The signed permutation group on Feynman graphs

Humboldt-Universität zu Berlin
Mathematisch-Naturwissenschaftliche Fakultät I
Institut für Physik

Masterarbeit
zur Erlangung des akademischen Grades
Master of Science (M.Sc.)
im Fach Physik

eingereicht von: Julian Purkart
geboren am: 28. April 1990 in Berlin

1. Gutachter/in: Prof. Dr. Dirk Kreimer
2. Gutachter/in: Dr. Christian Bogner

eingereicht am: 22.09.2015

Das Verständnis der bunten Mannigfaltigkeit der Erscheinungen soll also dadurch zustande kommen, daß wir in ihr einheitliche Formprinzipien erkennen, die in der Sprache der Mathematik ausgedrückt werden können. Damit wird auch ein enger Zusammenhang zwischen dem Verständlichen und dem Schönen hergestellt. Denn wenn das Schöne als Übereinstimmung der Teile untereinander und mit dem Ganzen erkannt wird und wenn andererseits alles Verständnis erst durch diesen formalen Zusammenhang zustande kommen kann, so wird das Erlebnis des Schönen fast identisch mit dem Erlebnis des verstandenen oder wenigstens geahnten Zusammenhangs.

Werner Heisenberg. „Das Schöne in der exakten Naturwissenschaft“.

Abstract

The present thesis deals with the L-linear term of the renormalized Feynman rules and examines if there are special combinations of Feynman graphs such that the angle-dependence drops out. After giving a short introduction to quantum field theory some graph theoretical definitions are provided preliminarily and the Hopf algebra of rooted trees is discussed. In order to set the stage for the parametric representation of Feynman integrals and their renormalization, the Symanzik polynomials are introduced, which are directly connected to the combinatorics of the graph. Using the forest formula and rescaling the parametric integral indicates that the renormalized Feynman rules can be written as a polynomial in the scaling parameter L. Subsequently, the linear term of the renormalized Feynman rules is considered for total antisymmetric permutations of nested graph insertions with the result that it is independent of the scattering angles. The thesis is finalized by the derivation of a general formula that allows to compute the L-linear term of the renormalized Feynman rules for the aforementioned combination of graphs regardless of the number of graphs inserted into each other.

Zusammenfassung

In der vorliegenden Arbeit wird der L-lineare Term der renormierten Feynman-Regeln betrachtet und es wird untersucht, ob spezielle Kombinationen von Feynman-Graphen zu einem Wegfallen der Winkelabhängigkeit führen. Nach einer kurzen Einführung in die Quantenfeldtheorie werden präliminar einige graphtheoretische Definitionen gegeben und es wird die Hopf-Algebra von Wurzelbäumen besprochen. Um den Weg für die parametrische Darstellung von Feynman-Integralen und deren Renormierung zu ebnen, werden zudem die mit der Kombinatorik des Graphen verknüpften Symanzik Polynome eingeführt. Durch Reskalierung des parametrischen Integrals und unter Verwendung der Waldformel zeigt sich, dass die renormierten Feynman-Regeln als Polynom in dem Skalierungsparameter L geschrieben werden können. Anschließend wird der lineare Term der renormierten Feynman-Regeln für total antisymmetrische Permutationen von sukzessiv ineinander eingesetzten Feynman-Graphen betrachtet, mit dem Ergebnis, dass dieser unabhängig von den Streuwinkeln ist. Den Abschluss dieser Arbeit bildet die Herleitung einer allgemeinen Formel, die es erlaubt, den L-linearen Term der renormierten Feynman-Regeln für die gefundene Kombination von Graphen zu berechnen, unabhängig von deren Anzahl.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The idea that any kind of matter is composed of a finite construction kit of indivisible elements is about 2500 years old and was first mentioned by philosophers like Empedokles, Demokrit, and Platon in the fifth and fourth century before Christ. Since the late 19th century, this conjecture has been reinforced by several experiments as for instance the discovery of the electron by J. J. Thomson or the scattering experiments performed by E. Rutherford.
In the mid-seventies physicists developed the so-called standard model (SM) of particle physics, based on the experimental data primarily obtained from scattering experiments. This model describes the weak, strong, and electromagnetic interaction between all elementary particles known to us. Several attempts to incorporate gravity into the standard model as well failed until now. Therefore, an extension or modification of the model is expected.
Nevertheless, the model yields remarkable results in a number of experiments and i.a. predicted the existence of some fundamental particles before they were observed. Moreover, this theory provides a good service on the computation of cross sections which can be measured in scattering experiments with high accuracy nowadays. The probably most popular particle accelerator performing such experiments is the Large Hadron Collider (LHC) at CERN in Geneva.
The increasing precision of such experiments requires equally increasing accuracy in the theoretical calculation and prediction of their outcome. The mathematical framework behind the standard model which is used to perform these calculations is quantum field theory (QFT) or more precisely perturbative quantum field theory (pQFT). The latter can be used to compute scattering amplitudes for particle collisions as a perturbation series in which the contributory elements are given by so-called Feynman integrals. In most cases, solving these integrals is anything but trivial not least because one has to deal with divergences commonly occurring in the integrals. In order to extract physically interesting, that is finite results from divergent Feynman integrals, one has to renormalize them. In the case of scalar quantum field theory the occurring integrals in momentum-space can be rewritten as integrals over positive real parameters. [1] The polynomials occurring in the integrand of such integrals are directly connected to the combinatorics of the graph that visualizes the underlying scattering process. These graphs associated to the respective Feynman integrals are called Feynman graphs.
A novel and illuminating formulation of the intricate problem of renormalization has been presented by Dirk Kreimer and collaborators and is based on the Hopf algebraic

[1] Indeed, it is also possible to extend the parametric representation of the integrals from scalar to gauge theories as it was shown in [20] for quantum electrodynamics and Yang-Mills-theories.

structure of Feynman graphs.

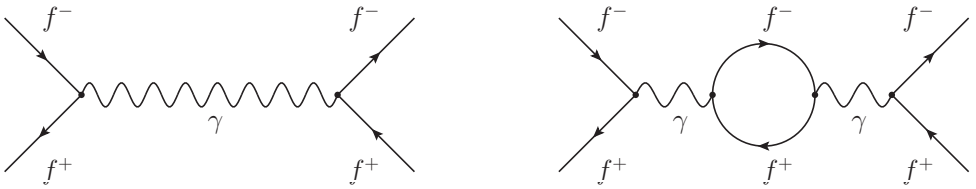
The aim of this thesis is to determine specific combinations of graphs in a scalar quantum field theory that lead to a remarkable simplification of the first non-trivial term in the perturbation series. It will be seen that the result is independent of our renormalization scheme to the effect that it is invariant under the choice of the reference point. To achieve that goal we will utilize the parametric representation of scalar Feynman integrals as well as the Hopf algebraic structure of the Feynman graphs under consideration. Moreover, we will present a formula which reduces the effort of computing the first-order term in the perturbation series for the specific combination of graphs to a minimum.

1.1 Background

1.1.1 Quantum field theory and Feynman integrals

In the following we want to give a brief overview on quantum field theory and Feynman integrals. The latter should not be confused with path integrals which also go back to Richard Feynman and can be used to give an alternative formulation on QFT besides the canonical quantization of the fields.
There are different approaches to introduce the topic of quantum field theory, and we will choose a more physics-based pictorial concept rather than a strict mathematical derivation. At this point we want to state that the focus of this section is to give a general idea of what QFT is about. For a detailed introduction of QFT the reader is referred to one of the well-established books on this topic like [9] and [18].
To start with, we think of two particles A and B brought to collision in a particle accelerator. In the aftermath of the interaction process in the region of collision, two particles C and D are measured by the detector, where the sum of energies before and after the collision is conserved. Now we think of a point in time long before the collision ($t \to -\infty$), where the particles A and B are far apart from each other, and we can disregard the interaction between them. All information on the incoming particles we have at this time should be encoded by the initial state vector $|i\rangle$ in Hilbert space. Similarly, we assume that the final state vector $\langle f|$ provides all information about the outgoing particles C and D at a subsequent point in time ($t \to \infty$) at which the interaction between them can be neglected. In order to calculate the probability for the transition $A + B \to C + D$, we have to collide beams of particles A and B respectively and count how often the result $C + D$ occurs. Physically, this quantity is given by the so-called cross section. If we want to investigate the cross section theoretically, we are facing the problem that the only information we have is about the initial and final state of the collision. It is impossible to make a clear statement about events within the region of interaction. As a consequence, we have to consider all possible processes allowed by the underlying quantum field theory. Particularly, we do not only have to consider real particles, that is actually measurable particles appearing in the initial or final state of the collision, but also virtual particles emerging only within the interaction region. The reason for this lies in the fact that the transition from $A+B$ to $C+D$ could theoretically take place over an infinite number of intermediate states, caused by the creation and annihilation of virtual particles that do not show up in the final state. Assuming that this process is described by the transition operator H, we have to calculate the scalar product $\langle f|H|i\rangle$ in order to determine the cross section of the reaction. However,

supposing that the amount of interaction energy is small compared to the kinetic energy of the colliding particles allows us to expand the transition operator H as a power series in the coupling g, with g scaling the interaction energy in the process under consideration. In the case $0 < g \ll 1$ it is therefore sufficient to consider only a finite number of processes which means that we can truncate the power series at finite order. Pictorially, these processes are illustrated by Feynman graphs which can be translated into Feynman integrals in accordance with the Feynman rules. Those Feynman graphs consist of edges and vertices, with the edges corresponding to propagating particles whose interaction is described by the vertices. The different types of particles are represented by several kinds of edges, e.g. in quantum electrodynamics (QED) straight lines are used to represent fermions and wiggly lines to represent photons. Figure 1.1 shows the annihilation and creation of a fermion - anti-fermion pair. In 1.1(a) a virtual photon is created and annihilated, and in 1.1(b) a virtual photon decays into a virtual fermion - anti-fermion pair which is annihilated, again generating a virtual photon which in turn decays into the final state particles. The arrows on the edges



(a) Second order Feynman diagram for the process $f^- + f^+ \to f^- + f^-$.

(b) Fourth order Feynman diagram for the process $f^- + f^+ \to f^- + f^-$.

Figure 1.1: Feynman diagrams for the annihilation and creation of a fermion - anti-fermion pair in QED.

are representing the negative charge flow. Thereby, a reversed arrow corresponds to a positive charge and therefore an antiparticle. Since each vertex in QED contributes a factor g, the order of the diagram is determined by its number of vertices. Hence, the diagram in figure 1.1(a) is of order g^2 and that in figure 1.1(b) of order g^4.

1.1.2 Divergences and renormalization

Before we introduce some of the basic definitions and mathematical tools applied throughout this thesis, we first want to say a few more words about divergences and renormalization of Feynman integrals in general. As mentioned previously, it is possible to translate Feynman graphs into Feynman integrals. To do so, we need a set of translation regulations, the so-called Feynman rules which can be understood from every common textbook on QFT as [9] and [18] and will not be the subject of this thesis.

In compliance with Heisenberg's uncertainty principle the space-time point of a particle and its momentum cannot be determined precisely at once. Therefore, we have to decide on which of the quantities our attention is focused. Throughout this thesis we will work in momentum space which is the most prevalent choice. The integrals arising from the Feynman rules (in momentum space) are of the sort

$$I\left(\{p_e\},\{m_e\}\right) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^4 k_i}{(2\pi)^4} N\left(\{k_i, p_e, m_e\}\right) \prod_{j=1}^{E_{int}} \frac{1}{P_j} \tag{1.1.1}$$

where L is the loop-number, k_i the momentum of the i-th loop and the p_e are the external momenta. The numerator N is a function of the loop momenta as well as the masses and momenta of the external particles and the inverse propagator is defined by $P_j = q_j^2 - m_j^2$, with m_j the mass and q_j the momentum of the internal particle or edge labeled by j. The q_j are linear combinations of the external momenta p_e and the loop momenta k_i. Since the k_i correspond to virtual particles that do not show up in the initial or final state we have to integrate over all possible values. The problem arising from the Feynman rules is that the resulting integrals are not ensured to be convergent and well-defined. Plenty of them are divergent, for which we have to distinguish two kinds of divergences:

- The whole integrand including $\prod_{i=1}^{l} \mathrm{d}^4 k_i$ diverges if one or more $|k_i| \to \infty$. In this case, I is ultraviolet (UV) divergent.
- One or more of the inverse propagators P_j have zeros in the region of integration, i.e. the integrand has singularities. In this case, the loop momenta k_i are considered in the complex plain and we try to deform the integration path around the singularity. If this is not possible, our integral cracks and/or is divergent. In this particular case we say that the integral is infrared (IR) divergent.

