Today: the Connes-Moscovici sub-Hopf algebra $H_{CM} \subsetneq H$

It is generated by δ_n, $n \geq 0, \in \mathbb{N}$. $\delta_0 = \mathbb{1}$.

To be defined below.

H is a free commutative Hopf algebra.

$$H_{CM} \equiv H_{CM}(m, \mathbb{1}, \hat{\mathbb{1}}, \Delta, S)$$

Same maps as in H, only $H_{CM} \subsetneq H$.

To introduce δ_n, use a map

$$N : H \to H$$

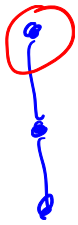
$$N(h_1 h_2) = h_1 N(h_2) + N(h_1) h_2$$

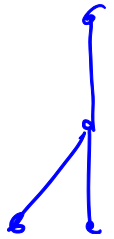
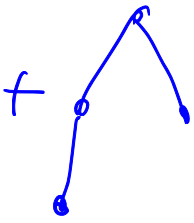
and set

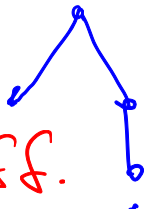
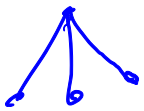
$N(T)$ "is T with an extra edge and vertex emanating from any vertex of T"

$N(\mathbb{I}) := \bullet = \delta_1$

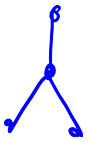
$N(\bullet) =$  $= \delta_2$

$N($  $=$  $+$  $= \delta_3$

$N($  $+$  $) =$  $+$  $+$  $+$ 

$+$  $+$ 

cm-coeff.

$=$  $+ 3$  $+$  $+$  $= \delta_4$

So, we define

$$\delta_n = N^n(\mathbb{1}), \quad \delta_1(\mathbb{1}) = \bullet$$

$$|\delta_n| = n$$

H_{cn} is a sub-algebra of H as any polynomial in δ_k's $\subseteq H_{cn}$.

Is it a sub-Hopf algebra?

$$\Delta \delta_n \subset H_{cn} \otimes H_{cn},$$

$$\tilde{\Delta} \delta_2 = \bullet \otimes \bullet = \delta_1 \otimes \delta_1$$

$$\tilde{\Delta} \delta_3 = \tilde{\Delta}\left(\begin{tikzcd}[column sep=small] \bullet \arrow[d] \\ \bullet \end{tikzcd} + \begin{tikzcd}[column sep=small] & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd} \right) = + \overset{\delta_1}{\bullet} \otimes \overset{\delta_2}{\begin{tikzcd}[column sep=small] \bullet \arrow[d] \\ \bullet \end{tikzcd}} + \begin{tikzcd}[column sep=small] \bullet \arrow[d] \\ \bullet \end{tikzcd} \otimes \bullet$$

$$+ 2 \cdot \otimes \begin{tikzcd}[column sep=small] \bullet \arrow[d] \\ \bullet \end{tikzcd} + \bullet \otimes \bullet$$

Thus H_{cn} is a sub-Hopf alg. of H.

Proof.

δ_n is by construction a sum of trees $t_i \in H^{(n)}$,

$$\delta_n = \sum_i t_i$$

$$\Delta \delta_n = \mathbb{1} \otimes \delta_n + \delta_n \otimes \mathbb{1} + \sum_i \sum_{\substack{\text{all cuts} \\ c_i \\ \text{of } t_i}} P^{c_i}(t_i) \otimes R^{c_i}(t_i).$$

