Renormalization, Hopf algebras and Mellin transforms

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Abstract. This article aims to give a short introduction into Hopf-algebraic aspects of renormalization, enjoying growing attention for more than a decade by now. As most available literature is concerned with the minimal subtraction scheme, we like to point out properties of the kinematic subtraction scheme which is also widely used in physics (under the names of *MOM* or *BPHZ*).

In particular we relate renormalized Feynman rules ϕ_R in this scheme to the universal property of the Hopf algebra H_R of rooted trees, exhibiting a refined renormalization group equation which is equivalent to $\phi_R : H_R \to \mathbb{K}[x]$ being a morphism of Hopf algebras to the polynomials in one indeterminate.

Upon introduction of analytic regularization this results in efficient combinatorial recursions to calculate ϕ_R in terms of the Mellin transform. We find that different Feynman rules are related by a distinguished class of Hopf algebra automorphisms of H_R that arise naturally from Hochschild cohomology.

Also we recall the known results for the minimal subtraction scheme and shed light on the interrelationship of both schemes.

Finally we incorporate combinatorial Dyson-Schwinger equations to study the effects of renormalization on the physical meaningful correlation functions. This yields a precise formulation of the equivalence of the two different renormalization prescriptions mentioned before and allows for non-perturbative definitions of quantum field theories in special cases.

Motivation: The renormalization problem

Suppose we want to assign a value to the logarithmically divergent integral $\phi_s(\bullet) := \int_0^\infty \frac{\mathrm{d}x}{x+s}$, which we associate to the tree $\bullet$. Observing the (absolutely) integrable difference

$$\int_0^\infty \left[\frac{\mathrm{d}x}{x+s} - \frac{\mathrm{d}x}{x+\mu}\right] = -\ln\frac{s}{\mu} =: -\ell \tag{0.1}$$

allows for the definition of $\phi_{\mathrm{R},s}(\bullet) := \phi_s(\bullet) - \phi_\mu(\bullet) = -\ln\frac{s}{\mu} = -\ell$, which we call the *renormalized* value of the expression $\phi_s(\bullet)$. We need to choose the *renormalization point* μ to fix the constant not determined by (0.1). This natural *renormalization scheme* given by subtraction at a *reference scale* $s \mapsto \mu$ is commonly employed in quantum field theory (where similar divergent expressions occur as we briefly describe in section 7) and will be called kinematic subtraction scheme in the sequel.

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When we apply the same idea to multi-dimensional integrals, we have to take care of *subdivergences* as occuring in

$$\phi_s\left(\;\right) := \int_0^\infty \frac{\mathrm{d}x}{x+s}\left[\int_0^\infty \frac{\mathrm{d}y}{x+y}\cdot\int_0^\infty \frac{\mathrm{d}z}{x+z}\right] = \int_0^\infty \frac{\mathrm{d}x}{x+s}\left[\phi_x\left(\bullet\right)\right]^2 . \tag{0.2}$$

A single subtraction at $s=\mu$ is insufficient as the subintegrals over y and z remain divergent. This problem is circumvented by applying renormalization to these first:

$$\begin{aligned}\phi_{\mathrm{R},s}\left(\;\right) &:= \int_0^\infty \left[\frac{\mathrm{d}x}{x+s}-\frac{\mathrm{d}x}{x+\mu}\right]\left\{\int_0^\infty \left[\frac{\mathrm{d}y}{x+y}-\frac{\mathrm{d}y}{\mu+y}\right]\cdot\int_0^\infty\left[\frac{\mathrm{d}z}{x+z}-\frac{\mathrm{d}z}{\mu+z}\right]\right\}\\ &= \int_0^\infty \left[\frac{\mathrm{d}x}{x+s}-\frac{\mathrm{d}x}{x+\mu}\right]\left[\phi_{\mathrm{R},x}\left(\bullet\right)\right]^2 = \int_0^\infty \left[\frac{\mathrm{d}x}{x+s}-\frac{\mathrm{d}x}{x+\mu}\right]\left(\ln\frac{x}{\mu}\right)^2 = -\frac{\ell^3}{3}-\frac{\pi^2}{3}\ell.\end{aligned} \tag{0.3}$$

We want to summarise how this procedure is formulated in terms of Hopf algebras, study under which conditions it can be applied and reveal the main properties of the resulting maps $\phi_{\mathrm{R},s}$. In particular we will show that they are morphisms of Hopf algebras, taking values in the polynomials in ℓ.

For a quick start, we prove this analytically in section 2, along the ideas [**17**] originating from quantum field theory. Section 3 exploits an artificial regulator to rederive the same results in a more combinatorial setup, which is more common in the literature. Along this way we take the time to recall the common algebraic techniques and contrast both methods.

After this construction of renormalized Feynman rules, we study their algebraic properties in section 4 focusing on the renormalization group. Together with the Mellin transform we can derive compact recursion relations, allowing for efficient combinatorial calculations.

At this point we turn towards the *minimal subtraction scheme* in section 5. We summarize the known results and particularly relate the different realisations of the renormalization group equations in the two schemes, developing the duality between the concepts of *finiteness* in the subtraction scheme and *locality* in minimal subtraction.

Section 6 is devoted to Dyson-Schwinger equations, which link the combinatorics of the Hopf algebra to the physically meaningful correlation functions. In particular we observe how the change of renormalization scheme is equivalent to a redefinition of the coupling constant, proving the renormalization group equation in its physical form.

Finally we comment on the necessary modifications for generalizations of the model in different directions, like the presence of multiple parameters or higher degrees of divergence.

For reference and convenience of the reader, we collected the required features of the Hopf algebras H_R of rooted trees and $\mathbb{K}[x]$ of polynomials in the appendix. We also added a collection of well-known results on the *Dynkin operator* $S\star Y$ which plays a prominent role in renormalization theory and shows up also in this text at various places.

1. Notations and preliminaries

The essential structure behind perturbative renormalization is the Hopf algebra as discovered in [**15**]. As the literature grew comprehensive already, we content ourselves with fixing notation and recommend [**21, 22**] for extended accounts of these concepts with a particular focus on their application to renormalization.

1.1. Hopf algebras. Throughout we consider associative, coassociative, commutative, unital and counital Bialgebras $(H, m, u, \Delta, \varepsilon)$ given a connected ($H_0 = \mathbb{K} \cdot \mathbb{1}$) grading $H = \bigoplus_{n \geq 0} H_n$. For homogeneous $0 \neq x \in H_n$, write $|x| := n$ while the induced *grading operator* $Y \in \operatorname{End}(H), x \mapsto Yx := |x| \cdot x$ exponentiates to the one-parameter group $\mathbb{K} \ni t \mapsto \theta_t$ of Hopf algebra automorphisms

$$\theta_t := \exp(tY) = \sum_{n \in \mathbb{N}_0} \frac{(tY)^n}{n!}, \quad \forall n \in \mathbb{N}_0: \quad H_n \ni x \mapsto \theta_t(x) = e^{t|x|}x = e^{nt}x. \tag{1.1}$$

Algebras $(\mathcal{A}, m_{\mathcal{A}}, u_{\mathcal{A}})$ are unital, associative and commutative, giving rise to the associative *convolution product* on $\operatorname{Hom}(H, \mathcal{A})$ with unit given by $e := u_{\mathcal{A}} \circ \varepsilon$:

$$\operatorname{Hom}(H, \mathcal{A}) \ni \phi, \psi \mapsto \phi \star \psi := m_{\mathcal{A}} \circ (\phi \otimes \psi) \circ \Delta \in \operatorname{Hom}(H, \mathcal{A}).$$

As $H = \mathbb{K} \cdot \mathbb{1} \oplus \ker \varepsilon = \operatorname{im} u \oplus \ker \varepsilon$ splits into the scalars and the *augmentation ideal* $\ker \varepsilon$, we obtain a projection $P := \mathrm{id} - u \circ \varepsilon\colon\ H \twoheadrightarrow \ker \varepsilon$ and use Sweedler's [**23**] notation $\Delta(x) = \sum_x x_1 \otimes x_2$ and $\widetilde{\Delta}(x) = \sum_x x' \otimes x''$ to abbreviate the reduced coproduct $\widetilde{\Delta} := \Delta - \mathbb{1} \otimes \mathrm{id} - \mathrm{id} \otimes \mathbb{1}$. The connectedness implies:

(1) Under $\star$, the *characters* (morphisms of unital algebras) form a group $G^H_{\mathcal{A}} := \{\phi \in \operatorname{Hom}(H, \mathcal{A})\colon\ \phi \circ u = u_{\mathcal{A}} \text{ and } \phi \circ m = m_{\mathcal{A}} \circ (\phi \otimes \phi)\}$.
(2) These biject along $\exp_\star\colon\ \mathfrak{g}^H_{\mathcal{A}} \to G^H_{\mathcal{A}}$ with inverse $\log_\star\colon\ G^H_{\mathcal{A}} \to \mathfrak{g}^H_{\mathcal{A}}$ to the *infinitesimal characters* $\mathfrak{g}^H_{\mathcal{A}} := \{\phi \in \operatorname{Hom}(H, \mathcal{A})\colon\ \phi \circ m = \phi \otimes e + e \otimes \phi\}$, using the pointwise finite series

$$\exp_\star(\phi) := \sum_{n \in \mathbb{N}_0} \frac{\phi^{\star n}}{n!} \quad \text{and} \quad \log_\star(\phi) := \sum_{n \in \mathbb{N}} \frac{(-1)^{n+1}}{n} (\phi - e)^{\star n}. \tag{1.2}$$

(3) The unique inverse $S := \mathrm{id}^{\star -1} \in G^H_H$ is called *antipode* and reveals H as Hopf algebra. For all $\phi \in G^H_{\mathcal{A}}$ we have $\phi^{\star -1} = \phi \circ S$.

In general we assume the ground field $\mathbb{K}$ to be $\mathbb{R}$, though the reader will easily recognize that the majority of results allows for more generality (often characteristic zero suffices). Note that by $\operatorname{Hom}(\cdot, \cdot)$ and $\operatorname{End}()$ we always denote $\mathbb{K}$-linear maps and explicitly spell out if more structure should enjoy preservation by a morphism. Finally, $\operatorname{lin} M$ denotes the linear span of M.

1.2. Hochschild cohomology. The Hochschild cochain complex [**8, 1, 22**] we associate to H contains the functionals $H' = \operatorname{Hom}(H, \mathbb{K})$ as zero-cochains. One-cocycles $L \in \mathrm{HZ}^1_\varepsilon(H) \subset \operatorname{End}(H)$ are linear maps such that $\Delta \circ L = (\mathrm{id} \otimes L) \circ \Delta + L \otimes \mathbb{1}$ and the differential

$$\delta : H' \to \mathrm{HZ}^1_\varepsilon(H), \alpha \mapsto \delta\alpha := (\mathrm{id} \otimes \alpha) \circ \Delta - u \circ \alpha \in \mathrm{HB}^1_\varepsilon(H) := \delta\left(H'\right) \tag{1.3}$$

determines the first cohomology group by $\mathrm{HH}^1_\varepsilon(H) := \mathrm{HZ}^1_\varepsilon(H)/\mathrm{HB}^1_\varepsilon(H)$.

Lemma 1.1. *Cocycles $L \in HZ^1_\varepsilon(H)$ fulfil $\operatorname{im} L \subseteq \ker \varepsilon$ and $L(\mathbb{1}) \in \operatorname{Prim}(H) := \ker \widetilde{\Delta}$ is primitive. The map $HH^1_\varepsilon(H) \to \operatorname{Prim}(H)$, $[L] \mapsto L(\mathbb{1})$ is well-defined since $\delta\alpha(\mathbb{1}) = 0$ for all $\alpha \in H'$.*

2. Finiteness of renormalization by subtraction

Originally, perturbative quantum field theory assigns (divergent) expressions to combinatorial objects called *Feynman graphs*, as we will comment on in section 7. However the Hopf algebra H_R of rooted trees summarized in appendix A suffices to encode the structure of subdivergences [**15, 8, 9**] such that we can focus on *Feynman rules* of the form $\phi : H_R \to \mathcal{A}$ as above. The *target algebra* $\mathcal{A}$ has to sustain divergent expressions which only become finite after we accomplished the renormalization. Therefore we consider $\mathcal{A}$ as the integrands (differential forms) which for convenience we nevertheless write as integrals, keeping in mind that we do not evaluate them.

Guided by the examples (0.1) and (0.2) we make

Definition 2.1. *By virtue of A.3 let $\phi \in G_{\mathcal{A}}^{H_R}$ be the character determined through*

$$\phi_s\left(B_+(w)\right) := \int \frac{\mathrm{d}\zeta}{s} f\left(\frac{\zeta}{s}\right)\phi_\zeta(w) \quad \textit{for any} \quad w \in H_R. \tag{2.1}$$

As each node of a tree thus corresponds to an integration of the function given by its children, 2.1 ensures that all information about subdivergences of these Feynman rules ϕ is encoded in the coproduct of H_R.

Example 2.2. *For* (0.2), $\widetilde{\Delta}\left(\ \right) = 2\bullet \otimes \ + \bullet\bullet \otimes \bullet$ *informs about:*

1. *Two individual subdivergences ($\int \frac{\mathrm{d}y}{x+y}$ and $\int \frac{\mathrm{d}z}{x+z}$) of the type $\phi(\bullet)$ inside the integrals over the remaining variables: These are of the form $\phi\left(\ \right)$ like* $\int_0^\infty \frac{\mathrm{d}x}{x+s}\int_0^\infty \frac{\mathrm{d}y}{x+y}$.
2. *One subdivergence of type $\phi(\bullet\bullet)$ (when y and z approach ∞ jointly) inside the x-integration* $\phi(\bullet) = \int_0^\infty \frac{\mathrm{d}x}{x+s}$.

We allow for a general integration kernel $f(\zeta)$ replacing $\frac{1}{1+\zeta}$ of the introduction, assuming f to be bounded and piecewise differentiable on $[0, \infty)$. Hence the divergences can only occur at $\zeta \to \infty$ and are therefore called *ultraviolet*[1].

2.1. Subtraction scheme.

Note that the integrands ϕ depend on a remaining *external parameter* $s > 0$. Our goal is to replace them by integrable integrands to achieve well-defined functions of s. The approach exemplified in the introduction is to exploit the dependence of s to construct convergent integrands by means of a subtraction at $s \mapsto \mu$. Renormalizing the subdivergences first motivates

Definition 2.3. *Fixing a renormalization point $\mu > 0$ we define the character $\phi_R : H_R \to \mathcal{A}$ (again as an instance of A.3) by requiring*

$$\phi_{R,s}(B_+(w)) := \int \mathrm{d}\zeta \left[\frac{f(\frac{\zeta}{s})}{s} - \frac{f(\frac{\zeta}{\mu})}{\mu}\right]\phi_{R,\zeta}(w) \quad \textit{for any} \quad w \in H_R. \tag{2.2}$$

To achieve finiteness we need to constrain the growth of $f(\zeta)$ at $\zeta \to \infty$ to be not worse than ζ^{-1}, corresponding to a *logarithmic divergence* in

Theorem 2.4. *Assume $f(\zeta) \in \mathcal{O}\left(\zeta^{-1}\right)$ and with $c_{-1} := \lim_{\zeta\to\infty}[\zeta f(\zeta)]$ that also*

$$f(\zeta) - \frac{c_{-1}}{\zeta}, f(\zeta) + \zeta f'(\zeta) \in \mathcal{O}\left(\zeta^{-1-\varepsilon}\right) \quad \textit{for some} \quad \varepsilon > 0. \tag{2.3}$$

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[1] physically ζ corresponds to a momentum, so this limit means high energies

Then for any $w \in H_R$, the integral $\phi_{R,s}(w)$ is absolutely convergent and evaluates to a polynomial $\phi_{R,s}(w) \in \mathbb{K}[\ell]$ in $\ell := \ln \frac{s}{\mu}$.

PROOF. We proceed by induction: By definition $\phi_{\mathrm{R}}(\mathbb{1}) = 1$, also note that since ϕ_{R} is a character the claim for $\phi_{\mathrm{R}}(ab)$ follows from $\phi_R(a)$ and $\phi_R(b)$. Therefore we may assume the statement for some element $w \in H_R$ and only need to consider $t = B_+(w)$. But then the difference in brackets in (2.2) falls of like $\zeta^{-1-\varepsilon}$ while $\phi_{\mathrm{R},\zeta}(w)$ only grows like $\ln^N \zeta$ for the degree N of $\phi_{\mathrm{R}}(w)$. Hence (2.2) is absolutely convergent (the logarithmic singularities $\ln^N \zeta$ at $\zeta \to 0$ are integrable anyway) and thus $\phi_{\mathrm{R},s}(B_+(w))$ finite.

By (2.3) we can also interchange integration with ∂_ℓ in

$$-\partial_\ell \phi_{\mathrm{R},s}(t) = \int_0^\infty \mathrm{d}\zeta \left[\frac{f(\frac{\zeta}{s})}{s} + \frac{\zeta}{s}\frac{f'(\frac{\zeta}{s})}{s} \right] \phi_{\mathrm{R},\zeta}(w) = \int_0^\infty \mathrm{d}\zeta \left[f(\zeta) + \zeta f'(\zeta) \right] \phi_{\mathrm{R},\zeta s}(w).$$

Exploiting that $\phi_{\mathrm{R},\zeta s}$ is polynomial in $\ln \frac{\zeta s}{\mu}$ we can evaluate

$$\int_0^\infty \mathrm{d}\zeta \left[f(\zeta) + \zeta f'(\zeta) \right] \ln^n(\zeta\ell) = \sum_{i=0}^{n} \binom{n}{i} \ell^{n-i} c_{i-1} (-1)^i i! = \sum_{i=0}^{\infty} c_{i-1} \left(-\partial_\ell\right)^i \ell^n \tag{2.4}$$

upon defining the constants (which are *periods* [**14**] for algebraic functions f)

$$c_{n-1} := \int_0^\infty \mathrm{d}\zeta \left[f(\zeta) + \zeta f'(\zeta) \right] \frac{(-\ln \zeta)^n}{n!} \qquad \text{for any} \qquad n \in \mathbb{N}_0. \tag{2.5}$$

Thus linearity shows $\partial_{-\ell}\phi_{\mathrm{R},s}(t) \in \mathbb{K}[\ell]$ and we merely have to integrate once. □

We remark that conditions (2.3) are not very restrictive, especially they hold for all rational functions $f \in \mathcal{O}\left(\zeta^{-1}\right) \cap \mathbb{K}(\zeta)$.