In addition to the aforementioned divergences there are further IR divergent contributions that are revealed in the computation of transition amplitudes. The origin of these divergences are phase space integrations over terms that correspond to the radiation of massless particles like photons and gluons. These terms become divergent in the region of small momenta [2]. Nevertheless, it becomes apparent that, if we choose a suitable definition of the observables, the upper divergent contributions are exactly canceled by the infrared divergences arising from loop integrals as stated by the Kinoshita-Lee-Nauenberg- or KLN-theorem. Note that this statement is not true in general because there are quantum field theories not satisfying the theorem. However, it is true for QED as well as for the whole standard model that it is infrared finite.
In the following we will only focus on UV divergences. For each Feynman integral we can define a superficial degree of ultraviolet divergence (see equation (2.1.2)) that enables us to draw conclusions about the high energy behavior of the integral from it. [3]
Indeed, it is also possible to define a superficial degree of infrared divergence that differs from the one in equation (2.1.2). If the integral features UV divergences, it is, however, possible to extract finite results if the underlying theory is renormalizable. The act of applying two different methods to the integral, describes the regular procedure to deal with UV divergences, namely regularization and renormalization.
Regularization is based on the idea of considering a regularized integral $I_R(\varepsilon)$ instead of the divergent Feynman integral I. The new integral is a function of the regularization parameter ε, where the dependence is defined such that $\lim_{\varepsilon \to 0} I_R(\varepsilon) = I$ and that there exists a region of ε-values where $I_R(\varepsilon)$ is finite. Then, the result can be expanded as a Laurent series $I_R(\varepsilon) = \sum_i c_i \varepsilon^i$ in $\varepsilon = 0$.
One possible regularization strategy is:

- Introduce a cut-off parameter Λ for UV divergences $\int_{-\infty}^{\infty} \mathrm{d}^4 k \longrightarrow \int_{|k| \leq \Lambda} \mathrm{d}^4 k$ and, after computing the integral, take $\lim_{\Lambda \to \infty}$.

[2] Particles which carry small momentum are often referred to as soft particles.
[3] The idea is based on Dyson's power counting theorem which was first proven by Weinberg for the case of scalar Feynman integrals with euclidean metric.

- For IR divergences we introduce a small formal mass m_ε such that $\frac{1}{k^2} \longrightarrow \frac{1}{k^2-m_\varepsilon^2}$ and at the end take the limit $\lim_{\varepsilon\to 0}$.

The disadvantage of this method is that we need different regularization parameters and aside from that, the cut-off parameter Λ destroys the Lorentz invariance with regard to the loop momenta. Instead, we commonly use dimensional regularization, redefining the space-time by $D = 4 - 2\varepsilon$ with $\varepsilon \in \mathbb{C}$ and $D \to 4$ for $\varepsilon \to 0$. Afterwards, we evaluate the regularized integral under the assumption that D or rather ε lies in a region where the integral converges. The result $I_R(\varepsilon)$ again can be expanded as a Laurent series in ε. Instead of introducing new parameters, the subject of renormalization is redefining some of the pre-existing parameters like the coupling or the mass of the particles. This can be done in such a way that the divergences arising from the regularization are canceled or absorbed. The redefinition of the parameters leads to a new Lagrangian of the underlying theory, whose deviation from the old one manifests itself in so-called counter terms. Nevertheless, it is important to emphasize that the redefinition of the parameters has to be done for each order of the perturbation series.

Chapter 2

Preliminaries

As mentioned in the previous section, the coefficients in the perturbative expansion of the correlation function (or Green's function) are integrals, which can be interpreted as physical processes. Graphically, these processes can be represented via Feynman diagrams, which are the central objects of perturbative quantum field theory. To treat them in an adequate manner, it will be necessary to get familiar with some fundamental aspects and definitions of graph theory. Afterwards, we introduce polynomials associated with the respective graphs, called the first and second Symanzik polynomial. In section 3.1, these polynomials will also show up in the integrand of the Feynman integral when going from momentum to parametric space, using the so-called Schwinger trick. At the end of this chapter we establish an algebraic structure on the set of Feynman graphs and thereby give a brief insight in the Hopf algebra of rooted trees.
The definitions in section 2.1 basically follow [14], [22], and [23]. The first one, together with [3], was also the principal source of section 2.2.

2.1 Graph theoretical foundations

In this section, we want to acquaint ourselves with some basic definitions of graph theory and a bit of vocabulary needed when talking about Feynman graphs and graphs in general. Therefore, we first have to define:

Definition 2.1. (Graphs)
A graph $G = (E, V, \Phi)$ consists of a set of edges E, a set of vertices V, and a map Φ (incidence relation) from edges to pairs of vertices.

- *An edge $e \in E$ is said to be incident to $v, w \in V$ if $\Phi(e) = \{v, w\}$. The vertices v, w are called the endpoints of e.*
- *Two vertices $v, w \in V$ are adjacent if $\exists e \in E$ such that $\Phi(e) = \{v, w\}$.*
- *The valence of a vertex is given by the number of edges incident to it.*
- *An edge with equal endpoints is called a loop.*
- *A path (v, w) of length k from v to w is given by a subset $E_P = \{e_1, \dots, e_k\} \subseteq E$ such that any two edges e_i, e_{i+1} have one endpoint in common. If $v = w$, the path is called a cycle.*
- *If there exists a path (v, w) for any pair $v, w \in V$, the graph is called connected. A connected graph without loops is said to be simply connected.*

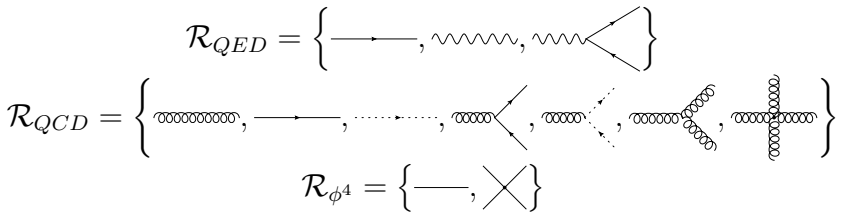


Figure 2.1: Sets of allowed vertices and edges for QED, QCD, and ϕ^4.

In addition to the graph theoretical definition given above, there are some features that come up when treating Feynman graphs instead of standard graphs. In general, the edges and vertices of a Feynman graph are labeled, that is assigning information of physical interest to them like the momentum and mass of the particle represented by the edge. Furthermore, Feynman graphs are constructed from a particular set of edges and vertices we will denote by $\mathcal{R} = \mathcal{R}_E \cup \mathcal{R}_V$, following [12] and [22]. While $\mathcal{R}_E$ corresponds to the type of quantum particles, $\mathcal{R}_V$ determines the type of interaction between those particles, respectively. Generally, the sets $\mathcal{R}_E$ and $\mathcal{R}_V$ are dictated and restricted by the quantum field theory we are looking at. In some theories the edges also get an orientation, corresponding to the charge flow of the particles[4]. In figure 2.1 the sets of vertices and edges are given for quantum electrodynamics (QED), quantum chromodynamics (QCD), and ϕ^4-theory [5] in $D = 4$ dimensions of space-time.
In the following we will denote such Feynman graphs by Γ with vertex set $\Gamma^{[0]}$ and edge set $\Gamma^{[1]}$. In contrast to standard graph theory, we have to distinguish between internal and external edges. An edge is called internal if it connects two vertices whereas an external edge connects only to one vertex, that is to say it has only one endpoint. The set of edges then is given by the union $\Gamma^{[1]} = \Gamma^{[1]}_{\text{int}} \cup \Gamma^{[1]}_{\text{ext}}$.

Definition 2.2. (Feynman graphs)
A Feynman graph $\Gamma = (G, \text{res})$ *is given by a graph* G *and a map* res

$$\text{res} : \Gamma^{[0]} \cup \Gamma^{[1]} \to \mathcal{R}_V \cup \mathcal{R}_E \tag{2.1.1}$$

which assigns to each vertex and edge in Γ *an element from a set of allowed types of edges and vertices. The elements* $r \in \mathcal{R}$ *are called the allowed residues of the theory. For any connected Feynman graph* Γ *we let* $\text{res}(\Gamma)$ *be the graph* Γ *when all its internal edges shrink to one point. Then,* $\text{res}(\Gamma)$ *is just the residue of the graph, defining its external structure.*

The allowed residues of a theory form the set of building blocks such that each Feynman graph of the theory can be built up out of it. In literature, the terms Feynman graph/diagram and graph/diagram are often used interchangeably and so will we do in the following. To which kind of graphs we refer to will always be clear from the context. Moreover, we will only consider a special kind of graphs, called one-particle irreducible graphs, throughout this thesis.

[4] Clearly, only edges corresponding to charged particles (like fermions) are oriented and others are not.
[5] ϕ^k-theories are scalar field theories treating only one kind of particles with spin zero represented by the one-component scalar field ϕ. Those particles self-interact in groups of k which means that all vertices are k-valent.

graph	power counting weight ω
	1
	2
	0
	2
	1
	0
	0
	−1
	0
	2
	0

Table 2.1: The power counting weights ω for all vertex and edge types in QED, QCD, and ϕ^4-theory.

Definition 2.3. (One-particle irreducible graphs)
A connected Feynman graph Γ is said to be one-particle irreducible (1PI) if it is still connected after removing one of its internal edges. Depending on the number of external edges, there are several kinds of 1PI graphs:

- *If Γ has no external edges, it is called a vacuum graph or vacuum bubble.*
- *For $|\Gamma^{[1]}_{ext}| = 1$ the graph is called tadpole.*
- *If $|\Gamma^{[1]}_{ext}| = 2$, we call Γ a propagator or self-energy graph.*
- *All other graphs with $|\Gamma^{[1]}_{ext}| \geq 3$ are said to be vertex graphs.*

As we already know, Feynman rules can be used to translate a given graph into a Feynman integral. If the resulting integral converges, it can be solved and we are done. Otherwise, we have to renormalize it. Since we are only interested in ultraviolet divergent graphs, it would be convenient to investigate a method, how to deduce the degree of divergence of the integral directly from the corresponding graph. Therefore, we introduce the weight of a certain vertex or edge of Γ, given by the map $\omega : \mathcal{R}_E \cup \mathcal{R}_V \to \mathbb{Z}$, which assigns an integer to every element in $\mathcal{R}$. The weight's magnitude corresponds to the negative power of the momentum, the Feynman rules associate to every single $r \in \mathcal{R}$. An overview of the power counting weights for the different edge and vertex types in QED, QCD, and ϕ^4-theory (cf. figure 2.1) is given in table 2.1.

Assuming that the weights of all constituents of Γ are known, we can define a weight for the whole graph by

$$\omega_D(\Gamma) \coloneqq \sum_{e \in \Gamma^{[1]}_{\text{int}}} \omega(e) + \sum_{v \in \Gamma^{[0]}} \omega(v) - D \cdot L(\Gamma), \tag{2.1.2}$$

with D the dimension of spacetime and $L(\Gamma)$ the number of independent loops of the graph defined in equation (2.2.1). The weight $\omega_D(\Gamma)$ is also called the superficial degree of (ultraviolet) divergence. For $\omega_D(\Gamma) = 0, -1, -2, \ldots$ the graph is said to be (superficially) logarithmic, linear, quadratic, ... divergent. If $\omega_D(\Gamma) > 0$, the

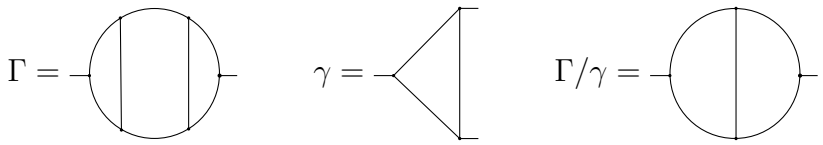


Figure 2.2: Example for a subgraph γ and the corresponding cograph Γ/γ of the three-loop graph Γ.

integral associated to the graph is (superficial) ultraviolet convergent. At this point it is important to emphasize that the convergence is only superficial. Although, a graph is superficially convergent, it is actually possible that the graph contains one or more subgraphs, being divergent.

Definition 2.4. (Sub- and cographs)
A graph $\gamma \subseteq \Gamma$ is called subgraph of Γ if $\gamma^{[0]} \subseteq \Gamma^{[0]}$, $\gamma^{[1]} \subseteq \Gamma^{[1]}$, and the assignment of endpoints to edges in γ and Γ is the same. In the case that γ contains all vertices of Γ, i.e. $\gamma^{[0]} = \Gamma^{[0]}$, γ is said to be a spanning subgraph of Γ.
The cograph Γ/γ is obtained from Γ by shrinking all internal edges of γ in Γ to length zero, i.e. to a single point, such that the external leg structure is not affected, $\mathrm{res}\,(\Gamma/\gamma) = \mathrm{res}\,(\Gamma)$. *The operation "/", which can be used to reverse graph insertions, is called contraction. Using this notion, the map* res *acting on a connected graph Γ, can be seen as the maximal contraction Γ/Γ.*

An example for a sub- and cograph is given in figure 2.2. However, the notion of the superficial degree of divergence is very useful when studying renormalization theory, e.g. it can be used to verify if a theory is renormalizable or not [6].
To get ahead we will need two other types of graphs in the following.