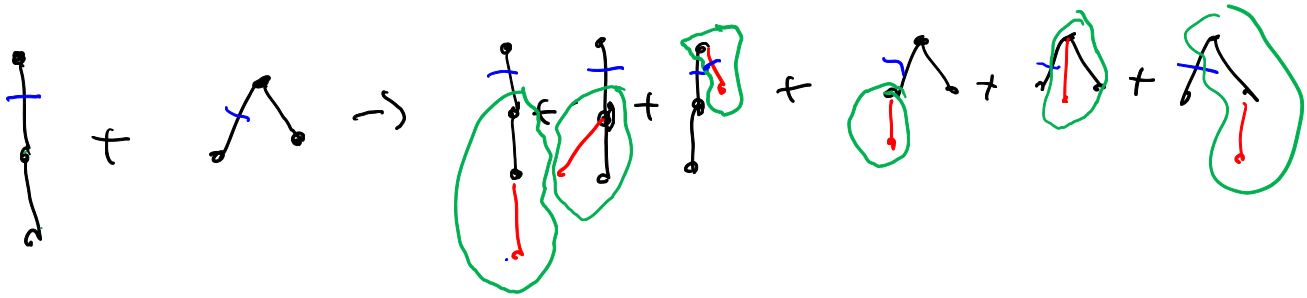
$$\Delta \delta_{n+1} = \mathbb{1} \otimes \delta_{n+1} + \delta_{n+1} \otimes \mathbb{1}$$

$$+ \sum_i \sum_{\substack{\text{all cuts} \\ c_i \text{ of } t_i}} \Big\{ N(P^{c_i}(t_i)) \otimes R^{c_i}(t_i) +$$

$$+ P^{c_i}(t_i) \otimes N(R^{c_i}(t_i)) \Big\}$$

$$+ n\, \delta_1 \otimes \delta_n + \sum_i \delta_1 P^{c_i}(t_i) \otimes |R^{c_i}(t_i)|^{c_i}_{R}(t_i)$$

So we decomposed δ_{n+1} or be the $\Delta\delta_{n+1}$ in terms of adm. cuts from t_i of δ_n.



$\Rightarrow$ H_{CM} is sub Hopf.

What is purpose of H_{CM}?

CM where interested in diffeomorphisms of a non-commutative mfd.

Consider <u>formal</u> diffeomorphisms.

$$x \to x + \sum_{k=2}^{\infty} a_k x^k \qquad a_k \in \mathbb{R}, \text{ or } \mathbb{C}$$

$x + O(x^2)$ "diffeomorphism tangent to 1"

consider

$$\varphi(x) = x + \sum_{k\geq 2} a_k x^k, \quad \psi(x) = x + \sum_{k\geq 2} b_k x^k$$

two diffeo's.

What is $\varphi\circ\psi$, or $\psi\circ\varphi$ $\quad \varphi\circ\psi \neq \psi\circ\varphi$.

$$\varphi\circ\psi = \psi(x) + \sum_{k\geq 2} a_k \psi^k(x)$$

What are the Taylor coeff's of $\varphi\circ\psi$?

Set

$$\delta^n_\varphi := \log^{(n)}(\varphi'(x))\Big|_{x=0} \quad \text{(n-th derivative of } \log\varphi'\text{)}$$

$$\delta^n_\psi := \quad \cdots$$

What is

$$\delta^n_{\varphi\circ\psi} \quad ?$$

Define $\bar{\varphi} : H_{cn} \to \mathbb{R}$, or $\mathbb{C}$

$\bar{\psi} : \quad \cdots$

by $\bar{\varphi}(\delta_n) = \delta^n_\varphi \qquad \bar{\psi}(\delta_n) = \delta^n_\psi$

and set $\bar{\varphi}(\delta_{i_1}\cdots\delta_{i_k}) = \bar{\varphi}(\delta_{i_1})\cdots\bar{\varphi}(\delta_{i_k})$

..

Thm. $\delta^n_{\varphi\circ\psi} = m\circ(\bar{\psi}\otimes\bar{\varphi})\circ\Delta(\delta_n)$.

