gamma/f}(\Theta_0)\psi_f + S_0\phi_f(\Theta_0)\psi_{\Gamma/f}}}{\psi^2_{\Gamma/f}\psi^2_f}\Omega_\Gamma. \tag{17}$$

Summing over all forests including the empty one delivers the renormalized integrand as the homogeneous of degree zero form

$$\Phi_\Gamma^R := \sum_f^{\emptyset} (-1)^{|f|} M_f^\Gamma. \tag{18}$$

renormalized integrand

Φ_Γ^R is an integrand which can, this is just a rewriting of the forest formula, be integrated against $\mathbb{P}^{|\Gamma^{[1]}|-1}(\mathbb{R}_+)$. An explicit proof from scratch is given below though, after we decomposed Feynman rules suitably.

Any dangerous sector has a remainder for which vanish in that sector and then cancels against some corresponding term from the forest formula.