Not only did we achieve our goal of renormalization, but we found an explicit recursion (2.4) determining ϕ_{R} completely using the universal property A.3 in

Corollary 2.5. *The constants $c_\cdot$ of (2.5) determine the renormalized Feynman rules $\phi_R \in G^{H_R}_{\mathbb{K}[\ell]}$ completely through the universal property A.3 by*

$$\phi_R \circ B_+ = F(-\partial_\ell)\circ\phi_R, \quad \textit{where} \quad F(-\partial_\ell) := P \circ \sum_{n \geq -1} c_n (-\partial_\ell)^n \in \operatorname{End}\left(\mathbb{K}[\ell]\right). \tag{2.6}$$

For convenience we write $(-\partial_\ell)^{-1} := -\int_0$ for the integral operator (note that the projection $P : \mathbb{K}[\ell] \twoheadrightarrow \ell\,\mathbb{K}[\ell]$ annihilates any constants).

In section 4 we will see that (2.6) implies the *renormalization group* upon realizing that $F(-\partial_\ell) \in \mathrm{HZ}^1_\varepsilon(\mathbb{K}[\ell])$ is a Hochschild-1-cocycle. But before let us review the

2.2. Algebraic renormalization process. Renormalization of a character $\phi \in G^H_{\mathcal{A}}$ can be described as a *Birkhoff decomposition* into the *renormalized* $\phi_R := \phi_+ \in G^H_{\mathcal{A}}$ and the *counterterms* $\phi_- \in G^H_{\mathcal{A}}$ subject to the conditions that

$$\phi = \phi_-^{\star -1} \star \phi_+ \quad \text{and} \quad \phi_\pm\left(\ker \varepsilon\right) \subseteq \mathcal{A}_\pm. \tag{2.7}$$

It depends on a splitting $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ of the target algebra, determining the *renormalization scheme* which we identify with the corresponding projection $R : \mathcal{A} \twoheadrightarrow \mathcal{A}_-$ along $\mathcal{A}_+$.

Theorem 2.6 ([**9, 21, 22**])**.** *A unique Birkhoff decomposition* (2.7) *exists given that R is a Rota-Baxter map, meaning*

$$m \circ (R \otimes R) = R \circ m \circ [R \otimes \mathrm{id} + \mathrm{id} \otimes R - \mathrm{id} \otimes \mathrm{id}] \,. \tag{2.8}$$

On the augmentation ideal $\ker \varepsilon$ *it may be computed inductively by*

$$\phi_-(x) = -R \circ \bar{\phi}(x) \qquad \textit{and} \qquad \phi_+(x) = (\mathrm{id} - R) \circ \bar{\phi}(x), \tag{2.9}$$

using the Bogoliubov character $\bar{\phi}$ *(also* $\bar{R}$-operation*) which is defined as*

$$\bar{\phi}(x) := \phi(x) + \sum_x \phi_-(x')\phi(x'') = \phi_+(x) - \phi_-(x). \tag{2.10}$$

Definition 2.7. *The* kinematic subtraction scheme R_μ *by evaluation at $s \mapsto \mu$ is defined as*

$$\mathrm{End}(\mathcal{A}) \ni R_\mu := \mathrm{ev}_\mu = \Big(\mathcal{A} \ni f \mapsto f|_{s=\mu}\Big) \tag{2.11}$$

and splitts $\mathcal{A}$ into $\mathrm{im}\, R_\mu = \mathcal{A}_-$ *(s-independent integrals) and* $\ker R_\mu = \mathcal{A}_+$*, those integrals that vanish at $s = \mu$.*

As R_μ is a character of $\mathcal{A}$, it not only fulfils (2.8) and we obtain a unique Birkhoff decomposition, but also simplifies the recursion (2.9) to just

$$\phi_- = R_\mu \circ \phi \circ S = \phi_\mu \circ S = \phi_\mu^{\star -1} \qquad \text{and} \qquad \phi_+ = \phi_\mu^{\star -1} \star \phi_s. \tag{2.12}$$

Example 2.8. *In accordance with* (0.1) *we find*

$$\phi_{R,s}(\bullet) = \phi_{+,s}(\bullet) = \phi_-(\bullet) + \phi_s(\bullet) = \int_0^\infty \left[- \left.\frac{\mathrm{d}x}{x+s}\right|_{s \mapsto \mu} + \frac{\mathrm{d}x}{x+s} \right],$$

and $\bar{\phi}\left(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\right) \underset{(2.9)}{=} \phi_s\left(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\right) + 2\phi_-(\bullet)\,\phi_s\left(\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}\right) + \phi_-(\bullet\bullet)\,\phi_s(\bullet)$ *indeed agrees with* (0.3) *using* $\phi_R\left(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\right) = (\mathrm{id} - R_\mu)\bar{\phi}\left(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\right)$ *after rearranging the terms*[2]

$$\bar{\phi}\left(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\right) = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}z \left[\frac{1}{(s+x)(x+y)(x+z)} - \frac{1}{\mu+y}\frac{1}{(s+x)(x+z)} \right.$$
$$\left. - \frac{1}{\mu+z}\frac{1}{(s+x)(x+y)} + \frac{1}{\mu+y}\frac{1}{\mu+z}\frac{1}{s+x} \right].$$

We remark that the recursion (2.9) makes explicit reference to the divergent counterterms ϕ_-. In (2.2) we anticipated the much more practical formula resulting from the special structure A.3 of the Feynman rules ϕ of (2.1) in

Theorem 2.9. *Let the character $\phi : H_R \to \mathcal{A}$ be subject to $\phi \circ B_+ = L \circ \phi$ for some $L \in \mathrm{End}(\mathcal{A})$ and the renormalization scheme $R \in \mathrm{End}(\mathcal{A})$ such that it ensures*

$$L \circ m_{\mathcal{A}} \circ (\phi_- \otimes \mathrm{id}) = m_{\mathcal{A}} \circ (\phi_- \otimes L), \tag{2.13}$$

linearity of L over the counterterms. Then we have

$$\bar{\phi} \circ B_+ = L \circ \phi_+ \qquad \textit{and therefore} \qquad \phi_+ \circ B_+ = (\mathrm{id} - R) \circ L \circ \phi_+ \tag{2.14}$$

Proof. This is a straightforward consequence of the cocycle property of B_+:

$$\bar{\phi} \circ B_+ = (\phi_- \star \phi - \phi_-) \circ B_+ = m_{\mathcal{A}} \circ (\phi_- \otimes \phi) \circ [(\mathrm{id} \otimes B_+) \circ \Delta + B_+ \otimes \mathbb{1}] - \phi_- \circ B_+$$
$$= \phi_- \star (\phi \circ B_+) = \phi_- \star (L \circ \phi) \underset{(2.13)}{=} L \circ (\phi_- \star \phi) = L \circ \phi_+. \qquad \square$$

[2]Note that we need to track the correspondence of variables and nodes.

As for R_μ the counterterms $\phi_-(x) \in \mathcal{A}_-$ are independent of s, they separate from the integration in (2.1) and (2.13) is fulfilled indeed. This is a general feature of quantum field theories: The counterterms to not depend on any external variables[3].

The significance of (2.14) lies in the expression of the renormalized $\phi_+(t)$ for a tree $t = B_+(w)$ only in terms of the renormalized value $\phi_+(w)$. This allows for inductive proofs like 2.4 on properties of $\phi_{\mathrm{R}} = \phi_+$, without having to consider the unrenormalized Feynman rules or their counterterms (both of which are divergent) at all.

Summarizing, we proved in 2.4 that for any forest $w \in H_R$, the expression $\phi_+(w) \in \mathcal{A}_+$ is actually integrable and may be directly written as a convergent integral using (2.2).

3. Regularization and Mellin transforms

A technique often applied prior the renormalization is the introduction of a *regulator* to assign finite values also to divergent expressions. Popular methods include

(1) Confine integrations to the bounded interval $[0, \Lambda]$ for a *cut-off* $\Lambda > 0$. Then all integrals converge but acquire a dependence on Λ, which will in general diverge in the *physical limit* $\Lambda \to \infty$ resembling the original situation. After renormalization however, this limit will be finite.
(2) Variations of mixed Hodge structures [**3**] also vary the chain of integration to avoid singularities.
(3) Choose an *analytic regulator* $0 < z < 1$ and replace each $\int_0^\infty \mathrm{d}x$ with $\int_0^\infty x^{-z}\mathrm{d}x$. This increases the decay of the integrand at $x \to \infty$ and we again get finite results which depend on z. As for the cut-off, these typically diverge in the physical limit $z \to 0$, unless we renormalize.
(4) *Dimensional regularization* is a similar method introducing a complex regulator $z \neq 0$ associated to the dimenson $D = 4 - 2z$ of space-time. It is tailor made for Feynman integrals in quantum field theory and we refer to [**7**] for its definition and examples.

We study the analytic regularization in detail, as it allows for the simplest algebraic description: Due tue the regulator all integrals converge and give functions of both s and z that lie in the target algebra $\mathcal{A} = \mathbb{K}[z^{-1}, z]][s^{-z}]$ by proposition 3.2.

Definition 3.1. *The analytically regularized Feynman rules ${}_z\phi \in G_{\mathcal{A}}^{H_R}$ are given through the universal property A.3 by requiring*

$$
{}_z\phi_s \circ B_+ = \int_0^\infty \frac{f(\frac{\zeta}{s})\zeta^{-z}}{s}\ {}_z\phi_\zeta\ \mathrm{d}\zeta = \int_0^\infty f(\zeta)(s\zeta)^{-z}\ {}_z\phi_{s\zeta}\ \mathrm{d}\zeta. \tag{3.1}
$$

All integrals can be evaluated in terms of the coefficients c_n of the *Mellin transform*[4]

$$
F(z) := \int_0^\infty f(\zeta)\zeta^{-z}\ \mathrm{d}\zeta = \sum_{n=-1}^{\infty} c_n z^n \in z^{-1}\mathbb{K}[[z]], \tag{3.2}
$$

[3]Even if the divergence of a Feynman graph does depend on external momenta as happens for higher degrees of divergence, the Hopf algebra is defined such that the counterterms are evaluations on certain *external structures*, given by distributions in [**9**]. So in any case, ϕ_- maps to constants.

[4]Conditions (2.3) suffice to prove that $F(z)$ is a Laurent series of this form.

which we already encountered in (2.5) since a partial integration yields

$$c_{n-1}n! \underset{(2.5)}{=} \int_0^\infty \mathrm{d}\zeta \left[f(\zeta)+\zeta f'(\zeta)\right](-\ln\zeta)^n = \left.\frac{\partial^n}{\partial z^n}\right|_{z=0} \int_0^\infty \mathrm{d}\zeta \left[f(\zeta)+\zeta f'(\zeta)\right]\zeta^{-z}$$
$$= \left.\frac{\partial^n}{\partial z^n}\right|_{z=0} \left\{ \left[f(\zeta)\zeta^{1-z}\right]_{\zeta=0}^{\infty} - \int_0^\infty \mathrm{d}\zeta \left[f(\zeta)+(1-z)\,f(\zeta)\right]\zeta^{-z} \right\} = \left.\frac{\partial^n}{\partial z^n}\right|_{z=0} \left\{zF(z)\right\}.$$

Proposition 3.2. *For any forest $w\in\mathcal{F}$ we have (called* BPHZ model *in* [**4**]*)*

$$ {}_z\phi_s(w) = s^{-z|w|}\prod_{v\in V(w)} F\left(z\left|w_v\right|\right). \tag{3.3}$$

Proof. As both sides of (3.3) are clearly multiplicative, it is enough to inductively assume the claim for a forest $w\in\mathcal{F}$ and prove it for the tree $t=B_+(w)$:

$$ {}_z\phi_s(t) \underset{(3.1)}{=} \int_0^\infty (s\zeta)^{-z} f(\zeta)\ {}_z\phi_{s\zeta}(w)\ \mathrm{d}\zeta = \int_0^\infty (s\zeta)^{-z} f(\zeta)(s\zeta)^{-z|w|}\prod_{v\in V(w)} F\left(z\left|w_v\right|\right)\ \mathrm{d}\zeta$$
$$= s^{-z|B_+(w)|}\left[\prod_{v\in V(w)} F\left(z\left|w_v\right|\right)\right] F\left(z\left|B_+(w)\right|\right) = s^{-z|t|}\prod_{v\in V(t)} F\left(z\left|t_v\right|\right). \qquad \square$$

Example 3.3. *Using* (3.3)*, we can directly write down the Feynman rules like*

$${}_z\phi_s\left(\bullet\right) = s^{-z}F(z), \quad {}_z\phi_s\left(\substack{\bullet\\ \bullet}\right) = s^{-2z}F(z)F(2z) \quad \text{and} \quad {}_z\phi_s\left(\substack{\bullet\\ \bullet\,\bullet}\right) = s^{-3z}[F(z)]^2F(3z).$$

Many examples (choices of F) are discussed in [**4**], the particular case of the one-loop propagator graph γ of Yukawa theory is in [**5**] and for scalar Yukawa theory in six dimensions one has $F(z)=\frac{1}{z(1-z)(2-z)(3-z)}$ as in [**22**]. Already noted in [**16**], the highest order pole of ${}_z\phi_s(w)$ is independent of s and just the tree factorial

$$ {}_z\phi_s(w) \in s^{-z|w|}\prod_{v\in V(w)}\left\{\tfrac{c_{-1}}{z|w_v|}+\mathbb{K}[[z]]\right\} \underset{(A.4)}{\subset} \frac{1}{w!}\left(\tfrac{c_{-1}}{z}\right)^{|w|} + z^{1-|w|}\mathbb{K}[[z]]. \tag{3.4}$$

3.1. Finiteness. Using (3.3) and (2.12) we can quickly write down the renormalized functions like

Example 3.4. *We find* ${}_z\phi_{R,s}\left(\bullet\right) = \left(s^{-z}-\mu^{-z}\right)F(z)$ *and* $S\left(\substack{\bullet\\ \bullet}\right) = -\substack{\bullet\\ \bullet} + \bullet\bullet$ *results in*

$$ {}_z\phi_{R,s}\left(\substack{\bullet\\ \bullet}\right) = \left(s^{-2z}-\mu^{-2z}\right)F(z)F(2z) - \left(s^{-z}-\mu^{-z}\right)\mu^{-z}F^2(z). \tag{3.5}$$

As the *physical limit* $z\to 0$ reconstructs the original (unregularized) Feynman rules (2.1), the finiteness of theorem 2.4 is equivalent (by Lebesgue's theorem on dominated convergence) to the existence of the limit

$$\phi_{\mathrm{R}} := \lim_{z\to 0} {}_z\phi_R. \tag{3.6}$$

Corollary 3.5. *The renormalized regularized Feynman rules are holomorphic, that is they map into* $\mathrm{im}\left({}_z\phi_{R,s}\right)\subset\mathbb{K}[[z]]$.

Example 3.6. *Indeed we find* ${}_z\phi_{R,s}\left(\bullet\right)\in -c_{-1}\ln\frac{s}{\mu}+z\mathbb{K}[[z]]$. *For* (3.5) *check*

$$\phi_R\left(\substack{\bullet\\ \bullet}\right) \underset{(3.6)}{=} \lim_{z\to 0}\left\{-\left[-z\ln\tfrac{s}{\mu}+\tfrac{z^2}{2}\left(\ln^2 s+2\ln s\ln\mu-3\ln^2\mu\right)\right]\cdot\left[\tfrac{c_{-1}^2}{z^2}+2\tfrac{c_{-1}c_0}{z}\right]\right.$$
$$\left. + \left[-2z\ln\tfrac{s}{\mu}+2z^2\left(\ln^2 s-\ln^2\mu\right)\right]\cdot\left[\tfrac{c_{-1}^2}{2z^2}+\tfrac{3c_0c_{-1}}{2z}\right]\right\} = \frac{c_{-1}^2}{2}\ln^2\tfrac{s}{\mu} - c_{-1}c_0\ln\tfrac{s}{\mu},$$

where all poles in z perfectly cancel.

Observe that we proved the now purely combinatorial statement 3.5 of the cancellation of all pole terms in ${}_z\phi_R$ analytically by estimates on the asymptotic growths in theorem 2.4. As we absorbed all analytic input of the integrands in $F(z) \in z^{-1}\mathbb{K}[[z]]$ in (3.3) we can give a completely combinatorial proof in lemma 3.8.