We come back to that when we study the renormalization group.

$$\varphi(x) = x + a_2 x^2 \qquad \varphi' = 1 + 2a_2 x$$

$$\psi(x) = x + b_2 x^2 \qquad 2a_2 x - 2a_2^2 x^2$$

$$\varphi \circ \psi(x) = (x + b_2 x^2) + a_2 (x + b_2 x^2)^2$$

$$= x + x^2(b_2 + a_2) + x^3(2a_2 b_2) + a_2 b_2^2 x^4$$

$$\to \quad 1 + 2x(b_2 + a_2) + 3x^2(2a_2 b_2) + 4x^3 a_2 b_2^2$$

$$\to \quad x(2b_2 + 2a_2) - x^2\left(2(b_2 + a_2)^2 - 3(2a_2 b_2)\right) + \dots$$

$$m(\bar\varphi \otimes \bar\varphi)(\delta_1 \otimes \mathbb{1} + \mathbb{1} \otimes \delta_1) \qquad \Delta\delta_2 = \delta_2 \otimes \mathbb{1} + \mathbb{1} \otimes \delta_2 + \delta_1 \otimes \delta_1$$

$$\Rightarrow \bar\varphi(\delta_1) + \varphi(\delta_1) \Rightarrow 2a_2 + 2b_2$$

$$\bar\varphi(\delta_2) + \bar\varphi(\delta_2) + \bar\varphi(\delta_1)\,\bar\varphi(\delta_1)$$

$$= -2b_2^2 - 2a_2^2 + 4a_2 b_2 \quad \checkmark$$

Now assume H (rooted trees) "gives" you H_{FG}. "Feynman graphs"

Then, H_{CM} should give you a sub Hopf algebra of FG which covers "diffeographisms".

which under Feynman rule become diffeomorphisms of physical parameters.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\underline{m^2}\phi^2 + \underline{\frac{g}{3!}}\phi^3$$