Definition 2.5. (Tree)
A connected and simply connected (no cycles) graph is called a tree T with vertex set $T^{[0]}$ and edge set $T^{[1]}$.

- *A rooted tree is a tree T with a distinguished vertex $r \in T^{[0]}$, which is called the root, such that all edges are oriented away from it.*
- *The weight $|T|$ of a tree is given by its number of vertices.*
- *Let $\mathcal{T}_r$ be the set of all rooted trees and $\mathcal{T}_r^{(i)}$ the subset of all rooted trees with weight $|T| = i$, $\forall\, T \in \mathcal{T}_r^{(i)}$, then we can write $\mathcal{T}_r = \bigcup_i \mathcal{T}_r^{(i)}$.*
- *A rooted tree is said to be decorated if there exists a finite set D of decorations and a surjective map $c : D \to T^{[0]}$, which assigns to each vertex $v \in T^{[0]}$ an element $d \in D$.*

[6] A theory is renormalizable in D dimensions if $\omega_D\,(\Gamma) = \omega_D\,(\mathrm{res}\,(\Gamma))$. This statement is equivalent to the requirement that the dimension of the coupling parameter in a renormalizable theory is lower or equal zero.

Definition 2.6. (Forest [7])
Let Γ be a Feynman graph and $f \coloneqq \{\gamma_i\}$ a subset of divergent 1PI proper subgraphs $\gamma_i \subsetneq \Gamma$ such that for any $\gamma, \gamma' \in f$ one of the following conditions is fulfilled:

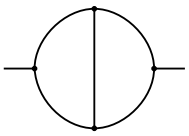
$$\textit{(i)}\ \gamma \subset \gamma', \qquad \textit{(ii)}\ \gamma' \subset \gamma\ , \quad \textit{or} \qquad \textit{(iii)}\ \gamma \cap \gamma' = \emptyset. \tag{2.1.3}$$

That is, the elements of f are either disjoint or contained in each other. Then, f is called a forest and $\mathcal{F}(\Gamma)$ denotes the set of all forests of the graph.

- *A forest f of a Feynman graph Γ is said to be maximal if and only if the cograph $\Gamma/f = \Gamma/\cup_{\gamma\in f}\gamma$ is a 1PI graph, not containing any divergent proper 1PI subgraphs. Such graphs are called primitive.*
- *A maximal forest f of Γ is complete if any $\gamma \in f$ is either primitive or there exists a proper subgraph $\gamma' \in f$ of γ such that the cograph γ/γ' is primitive.*
- *If f consists of k connected components, it is called a k-forest. A 1-forest is a tree.*
- *The union of rooted trees gives a rooted forest and its set is denoted by $\mathcal{F}_r$.*

Remark 2.1. *Hereafter, we will often restrict ourselves to trees and forests which are spanning subgraphs of the considered graph. In this case, we call them spanning trees and spanning forests respectively[8]. It is also important not to confuse spanning and rooted trees and forests. While the sets of spanning trees and forests of a graph will be used to define the graph polynomials in section 2.2, the sets of rooted trees and forests do not correspond to a specific graph even though one or more elements of $\mathcal{T}_r$ can be associated to a graph, representing its subgraph structure, as we will see. Moreover, we will set up a Hopf algebra structure on the set of rooted trees in section 2.3.*

As an example, we consider the two-loop graph



(2.1.4)

whose spanning trees and spanning 2-forests are given in figure 2.3 and 2.4 respectively.
Assume that $f = \{\gamma, \gamma'\}$ is a complete forest of Γ with primitive elements γ/γ' and γ'. Then, we can write f as a sequence of subsets

$$\gamma' \subsetneq \gamma \subsetneq \Gamma \tag{2.1.5}$$

to show how the graph and the subgraphs are nested. Using the notion of forests and trees, we can associate a decorated rooted tree to each complete forest of a graph Γ.

[7] It should be pointed out that there are two different definitions of the notion of a forest. In the present case we define the forest (of subdivergences) in the context of renormalization and Hopf algebra. This definition is also in accordance with the forest formula introduced in section 3.2. Within the framework of graph polynomials (cf. section 2.2) the forest (or k-forest) is defined as a graph without cycles/loops consisting of k connected components. That is, a k-forest is given by the disjoint union of k trees. For example, the forest set in figure 2.4 follows this definition.

[8] Note that in literature, especially when talking about graph polynomials, the terms "tree" and "forest" are used as they would imply that the respective subgraph is spanning. This is not true and we will always prefix the supplement "spanning" if we assume that the graph and the subgraph share the same set of vertices.

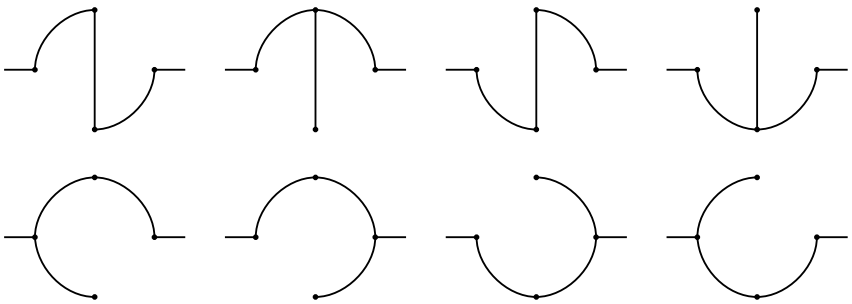
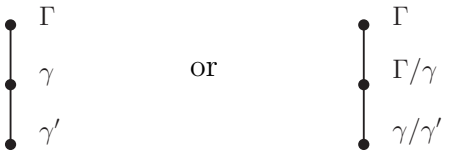


Figure 2.3: Set of spanning trees $\mathcal{F}_s^{(1)}$ for the two-loop graph in (2.1.4).

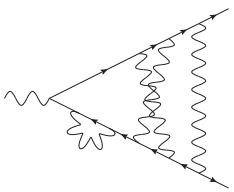
Taking the complete forest in equation (2.1.5), the corresponding decorated rooted tree is given by



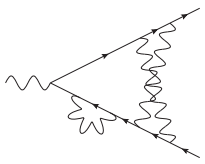
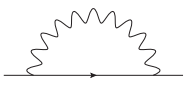
(2.1.6)

It becomes apparent that each Feynman diagram Γ furnishes a tree whose decorations are the elements of the complete forest. The rooted tree of a graph can also be read off from the box system as one can see in the example above, in which each box contains a divergent subgraph of the graph and corresponds to a leaf of the tree. The root is given by the whole graph (the outermost box). Like the elements in the complete forest, the boxes are not allowed to overlap, but rather are nested or disjoint.

Example 2.1.
We consider the 4*-loop vertex graph*

$\Gamma =$  (2.1.7)

from QED. The diagram shows the decay of a photon into a fermion - anti-fermion pair. Recalling the definition of the superficial degree of UV divergence from equation (2.1.2) and the weights of the vertices and edges listed in table 2.1, we can identify two UV divergent subgraphs, namely

$\gamma_1 =$  *and* $\gamma_2 =$  (2.1.8)

with superficial degree of divergence

$$\omega_4(\gamma_1) = 0 \quad \textit{and} \quad \omega_4(\gamma_2) = -1. \tag{2.1.9}$$

Thus, γ_1 *is logarithmic and* γ_2 *linear UV divergent. The rooted tree of the graph* Γ*, representing its subgraph structure, can now be obtained by taking the whole graph and*

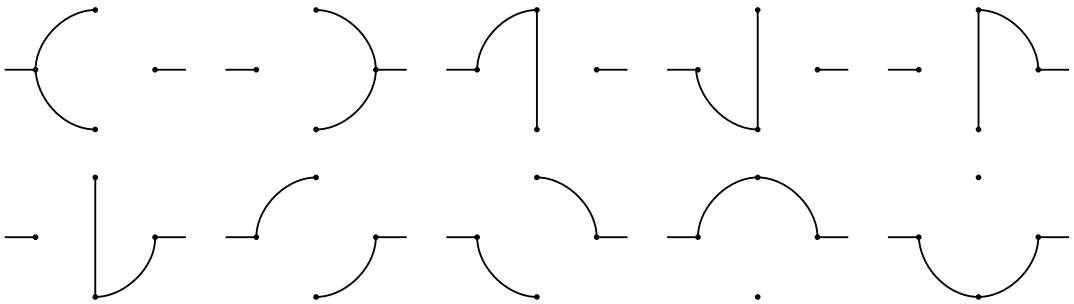
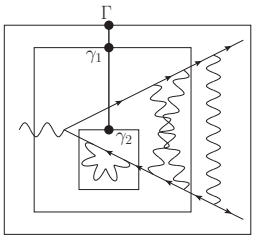


Figure 2.4: Set of spanning 2-forests $\mathcal{F}_s^{(2)}$ for the two-loop graph in (2.1.4).

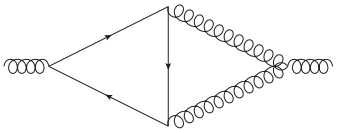
drawing a box around each divergent subgraph. Also, we have to draw a box around the entire graph since Γ *is logarithmic UV divergent as well. Afterwards, we draw a vertex at the top of each box labeled by the (sub-)graph and connect these vertices by edges, starting at the outermost box and draw edges from the outer boxes to all of their inner boxes at the same level. In the present case this procedure yields*



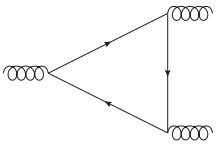
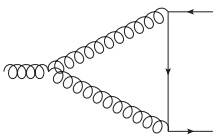
and hence

(2.1.10)

is the rooted tree associated to Γ*, representing its subgraph structure.*
In contrast to the foregoing example, there can be more than one rooted tree associated to the graph wherefore the choice of the tree is not unique anymore. For example, this can be seen by looking at the 2*-loop gluon self-energy graph*

$\tilde{\Gamma} =$  (2.1.11)

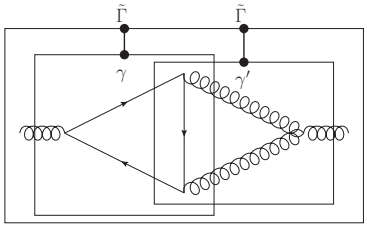
from QCD. The divergent subgraphs are

$\gamma =$  *and* $\gamma' =$  (2.1.12)

with power counting weights

$$\omega_4(\gamma) = -1 \quad \textit{and} \quad \omega_4(\gamma') = 0. \tag{2.1.13}$$

The graph as a whole is quadratic UV divergent. As before, we draw boxes around the subgraphs and connect the boxes by edges, yielding



. (2.1.14)

The problem arising from this graph is that we have overlapping sub-divergences. Therefore, we have to choose one of the rooted trees. Nevertheless, for theories we are interested in it can be shown that for $D = 4$ dimensions of space-time every complete forest has the same number of elements, i.e. every rooted tree has the same structure and number of vertices wherefore the choice of the tree does not matter. This is not invariably the case. For example in ϕ_3^6-theory this statement fails.