For this note that the analytic regularization yields a very simple dependence on the parameter s: Setting $\mathcal{A} := \mathbb{C}[z^{-1}, z]]$ and ${}_z\phi := {}_z\phi_1 = {}_z\phi|_{s=1} \in G^{H_R}_{\mathcal{A}}$, (3.3) fixes the scale dependence ${}_z\phi_s = {}_z\phi \circ \theta_{-z \ln s}$ completely. As this allows to write (3.7)

$$ {}_z\phi_{R,s} = {}_z\phi_\mu{}^{\star -1} \star {}_z\phi_s = {}_z\phi \circ [(S \circ \theta_{-z \ln \mu}) \star \theta_{-z \ln s}] = {}_z\phi \circ (S \star \theta_{-z \ln \frac{s}{\mu}}) \circ \theta_{-z \ln \mu}, $$

finiteness of the physical limit (3.6) can be rephrased in

Proposition 3.7. *For any character ${}_z\phi \in G^{H_R}_{\mathcal{A}}$, the following are equivalent:*

(1) *The physical limit $\phi_R := \lim_{z\to 0} {}_z\phi_R$ exists*
(2) *For any $\ell \in \mathbb{K}$, ${}_z\phi^{\star -1} \star ({}_z\phi \circ \theta_{-\ell z}) = {}_z\phi \circ (S \star \theta_{-\ell z})$ maps into $\mathbb{C}[[z]]$.*
(3) *For every $n \in \mathbb{N}_0$, ${}_z\phi^{\star -1} \star ({}_z\phi \circ Y^n) = {}_z\phi \circ (S \star Y^n)$ maps into $z^{-n}\mathbb{C}[[z]]$.*
(4) *${}_z\phi^{\star -1} \star ({}_z\phi \circ Y) = {}_z\phi \circ (S \star Y)$ maps into $\frac{1}{z}\mathbb{C}[[z]]$, equivalently the limit $\lim\limits_{z\to 0} {}_z\phi^{\star -1} \star ({}_z\phi \circ zY)$ exists.*

Proof. From (3.7), (1) $\Leftrightarrow$ (2) is just composition with the holomorphic $\theta_{-z \ln \mu}$ or $\theta_{z \ln \mu} = \theta^{-1}_{-z \ln \mu}$ while (2) $\Leftrightarrow$ (3) merely expands $\theta_{-\ell z} = \sum_{n \geq 0} \frac{(-\ell z Y)^n}{n!}$. It remains to prove (4) $\Rightarrow$ (3) inductively with

$$ {}_z\phi \circ \left(S \star Y^{n+1}\right) = {}_z\phi \circ (S \star Y^n) \circ Y + [{}_z\phi \circ (S \star Y)] \star [{}_z\phi \circ (S \star Y^n)], $$

exploiting $(S \circ Y) \star \mathrm{id} = -S \star Y$ in the formula (α arbitrary)

$$ S \star (\alpha \circ Y) - (S \star \alpha) \circ Y = -(S \circ Y) \star \alpha = -[(S \circ Y) \star \mathrm{id}] \star S \star \alpha = S \star Y \star S \star \alpha. \qquad \square $$

Lemma 3.8. *Let ${}_z\phi \in G^{H_R}_{\mathcal{A}}$ be the character defined by* (A.2) *with*

$$ {}_z\phi \circ B_+(w) = {}_z\phi(w) \cdot F(z\,|B_+(w)|) \qquad \text{for any fixed} \qquad F(z) \in z^{-1}\mathbb{K}[[z]]. \tag{3.8} $$

Then ${}_z\phi$ fulfils the conditions of proposition 3.7. In particular, $\operatorname{im}({}_z\phi_R) \subseteq \mathbb{K}[[z]]$ *allows the finite physical limit $\phi_R = \lim_{z\to 0} {}_z\phi_R \subseteq \mathbb{K}[\ell, c.]$ taking values in the polynomials in $\ell = \ln \frac{s}{\mu}$ and the coefficients c_n of the series $F(z)$.*

Proof. We show (2) of 3.7 inductively along the grading of H_R. So let it be true on $H_{R,m}$, then by the multiplicativity of ${}_z\phi \circ (S \star \theta_{-z\ell})$ it holds for all products in $H_{R,m+1}$ and we only need consider trees $t = B_+(w)$ for some $w \in H_{R,m}$. For any $k \in \mathbb{N}$ observe holomorphy of $\partial^k_{-\ell}\big|_{\ell=0} {}_z\phi \circ (S \star \theta_{-z\ell})$ through

$$ \begin{aligned} {}_z\phi \circ (S \star [zY]^k)(t) &\underset{(\mathrm{A.1})}{=} {}_z\phi \circ \left\{S \star ([zY]^k \circ B_+)\right\}(w) = {}_z\phi^{\star -1} \star ({}_z\phi \circ B_+ \circ [z(Y + \mathrm{id})]^{k+1})(w) \\ &\underset{(3.8)}{=} \sum_{n \geq -1} c_n \cdot {}_z\phi \circ \left\{S \star [z(Y + \mathrm{id})]^{n+k}\right\}(w) \\ &= \sum_{n \geq -1} c_n \sum_{j=0}^{n+k} \binom{n+k}{j} z^{n+k-j}\, \partial^j_{-\ell}\Big|_{\ell=0} {}_z\phi \circ (S \star \theta_{-z\ell})(w) \in \mathbb{K}[[z]], \end{aligned} \tag{3.9} $$

while for $k = 0$ we use $S \star [zY]^0 = S \star \mathrm{id} = e$ and $e \circ B_+ = 0$. $\square$

3.2. Feynman rule recursion from Mellin transforms. In fact this serves an alternative prove of the recursion (2.6), as in the physical limit $z \to 0$ only the contributions of $j = n + k$ in (3.9) survive:

(3.10)
$$\phi_{\mathrm{R}} \circ B_+ \underset{(3.7)}{=} \sum_{k \in \mathbb{N}} \frac{(-\ell)^k}{k!} \lim_{z \to 0} {}_z\phi \circ (S \star [zY]^k) \circ B_+ \underset{(3.9)}{=} \sum_{\substack{k \in \mathbb{N}\\ n \geq -1}} c_n \frac{(-\ell)^k}{k!} \left[\partial_{-\ell}^{n+k} \phi_{\mathrm{R}}\right]_{\ell=0} = P \circ F(-\partial_\ell) \circ \phi_{\mathrm{R}}.$$

Recall that $P = \mathrm{id} - \mathrm{ev}_0 : \mathbb{K}[\ell] \twoheadrightarrow \ker \varepsilon = \ell\, \mathbb{K}[\ell]$ projects out the constant terms and we defined $\partial_\ell^{-1} := \int_0$. This delivers and efficient recursion to calculate ϕ_R combinatorially in terms of the Mellin transform coefficients $c_\cdot$ without any need for series expansions in z as in example 3.6 or integrations like in (2.2):

Example 3.9. *Applying* (2.6) *we can reproduce example 3.6 as*

$$\phi_R(\bullet) = \phi_R \circ B_+(\mathbb{1}) = F(-\partial_\ell)\phi_R(1) = F(-\partial_\ell)(1) = P \circ \left[-c_{-1} \textstyle\int_0 1 + c_0\right] = -c_{-1}\,\ell$$
$$\phi_R\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\right) = \phi_R \circ B_+(\bullet) = F(-\partial_\ell) \circ \phi_R(\bullet) = P \circ \left[-c_{-1} \textstyle\int_0 + c_0 - c_1 \partial_\ell\right] \left(-c_{-1}\ell\right)$$
$$= c_{-1}^2 \frac{\ell^2}{2} - c_{-1} c_0\, \ell,$$
$$\phi_R\left(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\right) = \phi_R \circ B_+(\bullet\bullet) = F(-\partial_\ell) \circ \phi_R(\bullet\bullet) = P \circ \left[-c_{-1} \textstyle\int_0 + c_0 - c_1 \partial_\ell\right] \left\{\left(-c_{-1}\,\ell\right)^2\right\}$$
$$= -c_{-1}^3 \frac{\ell^3}{3} + c_{-1}^2 c_0\, \ell^2 - 2 c_{-1}^2 c_1\, \ell.$$

Here we can insert $c_{-1} = 1$, $c_0 = 0$ *and* $c_1 = \zeta(2) = \frac{\pi^2}{6}$ *to finally verify* (0.3) *from the introduction, where the choice* $f(\zeta) = \frac{1}{1+\zeta}$ *results in the beta function*

$$F(z) \underset{(3.2)}{=} B(z, 1-z) = \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \in z^{-1} + \frac{\pi^2}{6} z + \frac{7\pi^4}{360} z^3 + \mathcal{O}\left(z^5\right).$$

Corollary 3.10. *As in* $F(-\partial_\ell)$ *only* $-c_{-1} \int_0$ *increases the degree in* ℓ*, the highest order contribution (called* leading log*) of* ϕ_R *is the tree factorial we already saw in* (3.4)*: For any forest* $w \in \mathcal{F}$*,*

$$\phi_R(w) \in \left[-c_{-1} \textstyle\int_0\right]^{\rho(w)} + \mathcal{O}\left(x^{|w|-1}\right) \underset{A.5}{=} \frac{\left(-c_{-1}x\right)^{|w|}}{w!} + \mathbb{K}[x]_{<|w|}. \tag{3.11}$$

4. Hopf algebra morphisms and the renormalization group

From now on we identify $\phi_{\mathrm{R}} : H_R \to \mathbb{K}[x]$ with the polynomials that evaluate to the renormalized Feynman rules $\phi_{\mathrm{R},s} = \mathrm{ev}_\ell \circ \phi_{\mathrm{R}}$ at $x \mapsto \ell = \ln \frac{s}{\mu}$. In (2.6) and (3.10) we independently proved

Corollary 4.1. *As* $F(-\partial_x) \in HZ_\varepsilon^1(\mathbb{K}[x])$ *in* (2.6) *is a Hochschild-1-cocycle by* (B.1) *and* (B.3)*, theorem A.3 implies that* $\phi_R : H_R \to \mathbb{K}[x]$ *is a morphism of Hopf algebras.*

Therefore $\Delta \circ \phi_{\mathrm{R}} = (\phi_{\mathrm{R}} \otimes \phi_{\mathrm{R}}) \circ \Delta$ and the induced map $G_{\mathbb{K}}^{\mathbb{K}[x]} \to G_{\mathbb{K}}^{H_R}$ given by $\mathrm{ev}_\ell \mapsto \phi_{\mathrm{R}}|_\ell := \mathrm{ev}_\ell \circ \phi_{\mathrm{R}}$ becomes a morphism of groups, implying

Corollary 4.2. *Using* (B.3) *we obtain the* renormalization group equation *(called* Chens lemma *in* [**16**]*)*

$$\phi_R|_\ell \star \phi_R|_{\ell'} = \phi_R|_{\ell+\ell'}, \quad \textit{for any} \quad \ell, \ell' \in \mathbb{K}. \tag{4.1}$$

Before we obtain the generator of this one-parameter group in 4.4, note how this result imposes non-trivial relations between individual trees like

$$\begin{aligned}\phi_{\mathrm{R},\ell}\star\phi_{\mathrm{R},\ell'}\left(\overset{\bullet}{\underset{\bullet}{|}}\right) &\underset{(4.1)}{=} \phi_{\mathrm{R},\ell}\left(\overset{\bullet}{\underset{\bullet}{|}}\right)+\phi_{\mathrm{R},\ell}(\bullet)\,\phi_{\mathrm{R},\ell'}(\bullet)+\phi_{\mathrm{R},\ell'}\left(\overset{\bullet}{\underset{\bullet}{|}}\right)\\ &\underset{(3.9)}{=} c_{-1}^2\frac{\ell^2+\ell'^2}{2}-c_{-1}c_0\,(\ell+\ell')+c_{-1}^2\ell\ell' \underset{(3.9)}{=} \phi_{\mathrm{R},\ell+\ell'}\left(\overset{\bullet}{\underset{\bullet}{|}}\right).\end{aligned}$$

Proposition 4.3. *Let $\phi\colon H\to\mathbb{K}[x]$ be a morphism of bialgebras*[5]*, then $\log_\star\phi$ is precisely the monomial linear in x:*

$$\log_\star\phi = x\cdot[\partial_0\circ\phi]\,. \tag{4.2}$$

Proof. Letting $\phi\colon C\to H$ and $\psi\colon H\to\mathcal{A}$ denote morphisms of coalgebras and algebras, exploiting $(\psi\circ\phi-u_{\mathcal{A}}\circ\varepsilon_C)^{\star n}=\psi\circ(\phi-u_H\circ\varepsilon_H)^{\star n}=(\psi-u_{\mathcal{A}}\circ\varepsilon_H)^{\star n}\circ\phi$ in (1.2) proves $(\log_\star\psi)\circ\phi=\log_\star(\psi\circ\phi)=\psi\circ\log_\star\phi$. Now set $\psi=\mathrm{ev}_a$ and use lemma B.5. □

Definition 4.4. *The* anomalous dimension γ *of ϕ_R is the infinitesimal character*

$$H'_R\supset\mathfrak{g}_{\mathbb{K}}^{H_R}\ni\gamma:=-\partial_0\,\circ\,\phi_R\underset{(4.2)}{=}-\frac{1}{x}\log_\star\phi_R. \tag{4.3}$$

It completely determines all higher powers of x by means of

$$\phi_R=\exp_\star(-x\cdot\gamma)=\sum_{n\in\mathbb{N}_0}\frac{\gamma^{\star n}}{n!}(-x)^n. \tag{4.4}$$

Example 4.5. *Reading off $\gamma(\bullet)=c_{-1}$, $\gamma\left(\overset{\bullet}{\underset{\bullet}{|}}\right)=c_{-1}c_0$ and $\gamma\left(\underset{\bullet\,\bullet}{\overset{\bullet}{\wedge}}\right)=2c_{-1}^2c_1$ from the example 3.9 above,* (4.4) *correctly determines the higher powers of x through*

$$\begin{aligned}\phi_R\left(\overset{\bullet}{\underset{\bullet}{|}}\right) &\underset{(1.2)}{=}\left[e-x\gamma+x^2\frac{\gamma\star\gamma}{2}\right]\left(\overset{\bullet}{\underset{\bullet}{|}}\right)=0-x\gamma\left(\overset{\bullet}{\underset{\bullet}{|}}\right)+x^2\frac{\gamma^2(\bullet)}{2}=-c_{-1}c_0\,x+c_{-1}^2\frac{x^2}{2},\\ \phi_R\left(\underset{\bullet\,\bullet}{\overset{\bullet}{\wedge}}\right) &=0-x\gamma\left(\underset{\bullet\,\bullet}{\overset{\bullet}{\wedge}}\right)+x^2\frac{\gamma\otimes\gamma}{2}\left(2\bullet\otimes\overset{\bullet}{\underset{\bullet}{|}}+\bullet\bullet\otimes\bullet\right)-x^3\frac{\gamma\otimes\gamma\otimes\gamma}{6}\left(2\bullet\otimes\bullet\otimes\bullet\right)\\ &=-\gamma^3(\bullet)\,\frac{x^3}{3}+x^2\gamma(\bullet)\,\gamma\left(\overset{\bullet}{\underset{\bullet}{|}}\right)-2c_{-1}^2c_1\,x=-c_{-1}^3\frac{x^3}{3}+c_{-1}^2c_0\,x^2-2c_{-1}^2c_1\,x.\end{aligned}$$

Note how the fragment $\bullet\bullet\otimes\bullet$ of $\Delta\left(\underset{\bullet\,\bullet}{\overset{\bullet}{\wedge}}\right)$ does not contribute to the quadratic terms $\frac{x^2}{2}\gamma\star\gamma$, as γ vanishes on products. We will exploit this in (6.7) of section 6.1 on Dyson-Schwinger equations.

Example 4.6. *In the leading*-log *case* (A.3) *we read off $\partial_0\circ\varphi=Z_\bullet\in\mathfrak{g}_{\mathbb{K}}^{H_R}$ where $Z_\bullet(w):=\delta_{w,\bullet}$. Comparing $\varphi=\exp_\star(xZ_\bullet)$ with* (A.3) *shows $|w|!=w!\cdot Z_\bullet^{\star|w|}(w)$, implying the following combinatorial relation among tree factorials noted in* [**16**]*:*

$$\frac{|w|}{w!}=\frac{1}{(|w|-1)!}\sum_w Z_\bullet(w_1)Z_\bullet^{\star|w|-1}(w_2)=\sum_{w\colon\ w_1=\bullet}\frac{1}{|w_2|!}Z_\bullet^{\star|w_2|}(w_2)=\sum_{w\colon\ w_1=\bullet}\frac{1}{w_2!}.$$

[5] This already implies ϕ to be a morphism of Hopf algebras.

4.1. The regularized viewpoint. We can obtain these results also by exploiting the regulator as in [**10**]:

Lemma 4.7. *For* ${}_z\phi \in G^H_{\mathcal{A}}$ *subject to 3.7, the* anomalous dimension

$$\gamma := -\partial_\ell|_0 \phi_R = \lim_{z\to 0} z \cdot {}_z\phi \circ (S \star Y) = \operatorname{Res} {}_z\phi \circ (S \star Y) \in \mathfrak{g}^H_{\mathbb{K}} \tag{4.5}$$

is the residue (coefficient of $\frac{1}{z}$*) of* ${}_z\phi \circ (S \star Y) \in z^{-1}\mathbb{K}[[z]]$ *and fulfils*

$$-\frac{\partial}{\partial \ell}\phi_R = \gamma \star \phi_R, \quad \textit{therefore} \quad \phi_R = \exp_\star(-\ell\gamma) \star \phi_R|_{\ell=0} = \exp_\star(-\ell\gamma). \tag{4.6}$$

Proof. Proposition C.2 renders $\gamma \in \mathfrak{g}^H_{\mathbb{K}}$ immediate. We employ the coderivation property $\Delta \circ Y = (\mathrm{id} \otimes Y + Y \otimes \mathrm{id}) \circ \Delta$ and $S \circ Y = -S \star Y \star S$ in

$$\begin{aligned} &-\frac{\partial}{\partial \ell} \lim_{z\to 0} {}_z\phi \circ (S \star \theta_{-z\ell}) = \lim_{z\to 0} z \cdot {}_z\phi \circ (S \star [\theta_{-z\ell} \circ Y]) \\ &= \lim_{z\to 0} \Big\{ z \cdot {}_z\phi \circ (S \star \theta_{-z\ell}) \circ Y \Big\} + \lim_{z\to 0} \Big\{ z \cdot {}_z\phi \circ (S \star Y) \star {}_z\phi \circ (S \star \theta_{-z\ell}) \Big\}. \end{aligned}$$

The first term vanishes by the existence of $\lim_{z\to 0} {}_z\phi \circ (S \star \theta_{-z\ell})$, while the second factorizes as desired. It remains to observe $\phi_{\mathrm{R}}|_{\ell=0} = {}_z\phi \circ (S \star \mathrm{id}) = {}_z\phi \circ e = e$. ☐

Clearly we can easily rewrite this in the form of (4.1) since

$$\phi_{\mathrm{R},\ell} \star \phi_{\mathrm{R},\ell'} = \exp_\star(-\ell\gamma) \star \exp_\star(-\ell'\gamma) = \exp_\star(-(\ell + \ell')\gamma) = \phi_{\mathrm{R},\ell+\ell'}. \tag{4.7}$$

Note how this reasoning fails if ϕ_{R} disrespects the coproduct: Then

$$\log_\star \phi_{\mathrm{R}} = \sum_{n\in\mathbb{N}} \frac{\gamma_n}{n!} x^n \tag{4.8}$$

would contain higher powers in x and a family $\gamma_n \in H'$ of functionals. As these do not necessarily commute under $\star$, also $\mathrm{ev}_\ell \circ \log_\star(\phi_{\mathrm{R}})$ and $\mathrm{ev}_{\ell'} \circ \log_\star(\phi_{\mathrm{R}})$ do not commute such that (4.7) is not applicable.