really $g = g(\mu^2)$

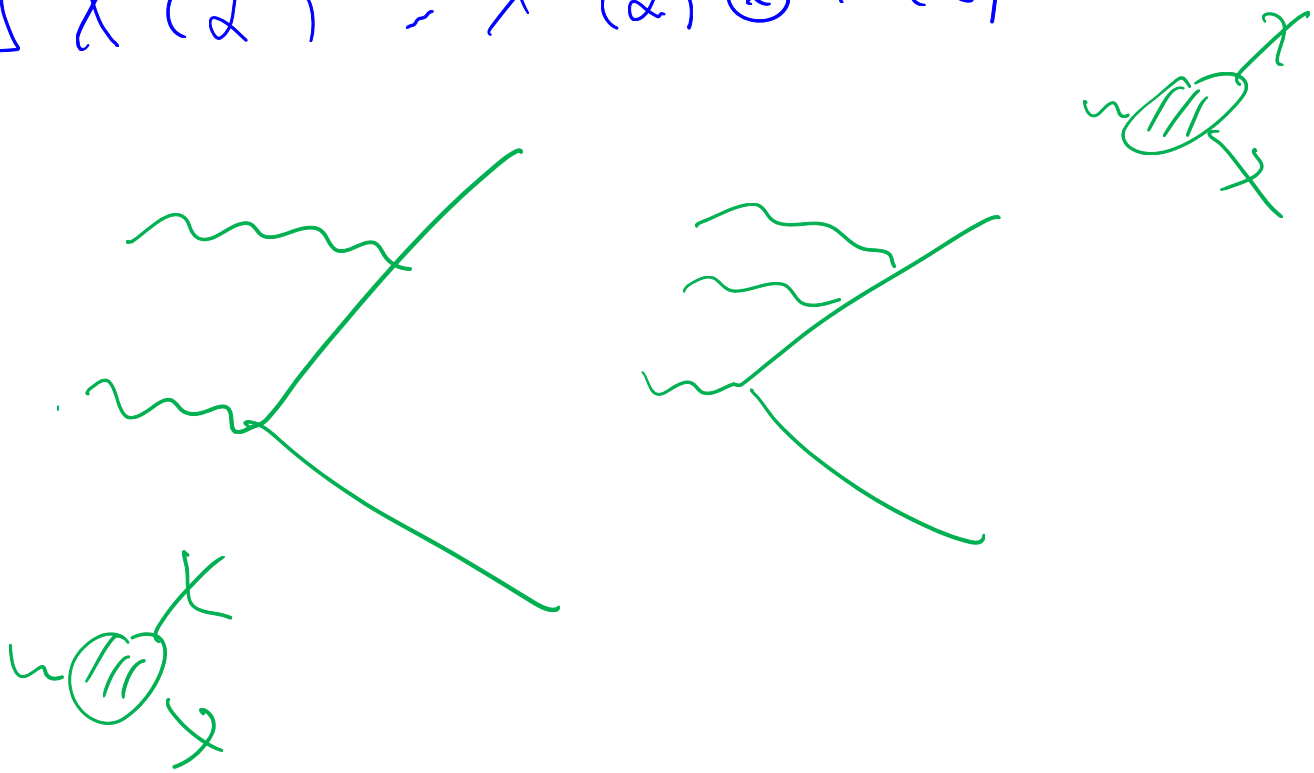
so $\mathcal{L} = \mathcal{L}(\mu^2)$

What is $\tilde{\mathcal{L}} \equiv \mathcal{L}(\lambda\mu^2)$?

To get there:

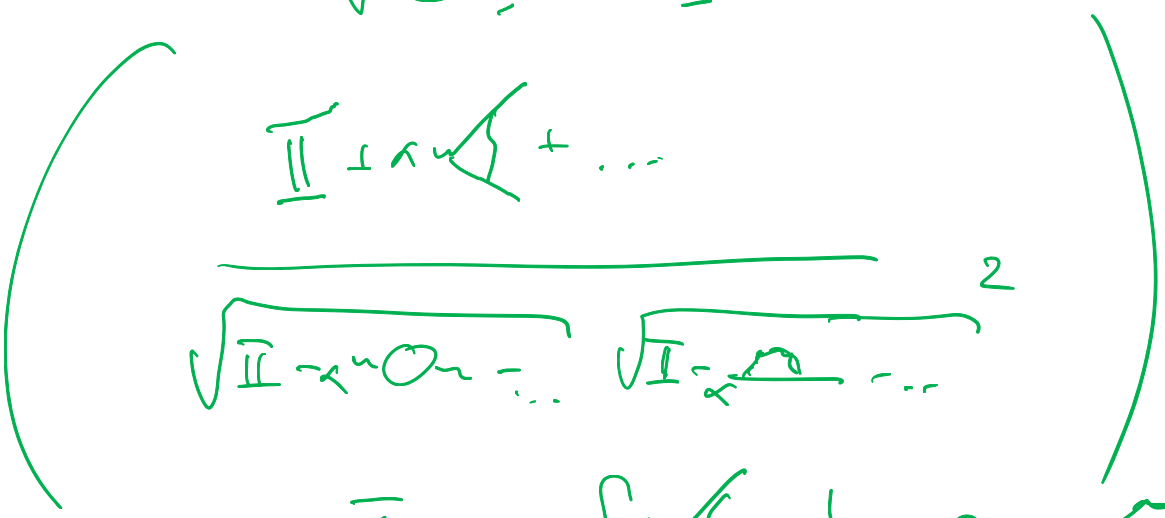
- understand universality of $\mathcal{H}$ such that H defines $\mathcal{H}_G$ for Feynman graphs of a renormalizable field theory.
- get renormalizable field theories?
- use $\mathcal{H}$ to derive sub-Hopf alg. and DSE.
- RGE.

$$\Delta X(\alpha) = X(\alpha) \otimes X(\alpha)$$



$$\alpha \to \alpha + c_2 \alpha^2 + \dots \qquad \lg \frac{q^2}{\mu^2}$$

$$[L]$$



$$= \mathbb{1} + \alpha \underbrace{\left[\text{diagram} - \frac{1}{2} \text{diagram} - \text{diagram} \right]}_{\delta_1} + \dots$$

$$\delta_2$$