2.2 Feynman graph polynomials

In the last section we presented some fundamentals of graph theory which will be needed to introduce the Feynman graph polynomials. These polynomials, known as the first and second Symanzik polynomial, have many special properties and can be read off directly from the corresponding graph. Also, they play a crucial role in the computation of Feynman loop integrals since they are directly related to the integrand of such integrals. From the variety of methods to determine the graph polynomials [3], we will only consider one by interpreting the polynomials in terms of spanning trees and spanning forests. Likewise, it is also possible to compute them with the aid of matrices, associated to the graph. This approach suits well when performing computer algebra since, after the particular matrices are known, the only thing left to do is computing the determinant of a matrix. The basic principle of this approach is the matrix-tree theorem, invented by Gustav Kirchhoff, which exhibits the possibility to compute the number of a graph's spanning trees as the determinant of a matrix derived from the graph. In addition to his contributions to the fundamental understanding of electric circuits and spectroscopy, Kirchhoff was also the one who invented the notion of graph polynomials.
Throughout this thesis let Γ be a connected graph with $E_\Gamma := \left|\Gamma^{[1]}_{\text{int}}\right|$ internal edges, $V_\Gamma := \left|\Gamma^{[0]}\right|$ vertices, and loop number $L(\Gamma)$ defined by

$$L(\Gamma) = E_\Gamma - V_\Gamma + 1. \tag{2.2.1}$$

This number is also called the first Betti number or the cyclomatic number of the graph. For disconnected graphs we have to replace 1 by k, with k the number of connected components of the graph [9] .
Furthermore, let $\mathcal{F}_s^{(k)}$ be the set of all spanning k-forests (see definition 2.6) and $\mathcal{F}_s$ be the set of all spanning forests of the graph Γ, given by

$$\mathcal{F}_s = \bigcup_k \mathcal{F}_s^{(k)}. \tag{2.2.2}$$

Then $f \in \mathcal{F}_s^{(k)}$ can be obtained from Γ by deleting $L + k - 1$ of its internal edges. The elements of a spanning k-forest are composed of the connected components T_i of $\mathcal{F}_s^{(k)}$, which are necessary trees, and will be denoted by

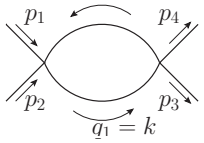
$$\bigcup_{i=1}^{k} T_i = (T_1, T_2, \ldots, T_k) \in \mathcal{F}_s^{(k)}. \tag{2.2.3}$$

From now on we will only consider scalar Feynman graphs in D dimensions of spacetime. The edges of the graph are associated with particles of mass m_e. Following the

[9] Note that, not only here but also in literature, the variable k also denotes the connected components of a forest. However, this will not cause any confusion since we are only dealing with connected graphs. Thus, in the following k always refers to the components of a k-forest.

conventions, the momenta of the particles will be denoted by p_e for $e \in \Gamma^{[1]}_{\text{ext}}$ and q_e for $e \in \Gamma^{[1]}_{\text{int}}$. We impose momentum conservation at each vertex, i.e. the sum of the momenta flowing into the vertex equals the sum of all outgoing momenta. Particularly, with regard to the external momenta it follows that $\sum_{e\in\Gamma^{[1]}_{\text{ext}}} p_e = 0$ if the momenta are taken to flow outwards. Therefore, the internal momenta q_e of a tree graph are completely determined by the external momenta p_e. To determine the internal momenta of a loop graph uniquely we have to add $L(\Gamma)$ internal momenta k_l, with l labeling the independent loops of the graph Γ. Since these loop-momenta cannot be observed physically we have to integrate over all possible values, and therefore the amplitude does not depend on the k_l after all. For example, the momenta of the one-loop graph of ϕ^4-theory can be labeled as follows

$$k-(p_1+p_2)=q_2=k-(p_3+p_4)$$



. (2.2.4)

We will now come back to the topic of this section and define the two Symanzik polynomials.

Definition 2.7. (First Symanzik polynomial)
Let $\mathcal{F}^{(1)}_s$ be the set of all spanning trees of Γ, such that $T \in \mathcal{F}^{(1)}_s$ is obtained from Γ by deleting L of its internal edges. We introduce parameters $\alpha_e \in \mathbb{R}_+$ associated to the internal edges $e \in \Gamma^{[1]}_{int}$ of the graph Γ. Then, the first Symanzik polynomial is defined by

$$\psi_\Gamma = \sum_{T\in\mathcal{F}^{(1)}_s} \prod_{e\notin T^{[1]}} \alpha_e \tag{2.2.5}$$

where the sum is over all spanning trees of Γ and $T^{[1]}$ denotes the edge set of T.

The parameters α_e are the so-called Schwinger parameters, which will also show up in the parametric representation of Feynman integrals in section 3.1.

Remark 2.2. *There exists another graph polynomial which is closely linked to the first Symanzik polynomial, called the the Kirchhoff polynomial, defined by*

$$\mathcal{K}_\Gamma = \sum_{T\in\mathcal{F}^{(1)}_s} \prod_{e\in T^{[1]}} \alpha_e. \tag{2.2.6}$$

This definition is quite similar to the previous one in equation (2.2.5) since we only have to replace the product of the edges which do not belong to the tree by the product of the tree's edges. Thus, there is a simple relation between the two graph polynomials

$$\psi_\Gamma(\{\alpha_e\}) = \prod_{e\in\Gamma^{[1]}_{int}} \alpha_e \cdot \mathcal{K}_\Gamma\left(\left\{\alpha_e^{-1}\right\}\right). \tag{2.2.7}$$

Although the Kirchhoff polynomial will not be considered anymore throughout this thesis, it plays an important role, for example, when going from a graph to its dual. To the reader interested in this topic, we strongly recommend [3] for a review.

Definition 2.8. (Second Symanzik polynomial)
Let $\mathcal{F}_s^{(2)}$ be the set of all spanning 2-forests of Γ, such that $(T_1, T_2) \in \mathcal{F}_s^{(2)}$ is obtained from Γ by deleting $L+1$ of its internal edges. The mass of the particles associated with the edges of the graph Γ will be denoted by m_e. Then, the second Symanzik polynomial is defined by

$$\phi_\Gamma = \varphi_\Gamma + \psi_\Gamma \sum_{e \in \Gamma_{int}^{[1]}} \alpha_e m_e^2 \tag{2.2.8}$$

with

$$\varphi_\Gamma = - \sum_{(T_1,T_2) \in \mathcal{F}_s^{(2)}} Q(T_1) \cdot Q(T_2) \prod_{e \notin T_1^{[1]} \cup T_2^{[1]}} \alpha_e. \tag{2.2.9}$$

The sum is over all spanning 2-forests of the graph Γ, and $Q(T_i)$ denotes the sum of all euclidean momenta flowing into the tree T_i.

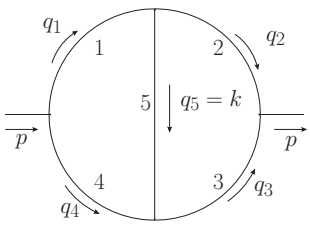
By momentum conservation it is clear that the sum of all incoming momenta of T_i and the sum of all momenta flowing outwards only differ in the sign. Therefore, the product $Q(T_1) \cdot Q(T_2)$ is equal to minus the square of the sum of the momenta flowing through the cut lines from one tree to the other. Obviously, it is generally valid that $Q(T_1) = -Q(T_2)$ and hence $Q(T_1) \cdot Q(T_2) < 0$.
Having defined the Symanzik polynomials, we can collect some of their elementary properties:

- The dependence on masses and external momenta is solely given by ϕ_Γ whereas ψ_Γ is independent of physical quantities.
- Both Symanzik polynomials are homogeneous in the Schwinger parameters. The degree of ψ_Γ is L and that of ϕ_Γ is $L+1$.
- ψ_Γ and φ_Γ are linear in every single α_e. ϕ_Γ is at most quadratic in the Schwinger parameters (if $m_e \neq 0$).
- For a product of graphs $\Gamma = \prod_i \gamma_i$ the polynomials ψ_Γ and φ_Γ can be decomposed as follows:

$$\psi_\Gamma = \prod_i \psi_{\gamma_i} \qquad \text{and} \qquad \varphi_\Gamma = \sum_i \varphi_{\gamma_i} \prod_{\substack{j \\ j \neq i}} \psi_{\gamma_j}. \tag{2.2.10}$$

Note that in literature the graph polynomials are also denoted as $\mathcal{U} = \psi_\Gamma$ and $\mathcal{F} = \phi_\Gamma$.
We will close this review by giving an example.

Example 2.2.
Consider the graph



(2.2.11)

we already mentioned in the last section. The edges are labeled as in the diagram above and carry momentum q_e and mass m_e. We impose momentum conservation at each vertex, and the momenta are assumed to flow from left to the right. The first Symanzik polynomial (2.2.5) with respect to the set of spanning trees of the graph (figure 2.3) is given by

$$\psi_{\text{◫}} = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3) + \alpha_5(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4). \tag{2.2.12}$$

The single terms in the upper sum are composed of the parameters α_e corresponding to the edges we have to delete to get one of the spanning trees of the graph. Obviously, there are two possible ways to construct a spanning tree out of the graph: Either we delete one edge on the left (1 or 4) and on the right (2 or 3) side, respectively, or we delete the fifth and any of the other edges. This fact is also reflected in the structure of $\psi_{\text{◫}}$ as one can see. The second Symanzik polynomial (2.2.8) is based on the spanning 2-forests of the graph given in figure 2.4. There are ten spanning 2-forests, though only eight of them contribute to $\phi_{\text{◫}}$. The last two of them do not show up in the polynomial since the sum of the momenta flowing from one tree to the other is zero. Thus,

$$\begin{aligned}\phi_{\text{◫}} = &[(\alpha_1 + \alpha_4)\alpha_2\alpha_3 + (\alpha_2 + \alpha_3)\alpha_1\alpha_4 \\ &+(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)\alpha_5]\, s + \psi_{\text{◫}} \sum_{e=1}^{5} \alpha_e m_e^2\end{aligned} \tag{2.2.13}$$

where we used $q_1 + q_4 = -(q_2 + q_3)$ (momentum conservation), and $s = (q_1 + q_4)^2 = (q_2 + q_3)^2$ denotes the center of mass energy. [10]

2.3 The Hopf algebra of rooted trees

The aim of this section is to establish an algebra on the set of Feynman graphs. It was discovered in [13] that the fundamental mathematical structure on which perturbative renormalization is based on is the Hopf algebra.
At first, we want to give some basic definitions and afterwards introduce the Hopf algebra of rooted trees, following [1], [7], [10], and [16]. What we aim at with this section is to give a brief overview of this topic and not a full mathematical description.
Let $\mathbb{K}$ be a field of characteristic zero, V_1 and V_2 two vector spaces, and $\tau_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ the flip map that interchanges the elements in a tensor product $\tau(v_1 \otimes v_2) = v_2 \otimes v_1$.

Definition 2.9. (Algebra)
An associative $\mathbb{K}$-algebra (A, m) is a $\mathbb{K}$-vector space A together with a linear map $m : A \otimes A \to A$, called product, such that

$$m \circ (\mathrm{id} \otimes m) = m \circ (m \otimes \mathrm{id}). \tag{2.3.1}$$

If there exists a linear map $\mathbb{I} : \mathbb{K} \to A$ fulfilling

$$m \circ (\mathrm{id} \otimes \mathbb{I}) = \mathrm{id} = m \circ (\mathbb{I} \otimes \mathrm{id}), \tag{2.3.2}$$

the algebra $(A, m, \mathbb{I})$ is said to be unital, and $\mathbb{I}$ is called the unit map.
For $m \circ \tau = m$ the algebra is commutative.

[10] This notation is very common when treating scattering processes in particle physics. s is one of the Mandelstam variables. The other two are $u = (q_1 - q_2)^2 = (q_3 - q_4)^2$ and $t = (q_1 - q_3)^2 = (q_2 - q_4)^2$.

The conditions (2.3.1) and (2.3.2) are the same as demanding that the diagrams

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{m\otimes \mathrm{id}} & A\otimes A \\ {\scriptstyle \mathrm{id}\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A\otimes A & \xrightarrow{\quad m\quad} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} \mathbb{K}\otimes A & \xrightarrow{\mathbb{I}\otimes \mathrm{id}} & A\otimes A & \xleftarrow{\mathrm{id}\otimes \mathbb{I}} & A\otimes \mathbb{K} \\ & \searrow^{\cong} & \downarrow{\scriptstyle m} & \swarrow_{\cong} & \\ & & A & & \end{array} \tag{2.3.3}$$

commute. By reversing the arrows of the diagrams, one can derive objects which are somehow dual to algebras, namely coalgebras.

Definition 2.10. (Coalgebra)
A coassociative $\mathbb{K}$-coalgebra (C, Δ) consists of a $\mathbb{K}$-vector space C and a linear map $\Delta : C \to C \otimes C$, called coproduct, such that coassociativity is fulfilled

$$(\mathrm{id}\otimes\Delta)\circ\Delta = (\Delta\otimes\mathrm{id})\circ\Delta. \tag{2.3.4}$$

If there exists a linear map $\hat{\mathbb{I}} : C \to \mathbb{K}$ with

$$\left(\hat{\mathbb{I}}\otimes\mathrm{id}\right)\circ\Delta = \mathrm{id} = \left(\mathrm{id}\otimes\hat{\mathbb{I}}\right)\circ\Delta, \tag{2.3.5}$$

the coalgebra $(C, \Delta, \hat{\mathbb{I}})$ is said to be counital, and $\hat{\mathbb{I}}$ is called the counit map.
For $\tau\circ\Delta = \Delta$ the coalgebra is cocommutative.