While the renormalization group allows us to reduce all computations to the linear terms γ, in our setup (3.1) we can give a simple recursion for γ itself in term of the Mellin transform coefficients in

Corollary 4.8. *From* $\gamma \circ B_+ \underset{(2.6)}{=} -\partial_0 \circ F(-\partial_\ell) \circ \phi_R \underset{(4.4)}{=} \mathrm{ev}_0 \circ [zF(z)]_{-\partial_x} \circ \exp_\star(-x\gamma)$ *we obtain the inductive formula* $\gamma \circ B_+ = \sum_{n\in\mathbb{N}_0} c_{n-1}\gamma^{\star n} = [zF(z)]_{z\mapsto\gamma}$.

As $\gamma \in \mathfrak{g}^H_{\mathbb{K}}$ vanishes on products, evaluating it on trees is all we need such that 4.8 is sufficient to determine γ.

Example 4.9. *Starting with* $\gamma(\bullet) = c_{-1}\varepsilon(\mathbb{1}) = c_{-1}$ *we can recursively calculate*

$$\begin{aligned} \gamma\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right) &= c_{-1}\varepsilon(\bullet) + c_0\gamma(\bullet) = c_{-1}c_0, \\ \gamma\left(\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix}\right) &= c_{-1}\varepsilon\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right) + c_0\gamma\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right) + c_1\gamma\star\gamma\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right) = c_{-1}c_0^2 + c_1\left[\gamma(\bullet)\right]^2 = c_{-1}c_0^2 + c_{-1}^2c_1, \\ \gamma\left(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet\end{smallmatrix}\right) &= c_{-1}\varepsilon(\bullet\bullet) + c_0\gamma(\bullet\bullet) + c_1\gamma\star\gamma(\bullet\bullet) = 2c_1\left[\gamma(\bullet)\right]^2 = 2c_{-1}^2c_1 \quad \textit{and so on.} \end{aligned}$$

5. Locality, finiteness and minimal subtraction

In the presence of a regulator z, considering a general character ${}_z\phi \in G^H_{\mathcal{A}}$ for $\mathcal{A} = \mathbb{K}[z^{-1}, z]]$ and scale dependence fixed by ${}_z\phi_s = {}_z\phi \circ \theta_{-z\ell}$ naturally leads to a different idea of renormalization in

Definition 5.1. *The* minimal subtraction scheme R_{MS} *splits* $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ *by projection on the poles* $\mathcal{A}_- := z^{-1}\mathbb{K}[z^{-1}]$ *along the holomorphic* $\mathcal{A}_+ := \mathbb{K}[[z]]$.

One easily checks (2.8) *and therefore obtains a unique Birkhoff decomposition* ${}_z\phi_+ = {}_z\phi_- \star ({}_z\phi \circ \theta_{-z\ell})$*, with physical limit* $\phi_+ := \lim_{z\to 0} {}_z\phi_+$.

Note that this scheme does not specify a subtraction point μ, but we included μ as an arbitrary scale inside $\ell = \ln \frac{s}{\mu}$ when we agreed on ${}_z\phi_s = {}_z\phi \circ \theta_{-z\ell}$. Physically this is necessary to obtain the dimensionless argument $\frac{s}{\mu}$ in the logaritm instead of expressions like $\ln s$ alone, as s is a quantity carrying a unit (typically momentum or energy). In any case, replacing $\ln s$ by ℓ is nothing but a rescaling of s.

This renormalization scheme R_{MS} and the resulting Birkhoff decomposition differ from the kinematic subtraction scheme R_μ, compare example 3.9 with

Example 5.2. *For* ${}_z\phi$ *arising from* (3.8)*, minimal subtraction yields*

$$
\begin{aligned}
{}_z\phi_+(\bullet) &= (\mathrm{id} - R_{MS})\,{}_z\phi_s(\bullet) = (\mathrm{id} - R_{MS}) e^{-z\ell}F(z) = e^{-z\ell}F(z) - \frac{c_{-1}}{z} \\
\phi_+(\bullet) &= \lim_{z\to 0} {}_z\phi_+(\bullet) = c_0 - c_{-1}\ell \qquad (5.1)\\
{}_z\phi_+\left(\overset{\bullet}{\underset{\bullet}{|}}\right) &= (\mathrm{id} - R_{MS})\left[{}_z\phi_s\left(\overset{\bullet}{\underset{\bullet}{|}}\right) + {}_z\phi_-(\bullet)\,{}_z\phi_s(\bullet)\right] \\
&= (\mathrm{id} - R_{MS})\left[e^{-2z\ell}F(z)F(2z) - \frac{c_{-1}}{z}e^{-z\ell}F(z)\right] \\
&= e^{-2z\ell}F(z)F(2z) - \frac{c_{-1}}{z}e^{-z\ell}F(z) + \frac{c_{-1}^2}{2z^2} - \frac{c_{-1}c_0}{2z} \\
\phi_+\left(\overset{\bullet}{\underset{\bullet}{|}}\right) &= \frac{c_{-1}^2}{2}\ell^2 - 2c_{-1}c_0\ell + c_0^2 + \frac{3}{2}c_{-1}c_1. \qquad (5.2)
\end{aligned}
$$

Note that by this choice of $\mathcal{A}_+ = \mathbb{K}[[z]]$, the finiteness of $\phi_+ := \lim_{z\to 0} {}_z\phi_+$ is automatic such that we can finitely renormalize any ${}_z\phi \in G^H_{\mathcal{A}}$ using R_{MS}. This seems preferable considering that the kinematic subtraction scheme only yields finite results under the conditions of proposition 3.7. However, the physics of local field theory requires *local* counterterms that are constants, independent of the external parameters (in our setup s)[6].

Definition 5.3 ([**10**])**.** *A Feynman rule* ${}_z\phi \in G^{H_R}_{\mathcal{A}}$ *is called* local *iff the minimal subtraction counterterm* ${}_z\phi_{-,s} = ({}_z\phi \circ \theta_{-z\ell})_-$ *is independent of* $\ell \in \mathbb{K}$.

By definition 2.7, counterterms in the kinematic subtraction scheme are s-independent a priori. For the minimal subtraction scheme R_{MS}, locality is a true condition and the study and characterization of local Feynman rules in this setting is a main theme of [**10, 21**]. It is therefore illuminating to find

Proposition 5.4. *Locality of* ${}_z\phi \in G^{H_R}_{\mathcal{A}}$ *in the* R_{MS} *scheme is equivalent to the finiteness conditions of proposition 3.7 in the kinematic subtraction scheme.*

---

[6]See also section 7: Counterterms may be allowed to depend polynomially on parameters, but not logarithmically as is the case for $\mathrm{im}\left({}_z\phi_{-,s}\right) \subseteq z^{-1}\mathbb{K}[z^{-1}, \ell]$.

Proof. Given condition 3.7 (2), ${}_z\phi_+ \star [{}_z\phi^{\star -1} \star ({}_z\phi \circ \theta_{-z\ell})]$ maps to $\mathbb{K}[[z]]$ so

$${}_z\phi \circ \theta_{-z\ell} = {}_z\phi_-{}^{\star -1} \star \left\{ {}_z\phi_+ \star [{}_z\phi^{\star -1} \star ({}_z\phi \circ \theta_{-z\ell})] \right\}$$

is a Birkhoff decomposition and its uniqueness implies the locality $({}_z\phi \circ \theta_{-z\ell})_- = {}_z\phi_-$. Conversely, for local ${}_z\phi$ we have ${}_z\phi_- = ({}_z\phi \circ \theta_{-z\ell})_-$ wherefore

$${}_z\phi_+{}^{\star -1} \star ({}_z\phi \circ \theta_{-z\ell})_+ = {}_z\phi^{\star -1} \star {}_z\phi_-{}^{\star -1} \star ({}_z\phi \circ \theta_{-z\ell})_- \star ({}_z\phi \circ \theta_{-z\ell}) = {}_z\phi^{\star -1} \star ({}_z\phi \circ \theta_{-z\ell})$$

shows 3.7 (2) as the left hand side maps to $\mathcal{A}_+ = \mathbb{K}[[z]]$. □

We have seen how algebraically the problems of finiteness in the kinematic subtraction scheme and locality in minimal subtraction coincide. Both finite renormalization and local counter terms are simultaneously only achieved under the conditions of proposition 3.7, no matter which of the schemes $\{R_\mu, R_{\mathrm{MS}}\}$ is chosen.

5.1. Renormalization group. We again identify ϕ_+ with the polynomials in $\mathbb{K}[x]$ such that $\phi_{+,s} = \mathrm{ev}_\ell \circ \phi_+$, but in contrast to ϕ_{R} from the kinematic subtraction scheme, these feature constant terms $\phi_{+,\mu} = \phi_+\big|_{x=0} = \mathrm{ev}_0 \circ \phi_+ = \varepsilon \circ \phi_+$ as we observed in the examples 5.2. Therefore the renormalization group equation (4.4) can not hold, but is instead replaced by

Corollary 5.5. *For local ${}_z\phi \in G^H_{\mathcal{A}}$, the* beta functional [**10**]

$$\beta := z \cdot {}_z\phi_-^{\star -1} \circ (S \star Y) = -\operatorname{Res} \circ {}_z\phi_- \circ Y = -\operatorname{Res} \circ {}_z\phi_- \circ (S \star Y) \in \mathfrak{g}^H_{\mathbb{K}} \tag{5.3}$$

completely dictates the scale dependence of the physical limit ϕ_+ through

$$\phi_{+,s} = \lim_{z \to 0} ({}_z\phi \circ \theta_{-z\ell})_+ = \exp_\star(-\ell\beta) \star \left(\varepsilon \circ \phi_+\right). \tag{5.4}$$

Proof. As ${}_z\phi_-^{\star -1}$ is local by lemma 5.6, proposition 3.7 allows to invoke 5.4 which proves the finiteness of (5.3): $\operatorname{im}\left({}_z\phi_-^{\star -1} \circ (S \star Y)\right) \subseteq z^{-1}\mathbb{K}[[z]] \cap \mathcal{A}_- = \mathbb{K} \cdot z^{-1}$,

$$\beta = \lim_{z \to 0} \left[z \cdot {}_z\phi_-^{\star -1} \circ (S \star Y) \right] = z \cdot {}_z\phi_-^{\star -1} \circ (S \star Y) \in \mathfrak{g}^H_{\mathbb{K}} \tag{5.5}$$

converges and (4.4) applies to give $\lim_{z \to 0} {}_z\phi_-^{\star -1} \circ (S \star \theta_{-z\ell}) = \exp_\star(-\ell\beta)$. Inserting this into the Birkhoff decomposition (here we set ${}_z\phi_+ = {}_z\phi_{+,\mu} = {}_z\phi_+\big|_{\ell=0}$)

$$\begin{aligned} {}_z\phi_{+,s} &= ({}_z\phi \circ \theta_{-z\ell})_- \star ({}_z\phi \circ \theta_{-z\ell}) = {}_z\phi_- \star \left[({}_z\phi_-^{\star -1} \star {}_z\phi_+) \circ \theta_{-z\ell} \right] \\ &= \left[{}_z\phi_+^{\star -1} \circ (S \star \theta_{-z\ell}) \right] \star \left({}_z\phi_+ \circ \theta_{-z\ell} \right) \end{aligned}$$

and exploiting $\lim_{z \to 0} \left({}_z\phi_+ \circ \theta_{-z\ell} \right) = \lim_{z \to 0} {}_z\phi_+ = \phi_+\big|_{\ell=0}$ shows (5.4). The relations in (5.3) follow from $S(w) = -w \mod (\ker \varepsilon)^2$, $S \star Y(w) = Y(w) \mod (\ker \varepsilon)^2$ and the fact that $\operatorname{Res} \circ {}_z\phi_- \in \mathfrak{g}^H_{\mathbb{K}}$ vanishes on products, because for any $w, w' \in \ker \varepsilon$

$${}_z\phi_-(w \cdot w') = {}_z\phi_-(w) \cdot {}_z\phi_-(w') \in \mathcal{A}_-^2 = z^{-2}\mathbb{K}[[z]]$$

has no pole of first order. □

Lemma 5.6. *Let ${}_z\phi \in G^H_{\mathcal{A}}$ be local, then ${}_z\phi_-^{\star -1} \in G^H_{\mathcal{A}}$ is local as well.*

Proof. As in [**21**], from ${}_z\phi_+, {}_z\phi_{+,s} \in G^H_{\mathcal{A}_+}$ we deduce that

$${}_z\phi_-^{\star -1} \circ \theta_{-z\ell} = \left({}_z\phi \star {}_z\phi_+^{\star -1} \right) \circ \theta_{-z\ell} = {}_z\phi_s \star \left({}_z\phi_+^{\star -1} \circ \theta_{-z\ell} \right) = {}_z\phi_-^{\star -1} \star {}_z\phi_{+,s} \star \left({}_z\phi_+^{\star -1} \circ \theta_{-z\ell} \right)$$

is a Birkhoff decomposition and read off $\left({}_z\phi_-^{\star -1} \circ \theta_{-z\ell} \right)_- = {}_z\phi_-$ by uniqueness. □

In the minimal subtraction scheme we can rephrase the renormalization group (5.4) as expressing all z-poles of ${}_z\phi_- : H \to \mathbb{K}[z^{-1}]$ in terms of the first order poles only [**10**]. This comes about as β captures all information on the character ${}_z\phi_-^{\star-1} \in \mathfrak{g}_{\mathcal{A}}^H$ since $\operatorname{im}(S \star Y)$ generates the full Hopf algebra by C.3. We shall demonstrate this in

Example 5.7. *First we take example 5.2 to read of the counterterms*

$$\begin{aligned} {}_z\phi_-^{\star-1}(\bullet) &= {}_z\phi_-(-\bullet) = -R_{MS}\left[-e^{-z\ell}F(z)\right] = \frac{c_{-1}}{z} \\ {}_z\phi_-^{\star-1}\left(\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right) &= {}_z\phi_-\left(-\genfrac{}{}{0pt}{}{\bullet}{\bullet} + \bullet\bullet\right) = -\frac{c_{-1}^2}{2z^2} + \frac{c_{-1}c_0}{2z} + \left[\frac{c_{-1}}{z}\right]^2 = \frac{c_{-1}^2}{2z^2} + \frac{c_{-1}c_0}{2z} \end{aligned}$$

and apply (5.3) *the get hold of the residues* $\beta(\bullet) \underset{(5.3)}{=} z \cdot {}_z\phi_-^{\star-1}(\bullet) = c_{-1}$ *as well as*

$$\beta\left(\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right) \underset{(5.3)}{=} z \cdot {}_z\phi_-^{\star-1}\left(2\genfrac{}{}{0pt}{}{\bullet}{\bullet} - \bullet\bullet\right) = c_{-1}c_0.$$

Observe how for $S \star Y\left(\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right) = 2\genfrac{}{}{0pt}{}{\bullet}{\bullet} - \bullet\bullet$ *we indeed obtained only a first order pole in* ${}_z\phi_-^{\star-1}$*, contrary to* $\genfrac{}{}{0pt}{}{\bullet}{\bullet}$ *itself entailing a second order pole as well. This one we can now predict using* (C.6) *applied to* ${}_z\phi_-^{\star-1} \circ (S \star Y) = \frac{\beta}{z}$*:*

$${}_z\phi_-^{\star-1}\left(\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right) = \left\{e + \frac{\beta \circ Y^{-1}}{z} + \frac{\left[(\beta \circ Y^{-1}) \star \beta\right] \circ Y^{-1}}{z^2}\right\}\left(\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right) = \frac{\beta\left(\frac{1}{2}\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right)}{z} + \frac{[\beta(\bullet)]^2}{2z^2} = \frac{c_{-1}c_0}{2z} + \frac{c_{-1}^2}{2z^2}.$$

5.2. Relating R_μ with R_{MS}.

Though both schemes seem so different, already [**4**] exploited their interaction

Lemma 5.8. *For local* ${}_z\phi \in G_{\mathcal{A}}^H$*, the scale dependence of* ${}_z\phi_{+,s}$ *(in the* R_{MS} *scheme) is dictated by* ${}_z\phi_{R,s}$ *(kinematic subtraction scheme) through*

$${}_z\phi_{+,s} = \left(R_\mu \circ {}_z\phi_{+,s}\right) \star {}_z\phi_{R,s}. \tag{5.6}$$

Proof. Locality of the counterterms ${}_z\phi_-$ implies $R_\mu \circ {}_z\phi_- = {}_z\phi_-$, hence

$$\begin{aligned} \left(R_\mu \circ {}_z\phi_{+,s}\right) \star {}_z\phi_{R,s} &= \left[R_\mu \circ \left({}_z\phi_- \star {}_z\phi_s\right)\right] \star \left(R_\mu \circ {}_z\phi_s\right)^{\star-1} \star {}_z\phi_s \\ &= \left[R_\mu \circ \left({}_z\phi_- \star {}_z\phi_s \star {}_z\phi_s^{\star-1}\right)\right] \star {}_z\phi_s = \left(R_\mu \circ {}_z\phi_-\right) \star {}_z\phi_s = {}_z\phi_{+,s}. \end{aligned}$$ ∎

