Today: the Connes-Moscovici sub-Hopf algebra $H_{co} \subseteq H$

It is generated by $S_n$, $n \geq 0$, i.e., $S_0 = \overline{\mathbb{I}}$.
To be defined below.

$H$ is a free commutative Hopf algebra.

$H_{co} \equiv H_{co} \langle m, \overline{\mathbb{I}}, \overline{\mathbb{I}}, \Delta, S \rangle$

Same maps as in $H$,
only $H_{co} \subseteq H$.

To introduce $S_n$,
use a map

$N : H \to H$

$N(h_1 h_2) = h_1 N(h_2) + N(h_1) h_2$

and set
\( \text{N}(\overline{T}) \) is a tree with an external edge and vertex emanating from any vertex of \( T \).

\[ \text{N}(\overline{e}) := \cdot = \delta_1 \]

\[ \text{N}(\cdot) = ! = \delta_2 \]

\[ \text{N}(\bigcirc) = \bullet + \bullet = \delta_3 \]

\[ \text{N}(\bigcirc + \bullet) = \bullet + \bigcirc + \bigcirc + \bigcirc = \delta_4 \]

\[ \text{cm} - \text{coeff.} \]

\[ = \bigcirc + 3 \bigcirc + \bigcirc + \bigcirc = \delta_4 \]
So we define
\[ \delta_n = N^n(\mathbb{I}), \quad \delta_1(\mathbb{I}) = \cdot \]
\[ |\delta_n| = n \]
then is a subalgebra of \( H \) as any polynomial in \( \delta_k \)'s
\[ \subset H_{cn} \]
is it a sub- \( H \) of \( \delta \) algebra
\[ \Delta \delta_n \subset H_{cn} \otimes H_{cn}, \]
\[ \Delta \delta_2 = \cdot \otimes \cdot = \delta_1 \otimes \delta_1, \]
\[ \Delta \delta_3 = \Delta(1 + \Lambda) = \delta_1 \otimes \delta_2 + \delta_2 \otimes \delta_1 + 2\delta_1 \otimes 1 \]
\[ + 2 \delta_1 \otimes \varnothing \]
Thus \( H_{cn} \) is a sub- \( H \) of \( H \).
Proof.

$\Delta S_n$ is by construction a sum of trees $t_i \in H_n^C$

$S_n = \sum t_i$

$\Delta S_n = \Pi \otimes S_n + S_n \otimes \Pi + \sum \sum P^C(t_i)$

$\triangleleft \otimes R^C(t_i)$

$\sum \sum \{ N C P^C(t_i) \otimes R^C(t_i) + P^C(t_i) \otimes N C R^C(t_i) \}$

$+ \Delta S_i \otimes S_n + \sum \delta_i P^C(t_i) \otimes R^C(t_i)$
So we decomposed $SU(1)$ a little $SU(1)$ in terms of ad. reps from $ti$ of $SU$.

\[ \begin{array}{c}
\text{\textbf{Diagram}} \\
\end{array} \]

\[ \Rightarrow \text{Here is sub Hopf.} \]

What is purpose of $Hopf$?

\[ (\text{I)) \text{ were interested in diffeomorphs of a non-commutative manifold.} \]

Consider formal diffeomorphisms.

\[ x \mapsto x + \sum_{k=2}^{\infty} a_k x^k \quad \text{where} \quad a_k \in \mathbb{R}, a_k \in \mathbb{C} \]

\[ x \in \sigma (x^2) \quad \text{and diffeomorphm ought to } 1 \]
Consider
\[ \varphi(x) = x + \sum_{n \geq 1} a_n x^n, \quad \psi(x) = x + \sum_{n \geq 1} b_n x^n \]

two diffeos.

What is \( \varphi \circ \psi \circ \varphi \) ?
\[ \varphi \circ \psi = \psi(x) + \sum_{k \geq 2} a_k \psi^k(x) \]

What are the Taylor coeffs of \( \varphi \circ \psi \)?

Set
\[ S^n_\varphi := \log \left( \frac{\varphi'(x)}{x} \right) \bigg|_{x=0} \quad \text{and} \quad S^n_\psi := \ldots \]

What is
\[ S^n_{\varphi \circ \psi} ? \]

Define \( \overline{\varphi} : H_n \to IR, \quad \overline{\varphi} \)
\[ \overline{\varphi} := \ldots \]

By
\[ \overline{\varphi}(S_n) = S^n_\varphi, \quad \overline{\psi}(S_n) = S^n_\psi \]

and set
\[ \overline{\varphi}(S_{i_1}, \ldots, S_{i_k}) = \overline{\varphi}(S_{i_1}) \cdots \overline{\varphi}(S_{i_k}) \ldots \]

Then
\[ S^n_{\varphi \circ \psi} = m^*(\overline{\varphi} \circ \overline{\varphi}) \circ \Delta(S_n). \]
We come back to (2.1) when we study the renormalization group.

\[ \varphi(x) = x + a_2^1 x^2 \quad \varphi' = \frac{1 + 2 a_2 x}{2 a_2 x - 2 a_2^2 x^2} \]

\[ \psi(x) = x + \ell_2^1 x^2 \]

\[ \varphi \circ \psi(x) = (x + \ell_2^1 x^2)^2 + a_2^1 (x + \ell_2^1 x^2)^2 \]

\[ = x + \ell_2^1 (\ell_2^1 + a_2^1) + x^3 (2 a_2 \ell_2^1) + a_2^1 \ell_2^2 x^4 \]

\[ \Rightarrow 1 + 2 x (\ell_2^1 + a_2^1) + 3 x^2 (2 a_2 \ell_2^1) + 4 x^3 a_2^1 \ell_2^2 \]

\[ \Rightarrow x (2 \ell_2^1 + 2 a_2^1) - \frac{(2 (\ell_2^1 + a_2^1) - 3 (2 a_2 \ell_2^1))}{1 - x} \]

\[ \mu (\varphi \circ \bar{\varphi}) (\delta_1 \otimes \delta_1^\ast) = \Delta \delta_2 = \delta_2 \otimes \delta_2^\ast + \delta_2^\ast \otimes \delta_2 \]

\[ = \bar{\varphi} (\delta_1) + \varphi (\delta_1) = 2 a_2^1 + 2 \ell_2^1 \]

\[ \bar{\psi} (\delta_1) + \bar{\varphi} (\delta_2) + \bar{\varphi} (\delta_1) \bar{\psi} (\delta_1) \]

\[ = -2 \ell_2^1 + 2 a_2^1 + 4 a_2^1 \ell_2^1 \]

Now assume \( H \) (graded tensors) gives you \( H_{FG} \), a Feynman graph.

Then, \( H_{CP} \) should give you a sub Hopf algebra of \( FG \) which covers a "diffeo-group..."
which under Feynman rule
become differential of
physical parameters.

$$L = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 + \frac{g}{3!} \phi^3$$

really $g = g (\nu^2)$

so $L = L (\nu^2)$

What is $L = L (\nu^2)$?

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To get there:

- understand universality of $H$
- such that $H$ defines $H_{\nu}$
  - Feynman graphs of a renormalizable
  - field theory

- get renormalizable field theories?
  - use $\beta$ to derive $g (\nu^2)$
    - RG Eq.