As we mentioned before, the properties (2.3.4) and (2.3.5) are equivalent to the commutativity of the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\quad\Delta\quad} & C\otimes C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ C\otimes C & \xrightarrow{\Delta\otimes\mathrm{id}} & C\otimes C\otimes C \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} \mathbb{K}\otimes C & \xleftarrow{\hat{\mathbb{I}}\otimes\mathrm{id}} & C\otimes C & \xrightarrow{\mathrm{id}\otimes\hat{\mathbb{I}}} & C\otimes\mathbb{K} \\ & \searrow^{\cong} & \uparrow{\scriptstyle \Delta} & \swarrow_{\cong} & \\ & & C & & \end{array} \tag{2.3.6}$$

which are dual to those in (2.3.3). More generally, we will extend the definition of the coproduct to that of the iterated coproduct $\Delta^n : C \otimes C^{\otimes(n+1)}$ by

$$\Delta^0 \coloneqq \mathrm{id} \qquad \text{and} \qquad \Delta^{n+1} \coloneqq \left(\Delta\otimes\mathrm{id}^{\otimes n}\right)\circ\Delta^n \text{ for } n\in\mathbb{N}_0. \tag{2.3.7}$$

Clearly, the recursive definition above is invariant under a variation of the order in which the coproduct is applied. This fact follows from the coassociativity of Δ (see equation (2.3.4)) and therefore

$$\Delta^{n+1} = \left(\mathrm{id}^{\otimes m}\otimes\Delta\otimes\mathrm{id}^{\otimes(n-m)}\right)\circ\Delta^n \quad \forall m, n\in\mathbb{N}_0,\ m\le n. \tag{2.3.8}$$

Remark 2.3. *In literature it is very common to use Sweedlers notation for the coproduct $\Delta(x) = x'\otimes x''$ with $x\in C$, which is shorthand for $\Delta(x) = \sum_i x'_{(i)}\otimes x''_{(i)}$.*

Definition 2.11. (Algebra and coalgebra morphism)
Consider two algebras (A_1, m_1) and (A_2, m_2). The linear map $\phi : A_1 \to A_2$ is an algebra morphism if

$$\phi\circ m_1 = m_2\circ(\phi\otimes\phi) \qquad \textit{and} \qquad \phi\circ\mathbb{I}_1 = \mathbb{I}_2 \tag{2.3.9}$$

in the case of unital algebras.
For coalgebras (C_1, Δ_1) and (C_2, Δ_2), the linear map $\tilde{\phi} : C_1 \to C_2$ is an coalgebra morphism if

$$\Delta_2 \circ \tilde{\phi} = \left(\tilde{\phi} \otimes \tilde{\phi}\right) \circ \Delta_1 \qquad \textit{and} \qquad \hat{\mathbb{I}}_2 \circ \tilde{\phi} = \hat{\mathbb{I}}_1 \tag{2.3.10}$$

is fulfilled. The latter only holds for the counital case.

Before we come to the notion of Hopf algebras, we first need to merge algebras and coalgebras to bialgebras as described in the following definition.

Definition 2.12. (Bialgebra)
A $\mathbb{K}$-vector space B together with a unital $\mathbb{K}$-algebra structure $(m, \mathbb{I})$ and a counital $\mathbb{K}$-coalgebra structure $\left(\Delta, \hat{\mathbb{I}}\right)$ is called a (unital and counital) $\mathbb{K}$-bialgebra $\left(B, m, \mathbb{I}, \Delta, \hat{\mathbb{I}}\right)$ if one of the following conditions hold:

(i) The linear maps $(m, \mathbb{I})$ are morphisms of coalgebras, or

(ii) the linear maps $\left(\Delta, \hat{\mathbb{I}}\right)$ are morphisms of algebras.

Note, that the requirements (i) and (ii) in the definition above are equivalent, as it was proven in [10]. Therefore, it suffices if only one of the conditions is fulfilled.
Since we will always assume (co-)algebras to be (co-)unital and bialgebras to be both of it, we can conveniently waive this prefix and just refer to them as (bi-, co-)algebras.
Motivated by the coproduct, there is another coassociative map one can define on bialgebras by

$$\tilde{\Delta} : B \to B \otimes B \qquad \text{and} \qquad \tilde{\Delta} \coloneqq \Delta - (\mathrm{id} \otimes \mathbb{I} + \mathbb{I} \otimes \mathrm{id}). \tag{2.3.11}$$

The map $\tilde{\Delta}$ is called the reduced coproduct, and the space $\mathrm{Prim}(B)$ of primitive elements is given by the kernel of $\tilde{\Delta}$

$$\mathrm{Prim}(B) \coloneqq \ker \tilde{\Delta} = \{b \in B : \Delta(b) = b \otimes \mathbb{I} + \mathbb{I} \otimes b\}. \tag{2.3.12}$$

Analogous to the iterated coproduct in equations (2.3.7) and (2.3.8), we define the iterated reduced coproduct recursively by the following definition.

$$\tilde{\Delta}^0 \coloneqq \mathrm{id} \quad \text{and} \quad \tilde{\Delta}^{n+1} \coloneqq \left(\mathrm{id}^{\otimes m} \otimes \tilde{\Delta} \otimes \mathrm{id}^{\otimes(n-m)}\right) \circ \tilde{\Delta}^n \quad \forall m, n \in \mathbb{N}_0,\ m \leq n \tag{2.3.13}$$

since $\tilde{\Delta}$ itself is coassociative, too. Now we will extend the notion of a bialgebra to that of a Hopf algebra.

Definition 2.13. (Hopf algebra)
A Hopf algebra $\left(H, m, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S\right)$ is a $\mathbb{K}$-bialgebra together with an endomorphism $S : H \to H$, called the antipode, satisfying

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \mathbb{I} \circ \hat{\mathbb{I}} = m \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{2.3.14}$$

Remark 2.4. *Consider an algebra $(A, m, \mathbb{I})$ and a coalgebra $\left(C, \Delta, \hat{\mathbb{I}}\right)$. Then, one can define an algebra $(\mathrm{Hom}_{\mathbb{K}}(C, A), *, e)$, consisting of the vector space $\mathrm{Hom}_{\mathbb{K}}(C, A)$ of*

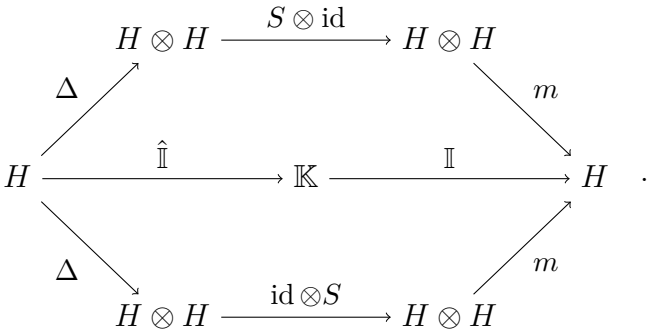
linear maps from C to A, a unit e, and a bilinear map $*$, called the convolution product, given by

$$e = \mathbb{I} \circ \hat{\mathbb{I}} \quad \text{and} \quad f * g = m \circ (f \otimes g) \circ \Delta \quad \forall f, g \in \mathrm{Hom}_{\mathbb{K}}(C, A). \tag{2.3.15}$$

Taking a Hopf algebra $\left(H, m, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S\right)$, the antipode $S \in \mathrm{Hom}_{\mathbb{K}}(H, H)$ on H can be defined by

$$S * \mathrm{id}_H = \mathrm{id}_H * S = e. \tag{2.3.16}$$

Let H be a bialgebra with antipode S. Then, the requirement for H being a Hopf algebra can be expressed by the commutativity of the following diagram



. (2.3.17)

The Hopf algebra of rooted trees, we will introduce soon, has the property to be connected and graded. Therefore, we have to clarify these terms first of all by

Definition 2.14. (Connectivity and graduation)
A Hopf algebra H over a field $\mathbb{K}$ is graded and connected if there exist subspaces H_i such that the following conditions hold

$$H = \bigoplus_{n \in \mathbb{N}_0} H_n, \qquad H_0 \simeq \mathbb{K}, \qquad H_i \equiv 0 \ \ \forall i < 0, \tag{2.3.18}$$

and

$$\begin{aligned} m(H_n \otimes H_m) &= H_n H_m \subseteq H_{n+m}, \\ \Delta H_n &\subseteq \bigoplus_{i+j=n} H_i \otimes H_j = \bigoplus_{i=0}^{n} H_i \otimes H_{n-i}, \\ S(H_n) &\subseteq H_n \end{aligned} \tag{2.3.19}$$

for any $n, m \in \mathbb{N}_0$.

In section 2.1 we already introduced the concept of rooted trees and denoted its set by $\mathcal{T}_r$, while $\mathcal{F}_r$ is the set of all rooted forests, i.e. the set of all disjoint unions of rooted trees. The empty tree [11] (or empty forest) is denoted by $\mathbb{I} := \emptyset$ and has weight zero $|\mathbb{I}| = 0$. We consider a Hopf algebra H_r over $\mathbb{Q}$ generated by the elements of $\mathcal{T}_r$ (including the empty tree $\mathbb{I}$) and define its Hopf algebra structure $\left(H, m, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S\right)$ as follows:

- For $T_1, T_2 \in \mathcal{T}_r$ the product $m(T_1 \otimes T_2) = T_1 T_2$ is given by the forest $T_1 \cup T_2$, that is the disjoint union of the graphs.

[11] Note that, by abuse of notation, $\mathbb{I}$ denotes the unit map as well as the empty tree.

- The unit map $\mathbb{I} : \mathbb{Q} \to H_r^{(0)}$ sends $q \in \mathbb{Q}$ to $q \cdot \mathbb{I} \in H_r^{(0)}$.
- The coproduct on a tree $T \in \mathcal{T}_r$ is defined through

$$\Delta(T) = \mathbb{I} \otimes T + T \otimes \mathbb{I} + \sum_{c \in \mathcal{C}(T)} P^c(T) \otimes R^c(T) \tag{2.3.20}$$

where the sum runs over all admissible cuts c of the tree, whose set $\mathcal{C}$ is given by [12]

$$\mathcal{C}(T) = \{c \subsetneq E(T) : |c \cap (r, v)| \leq 1 \ \forall v \in V(T), c \neq \emptyset\}. \tag{2.3.21}$$

By making a cut c, one or more edges of T are removed and the tree decomposes in a pruned part and a part still containing the root, denoted by $P^c(T)$ and $R^c(T)$, respectively (see example 2.3 below). For a product of trees, i.e. a forest $f = \cup_i T_i$, we have $\Delta(f) = \prod_i \Delta(T_i)$. The coassociativity of Δ was shown in [13].

- The counit map $\hat{\mathbb{I}} : H_r \to \mathbb{Q}$, defined by $\hat{\mathbb{I}}(T) = \begin{cases} 0, & T \neq \mathbb{I} \\ 1, & T = \mathbb{I} \end{cases}$, sends everything that is not the empty tree to zero.
- A recursive relation for the antipode S acting on a tree can be derived by using $S(\mathbb{I}) = \mathbb{I}$ and equations (2.3.14) and (2.3.20), obtaining

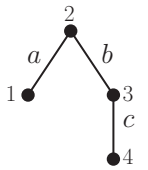
$$\begin{aligned} m\left(S \otimes \mathrm{id}_{H_r}\right) \Delta(T) &= S\left(\mathbb{I}\right) T + \mathbb{I} S(T) + \sum_{c \in \mathcal{C}(T)} S\left(P^c(T)\right) R^c(T) \\ &= \mathbb{I}\left(\hat{\mathbb{I}}(T)\right) = 0 \end{aligned} \tag{2.3.22}$$

and thus

$$S(T) = -T - \sum_{c \in \mathcal{C}(T)} S\left(P^c(T)\right) R^c(T) = -T - \sum_{c \in \mathcal{C}(T)} P^c(T) S\left(R^c(T)\right). \tag{2.3.23}$$

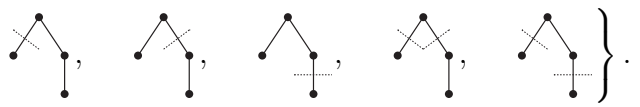
For a forest f, the antipode is given by $S(f) = S(T_1 \dots T_k) = S(T_k) \dots S(T_1)$.