Note how $R_\mu \circ {}_z\phi_{+,s} = {}_z\phi_+\big|_{\ell=0}$ reduces to the constants $\varepsilon \circ \phi_+ \in G_{\mathbb{K}}^H$ in the physical limit. There (5.6) takes the form of

Corollary 5.9. *The characters* $\phi_R, \phi_+ : H_R \to \mathbb{K}[x]$ *fulfil the relations*

$$\phi_+ = \left(\varepsilon \circ \phi_+\right) \star \phi_R, \quad \textit{equivalently} \quad \Delta \circ \phi_+ = \left(\phi_+ \otimes \phi_R\right) \circ \Delta. \tag{5.7}$$

Example 5.10. *After reading of the constants* $\varepsilon \circ \phi_+(\bullet) = c_0$ *and* $\varepsilon \circ \phi_+\left(\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right) = c_0^2 + \frac{3}{2}c_{-1}c_1$*, we can verify* (5.7) *against the examples 5.2 making use of 3.9:*

$$\begin{aligned} \phi_+(\bullet) &= \varepsilon \circ \phi_+(\bullet) + \phi_R(\bullet) = c_0 - c_{-1}\ell \\ \phi_+\left(\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right) &= \varepsilon \circ \phi_+\left(\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right) + \left[\varepsilon \circ \phi_+(\bullet)\right] \cdot \phi_R(\bullet) + \phi_R\left(\genfrac{}{}{0pt}{}{\bullet}{\bullet}\right) \\ &= c_0^2 + \frac{3}{2}c_{-1}c_1 - c_0c_{-1}\ell + \frac{c_{-1}^2}{2}\ell^2 - c_0c_{-1}\ell = \frac{c_{-1}^2}{2}\ell^2 - 2c_0c_{-1}\ell + c_0^2 + \frac{3}{2}c_{-1}c_1. \end{aligned}$$

Corollary 5.11. *Inserting both* (5.4) *and* (4.4) *into* (5.7) *reveals*

$$\beta \star \left(\varepsilon \circ \phi_+\right) = \left(\varepsilon \circ \phi_+\right) \star \gamma. \tag{5.8}$$

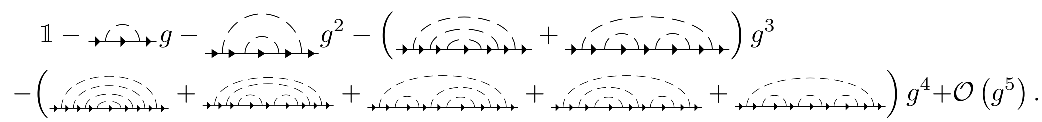
Hence β and γ differ only by conjugation with the character $\varepsilon \circ \phi_+ \in G^H_{\mathbb{K}}$ and therefore in particular agree on any cocommutative elements of H (actually on the entire maximal cocommutative Hopf subalgebra).

Example 5.12. *For the rooted trees H_R, the cocommutative elements include the* ladders $B^n_+(\mathbb{1})$. *In the examples 5.7 and 4.9 we explicitly checked the first two cases of $n \in \{1, 2\}$: $\beta(\bullet) = c_{-1} = \gamma(\bullet)$ as well as $\beta(\;) = c_{-1}c_0 = \gamma(\;)$.*

6. Dyson-Schwinger equations and correlation functions

Until now we considered the renormalized Feynman rules ϕ_{R} on their own, but these form only one ingredient to quantum field theory. The counterpart is the *perturbation series* $X(g)$ they are being applied to.

Example 6.1 ([**22**]). *In* Yukawa theory, *the propagation of a fermion is the superposition of infinitely many possible interactions with a scalar field each of which is represented by a Feynman diagram. Among these are contributions like*



The Feynman rules map each graph to an individual amplitude, but physically these are not distinguishable and need to be all summed up. Further, a coupling constant *g takes the strength of an interaction into account.*

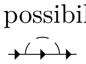
Definition 6.2. *A* perturbation series *is a formal power series*

$$X(g) = \sum_{n\in\mathbb{N}_0} x_n g^n \in H_R[[g]] \quad \textit{with} \quad x_0 = \mathbb{1} \tag{6.1}$$

taking values in the Hopf algebra H_R of rooted trees and indexed by the coupling constant *g. Evaluation of the renormalized Feynman rules $\phi_R \in G^{H_R}_{\mathbb{K}[\ell]}$ on $X(g)$ delivers the* correlation function

$$G(g) := \phi_R \circ X(g) = \sum_{n\in\mathbb{N}_0} \phi_R(x_n) g^n \in \mathbb{K}[\ell][[g]], \tag{6.2}$$

while the physical anomalous dimension *is $\widetilde{\gamma}(g) := \gamma \circ X(g) \underset{(4.3)}{=} -\partial_\ell|_0 G(g) \in \mathbb{K}[[g]]$.*

The crucial property of perturbation series is the possibility of insertions: In the above example, we started with the *primitive*[7] graph  and iteratively inserted it as a subdivergence into itself. Though the full perturbation series contains many more graphs, this illustrates how $X(g)$ may efficiently be described by means of recursive insertions. Those are represented by Hochschild-1-cocycles motivating

Definition 6.3. *To a parameter $\kappa \in \mathbb{K}$ and a family of cocycles $B_\cdot : \mathbb{N} \to HZ^1_\varepsilon(H_R)$ we associate the* combinatorial Dyson-Schwinger equation[8]

$$X(g) = \mathbb{1} + \sum_{n\in\mathbb{N}} g^n B_n\left(X^{1+n\kappa}(g)\right). \tag{6.3}$$

This type of equations is folklore in physics, but had not been cast into its pure algebraic form before [**1**]. Referring to [**11**] we recall the main results in

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[7] That means it is free of subdivergences.

[8] As $x_0 = \mathbb{1}$, for arbitrary p we define $[X(g)]^p := \sum_{n\in\mathbb{N}_0} \binom{p}{n} [X(g) - \mathbb{1}]^n \in H_R[[g]]$.

Lemma 6.4. *As* perturbation series (6.1)*, the equation* (6.3) *allows a unique solution which is determind recursively by*

$$x_k = \sum_{0 \le m+n \le k} \binom{1+\kappa n}{m} B_n \left(\sum_{\substack{i_1+\ldots+i_m+n=k \\ i_1,\ldots,i_m \ge 1}} x_{i_1} \cdots x_{i_m}. \right) \tag{6.4}$$

Most importantly, these coefficients generate a Hopf subalgebra $H_X := \langle \{x_n : n \in \mathbb{N}_0\} \rangle$ *(isomorphic to the Fàa di Bruno Hopf algebra when* $\kappa \neq 0$*). Explicitly we find*

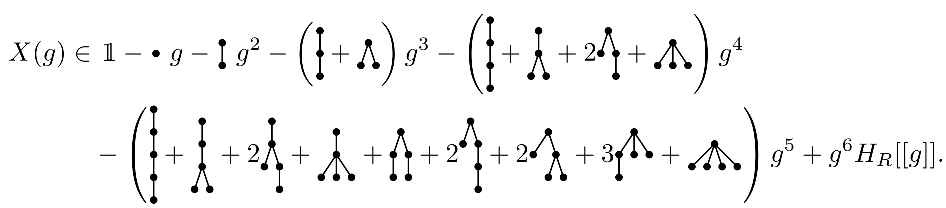
$$\Delta X(g) = \sum_{n \in \mathbb{N}_0} [X(g)]^{1+n\kappa} \otimes g^n x_n \in (H_R \otimes H_R)[[g]]. \tag{6.5}$$

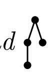
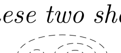
We learn that the solution of (6.3) has a very special property: The coproduct $\Delta x_n \in H_X \otimes H_X$ can be expressed by the coefficients alone, with (6.5) serving an explicit formula. The equations originating from quantum field theory are precisely of this type (6.3). Before we exploit this information on X let us give some examples.

Example 6.5. *When we set* $\kappa = 0$, $\Delta X(g) = X(g) \otimes X(g)$ *is grouplike such that* $\Delta x_n = \sum_{i+j=n} x_i \otimes x_j$*. The Dyson-Schwinger equation* $X(g) = \mathbb{1} + B_+ (X(g))$ *is linear and generates the cocommutative ladders* $x_n = B_+^n(\mathbb{1})$.

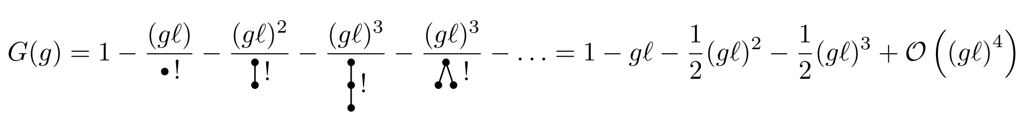
Recall that we take the trees in H_R as substitute for Feynman graphs, each node representing an insertion into some other graph.

Example 6.6. *In* [**5, 22**] *we find the equation* $X(g) = \mathbb{1} - gB_+\left(\frac{1}{X(g)}\right)$ *featuring* $\kappa = -2$ *which corresponds to the propagator example 6.1. The solution sums all trees with the factor counting the number of distinct ordered embeddings:*



The first factor of two arises from the different embeddings  *and*  *and correctly accounts for the fact that these two shall represent different graphs (though they are the same elements in* H_R*):*  *and* .

Example 6.7. *Let us consider the tree factorial Feynman rules* φ *from* (A.3) *applied to the above series. The anomalous dimension* $\widetilde{\gamma}(g) \underset{4.6}{=} -Z_{\bullet} \circ X(g) = g$ *is linear, while the first terms of the correlation function become*



6.1. Propagator coupling duality. The Hopf subalgebra of the perturbation series allows to calculate convolutions in

Lemma 6.8. *Let $\psi \in \mathfrak{g}_{\mathcal{A}}^{H_R}$ denote an infinitesimal character, $\Psi \in G_{\mathcal{A}}^{H_R}$ a character and $\lambda \in \operatorname{Hom}(H_R, \mathcal{A})$ a linear map. Then we obtain*

$$(6.6) \qquad (\Psi \star \lambda) \circ X(g) = [\Psi \circ X(g)] \cdot \lambda \circ X\left(g\, [\Psi \circ X(g)]^{\kappa}\right) := [\Psi \circ X(g)] \cdot \sum_{n \in \mathbb{N}_0} \lambda(x_n) \cdot \left(g\, [\Psi \circ X(g)]^{\kappa}\right)^n \in \mathcal{A}[[g]]$$

$$(6.7) \qquad (\psi \star \lambda) \circ X(g) = [\psi \circ X(g)] \cdot (\mathrm{id} + \kappa g \partial_g) [\lambda \circ X(g)] \in \mathcal{A}[[g]].$$

Proof. These are immediate consequences of lemma 6.4, for (6.7) consider

$$\psi\left([X(g)]^{1+n\kappa}\right) \cdot g^n = \sum_{i \in \mathbb{N}_0} \binom{1+n\kappa}{i} \psi\left([X(g) - \mathbb{1}]^i\right) g^n = \psi\left(X(g) - \mathbb{1}\right) \cdot (1 + n\kappa) g^n.$$

□

By combining (6.5) with the renormalization group equation we can calculate the correlation function out of the knowledge of $\widetilde{\gamma}$ only:

Example 6.9. *Continuing example 6.6, we can calculate the $\propto \ell^2$-term $Z_{\bullet}^{\star 2}(X(g)) = -g(1 - 2g\partial_g)(-g) = -g^2$ and all further convolution products*

$$Z_{\bullet}^{\star n+1}(X(g)) \underset{(6.7)}{=} -g^{n+1}(2n-1)(2n-3)\cdots(1) = -g^{n+1}\frac{(2n)!}{2^n n!},$$

proving $\varphi(x_{n+1}) = -2^{-n} C_n \ell^{n+1}$ with the Catalan numbers *C_n already noted in* [**20**]. *Combining their known generating function $2g \sum_{n \in \mathbb{N}_0} g^n C_n = 1 - \sqrt{1 - 4g}$ with $\varphi = \exp_{\star}(-\ell Z_{\bullet})$ allows us to completely determine the correlation function as $G(g) = \sqrt{1 - 2g\ell}$.*

Corollary 6.10. *As ϕ_R is a morphism of Hopf algebras by 4.1, for any $\ell, \ell' \in \mathbb{K}$ we can factorize the correlation function at $\ell + \ell'$ in two different ways*

$$(6.8) \quad G_{\ell+\ell'}(g) = (\phi_{R,\ell} \star \phi_{R,\ell'}) \circ X(g) \underset{(6.6)}{=} G_{\ell}(g) \cdot G_{\ell'}\left[g G_{\ell}^{\kappa}(g)\right] = G_{\ell'}(g) \cdot G_{\ell}\left[g G_{\ell'}^{\kappa}(g)\right].$$

At this point we like to briefly highlight the non-analytic nature of perturbative quantum field theory. The correlation function $G(g)$ at $g \neq 0$ (6.2) is a physical object that can in principle be measured through experiment. It is only by the nature of the perturbative method we apply that we are merely able to calculate the formal series expansion (6.2) of $G(g)$ around $g = 0$ with the help of the Feynman rules. The main issue is that in the interesting cases, the function $G(g)$ is not analytic at this point and the series (6.2) has zero radius of convergence[9].

However, in this perturbative approach we just deduced the functional equations (6.8) for the formal series. Therefore it is natural to impose these on the true correlation functions, such that we gain a non-perturbative handle on quantum field theory. We will continue to stress similar examples in this section.

But first observe how (6.8) takes the infinitesimal form

Corollary 6.11. *With the help of $-\frac{\mathrm{d}}{\mathrm{d}x}\, \phi_R = \gamma \star \phi_R = \phi_R \star \gamma$ or by differentiating* (6.8) *with respect to ℓ' at zero we find the differential equations*

$$(6.9) \qquad G_{\ell}(g) \cdot \widetilde{\gamma}\left[g G_{\ell}^{\kappa}(g)\right] \underset{(6.6)}{=} -\partial_{\ell} G_{\ell}(g) \underset{(6.7)}{=} \widetilde{\gamma}(g) \cdot (1 + \kappa g \partial_g)\, G_{\ell}(g).$$

[9] Hence the example 6.9 is still far away from quantum field theory as its correlation function is analytic at $g \to 0$.

The first of these equations generalizes the *propagator coupling duality* observed in [**5, 20**]. For any fixed coupling g, it expresses the correlation function as the solution of the first order ordinary differential equation

$$(6.10) \qquad -\frac{\mathrm{d}}{\mathrm{d}\ell}\ln G_\ell(g) = \widetilde{\gamma}\left[g e^{\kappa \ln G_\ell(g)}\right] \quad \text{with initial condition} \quad \ln G_0(g) = 0.$$

Note how this equation reconstructs $G_\ell(g)$ completely only from the input $\widetilde{\gamma}(g)$. This demonstrates the power of the renormalization group: Though G depends on g and ℓ, after imposing (4.1) only a one-dimensional degree of freedom is left. As before, (6.10) serves a non-perturbative relation and need not be restrained to the perturbative series alone.

Example 6.12. *The* leading-log *expansion takes only the highest power of ℓ in each g-order, so $\widetilde{\gamma}(g) = cg^n$ is a monomial for some $c \in \mathbb{K}$, $n \in \mathbb{N}$ (otherwise different g-powers would mix for a given order in ℓ). In this case* (6.10) *integrates to*

$$(6.11) \qquad G_{\textit{leading-}\mathrm{log}}(g) = \left[1 + cn\kappa\ell g^n\right]^{-\frac{1}{n\kappa}}.$$

As a special case we recover example 6.7 for $n = c = 1$ and $\kappa = -2$.

Example 6.13. *For the linear Dyson-Schwinger equation $\kappa = 0$,* (6.8) *states $G_{\ell+\ell'}(g) = G_\ell(g) \cdot G_{\ell'}(g)$ which is solved by the* scaling solution $G_\ell(g) = e^{-\ell\widetilde{\gamma}(g)}$ *of* (6.10)*, well-known from* [**18**].

Example 6.14. *The physical situation of vertex insertions as in* [**2**] *corresponds to $\kappa = 1$ and $G_{\ell+\ell'}(g) = G_{\ell'}(g) \cdot G_\ell\big[\widetilde{G}_{\ell'}(g)\big]$ can be interpreted as the running of the coupling constant $\widetilde{G} := g \cdot G$: A change in scale by ℓ' is (up to a multiplicative constant) equivalent to replacing the coupling g by $\widetilde{G}_{\ell'}(g)$.*

6.2. Running coupling. The idea of this last example 6.14 leads us to another form of the renormalization group equation, common to the physics literature like (7.3.15) and (7.3.21) in [**7**]. We introduce the *β-function*[10] $\beta(g) := -\kappa g\widetilde{\gamma}(g)$ and the *running coupling* $g(\mu)$ as the solution of

$$(6.12) \quad \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu) = \beta\big(g(\mu)\big), \quad \text{so} \quad \mu\frac{\mathrm{d}}{\mathrm{d}\mu}G\left(g(\mu), \ln\tfrac{s}{\mu}\right) \underset{(6.9)}{=} \widetilde{\gamma}\big(g(\mu)\big)G\left(g(\mu), \ln\tfrac{s}{\mu}\right).$$

Integration over μ results in a relation of the correlation functions corresponding to different choices of the renormalization point μ:

$$G\left(g(\mu_2), \ln\tfrac{s}{\mu_2}\right) = G\left(g(\mu_1), \ln\tfrac{s}{\mu_1}\right)\cdot\exp\left[\int_{\mu_1}^{\mu_2}\widetilde{\gamma}\big(g(\mu)\big)\tfrac{\mathrm{d}\mu}{\mu}\right] \underset{(6.12)}{=} G\left(g(\mu_1), \ln\tfrac{s}{\mu_1}\right)\cdot\left[\frac{g(\mu_2)}{g(\mu_1)}\right]^{-\frac{1}{\kappa}}.$$

This result is important from a conceptual point of view: To achieve renormalization, we introduced a parameter μ that is completely arbitrary, yet the shape of the correlation function is a measurable quantity wherefore it clearly has to be insensitive to the choice of μ.