Example 2.3.
Take the tree



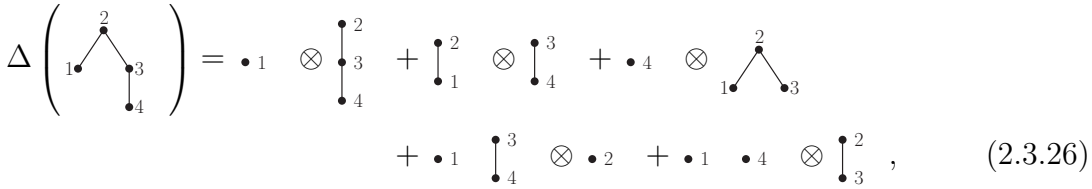
(2.3.24)

with edge set $T^{[0]} = \{a, b, c\}$ *and vertex set* $T^{[1]} = \{1, 2, 3, 4\}$. *The set of admissible cuts is given by* $\mathcal{C}(T) = \{a, b, c, (a, b), (a, c)\}$ *or pictorially*

$\mathcal{C}(T) =$  . (2.3.25)

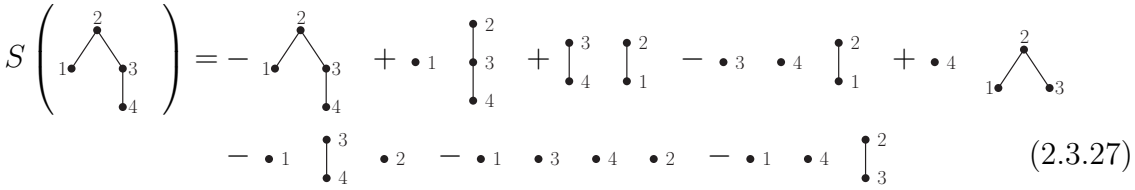
[12] For reminding the notation, see definition 2.1.

Therefore, the coproduct of the tree yields



(2.3.26)

and the antipode turns out to be



(2.3.27)

where we used that $S(T_2T_1) = S(T_1)S(T_2)$, $S(\bullet) = -\bullet$ *and* $S\left(\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}\right) = -\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix} + \bullet\bullet$.

It was, for example, shown in [19] that the upper definition of $\left(H, m, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S\right)$ gives a commutative, non-cocommutative, connected, and graded Hopf algebra with a natural grading given by the weight (= the node number) of the rooted trees. Taking $|T|$ to be the weight of the rooted tree T, the weight of a forest is just $|f| = \sum_i |T_i|$ for $f = \cup_i T_i$. Defining subspaces

$$H_r^{(n)} = \mathrm{span}_{\mathbb{Q}}\left\{f \in \mathcal{F}_r^{(n)}\right\} \qquad \text{with} \qquad \mathcal{F}_r^{(n)} = \{f \in \mathcal{F}_r : |f| = n\} \quad \forall n \in \mathbb{N}_0 \tag{2.3.28}$$

H_r decomposes as

$$H_r = \bigoplus_{n\in\mathbb{N}_0} H_r^{(n)} \tag{2.3.29}$$

which defines a grading on H_r. Another subspace of the Hopf algebra of rooted trees is the augmentation ideal

$$\mathrm{Aug}_{H_r} \coloneqq \bigoplus_{n\in\mathbb{N}} H_r^{(n)} = \bigoplus_{n=1}^{\infty} H_r^{(n)} = \ker \hat{\mathbb{I}} \tag{2.3.30}$$

given by the kernel of the counit. An important endomorphism of H_r is the grafting operator $B_+ : H_r \to \mathrm{span}_{\mathbb{Q}}(\mathcal{T}_r) \subset H_r$. This operator creates a new root and joins the roots of its arguments to it, returning a single tree:

$$B_+(\mathbb{I}) = \bullet \qquad \text{and} \qquad B_+(T_1 \dots T_n) = \quad \underbrace{\qquad\qquad}_{T_1\ T_2\ \cdots\ T_n} \quad . \tag{2.3.31}$$

The operator B_+ satisfies the relation

$$\Delta \circ B_+ = B_+ \otimes \mathbb{I} + (\mathrm{id} \otimes B_+) \circ \Delta \tag{2.3.32}$$

which can be regarded as a recursive definition of the coproduct since every tree can be written as $T = B_+(X)$ and $\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$.

Remark 2.5. *In fact, equation (2.3.32) implies that* B_+ *is a* 1*-cocycle in the Hochschild cohomology of* H_r*, see [6] or [8] for example.*

At the end of this section, we want to introduce a sub-Hopf algebra of H_r, namely the Hopf algebra of ladders, we will come back to, later. Generally, a sub-Hopf algebra of a graded Hopf algebra $\left(H, m, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S\right)$ with $H = \oplus_i H^{(i)}$ is defined as the subspace $\tilde{H} \subset H$, such that $\tilde{H}$ has a Hopf algebra structure $\left(\tilde{H}, m, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S\right)$ and a grading $\tilde{H} = \oplus_i \left(\tilde{H} \cap H^{(i)}\right)$. We define a ladder of weight k by $\lambda_k := (B_+)^k(\mathbb{I})$, which is the k-fold application of the grafting operator on the empty tree. Thus, ladders can be generated iteratively through $\lambda_k = B_+(\lambda_{k-1})$ with $\lambda_0 := \mathbb{I}$. Therefore, the diagrams take the form:

$\lambda_0 = \mathbb{I},\ \lambda_1 =$ $,\ \lambda_2 =$  $,\ \lambda_3 =$  $,\ \lambda_4 =$  $,\ \ldots,\ \lambda_k =$  k-times. (2.3.33)

The sub-Hopf algebra H_L, generated by the ladders, decomposes in the subspaces $H_L^{(n)} \subset H_r^{(n)}$ which consist of the elements of weight n. The coproduct on H_L is given by $\Delta(\lambda_k) = \sum_{j=0}^{k} \lambda_j \otimes \lambda_{k-j}$.

Chapter 3

Parametric renormalization

The Feynman rules Φ assign to every graph Γ an integral $I_\Gamma = \Phi(\Gamma)$ which is usually expressed in momentum space. Making use of the Schwinger trick, these integrals can be rewritten in parametric representation such that the integration variables we are left with are given by some abstract parameters α_e [13] . Its advantage is, as we will see, that the integrand of parametric integrals is directly connected to the combinatorics of the graph and can be constructed out of it.
In order to rescale the integral, one can introduce dimensionless scattering angles. If we then go from parametric to projective space and make use of the well-known forest formula, the integral can be written as a polynomial in the scaling parameter. Assuming, the scale of the process under consideration is very small, we can expand the integral and only consider the lowest orders in the scaling. Indeed, we will only look at the term linear in the scale parameter and ask for special combination of graphs such that the linear term is independent of the scattering angles.

3.1 Parametric representation

We now consider any Feynman diagram Γ in a scalar theory with arbitrarily oriented edges, meaning that each edge has a source and a target vertex. In general, the Feynman rules of the underlying theory assign the euclidean integral (cf. [17])

$$\Phi(\Gamma) = \prod_{e\in\Gamma_{\text{int}}^{[1]}} \int_{\mathbb{R}^D} \frac{\mathrm{d}^D k_e}{\pi^{D/2}} \frac{1}{\left(k_e^2 + m_e^2\right)^{a_e}} \prod_{v\in\Gamma^{[0]}\setminus\{v_0\}} \pi^{D/2}\delta^{(D)}\left(p(v) - \sum_{e\in\Gamma_{\text{int}}^{[1]}} \varepsilon_{ve} k_e\right) \tag{3.1.1}$$

to the graph, which is already dimensional regularized. The internal momenta k_e are partly determined through the external momenta p_e by momentum conservation expressed via the delta distribution. The incident matrix ε_{ve} is given by

$$\varepsilon_{ve} = \begin{cases} +1 \text{ if } v \text{ is the source vertex of } e, \\ -1 \text{ if } v \text{ is the target vertex of } e, \text{ and} \\ \ \ 0 \text{ if } e \text{ is not incident to } v. \end{cases} \tag{3.1.2}$$

$p(v)$ denotes the sum of all incoming external momenta p_e at the vertex v and obviously $\sum_{v\in\Gamma^{[0]}} p(v) = 0$. For internal vertices that are not connected to an external edge, we

[13] Indeed, these are the same parameters which already showed up in the definition of the two Symanzik polynomials.

have $p(v) = 0$. In the product of delta-functions we excluded one arbitrary vertex v_0 to get rid of an overall factor $\delta^{(D)}\left(\sum_{v\in\Gamma^{[0]}} p(v)\right)$.

Remark 3.1. *Actually, the integral obtained from the Feynman rules are not euclidean since the momentum vectors are elements of the Minkowski space. Thus, the scalar products of the momenta are defined with respect to the Minkowski metric. This problem can be remedied by a transformation of the time coordinate, namely*

$$k^0 = iK^0 \qquad \text{and} \qquad k^j = K^j \text{ for } j > 0. \tag{3.1.3}$$

Hence, $k^2 = -K^2$ is the ordinary euclidean scalar product and we can integrate over $\mathbb{R}^D$. The results for Minkowski space can be obtained from the computations in euclidean space by analytic continuation. The transformation (3.1.3) is referred to as Wick rotation, and can be looked up in every conventionally textbook about quantum field theory like [9] and [18].

In order to obtain the parametric representation of the integral (3.1.1), we have to apply the Schwinger [14] trick

$$\frac{1}{x^a} = \frac{1}{\Gamma(a)} \int_0^\infty \alpha^{a-1} e^{-\alpha x}\, \mathrm{d}\,\alpha \quad \text{for} \quad x, \mathrm{Re}(a) > 0 \tag{3.1.4}$$

to it, and substitute the delta function by its Fourier transform

$$(2\pi)^D \delta^{(D)}\left(p(v) - \sum_{e\in\Gamma^{[1]}_{\mathrm{int}}} \varepsilon_{ve} k_e\right) = \int \mathrm{d}^D\, y_v \exp\left[-iy_v\left(p(v) - \sum_{e\in\Gamma^{[1]}_{\mathrm{int}}} \varepsilon_{ve} k_e\right)\right]. \tag{3.1.5}$$

Remark 3.2. *The Schwinger trick can be obtained through reorganization from the integral representation of the Γ-function*

$$\Gamma(a) = \int_0^\infty \alpha^{a-1} e^{-\alpha}\, \mathrm{d}\,\alpha \overset{\alpha\to\alpha x}{=} \int_0^\infty \alpha^{a-1} x^a e^{-\alpha x}\, \mathrm{d}\,\alpha \quad \textit{for} \quad x, \mathrm{Re}(a) > 0. \tag{3.1.6}$$

Using (3.1.4), we can rewrite the product of propagators in (3.1.1) as a exponential function of the sum, and together with (3.1.5) this leads to the following representation of the scalar Feynman integral:

$$\begin{aligned}\Phi(\Gamma) = \prod_{e\in\Gamma^{[1]}_{\mathrm{int}}} \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\, k_e}{\pi^{D/2}} \int_0^\infty \frac{\alpha_e^{a_e-1}\, \mathrm{d}\,\alpha_e}{\Gamma(a_e)} \exp\left[-\alpha_e\left(k_e^2 + m_e^2\right)\right] \\ \times \prod_{v\in\Gamma^{[0]}\setminus\{v_0\}} \int_{\mathbb{R}^D} \frac{\mathrm{d}^D\, y_v}{(4\pi)^{D/2}} \exp\left[-iy_v\left(p(v) - \sum_{e\in\Gamma^{[1]}_{\mathrm{int}}} \varepsilon_{ve} k_e\right)\right].\end{aligned} \tag{3.1.7}$$

After interchanging the order of the integrals, the exponent can be rewritten in the form of matrices associated to the graph, which was done, for example, in [17]. Through completion of the square, we are left with Gaussian integrals over k_e and y_v afterwards.

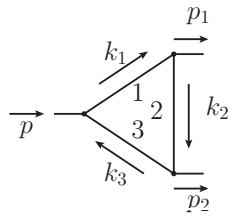
[14] Julian Seymour Schwinger (1918-94) was a theoretical physicist. Along with R. Feynman and S. Tomonaga, he was awarded with the Nobel Prize in Physics in 1965, for their fundamental work on quantum electrodynamics.

Solving these integrals, finally leads to the parametric representation of the integral (3.1.1)(cf. [17])

$$\Phi(\Gamma) = \prod_{e \in \Gamma_{\text{int}}[1]} \int_{\mathbb{R}_+} \frac{\alpha_e^{a_e - 1} \, \mathrm{d}\, \alpha_e}{\Gamma(a_e)} \cdot \frac{e^{-\phi_\Gamma / \psi_\Gamma}}{\psi_\Gamma^{D/2}}. \tag{3.1.8}$$

We will not give the explicit derivation here, but refer the reader to [17] for a very cute and compact version of this proof, or to [9] for a more detailed derivation.
The exponent in (3.1.8) only depends on the two Symanzik polynomials defined in (2.2.5) and (2.2.8). For a massless graph ϕ_Γ can be replaced by φ_Γ.

Example 3.1.
Consider the graph

$\Gamma =$ 

(3.1.9)

from ϕ_D^3-theory in D dimensions of spacetime with external momenta p_i. The internal edges are labeled from 1 to 3 and to the particles associated with them we assign a mass m_e and momentum $k_e \in \mathbb{R}^D$. According to equation (3.1.1), the Feynman rules assign the integral

$$\begin{aligned}\Phi(\Gamma) = \frac{1}{\pi^{D/2}} \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \frac{\mathrm{d}^D k_1 \, \mathrm{d}^D k_2 \, \mathrm{d}^D k_3}{(k_1^2 + m_1^2)(k_2^2 + m_2^2)(k_3^2 + m_3^2)} \\ \times \delta^{(D)}(p_1 - k_1 + k_2) \, \delta^{(D)}(p_2 - k_2 + k_3)\end{aligned} \tag{3.1.10}$$

to the graph. Performing the integrals over k_2 and k_3 gives

$$\Phi(\Gamma) = \frac{1}{\pi^{D/2}} \int_{\mathbb{R}^D} \frac{\mathrm{d}^D k_1}{(k_1^2 + m_1^2)((p_1 - k_1)^2 + m_2^2)((p_1 + p_2 - k_1)^2 + m_3^2)}. \tag{3.1.11}$$

In the case $D = 6$, this integral is ultraviolet logarithmically divergent since for $k_1 \to \infty$ the integrand decreases as $1/|k_1|$. In order to go from momentum to parametric space, we apply the Schwinger trick (3.1.4) to each of the three propagators separately and get

$$\begin{aligned}\Phi(\Gamma) = \frac{1}{\pi^{D/2}} \int_{\mathbb{R}_+} \mathrm{d}\, \alpha_1 \int_{\mathbb{R}_+} \mathrm{d}\, \alpha_2 \int_{\mathbb{R}_+} \mathrm{d}\, \alpha_3 \, e^{-\left(\alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_3^2\right)} \\ \times \int_{\mathbb{R}^D} \mathrm{d}^D k_1 \, e^{-\left[\alpha_1 k_1^2 + \alpha_2 (p_1 - k_1)^2 + \alpha_3 (p_1 + p_2 - k_1)^2\right]}.\end{aligned} \tag{3.1.12}$$

First, we focus on the k_1-integration and rewrite the exponent as follows

$$\begin{aligned}&\alpha_1 k_1^2 + \alpha_2 (p_1 - k_1)^2 + \alpha_3 (p_1 + p_2 - k_1)^2 \\ &\qquad = -(\alpha_1 + \alpha_2 + \alpha_3) \left[k_1 - \frac{\alpha_2 p_1 + \alpha_3 (p_1 + p_2)}{\alpha_1 + \alpha_2 + \alpha_3}\right]^2 \\ &\qquad\quad - \frac{\alpha_1 \alpha_2 p_1^2 + \alpha_1 \alpha_3 (p_1 + p_2)^2 + \alpha_2 \alpha_3 p_2^2}{\alpha_1 + \alpha_2 + \alpha_3}\end{aligned} \tag{3.1.13}$$

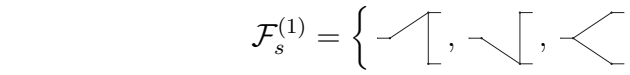
in which we completed the square on the right hand side. Thus, the k_1 integral is reduced to a Gaussian integral which yields

$$\int_{\mathbb{R}^D} \mathrm{d}^D\, k_1\, e^{-(\alpha_1+\alpha_2+\alpha_3)\left[k_1-\frac{\alpha_2 p_1+\alpha_3(p_1+p_2)}{\alpha_1+\alpha_2+\alpha_3}\right]^2} = \left(\frac{\pi}{\alpha_1+\alpha_2+\alpha_3}\right)^{D/2}. \tag{3.1.14}$$

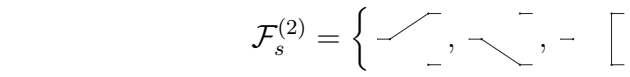
Hence, inserting everything back into (3.1.11), we are left with the integral

$$\Phi(\Gamma) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathrm{d}\,\alpha_1\,\mathrm{d}\,\alpha_2\,\mathrm{d}\,\alpha_3 \times \frac{e^{-(\alpha_1 m_1^2+\alpha_2 m_2^2+\alpha_3 m_3^2)-\frac{\alpha_1\alpha_2 p_1^2+\alpha_1\alpha_3(p_1+p_2)^2+\alpha_2\alpha_3 p_2^2}{\alpha_1+\alpha_2+\alpha_3}}}{(\alpha_1+\alpha_2+\alpha_3)^{D/2}}. \tag{3.1.15}$$

As we can see, the ultraviolet divergence in momentum space has turned into an infrared divergence in parametric space caused by the singularity at $0 \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ = \mathbb{R}_+^3$, where all the Schwinger parameters collectively vanish.
Now we want to rewrite the integrand in form of the graph polynomials. The spanning trees of the graph Γ in (3.1.9) are

$\mathcal{F}_s^{(1)} =$  , (3.1.16)

and the set of spanning 2-forests is given by

$\mathcal{F}_s^{(2)} =$  . (3.1.17)

Therefore, the two Symanzik polynomials of the graph take the form

$$\psi_\Gamma = \alpha_1+\alpha_2+\alpha_3 \qquad \textit{and} \tag{3.1.18}$$

$$\phi_\Gamma = \alpha_1\alpha_2 p_1^2 + \alpha_1\alpha_3(p_1+p_2)^2 + \alpha_2\alpha_3 p_2^2 + \psi_\Gamma \sum_{i=1}^{3} \alpha_i m_i^2. \tag{3.1.19}$$

These are exactly the expressions that show up in the integrand of (3.1.15) and, finally, the parametric integral becomes

$$\Phi(\Gamma) = \int_{\mathbb{R}_+^3} \mathrm{d}\,\alpha_1\,\mathrm{d}\,\alpha_2\,\mathrm{d}\,\alpha_3 \frac{e^{-\phi_\Gamma/\psi_\Gamma}}{\psi_\Gamma^{D/2}}. \tag{3.1.20}$$

Remark 3.3. *Another parametric representation in terms of Feynman parameters can be derived by using the so-called Feynman trick, see [2] or [18]. Furthermore, note that the integral (3.1.1) is given in momentum space. Applying the Schwinger trick to a position space integral yields the dual parametric representation of the integral. By analogy with the Fourier transform, one can use the Cremona transform to go from parametric to dual parametric space and vice versa.*

3.2 Rescaling, projective space and the forest formula

The parametric representation (3.1.8) of the scalar Feynman integral (3.1.1) introduced in the previous section is, in general, a function of the Schwinger parameters α_e, the

squared particle masses m_e^2, and the scalar products of the momenta $p_i \cdot p_j$ $(i,j \in \Gamma^{[1]}_{\text{ext}})$. Now, we want to rescale the Feynman rules $\Phi(\Gamma)$ by a parameter S. Therefore, we introduce dimensionless scattering angles $\{\Theta\} = \{\Theta_{ij}, \Theta_e\}$ given by the scaled variables

$$\Theta_{ij} = \frac{p_i \cdot p_j}{S} \quad \text{and} \quad \Theta_e = \frac{m_e^2}{S}. \tag{3.2.1}$$

The variable S sets the scale of the graph Γ defined by

$$S \coloneqq \sum_{e \in \Gamma^{[1]}_{\text{ext}}} p_e^2, \tag{3.2.2}$$

such that $S > 0$ and $S = 0$ only if all external momenta collectively vanish. The rescaled Feynman rules then can be written as a function of the scale variable and the angles

$$\Phi(\Gamma)\{S\Theta_{ij}, S\Theta_e\} \to \Phi(\Gamma)\{S, \Theta_{ij}, \Theta_e\}. \tag{3.2.3}$$

This rescaling only affects the second Symanzik polynomial ϕ_Γ in the integrand of $\Phi(\Gamma)$ whereas ψ_Γ is left unchanged since it only depends on the Schwinger parameters. We write

$$\phi_\Gamma \equiv \phi_\Gamma(S, \Theta) = S\phi_\Gamma(\Theta) \quad \text{and} \tag{3.2.4}$$

$$\phi_\Gamma(\Theta) = \frac{\varphi_\Gamma}{S} + \psi_\Gamma \sum_{e \in \Gamma^{[1]}_{\text{int}}} \alpha_e \frac{m_e^2}{S} = \varphi_\Gamma(\Theta) + \psi_\Gamma \sum_{e \in \Gamma^{[1]}_{\text{int}}} \alpha_e \Theta_e. \tag{3.2.5}$$

Then, the integral in equation (3.1.8) becomes

$$\Phi(\Gamma)\{S, \Theta\} = \prod_{e \in \Gamma^{[1]}_{\text{int}}} \int_{\mathbb{R}_+} \mathrm{d}\,\alpha_e \frac{e^{-S\frac{\phi_\Gamma(\Theta)}{\psi_\Gamma}}}{\psi_\Gamma^{D/2}} \quad \text{with} \quad a_e = 1 \ \forall e \in \Gamma^{[1]}_{\text{int}}. \tag{3.2.6}$$

In the following we will always set the exponents a_e equal to 1.
To carry out one of the integrations, we substitute $\alpha_e \to t\alpha_e$ which leads to [14]

$$\prod_{e \in \Gamma^{[1]}_{\text{int}}} \mathrm{d}\,\alpha_e \to t^{E_\Gamma - 1}\,\mathrm{d}\,t \wedge \Omega_\Gamma \tag{3.2.7}$$

where $\wedge$ denotes the exterior product (or wedge product) and the $(E_\Gamma - 1)$-form Ω_Γ defines the volume form $\Omega_\Gamma \coloneqq \sum_{i=1}^{E_\Gamma} (-1)^{i+1} \alpha_i\,\mathrm{d}\,\alpha_1 \wedge \cdots \wedge \widehat{\mathrm{d}\,\alpha_i} \wedge \cdots \wedge \mathrm{d}\,\alpha_{E_\Gamma}$ in projective space $\mathbb{P}_\Gamma \coloneqq \mathbb{P}^{E_\Gamma - 1}(\mathbb{R}_+)$. The circumflex accent $\widehat{\ }$ means that the argument is omitted. The Symanzik polynomials are homogeneous in the α_e's and of degree L and $L+1$, respectively. Thus, they transform like $\frac{\phi_\Gamma}{\psi_\Gamma} \to t\frac{\phi_\Gamma}{\psi_\Gamma}$ and $\psi_\Gamma^{D/2} \to t^{L\frac{D}{2}}\psi_\Gamma^{D/2}$. The projective integral finally takes the form

$$\Phi(\Gamma)\{S, \Theta\} = \int_{\mathbb{R}_+} \int_{\mathbb{P}_\Gamma} \frac{\mathrm{d}\,t}{t} \wedge \frac{e^{-tS\frac{\phi_\Gamma(\Theta)}{\psi_\Gamma}}}{t^{\omega_D/2}\psi_\Gamma^{D/2}} \Omega_\Gamma \tag{3.2.8}$$

with ω_D the superficial degree of divergence defined in equation (2.1.2).

Remark 3.4. *In ϕ_D^k-theory, the superficial degree of divergence is given by $\omega_D = 2 \cdot E_\Gamma - D \cdot L$ since the weight of the edges and vertices is $\omega(e) = 2$ and $\omega(v) = 0$, respectively.*

Now we want to perform the t-integration to get rid of the exponential. Therefore, we have to distinguish between the case $\omega_D > 0$, where the integral converges, and the case of ultraviolet divergence, i.e. $\omega_D \leq 0$. We are only interested in the latter case, thus we have to renormalize the integral. We apply kinetic renormalization conditions to $\Phi(\Gamma)$, that is to say that the renormalized amplitude of the graph Γ vanishes at a chosen reference or renormalization point $\{S_0, \Theta_0\}$, as well as all of its first ω_D derivatives in the Taylor expansion around that point.
In the logarithmic divergent case ($\omega_D = 0$), this condition can be implemented by modifying $\Phi(\Gamma)$ as follows

$$\Phi(\Gamma)\{S,\Theta\} \to \Phi(\Gamma)\{S,\Theta\} - \Phi(\Gamma)\{S_0,\Theta_0\}. \tag{3.2.9}$$

Hence, the overall divergence can be cured by a subtraction at the reference point. For $\omega_D < 0$, we also have to consider the derivatives which will not be done here but can be looked up, for example, in [4].
If Γ has only logarithmic subdivergences, the integrand can be renormalized by Zimmermanns well-known forest formula [4]

$$\Phi(\Gamma)\{S,\Theta\} \to \sum_{f\in\mathcal{F}(\Gamma)} (-1)^{\#f} \Phi(f)\{S_0,\Theta_0\}\,\Phi(\Gamma/f)\{S,\Theta\} \tag{3.2.10}$$

with $\#f$ denoting the number of connected components of the forest f. Assuming that Γ has not only logarithmic subdivergences but is additionally overall divergent, the forest formula yields [20]

$$\begin{aligned}\Phi_R(\Gamma)\{S,S_0,\Theta,\Theta_0\} = \sum_{f\in\mathcal{F}(\Gamma)} (-1)^{\#f}\Big[&\Phi(f)\{S_0,\Theta_0\}\,\Phi(\Gamma/f)\{S,\Theta\} \\ &- \Phi(f)\{S_0,\Theta_0\}\,\Phi(\Gamma/f)\{S_0,\Theta_0\}\Big]\end{aligned} \tag{3.2.11}$$

for the renormalized integrand, where the sum is over all forests f of the graph Γ, also including the empty one but excluding the forest containing Γ itself.