Indeed we see that a change from μ_1 to μ_2 affects the correlation function only by a constant overall factor and a redefinition of the coupling constant (which itself is a parameter of the theory). Hence the physical content of G is left invariant of the choice of renormalization point.

[10] This should not be confused with the β-functional of (5.3), though both are related.

Similar to (6.10) we can express G through a differential equation involving the running coupling $g(s)$ after choosing $\mu_1 = s$:

$$G_\ell(g) = \left[\frac{g}{g(s)}\right]^{-\frac{1}{\kappa}}, \quad \text{with } g(s) \text{ subject to} \quad \ell = \ln\frac{s}{\mu} = \int_g^{g(s)} \frac{\mathrm{d}g'}{\beta(g')}. \tag{6.13}$$

6.3. Relation to Mellin transforms. So far we exploited the renormalization group equation (4.4) and the Hopf subalgebra (6.5) of perturbation series. Now we like to take the special structure (2.6) of the Feynman rules from (2.2) into account. As different cocycles[11] represent different Feynman integrals, we allow for a family $F_\cdot : \mathbb{N} \to z^{-1}\mathbb{K}[[z]]$ of Mellin transforms such that $\phi_{\mathrm{R}} \circ B_n = P \circ F_n(-\partial_\ell) \circ \phi_{\mathrm{R}}$. Applying this to (6.3) results in

$$G_\ell(g) \underset{(6.3)}{=} 1 + \sum_{n\in\mathbb{N}} g^n \phi_{\mathrm{R}} \circ B_n \left(X(g)^{1+n\kappa}\right) \underset{(2.6)}{=} 1 + P \circ \sum_{n\in\mathbb{N}} g^n F_n\left(-\partial_\ell\right) G_\ell(g)^{1+n\kappa}$$

and taking a derivative brings us to the differential equation of

Corollary 6.15. *The power series $G_\ell(g) \in \mathbb{K}[\ell][[g]]$ is fully determined by*

$$\partial_{-\ell} G_\ell\left(g\right) \underset{(2.6)}{=} \sum_{n\in\mathbb{N}} g^n \left[zF_n(z)\right]_{z=-\partial_\ell} \left(G_\ell(g)^{1+n\kappa}\right) \quad \textit{and} \quad G_\ell(0) = 1. \tag{6.14}$$

Though (6.14) is always well defined for the formal power series (since the growing powers of g allow only finitely many contributions in each order), the differential operator $[zF(z)]_{z=-\partial_\ell}$ can be of infinite order which might hinder a non-perturbative interpretation of this equation. However we can proceed in a couple of interesting cases, allowing us to construct the full correlation function or anomalous dimension (and not just individual terms of the perturbation series):

Example 6.16. *Consider a single cocycle $F_k(z) = F(z)\delta_{k,n}$ and choose $F(z) = \frac{c_{-1}}{z}$. Then (6.14) reproduces the leading-log example (6.11) as it becomes*

$$\partial_{-\ell} G_\ell(g) = g^n c_{-1} G_\ell(g)^{1+n\kappa}. \tag{6.15}$$

More generally, for rational $F(z) = \frac{p(z)}{q(z)} \in \mathbb{K}(z)$ with polynomials $p(z), q(z) \in \mathbb{K}[z]$ we can apply $q(-\partial_\ell)$ on both sides of (6.14) resulting in a finite order differential equation

$$q(-\partial_\ell) G_\ell(g) = g^n p(-\partial_\ell) G_\ell(g)^{1+n\kappa}. \tag{6.16}$$

Enjoying this situation we can directly interpret it non-perturbatively (extending the algebraic $\partial_\ell \in \mathrm{End}(\mathbb{K}[\ell])$ to the analytic differential operator).

Example 6.17. *The single Mellin transform $F(z) = \frac{1}{z(1-z)}$ in the propagator-type equation ($\kappa = -2$ as in example 6.6) occurs in the first approximation to quantum electrodynamics and the Yukawa theory. The equation*

$$\frac{g}{G_\ell(g)} = \partial_{-\ell}\left(1 - \partial_{-\ell}\right) G_\ell(g) \underset{(6.9)}{=} \widetilde{\gamma}(g)\left(1 - 2g\partial_g\right)\left[1 - \widetilde{\gamma}(g)\left(1 - 2g\partial_g\right)\right] G_\ell(g),$$

evaluates at $\ell = 0$ to the compact form

$$g = \widetilde{\gamma}(g) - \widetilde{\gamma}(g)(1 - 2g\partial_g)\widetilde{\gamma}(g). \tag{6.17}$$

[11] This is most easily considered in the Hopf algebra of decorated rooted trees (section A.2) where each B_n inserts into a node of decoration n.

We stress how this equation determines the anomalous dimension non-perturbatively, in fact it can be expressed in terms of the complementary error function as analyzed in [**26, 5**]. *Reference* [**24**] *is devoted to a detailed study of this type of equations and also solves the case $\kappa = 1$ with the help of the Lambert W function.*

6.4. Variations of Mellin transforms. Consider a change of the Mellin transform F to a different $\widetilde{F}$ that keeps c_{-1} fixed but is free to alter the other coefficients c_n with $n \in \mathbb{N}_0$. Then by (B.2) the difference

$$\delta\alpha := P \circ \left[\widetilde{F} - F\right]_{-\partial_\ell} \in \mathrm{HB}^1_\varepsilon(\mathbb{K}[\ell]) \tag{6.18}$$

is a Hochschild-1-coboundary and (A.6) shows that we can relate the two resulting renormalized Feynman rules ϕ_{R} and $\widetilde{\phi_{\mathrm{R}}}$ by composition with a distinguished Hopf algebra automorphism:

$$\widetilde{\phi_{\mathrm{R}}} \underset{(2.6)}{=} {}^{P\circ\widetilde{F}(-\partial_\ell)}\rho = {}^{P\circ F(-\partial_\ell)+\delta\alpha}\rho \underset{(\mathrm{A.6})}{=} {}^{P\circ F(-\partial_\ell)}\rho \circ {}^{\left[\alpha\,\circ\, {}^{P\circ F(-\partial_\ell)}\rho\right]}\chi \underset{(2.6)}{=} \phi_{\mathrm{R}} \circ {}^{\left[\alpha\,\circ\,\phi_{\mathrm{R}}\right]}\chi.$$

This accentuates that the Feynman rules coming from (2.2) obey even more structure than just the renormalization group (4.4): The origin from the Mellin transform poses restrictions on the generator γ of ϕ_{R}. For illustration consider how example 4.9 implies

$$\gamma(\bullet) \cdot \left[2\gamma\left(\begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix}\right) - \gamma\left(\underset{\bullet\ \bullet}{\bullet}\right)\right] = 2c_{-1}^2 c_0^2 = 2\left[\gamma\left(\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}\right)\right]^2. \tag{6.19}$$

Example 6.18. *Assume that $c_{-1} = -1$, then we can relates ϕ_R to the tree factorial character $\varphi = \int_0 \rho$: The difference in Mellin transforms is*

$$\delta\alpha = \circ \sum_{n\in\mathbb{N}_0} c_n \partial_{-x}^n \in HB^1_\varepsilon(\mathbb{K}[x]), \quad \textit{therefore} \quad \alpha \underset{(\mathrm{B.2})}{=} \varepsilon \circ \sum_{n\in\mathbb{N}_0} c_n \partial_{-x}^n \in \mathbb{K}[x]' \tag{6.20}$$

such that we find $\alpha \circ \varphi(w) = \alpha\left(\frac{x^{|w|}}{w!}\right) = (-1)^{|w|}\frac{|w|!}{w!}c_{|w|}$. Now we can verify

$$\phi_R(\bullet) = x = \varphi(\bullet) \underset{A.8}{=} \varphi \circ {}^{\alpha\circ\varphi}\chi(\bullet), \qquad \phi_R\left(\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}\right) = \frac{x^2}{2} + c_0 x = \varphi\left\{\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} + \eta(1)\bullet\right\} \underset{A.8}{=} \varphi \circ {}^{\alpha\circ\varphi}\chi\left(\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}\right),$$

$$\phi_R\left(\begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix}\right) = \frac{x^3}{6} + x^2 c_0 + x(c_0^2 - c_1) = \varphi\left\{\begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix} + 2c_0 \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix} + \left[c_0^2 - c_1\right]\bullet\right\} \underset{A.8}{=} \varphi \circ {}^{\alpha\circ\varphi}\chi\left(\begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix}\right) \quad \textit{and}$$

$$\phi_R\left(\underset{\bullet\ \bullet}{\bullet}\right) = \frac{x^3}{3} + c_0 \cdot x^2 - 2c_1 \cdot x = \varphi\left\{\underset{\bullet\ \bullet}{\bullet} + c_0 \bullet\bullet - 2c_1 \bullet\right\} \underset{A.8}{=} \varphi \circ {}^{\alpha\circ\varphi}\chi\left(\underset{\bullet\ \bullet}{\bullet}\right).$$

Having related different Feynman rules by such automorphisms, we may ask how the actual correlation functions are influenced by such a change. Here we can observe a kind of rigidity of the Dyson-Schwinger equation in

Corollary 6.19. *The new correlation function $\phi_R \circ X = \varphi \circ \widetilde{X}$ equals the original Feynman rules φ applied to a modified perturbation series $\widetilde{X}(g)$, fulfilling an equivalent Dyson-Schwinger equation that merely differs in the cocycles by coboundaries. By* (A.7) *the leading logs coincide and explicitly*

$$\widetilde{X}(g) := {}^{\alpha.\circ\varphi}\chi \circ X(g) = \mathbb{1} + \sum_{n\in\mathbb{N}} g^n \left(B_n + \delta\alpha_n\right)\left(\widetilde{X}(g)^{1+n\kappa}\right).$$

This follows directly through application of ${}^{\alpha\circ\varphi}\chi$ to the original Dyson-Schwinger equation. Since we consider now many cocycles B_n, each of which corresponding to a different Mellin transform F_n, also the functionals α_n are now indexed by n.

6.5. Minimal subtraction. In section 6.2 we already understood precisely why the arbitrary choice of the renormalization point μ in the kinematic subtraction scheme does not influence the physical interpretation. Now we can find an equivalent relation for the minimal subtraction scheme in

Corollary 6.20. *Applying* (6.6) *to* (5.6) *expresses the correlation function* G_{MS} *of the* R_{MS}*-scheme in terms of* G *in the kinematic subtraction scheme by a redefinition of the coupling constant and an overall factor:*

$$G_{\mathrm{MS},\ell}(g) = G_{\mathrm{MS},0}(g) \cdot G_\ell\Big(g \cdot \left[G_{\mathrm{MS},0}(g)\right]^{\kappa}\Big). \tag{6.21}$$

7. Extensions towards quantum field theory

7.1. Feynman graphs and Feynman integrals. In the formulation of perturbative quantum field theory, the Hopf algebra H_R of rooted trees is replaced by the Hopf algebra H_{FG} of Feynman graphs [**9**]. Most importantly, it features insertion operators that act like Hochschild-1-cocycles on the relevant subspace of H_{FG}. Under the Feynman rules, these result in (divergent) subintegrals just as in (2.1) and the coproduct of H_{FG} again mirrors the structure of subdivergences by definition.

The integrals are multi-dimensional and entail algebraic functions as integrands. For massive theories, a *Wick rotation* to Euclidean space-time disposes of all singularities in these integrands such that divergences only occur from the integrations at infinity. These may be renormalized by suitable subtractions again, though one has to face two new issues: Multiple parameters and higher degrees of divergence.

7.1.1. *Dimensional regularization.* Tailor-made for the application to Feynman integrals is the dimensional regularization [**7**], which introduces a regulator $z \neq 0$ in a small punctured neighbourhood of zero like in section 3 and assigns a Laurent series ${}_z\phi$ in z to every Feynman graph.

By dimensional analysis, one obtains a scale dependence of the form

$${}_z\phi = {}_z\phi|_{s=\tilde{s}} \circ \theta_{-z\ell}, \tag{7.1}$$

where we employ the grading Y by *loop number* of H and $\ell = \ln \frac{s}{\tilde{s}}$ encodes the ratio of rescaling of all dimensionfull parameters. Therefore the techniques of section 3 become available in quantum field theory and many of those results were formulated in this very setting in [**10**].

7.2. Multiple parameters. Correlation functions of quantum field theory typically depend on many variables, namely the masses of internal particles and the momenta of external particles. For illustration consider a logarithmic divergence with two parameters (s,t): We can still renormalize

(7.2)
$$\phi_{\mathrm{R},(s,t)} = \int_0^\infty \left[\frac{x\,\mathrm{d}x}{(x+s)(x+t)} - \frac{x\,\mathrm{d}x}{(x+\tilde{s})(x+\tilde{t})}\right] = \frac{s\ln s - t\ln t}{t-s} - \frac{\tilde{s}\ln\tilde{s} - \tilde{t}\ln\tilde{t}}{\tilde{t}-\tilde{s}}$$

by a single subtraction at a reference point $(\tilde{s},\tilde{t})$ in the parameter space. However, the function is no longer a plain logarithm in a single variable ℓ. In fact the dependence of correlation functions on the parameters becomes indeed extremely complicated and is only fully understood completely for the simplest (one-loop)

Feynman graphs or slightly better in special situations like massless or supersymmetric theories, with on-shell conditions or in space-time dimensions different from four.

Crucially though the fundamental properties of a theory are described by asymptotic behaviour, and the renormalization group still persists: If we rescale all parameters simultaneously by a factor e^{ℓ}, then (7.2) simplifies drastically to $\phi_{\mathrm{R},(\tilde{s}\cdot e^{\ell},\tilde{t}\cdot e^{\ell})} = -\ell$.

Analogously to the single scale case we considered, the presence of subdivergences requests additional subtractions generating richer dependence on ℓ, which nevertheless stays polynomial throughout.

Explicitly, encode all parameters as multiples of a distinguished *scale* s and dimensionless ratios $\theta \in \Theta$ called *angles* and choose a renormalization point $(\tilde{s}, \tilde{\Theta})$ for the subtraction scheme $R_{(\tilde{s},\tilde{\Theta})}$ evaluating $(s,\Theta) \mapsto (\tilde{s},\tilde{\Theta})$. Then we can state (proof is provided in [**19**] for Feynman integrals) the replacement for (4.4) as

Theorem 7.1. *The renormalized Feynman rules* $\phi_R = \phi_R|_{\Theta=\tilde{\Theta}} \star \phi_R|_{s=\tilde{s}}$ *factorize into the angle-dependent part* $\phi_R|_{s=\tilde{s}}$ *(independent of* ℓ *and only a function of* Θ *and* $\tilde{\Theta}$*) and the scale-dependence* $\phi_R|_{\Theta=\tilde{\Theta}}$*. The later depends only on* $\ell = \frac{s}{\tilde{s}}$ *(and the fixed renormalization point angles* $\tilde{\Theta}$*) and defines a morphism of Hopf algebras*

$$\phi_R|_{\Theta=\tilde{\Theta}} = \exp_{\star}(-\ell\mathcal{P}): \quad H \to \mathbb{K}[\ell]. \tag{7.3}$$

Its generator $\mathcal{P} \in \mathfrak{g}_{\mathbb{K}}^{H}$*, commonly called* period*, is given by* $\mathcal{P} := -\partial_{\ell}|_{\ell=0}\, \phi_R|_{\Theta=\tilde{\Theta}}$.

We also recommend [**6**] for a different decomposition and detailed analysis of the angle- and scale dependence.

7.3. Higher degrees of divergence. So far we restricted ourselves to logarithmic divergences only. Recall from (0.1) that in this case though ϕ itself diverges, the derivative $\frac{\partial}{\partial s}\phi$ is convergent (by differentiating the integrand we obtain an integrable form).

In general one defines the *superficial degree of divergence* sdd by simple power counting of the integrand f such that $f(\zeta) \in \mathcal{O}\left(\zeta^{\mathrm{sdd}-n}\right)$, where n counts the number of variables that we integrate over and the asymptotics is to be understood as all of these variables approaching ∞ jointly (for rational functions f, sdd is plainly the degree of the numerator minus the degree of the denominator, less the number of variables in the integral).

In this situation we find that any derivative $\frac{\partial^{\mathrm{sdd}+k}}{\partial s^{\mathrm{sdd}+k}}\phi$ for $k \in \mathbb{N}$ is convergent. Hence we can renormalize and keep these derivatives intact by subtracting a polynomial in $\mathbb{K}[s]_{\leq \mathrm{sdd}}$ of degree $\leq$ sdd. In the logarithmic case, this freedom is precisely a single constant we parametrized by μ so far.

This renormalization scheme is common practice in quantum field theory under the name BPHZ and the involved analytic estimates on the integrands necessary to prove the finiteness (as we did in theorem 2.4 in the simple setup of definition 2.1) have been worked out in [**25, 27**]. Variants exist for massless theories as well, while minimal subtraction (in connection with dimensional regularization) is particularly popular to handle gauge theories.

Subtractions of different polynomials in the external parameters are encoded into the Hopf algebra H_{FG} by adjunction of auxiliary marked vertices and specification of *external structures* [**9**].

We close by only briefly mentioning the increasing freedom in the Feynman rules coming along with growing degrees of divergence. If the divergences can attain arbitrarily high degrees, infinitely many subtraction terms are necessary in the renormalization process and thus generate as many constants (like our subtraction point μ) to be fixed. Such a theory is unfortunately called *unrenormalizable*: Though it *is* renormalizable, it loses any predictive power due to the infinity of unknown constants.

Contrary, *renormalizable* theories adhere to an upper bound on the degree of divergences that occur. Therefore only finitely many parameters (*renormalization conditions*) have to be fixed through measurements, whereafter all other processes may in principle be predicted.

The divergences of scalar field theory together with their renormalization according to BPHZ is for example elaborated on in [**6**], though most textbooks contain at least a basic account of this theme.

7.4. Systems of Dyson-Schwinger equations. So far we only considered a single Dyson-Schwinger equation in section 6, though quantum field theorys typically involve different types of fields (like fermions, photons or scalars) and also a variety of couplings (*vertices*). Each of those is represented by an (mostly) infinite series over Feynman graphs and these series may be inserted into each other in many ways.

These changes can be incorporated combinatorially by considering systems of Dyson-Schwinger equations as in [**12, 13**].

8. Summary

We reviewed renormalization of logarithmic ultraviolet divergences in the Hopf algebraic framework working with the rooted trees H_R. In the kinematic subtraction scheme we arrived at the same renormalized Feynman rules ϕ_{R} either by direct integration or with an analytic regulator being present.

After renormalization, the physical limit revealed a very special structure as being a Hopf algebra morphism $\phi_{\mathrm{R}}\colon H_R \to \mathbb{K}[x]$. This is the *renormalization group property* and reduces the full character down to the linear terms γ only.

The minimal subtraction scheme does not allow for such a simple description: Though we could obtain the scale dependence, the constant terms in this scheme are not as easily understood.

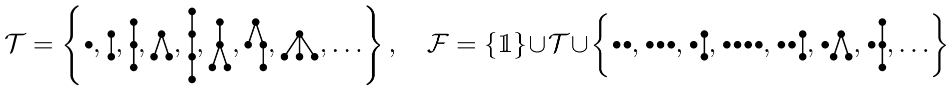
All along the case of Feynman rules that can be described by the Mellin transform is special in that it gives simple explicit recursions for the renormalization process.

In particular they make a non-perturbative investigations possible as we saw in section 6 studying the correlation functions, were we also understood the physical equivalence of different renormalization schemes.

Hochschild cohomology appeared ubiquitously in form of the universal property of rooted trees. governing most constructions we made. Its importance lies not only in the concept and power of Dyson-Schwinger equations alone, but also in the induced automorphisms of H_R which help to understand variations of Feynman rules from Mellin transforms.

Appendix A. The Hopf algebra of rooted trees

As an algebra, $H_R = S(\operatorname{lin}\mathcal{T}) = \mathbb{K}[\mathcal{T}]$ is free commutative[12] generated by the *rooted trees* $\mathcal{T}$ and spanned by their disjoint unions (products) called *rooted forests* $\mathcal{F}$:



Every $w \in \mathcal{F}$ is just the monomial $w = \prod_{t\in\pi_0(w)} t$ of its multiset of tree components $\pi_0(w)$, while $\mathbb{1}$ denotes the empty forest. The number $|w| := |V(w)|$ of nodes $V(w)$ induces the grading $H_{R,n} = \operatorname{lin}\mathcal{F}_n$ where $\mathcal{F}_n := \{w \in \mathcal{F}\colon\ |w| = n\}$.

Definition A.1. *The (linear)* grafting operator $B_+ \in \operatorname{End}(H_R)$ *attaches all trees of a forest to a new root, so for example* $B_+(\mathbb{1}) = \bullet$, $B_+(\bullet) =$ *and* $B_+(\bullet\bullet) =$.

B_+ is homogenous of degree one and restricts to a bijection $B_+\colon\ \mathcal{F} \to \mathcal{T}$. The coproduct Δ is defined to make B_+ a cocycle by requiring

$$\Delta \circ B_+ = B_+ \otimes \mathbb{1} + (\mathrm{id} \otimes B_+) \circ \Delta. \tag{A.1}$$

Lemma A.2. $B_+(\mathbb{1}) = \bullet \neq 0$ *implies that* $0 \neq [B_+] \in HH^1_\varepsilon(H_R)$ *is non-trivial.*

H_R is characterized through the well-known (theorem 2 of [**8**]) *universal property* of

Theorem A.3. *To an algebra* $\mathcal{A}$ *and* $L \in \operatorname{End}(\mathcal{A})$ *there exists a unique morphism* ${}^L\rho\colon H_R \to \mathcal{A}$ *of unital algebras such that*

$$ {}^L\rho \circ B_+ = L \circ {}^L\rho, \quad \textit{equivalently} \quad \begin{array}{ccc} H_R & \xrightarrow{{}^L\rho} & \mathcal{A} \\ {\scriptstyle B_+}\downarrow & & \downarrow{\scriptstyle L} \\ H_R & \xrightarrow[{}^L\rho]{} & \mathcal{A} \end{array} \quad \textit{commutes.} \tag{A.2}$$

In case of a bialgebra $\mathcal{A}$ *and a cocycle* $L \in HZ^1_\varepsilon(\mathcal{A})$, ${}^L\rho$ *is a morphism of bialgebras and even of Hopf algebras when* $\mathcal{A}$ *is Hopf.*

Note that ${}^L\rho = e$ trivializes if $L \in \mathrm{HB}^1_\varepsilon(\mathcal{A})$ is a coboundary.

Example A.4. *The action of* ${}^L\rho$ *is plainly replacement of* B_+ *by* L*:*

$$ {}^L\rho\left(- 3\bullet\right) = {}^L\rho\left\{B_+\left([B_+(\mathbb{1})]^2\right) - 3B_+(\mathbb{1})\right\} = L\left([L(\mathbb{1}_\mathcal{A})]^2\right) - 3L(\mathbb{1}_\mathcal{A}).$$

Example A.5. *The cocycle* $\int_0 \in HZ^1_\varepsilon(\mathbb{K}[x])$ *of appendix B induces the character*

$$\varphi := {}^{\int_0}\!\rho \in G^{H_R}_{\mathbb{K}[x]} \quad \textit{fulfilling} \quad \varphi(w) = \frac{x^{|w|}}{w!} \quad \textit{for any forest} \quad w \in \mathcal{F}, \quad \textit{using} \tag{A.3}$$

Definition A.6. *The* tree factorial $(\cdot)! \in G^{H_R}_{\mathbb{K}}$ *is determined by*

$$[B_+(w)]! = w! \cdot |B_+(w)|, \quad \textit{equivalently} \quad w! = \prod_{v\in V(w)} |w_v| \quad \textit{for all} \quad w \in \mathcal{F}. \tag{A.4}$$

[13]

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[12] We consider *unordered* trees and forests, sometimes called *non-planar*.

[13] By w_v we denote the subtree of w rooted at the node $v \in V(w)$.

A.1. Automorphisms of H_R. Applying the universal property to H_R itself, adding coboundaries to B_+ leads to

Definition A.7. *For any $\alpha \in H_R'$, theorem A.3 defines the Hopf algebra morphism*

$$ {}^{\alpha}\chi := {}^{B_+ + \delta\alpha}\rho\colon\ H_R \to H_R \quad \textit{such that} \quad {}^{\alpha}\chi \circ B_+ = [B_+ + \delta\alpha] \circ {}^{\alpha}\chi. \tag{A.5}$$

Example A.8. *The action on the simplest trees yields*

$$\begin{aligned}
{}^{\alpha}\chi\left(\bullet\right) &= {}^{\alpha}\chi \circ B_+(\mathbb{1}) = B_+(\mathbb{1}) + (\delta\alpha)(\mathbb{1}) = B_+(\mathbb{1}) = \bullet,\\
{}^{\alpha}\chi\left(\substack{\bullet\\ \bullet}\right) &= {}^{\alpha}\chi \circ B_+\left(\bullet\right) = (B_+ + \delta\alpha)\, {}^{\alpha}\chi\left(\bullet\right) = \substack{\bullet\\ \bullet} + \delta\alpha\left(\bullet\right) = \substack{\bullet\\ \bullet} + \alpha(\mathbb{1})\bullet,\\
{}^{\alpha}\chi\left(\substack{\bullet\\ \bullet\\ \bullet}\right) &= \substack{\bullet\\ \bullet\\ \bullet} + 2\alpha(\mathbb{1})\substack{\bullet\\ \bullet} + \left\{[\alpha(\mathbb{1})]^2 + \alpha\left(\bullet\right)\right\}\bullet \quad \textit{and} \quad {}^{\alpha}\chi\left(\overset{\bullet}{\bullet\,\bullet}\right) = \overset{\bullet}{\bullet\,\bullet} + 2\alpha\left(\bullet\right)\bullet + \alpha(\mathbb{1})\bullet\bullet.
\end{aligned}$$

These morphisms capture how ${}^{L}\rho$ reacts to variation of L by a coboundary in

Theorem A.9. *Let H denote a bialgebra, $L \in HZ^1_{\varepsilon}(H)$ a 1-cocycle and further $\alpha \in H'$ a functional. Then for ${}^{L}\rho, {}^{L+\delta\alpha}\rho\colon\ H_R \to H$ given through theorem A.3 and ${}^{\alpha\circ{}^{L}\rho}\chi\colon\ H_R \to H_R$ from definition A.7, we have*

$$ {}^{L+\delta\alpha}\rho = {}^{L}\rho \circ {}^{\left[\alpha\,\circ\,{}^{L}\rho\right]}\chi, \quad \textit{equivalently} \quad \begin{tikzcd} H_R \arrow[r, "{}^{L+\delta\alpha}\rho"] \arrow[d, "{}^{\alpha\circ{}^{L}\rho}\chi"'] & H \\ H_R \arrow[ur, "{}^{L}\rho"'] & \end{tikzcd} \quad \textit{commutes.} \tag{A.6}$$

Proof. As both sides of (A.6) are algebra morphisms, it suffices to prove it inductively for trees: Let it be true for a forest $w \in \mathcal{F}$, then it holds as well for the tree $B_+(w)$ by

$$\begin{aligned}
&{}^{L}\rho \circ {}^{\left[\alpha\circ{}^{L}\rho\right]}\chi \circ B_+(w) \underset{\text{(A.2)}}{=} {}^{L}\rho \circ \left[B_+ + \delta\left(\alpha \circ {}^{L}\rho\right)\right] \circ {}^{\left[\alpha\circ{}^{L}\rho\right]}\chi(w)\\
&= \left\{L \circ {}^{L}\rho + (\delta\alpha)\circ {}^{L}\rho\right\} \circ {}^{\left[\alpha\circ{}^{L}\rho\right]}\chi(w) = \{L + \delta\alpha\} \circ \underbrace{{}^{L}\rho \circ {}^{\left[\alpha\circ{}^{L}\rho\right]}\chi(w)}_{{}^{L+\delta\alpha}\rho(w)} \underset{\text{(A.2)}}{=} {}^{L+\delta\alpha}\rho \circ B_+(w).
\end{aligned}$$

We used $(\delta\alpha)\circ{}^{L}\rho = {}^{L}\rho\circ\delta\left(\alpha\circ{}^{L}\rho\right)$, following from ${}^{L}\rho$ being a morphism of bialgebras. $\square$

Theorem A.10. *The map ${}^{\cdot}\chi\colon\ H_R' \to \operatorname{End}_{Hopf}(H_R)$, taking values in the space of Hopf algebra endomorphisms of H_R, fulfils the following properties:*

(1) *For any $w \in \mathcal{F}$ and $\alpha \in H_R'$, ${}^{\alpha}\chi(w)$ differs from w only by lower order forests:*

$$ {}^{\alpha}\chi(w) \in w + H_R^{|w|-1} = w + \bigoplus_{n=0}^{|w|-1} H_{R,n}. \tag{A.7}$$

(2) *${}^{\cdot}\chi$ maps H_R' into the Hopf algebra automorphisms $\operatorname{Aut}_{Hopf}(H_R)$. Its image is closed under composition, as for any $\alpha, \beta \in H_R'$ we have ${}^{\alpha}\chi \circ {}^{\beta}\chi = {}^{\gamma}\chi$ taking*

$$ \gamma = \alpha + \beta \circ {}^{\alpha}\chi^{-1}. \tag{A.8}$$

(3) *The maps* $\delta\colon H'_R \to HZ^1_\varepsilon(H_R)$ *and* ${}^{\cdot}\chi\colon H'_R \to \mathrm{Aut}_{Hopf}(H_R)$ *are injective, thus the subgroup* $\mathrm{im}\,{}^{\cdot}\chi = \{{}^{\alpha}\chi\colon\ \alpha \in H'_R\} \subset \mathrm{Aut}_{Hopf}(H_R)$ *induces a group structure on* H'_R *with neutral element* 0 *and group law* $\triangleright$ *given by*

$$\alpha \triangleright \beta := {}^{\cdot}\chi^{-1}\left({}^{\alpha}\chi \circ {}^{\beta}\chi\right) \underset{\text{(A.8)}}{=} \alpha + \beta \circ {}^{\alpha}\chi^{-1} \quad \textit{and} \quad \alpha^{\triangleright -1} = -\alpha \circ {}^{\alpha}\chi. \tag{A.9}$$

Proof. Statement (A.7) is an immediate consequence of $\delta\alpha(H^n_R) \subseteq H^n_R$: Starting from ${}^{\alpha}\chi(\bullet) = \bullet$, suppose inductively (A.7) to hold for forests $w, w' \in \mathcal{F}$. Then it obviously also holds for $w \cdot w'$ as well and even so for $B_+(w)$ through

$${}^{\alpha}\chi \circ B_+(w) = [B_+ + \delta\alpha] \circ {}^{\alpha}\chi(w) \subseteq [B_+ + \delta\alpha]\left(w + H^{|w|-1}_R\right) \subseteq B_+(w) + H^{|w|}_R.$$

This already implies bijectivity of ${}^{\alpha}\chi$, but applying (A.6) to $L = B_+ + \delta\alpha$ and ${}^{\tilde\alpha}\chi$ for $\tilde\alpha := -\alpha \circ {}^{\alpha}\chi$ shows $\mathrm{id} = {}^{\alpha}\chi \circ {}^{\tilde\alpha}\chi$ directly. We deduce bijectivity of all ${}^{\alpha}\chi$ and thus ${}^{\alpha}\chi \in \mathrm{Aut}_{\mathrm{Hopf}}(H_R)$ with the inverse ${}^{\alpha}\chi^{-1} = {}^{\tilde\alpha}\chi$. Now (A.8) follows from

$${}^{\left[\alpha+\beta\,\circ\,{}^{\alpha}\chi^{-1}\right]}\chi = {}^{[B_+ + \delta\alpha] + \delta\left(\beta\,\circ\,{}^{\alpha}\chi^{-1}\right)}\rho \underset{\text{(A.6)}}{=} {}^{[B_+ + \delta\alpha]}\rho \circ {}^{\left[\beta\ \circ\ {}^{\alpha}\chi^{-1}\ \circ\ {}^{(B_+ + \delta\alpha)}\rho\right]}\chi = {}^{\alpha}\chi \circ {}^{\beta}\chi.$$

Finally consider $\alpha, \beta \in H'_R$ with ${}^{\alpha}\chi = {}^{\beta}\chi$, then $0 = ({}^{\alpha}\chi - {}^{\beta}\chi) \circ B_+ = \delta \circ (\alpha - \beta) \circ {}^{\alpha}\chi$ reduces the injectivity of ${}^{\cdot}\chi$ to that of δ. But if $\delta\alpha = 0$, for all $n \in \mathbb{N}_0$

$$0 = \delta\alpha\left(\bullet^{n+1}\right) = \sum_{i=0}^{n} \binom{n+1}{i} \alpha\left(\bullet^{i}\right) \bullet^{n+1-i} \quad \text{implies} \quad \alpha\left(\bullet^{n}\right) = 0.$$

Given an arbitrary forest $w \in \mathcal{F}$ and $n \in \mathbb{N}$, the expression

$$0 = \delta\alpha\left(\bullet^{n} w\right) = w \underbrace{\alpha\left(\bullet^{n}\right)}_{0} + \sum_{w} \sum_{i=0}^{n} \binom{n}{i} \bullet^{i} w' \alpha\left(\bullet^{n-i} w''\right) + \sum_{i=1}^{n} \binom{n}{i} \left[\bullet^{i} w \underbrace{\alpha\left(\bullet^{n-i}\right)}_{0} + \bullet^{i} \alpha\left(w \bullet^{n-i}\right)\right]$$

simplifies upon projection onto $\mathbb{K}\bullet$ to $\alpha\left(w\bullet^{n-1}\right) = -\frac{1}{n}\sum_{w\colon\ w'=\bullet} \alpha\left(\bullet^{n} w''\right)$. Iterating this formula exhibits $\alpha(w)$ as a scalar multiple of $\alpha\left(\bullet^{|w|}\right) = 0$ and proves $\alpha = 0$. □

A.2. Decorated rooted trees. Our observations generalize straight forwardly to the Hopf algebra $H_R(\mathcal{D})$ of rooted trees with decorations drawn from a set $\mathcal{D}$. In this case, the universal property assigns to each $\mathcal{D}$-indexed family $L_{\cdot}\colon\ \mathcal{D} \to \mathrm{End}(\mathcal{A})$ the unique algebra morphism

$${}^{L_{\cdot}}\rho\colon H_R(\mathcal{D}) \to \mathcal{A} \quad \text{such that} \quad {}^{L_{\cdot}}\rho \circ B^d_+ = L_d \circ {}^{L_{\cdot}}\rho \quad \text{for any } d \in \mathcal{D}.$$

For cocycles $\mathrm{im}\, L_{\cdot} \subseteq HZ^1_\varepsilon(\mathcal{A})$ this is a morphism of bialgebras and even of Hopf algebras (should $\mathcal{A}$ be Hopf). For a family $\alpha_{\cdot}\colon\ \mathcal{D} \to H'_R(\mathcal{D})$ of functionals, setting $L^{\alpha_{\cdot}}_d := B^d_+ + \delta\alpha_d$ yields an automorphism ${}^{\alpha_{\cdot}}\chi := {}^{L^{\alpha_{\cdot}}}\rho$ of the Hopf algebra $H_R(\mathcal{D})$. Theorems A.9 and A.10 generalize in the obvious way.

Appendix B. The Hopf algebra of polynomials

Lemma B.1. *Requiring* $\Delta(x) = x \otimes \mathbb{1} + \mathbb{1} \otimes x$ *induces a unique Hopf algebra structure on the polynomials* $\mathbb{K}[x]$*. It is graded by degree, connected, commutative and cocommutative with* $\Delta\left(x^n\right) = \sum_{i=0}^{n}\binom{n}{i} x^i \otimes x^{n-i}$ *and the primitive elements are* $\mathrm{Prim}\left(\mathbb{K}[x]\right) = \mathbb{K} \cdot x$.

The *integration operator* $\int_0 \colon x^n \mapsto \frac{1}{n+1}x^{n+1}$ is a cocycle $\int_0 \in \mathrm{HZ}^1_\varepsilon(\mathbb{K}[x])$ as

$$\begin{aligned}\Delta \textstyle\int_0 \left(\frac{x^n}{n!}\right) &= \Delta\left(\frac{x^{n+1}}{(n+1)!}\right) = \sum_{k=0}^{n+1} \frac{x^k}{k!} \otimes \frac{x^{n+1-k}}{(n+1-k)!} \\ &= \frac{x^{n+1}}{(n+1)!} \otimes \mathbb{1} + \sum_{k=0}^{n} \frac{x^k}{k!} \otimes \textstyle\int_0 \left(\frac{x^{n-k}}{(n-k)!}\right) = \left[\int_0 \otimes \mathbb{1} + \left(\mathrm{id} \otimes \int_0\right) \circ \Delta\right] \left(\frac{x^n}{n!}\right),\end{aligned}$$

and is not a coboundary since $\int_0 1 = x \neq 0$. In fact it generates the cohomology by

Theorem B.2. *$HH^1_\varepsilon(\mathbb{K}[x]) = \mathbb{K} \cdot [\int_0]$ is one-dimensional as the 1-cocycles are*

$$HZ^1_\varepsilon(\mathbb{K}[x]) = \mathbb{K} \cdot \textstyle\int_0 \,\oplus\, \delta\left(\mathbb{K}[x]'\right) = \mathbb{K} \cdot \int_0 \,\oplus\, HB^1_\varepsilon(\mathbb{K}[x]). \tag{B.1}$$

Proof. For an arbitrary cocycle $L \in \mathrm{HZ}^1_\varepsilon(\mathbb{K}[x])$, lemma 1.1 ensures $L(1) = xa_{-1}$ where $a_{-1} := \partial_0 L(1)$. Hence $\tilde{L} := L - a_{-1}\int_0 \in \mathrm{HZ}^1_\varepsilon$ fulfils $\tilde{L}(1) = 0$, so $L_0 := \tilde{L} \circ \int_0 \in \mathrm{HZ}^1_\varepsilon$ by

$$\Delta \circ L_0 = (\mathrm{id} \otimes \tilde{L}) \circ \Delta \circ \textstyle\int_0 + (\tilde{L} \otimes 1) \circ \int_0 = (\mathrm{id} \otimes L_0) \circ \Delta + L_0 \otimes 1 + \tilde{L}(1) \cdot \int_0 .$$

Repeating the argument inductively yields $a_n := \partial_0 L_n(1) = \partial_0 \circ L \circ \int_0^{n+1}(1) \in \mathbb{K}$ and $L_{n+1} := (L_n - a_n \int_0) \circ \int_0 \in \mathrm{HZ}^1_\varepsilon$, so for any $n \in \mathbb{N}_0$ we may read off from

$$L \circ \textstyle\int_0^n(1) = a_{-1}\int_0^{n+1}(1) + \ldots + a_{n-2}\int_0^2(1) + L_{n-1}(1) = a_{-1}\int_0\left(\int_0^n 1\right) + \sum_{j=0}^{n-1} a_j \int_0^{n-j}(1)$$

that indeed $L = a_{-1}\int_0 + \delta\alpha$ for the functional $\alpha := \partial_0 \circ L \circ \int_0$ with $\alpha(\frac{x^n}{n!}) = a_n$. ☐

Lemma B.3. *Up to subtraction $P = \delta\varepsilon = \mathrm{id} - \mathrm{ev}_0 \colon \mathbb{K}[x] \twoheadrightarrow \ker\varepsilon = x\mathbb{K}[x]$ of the constant part, direct computation exhibits $\delta\alpha$ as the differential operator*

$$\delta\alpha = P \circ \sum_{n \in \mathbb{N}_0} \alpha\left(\tfrac{x^n}{n!}\right) \partial^n \in \mathrm{End}(\mathbb{K}[x]) \quad \textit{for any} \quad \alpha \in \mathbb{K}[x]'. \tag{B.2}$$

Lemma B.4. *As any character $\phi \in G^{\mathbb{K}[x]}_{\mathbb{K}}$ of $\mathbb{K}[x]$ is fixed by $\lambda := \phi(x)$, they are the group $G^{\mathbb{K}[x]}_{\mathbb{K}} = \{\mathrm{ev}_\lambda \colon\ \lambda \in \mathbb{K}\}$ of evaluations (the counit $\varepsilon = \mathrm{ev}_0$ equals the neutral element)*

$$\mathbb{K}[x] \ni p(x) \mapsto \mathrm{ev}_\lambda(p) := p(\lambda) \quad \textit{with the product} \quad \mathrm{ev}_a \star \mathrm{ev}_b = \mathrm{ev}_{a+b}. \tag{B.3}$$

Proof. Note $[\mathrm{ev}_a \star \mathrm{ev}_b](x^n) = [\mathrm{ev}_a(1) \cdot \mathrm{ev}_b(x) + \mathrm{ev}_a(x) \cdot \mathrm{ev}_b(1)]^n = (b+a)^n$. ☐

Lemma B.5. *The isomorphism $(\mathbb{K}, +) \ni a \mapsto \mathrm{ev}_a \in G^{\mathbb{K}[x]}_{\mathbb{K}}$ of groups is generated by the functional $\partial_0 = \mathrm{ev}_0 \circ \partial \in \mathfrak{g}^{\mathbb{K}[x]}_{\mathbb{K}}$, meaning $\log_\star \mathrm{ev}_a = a\partial_0$ and $\mathrm{ev}_a = \exp_\star(a\partial_0)$.*

Proof. Expanding the exponential series reveals $\exp_\star(a\partial_0)(x^n) = a^n$ as a direct consequence of $\partial_0^{\star k} = \varepsilon \circ \partial^{\star k} = \varepsilon \circ \partial^k$:

$$\partial_0^{\star k}\left(\frac{x^n}{n!}\right) = \sum_{i_1+\ldots+i_k=n} \left(\partial_0 \tfrac{x^{i_1}}{i_1!}\right) \cdots \left(\partial_0 \tfrac{x^{i_k}}{i_k!}\right) = \sum_{i_1+\ldots+i_k=n} \delta_{1,i_1} \cdots \delta_{1,i_k} = \delta_{k,n} = \partial^k\big|_0 \left(\frac{x^n}{n!}\right). \quad ☐$$

Appendix C. The Dynkin operator $D = S \star Y$

Definition C.1. *For some fixed graduation* Y *of* H*, define operators* $D_Y := S \star Y$ *and* $\pi_Y := Y^{-1} \circ D_Y = D_Y \circ Y^{-1}$.

Note that each of $\{S, D_Y, \pi_Y\}$ commutes with Y and Y^{-1}.

Proposition C.2. $D_Y, \pi_Y \in \mathfrak{g}_H^H$ *are infinitessimal characters with* $\mathbb{K} \cdot \mathbb{1} \oplus (\ker \varepsilon)^2 = \ker D_Y = \ker \pi_Y$ *and* $D_Y - Y, \pi_Y - P$ *map into* $(\ker \varepsilon)^2$.

Proof. Clearly, $\mathbb{K} \cdot \mathbb{1} \oplus (\ker \varepsilon)^2 \subseteq \ker D_Y$ is an immediate consequence of

$$\begin{aligned} D_Y \circ m &= m \circ (S \otimes Y) \circ \Delta \circ m = m \circ (S \otimes Y) \circ (m \otimes m) \circ \tau_{2,3} \circ (\Delta \otimes \Delta) \\ &= m^3 \circ (S \otimes S \otimes Y \otimes \mathrm{id} + S \otimes S \otimes \mathrm{id} \otimes Y) \circ \tau_{1,2} \circ \tau_{2,3} \circ (\Delta \otimes \Delta) \\ &= m \circ [(S \star Y) \otimes (S \star \mathrm{id}) + (S \star \mathrm{id}) \otimes (S \star Y)] = D_Y \otimes \varepsilon + \varepsilon \otimes D_Y. \end{aligned}$$

The reverse inclusion follows from $D_Y(x) = Yx + \sum_x (Sx')(Yx'') = Yx \mod (\ker \varepsilon)^2$ for $x \in \ker \varepsilon$. □

Corollary C.3. $V_Y := \operatorname{im} D_Y = \operatorname{im} \pi_Y$ *generates* H *as an algebra and* $\operatorname{Prim}(H) \subseteq V_Y$.

Proof. $H_{n+1} \subseteq V_Y^{\cdot} := \sum_{n \geq 0} V_Y^n$ follows inductively from $H^n = \bigoplus_{i \leq n} H_i \subseteq V_Y^{\cdot}$ using $x \in \pi_Y(x) + m(H^n \otimes H^n)$ for any $x \in H_{n+1}$. A primitive p yields $D_Y(p) = S(p) \cdot 0 + S(\mathbb{1}) \cdot Y(p) = Y(p)$. □

Proposition C.4. $\pi_Y^2 = \pi_Y$ *is a projection, hence its image complements the square of the augmentation ideal:* $H = \mathbb{K} \cdot \mathbb{1} \oplus V_Y \oplus (\ker \varepsilon)^2, \ker \varepsilon = V_Y \oplus (\ker \varepsilon)^2$.

Proof. Expand $D_Y^2 = m \circ (S \otimes Y) \circ (m \otimes m) \circ \tau_{2,3} \circ (\Delta \otimes \Delta) \circ (S \otimes Y) \circ \Delta$ to

$$\begin{aligned} D_Y^2 &= m^3 \circ [S \otimes S \otimes Y \otimes \mathrm{id} + S \otimes S \otimes \mathrm{id} \otimes Y] \circ \tau_{1,2} \circ \tau_{2,3} \circ (\Delta \otimes \Delta) \circ (S \otimes Y) \circ \Delta \\ &= m \circ [(S \star Y) \otimes e + e \otimes (S \star Y)] \circ (S \otimes Y) \circ \Delta = D_Y \circ Y \end{aligned}$$ □

Proposition C.5. *From* $\Delta \circ D_Y = \mathbb{1} \otimes D_Y + [D_Y \otimes m \circ (S \otimes \mathrm{id})] \circ \tau_{1,2} \circ \Delta^2$ *we deduce that* $\mathbb{K} \cdot \mathbb{1} \oplus V_Y$ *is a right-coideal. Further,* π_Y *and* D_Y *map co-commutative elements to primitives as then* $\Delta \circ \pi_Y = \mathbb{1} \otimes \pi_Y + \pi_Y \otimes \mathbb{1}$.

Proof. Apply $S \star \mathrm{id} = e = \mathbb{1} \cdot \varepsilon$ and $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$ to

$$\begin{aligned} \Delta \circ D_Y &= (m \otimes m) \circ \tau_{2,3} \circ \tau_{1,2} \circ [S \otimes S \otimes Y \otimes \mathrm{id} + S \otimes S \otimes \mathrm{id} \otimes Y] \circ \Delta^3 \\ &= \Big\{ (S \star Y) \otimes [m \circ (S \otimes \mathrm{id})] \Big\} \circ \tau_{1,2} \circ \Delta^2 + \Big\{ (S \star \mathrm{id}) \otimes [m \circ (S \otimes Y)] \Big\} \circ \tau_{1,2} \circ \Delta^2. \end{aligned}$$ □

Corollary C.6. *For cocommutative* H, $\exp_\star(\pi_Y) \in G_H^H$ *is a character that coincides with* id *on the generating subspace* $\operatorname{im}(\pi_Y) = \operatorname{Prim}(H)$*, hence*

$$\exp_\star(\pi_Y) = \mathrm{id}, \qquad \textit{equivalently} \qquad \log_\star(\mathrm{id}) = \pi_Y. \tag{C.1}$$

In particular note that in this case $\pi_Y = \log_\star(\mathrm{id})$ does not depend on the choice of grading Y. Recalling that by the Milnor-Moore theorem for this case $H = S(\operatorname{Prim}(H))$ is just the symmetric algebra, π_Y is nothing but the projection on $\operatorname{Prim}(H)$ corresponding to

$$H = \bigoplus_{n \geq 0} \operatorname{Prim}(H)^{\otimes n}. \tag{C.2}$$

But also in the non-cocommutative case we have

Proposition C.7. *V_Y generates H as a free algebra: $S(V_Y) = H$ (as algebras).*

Proof. The inclusion $V_Y \hookrightarrow H$ induces a unique morphism $\nu : S(V_Y) \twoheadrightarrow H$ of algebras which is surjective by C.3. For $n, m \in \mathbb{N}_0$ and $v_1, \ldots, v_m \in H$,

$$(*) \qquad \pi_Y^{\star n}(v_1 \cdots v_m) = \sum_{i_1 + \ldots + i_m = n} \binom{n}{i_1 \cdots i_m} \pi_Y^{\star i_1}(v_1) \cdots \pi_Y^{\star i_m}(v_m)$$

results from iteration of $\pi_Y \circ m = m \circ (e \otimes \pi_Y + \pi_Y \otimes e)$ and proves

$$\pi_Y^{\star n}(V_Y^m) = 0 \quad \text{for any} \quad 0 \leq n < m$$

as in $(*)$ some i_k must vanish and $\pi_Y^{\star 0} = e$ annihilates $V_Y \subset \ker \varepsilon$. In the case $n = m$ we find $\pi_Y^{\star n}|_{V_Y^n} = n! \cdot \mathrm{id}|_{V_Y^n}$ by $i_1 = \ldots = i_n = 1$. Therefore a finite sum $0 = \sum_{n \geq 0} x_n \in H$ with $x_n \in V_Y^n$ implies $x_0 = \frac{1}{0!}\pi_Y^{\star 0}(x) = 0$, then $x_1 = \frac{1}{1!}\pi_Y^{\star 1}(x) = 0$ and hence iteratively $x_n = 0$ for any n.

Thus $H = \bigoplus_{n \geq 0} V_Y^n$ is a direct sum with $V_Y^n = \nu(S_n(V_Y))$ upon the decomposition $S(V_Y) = \bigoplus_{n \geq 0} S_n(V_Y)$ into the homogeneous polynomials $S_n(V_Y)$ of degree n. Since $\pi_Y^{\otimes n} \circ \Delta^{n-1}(v_1 \cdots v_n) = \sum_\sigma v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$ for $v_1, \ldots, v_n \in V_Y$ delivers an inverse to $\nu|_{S_n(V_Y)}$, ν is injective. □

Hence all commutative connected graded Hopf algebras are free, and $V_Y \supseteq \mathrm{Prim}(H)$ delivers a generating coideal. Note that rooted trees are usually defined as $S(\mathrm{lin}\,\mathcal{T})$, which algebraically is a bad choice of generators as the trees $\mathcal{T}$ develop elaborate coproducts and do only contain the single primitive $\bullet$.

V_Y is a much more special generator as it contains all primitives and hence roughly speaking many elements with simple coproducts. Especially in the cocommutative case V_Y delivers the simplest possible generator $V_Y = \mathrm{Prim}(H)$ and $S(V_Y) = H$ becomes an isomorphism of Hopf algebras.

Corollary C.8. *The map $\tilde{R} : G_{\mathcal{A}}^H \to \mathfrak{g}_{\mathcal{A}}^H, \varphi \mapsto \varphi^{\star -1} \star (\varphi \circ Y) = \varphi \circ D_Y$ defined in* [**21**] *is a bijection.*

Theorem C.9 (*scattering formula* from [**10**] in the form of [**21**])**.** *The inverse of $\tilde{R}$ is given by $\tilde{R}^{-1}(\beta) = \lim_{t\to\infty} e^{-tZ_0} e^{t(Z_0 + \beta)}$ for any $\beta \in \mathfrak{g}_{\mathcal{A}}^H$. This equation is to be understood in the Lie group associated to the semidirect sum $\mathfrak{g}_{\mathcal{A}}^H \rtimes \mathbb{C} \cdot Z_0$, adjoining the derivation $Z_0(\beta) := \beta \circ Y$.*

Another description of $\tilde{R}^{-1}$ can be given as follows: Denote by $\Psi \in \mathrm{End}\,(\mathrm{End}(H))$ the map $\mathrm{End}(H) \ni \alpha \mapsto \Psi(\alpha) := (\alpha \star D_Y) \circ Y^{-1}$ and exploit $\mathrm{id} = e + Y \circ Y^{-1}$ as well as $Y = \mathrm{id} \star S \star Y = \mathrm{id} \star D_Y$ repeatedly, then

$$\text{(C.3)} \qquad \mathrm{id} = e + (\mathrm{id} \star D_Y) \circ Y^{-1} = e + \left\{ \left[e + (\mathrm{id} \otimes D_Y) \circ Y^{-1} \right] \star D_Y \right\} \circ Y^{-1}$$

$$\text{(C.4)} \qquad = \cdots = \sum_{n \geq 0} \Psi^n(e).$$

Observe that $\Psi^n(e)$ vanishes on any element $x \in H$ of coradical degree less than n, wherefore this series is pointwise finite. Hence given $\beta := \tilde{R}(\varphi)$ we can reconstruct

$$\text{(C.5)} \qquad \varphi = \varphi \circ \mathrm{id} = e + \beta \circ Y^{-1} + \left[(\beta \circ Y^{-1}) \star \beta \right] \circ Y^{-1}$$

$$\text{(C.6)} \qquad + \left(\left\{ \left[(\beta \circ Y^{-1}) \star \beta \right] \circ Y^{-1} \right\} \star \beta \right) \circ Y^{-1} + \ldots.$$

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