Remark 3.5. *Note that for the empty forest $f = \{\emptyset\}$, the graph polynomials are defined as $\psi_\emptyset = 1$ and $\phi_\emptyset(\Theta) = 0$.*

In order to come back to our primary goal, namely performing the t-integration, we replace the Φ's in the renormalized Feynman rules Φ_R by their parametric integral representation (3.2.8) for $\omega_D = 0$

$$\begin{aligned}\Phi_R(\Gamma) = \int_{\mathbb{R}_+}\int_{\mathbb{P}_\Gamma} \sum_{f\in\mathcal{F}(\Gamma)} (-1)^{\#f} \frac{1}{\psi_f^{D/2}\psi_{\Gamma/f}^{D/2}} \Bigg\{&\exp\left[-t\left(S_0\frac{\phi_f^0}{\psi_f} + S\frac{\phi_{\Gamma/f}}{\psi_{\Gamma/f}}\right)\right] \\ &- \exp\left[-S_0 t\left(\frac{\phi_f^0}{\psi_f} + \frac{\phi_{\Gamma/f}^0}{\psi_{\Gamma/f}}\right)\right]\Bigg\}\frac{\mathrm{d}\,t}{t}\wedge\Omega_\Gamma\end{aligned} \tag{3.2.12}$$

in which we already combined the products of exponentials and used the short hand notation $\phi_\Gamma \equiv \phi_\Gamma(\Theta)$ and $\phi_\Gamma^0 \equiv \phi_\Gamma(\Theta_0)$. Due to the $\frac{\mathrm{d}\,t}{t}$-integration over $\mathbb{R}_+$, the upper integral is singular at $t = 0$. Nevertheless, it can be regularized by introducing a regulator c, using that, for sufficiently small $c > 0$,

$$\int_c^\infty \frac{e^{-tX}}{t}\,\mathrm{d}\,t = -\ln c - \ln X - \gamma_E + \mathcal{O}(c\ln c) \tag{3.2.13}$$

with $X > 0$ fixed and γ_E the Euler-Mascheroni constant. A proof of this formula is given in [14]. To take the limit $c \to 0$, we have to subtract the integral at X_0 which yields

$$\lim_{c\to 0} \int_c^\infty \left[e^{-tX} - {}^{-tX_0}\right] \frac{\mathrm{d}\, t}{t} = -\ln(X/X_0). \tag{3.2.14}$$

Therefore, we found a way to carry out the t-integration. Applying the formula above to the integral (3.2.12) finally delivers

$$\Phi_R\left(\Gamma\right) = -\int_{\mathbb{P}_\Gamma} \sum_{f\in\mathcal{F}(\Gamma)} (-1)^{\#f} \frac{\ln\left(\frac{S\phi_{\Gamma/f}\psi_f + S_0\phi_f^0\psi_{\Gamma/f}}{S_0\phi_{\Gamma/f}^0\psi_f + S_0\phi_f^0\psi_{\Gamma/f}}\right)}{\psi_{\Gamma/f}^{D/2}\psi_f^{D/2}} \Omega_\Gamma \tag{3.2.15}$$

for the renormalized Feynman rules in projective form.

3.3 L-linear term of the renormalized Feynman rules

So far, we have seen that the divergent Feynman integral (3.1.1) in a scalar theory can be rewritten in parametric space by making use of the so-called Schwinger trick. The resulting integrand in (3.1.8) is a function of the squared particle masses, the scalar products of the external momenta, and the Schwinger parameters which combine to the Symanzik polynomials of the graph. In order to renormalize the integral, we put some constrains to our Feynman rules and assume that $\Phi\left(\Gamma\right)$ vanishes at a chosen reference point. In the case of logarithmic divergence, this condition can be implemented by simply subtracting at the renomralization point. To deal with logarithmic subdivergences, we also have to consider the forests of the graph and the integrand turns into a sum over the forest-set of Γ (3.2.11), due to Zimmermanns forest formula. The number of integrations can be reduced by one, going from parametric to projective space. Performing the t-integration also helps us to get rid of the exponential in the integrand. Finally, we achieve that the renormalized Feynman integral (3.2.15) in a scalar quantum field theory can be written as a function of the energy variable $L = \ln(S/S_0)$ and the dimensionless scattering angles Θ, with S setting the scale of the process under consideration.
It was shown by Brown and Kreimer that the renormalized Feynman rules $\Phi_R\left(\Gamma\right)$ can be decomposed into angle- and scale-dependent parts, which was well discussed in [4]. Indeed, for an overall divergent Feynman graph with divergent subgraphs, Φ_R is a polynomial in L and can be written as

$$\Phi_R\left(\Gamma\right) = \sum_{j=0}^{\mathrm{cor}(\Gamma)} c_j^\Gamma\left(\Theta, \Theta_0\right) L^j \tag{3.3.1}$$

where $c_0^\Gamma\left(\Theta, \Theta_0\right)$ is a scale-independent function of the angles and $c_{\mathrm{cor}(\Gamma)}^\Gamma\left(\Theta, \Theta_0\right)$ is an angle-independent coefficient. The integer $\mathrm{cor}\left(\Gamma\right)$ is called the co-radical degree of Γ defined as the maximal number $j_{\max} \in \mathbb{N}$, such that

$$\tilde{\Delta}^{j_{\max}-1}\Gamma \neq 0 \quad \text{and} \quad \tilde{\Delta}^j\Gamma = 0, \ \forall j \geq j_{\max} \tag{3.3.2}$$

in which $\tilde{\Delta}^j$ is the iterated reduced coproduct defined in equation (2.3.13). In other words, the co-radical degree equals the weight $|T\left(\Gamma\right)|$ of the rooted tree T associated to the graph Γ, i.e. $T\left(\Gamma\right) \in \mathcal{H}^{(\mathrm{cor}(\Gamma))}$, or, roughly speaking, $\mathrm{cor}\left(\Gamma\right)$ gives the amount of divergent subgraphs nested in Γ.

Example 3.2.
Let Γ be a primitive graph, i.e. the complete forest of the graph is the empty set $f = \{\emptyset\}$. Then, the rooted tree corresponding to the graph is just $\bullet \in H^{(1)}$, and therefore the Feynman graph Γ evaluates to

$$\Phi_R(\Gamma) = c_0^\Gamma(\Theta, \Theta_0) + c_1^\Gamma(\Theta, \Theta_0) L. \tag{3.3.3}$$

On the other hand the renormalized Feynman integral associated to the graph can be deduced from (3.2.15). In projective space the integral is given by

$$\Phi_R(\Gamma) = -\int_{\mathbb{P}_\Gamma} \frac{\ln(\phi_\Gamma/\phi_\Gamma^0) + \ln(S/S_0)}{\psi_\Gamma^{D/2}} \Omega_\Gamma \tag{3.3.4}$$

which is indeed a polynomial in L. Through equating of the coefficients, we get that

$$c_0^\Gamma = -\int_{\mathbb{P}_\Gamma} \frac{\ln(\phi_\Gamma/\phi_\Gamma^0)}{\psi_\Gamma^{D/2}} \Omega_\Gamma \quad \text{and} \quad c_1^\Gamma = -\int_{\mathbb{P}_\Gamma} \frac{\Omega_\Gamma}{\psi_\Gamma^{D/2}} = p_\Gamma. \tag{3.3.5}$$

In contrast to c_0^Γ, the coefficient c_1^Γ is not only independent of the angles but a period [15] *. The L-linear term c_1^Γ is constant and thus renormalization scheme independent, that is c_1^Γ is invariant under a change of the renormalization point.*

We are interested in the coefficient $c_1^\Gamma(\Theta, \Theta_0)$ of the term of Φ_R linear in L. Therefore, we first have to differentiate (3.2.15) with respect to L

$$\frac{\partial}{\partial L}\Phi_R(\Gamma) = S\frac{\partial}{\partial S}\Phi_R(\Gamma) \tag{3.3.6}$$

which yields

$$\int_{\mathbb{P}_\Gamma} \sum_{f \in \mathcal{F}(\Gamma)} (-1)^{\#f} \frac{1}{\psi_{\Gamma/f}^2 \psi_f^2} \frac{\frac{S}{S_0}\phi_{\Gamma/f}\psi_f}{\frac{S}{S_0}\phi_{\Gamma/f}\psi_f + \phi_f^0 \psi_{\Gamma/f}} \Omega_\Gamma. \tag{3.3.7}$$

Taking the derivative at $S = S_0$, or equivalent $L = 0$, gives us the L-linear term of the renormalized Feynman rules we are looking for

$$\Phi_R^{(1)}(\Gamma) = c_1^\Gamma(\Theta, \Theta_0) = \int_{\mathbb{P}_\Gamma} \sum_{f \in \mathcal{F}(\Gamma)} (-1)^{\#f} \frac{1}{\psi_{\Gamma/f}^2 \psi_f^2} \frac{\phi_{\Gamma/f}\psi_f}{\phi_{\Gamma/f}\psi_f + \phi_f \psi_{\Gamma/f}} \Omega_\Gamma. \tag{3.3.8}$$

If we assume that L is very small, this term gives us the main contribution to Φ_R together with the L-independent term.

[15] These are complex numbers "whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients" [11, p. 773]. For example, the famous irrational number $\pi = 3.14\ldots$, given by the circumference of a circle of unit diameter, can be represented as a period by the integral $\pi = \iint_{x^2+y^2 \leq 1} \mathrm{d}\, x\, \mathrm{d}\, y$.

Chapter 4

Fun with flags

After we introduced the mathematical and physical background knowledge in the previous chapters, we want to present the results of this thesis now. Therefore, we will define a special class of graphs, called flags. For symmetric flags it was already shown that the angle-dependence drops out in the L-linear term of the renormalized Feynman rules, and we will prove that this is also true in the case of total antisymmetric flags. Moreover, we present a formula which allows us to compute all (angle-independent) terms surviving in the sum. Thereby, the problem of finding all forests of a graph is boiled down to the much more simple task of figuring out all possible decomposition of the co-radical degree of the graph into positive integers.

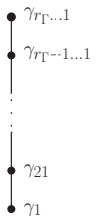
4.1 Flags

Consider a graph Γ that consists of nested insertions of primitive graphs γ_i into each other. Analogous to [15] we define:

Definition 4.1. (Flag)
A Hopf algebra element Γ of co-radical degree $\mathrm{cor}(\Gamma) = r_\Gamma$ is said to be a flag if there exists a sequence of primitive graphs γ_i with $1 \leq i \leq r_\Gamma$ such that

$$\tilde{\Delta}^{r_\Gamma - 1}\Gamma = \gamma_1 \otimes \cdots \otimes \gamma_{r_\Gamma}. \tag{4.1.1}$$

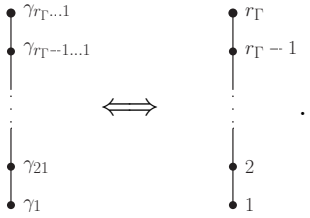
If Γ is a flag, the corresponding rooted tree is given by the (decorated) ladder

$\lambda_{r_\Gamma}^{(r_\Gamma,\ldots,1)} =$ 

(4.1.2)

of weight r_Γ. The expression $\gamma_{r_\Gamma\ldots1}$ is shorthand for the successive nested insertion $\gamma_{r_\Gamma} \leftarrow (\cdots \leftarrow \gamma_1)$, meaning that we start with γ_1, insert it into γ_2, insert the resulting graph γ_{21} into γ_3, and so on until we end up with inserting $\gamma_{r_\Gamma-1\ldots1}$ into γ_{r_Γ} receiving the graph $\Gamma = \gamma_{r_\Gamma\ldots1}$. In the following it will prove beneficial to label the vertices of the

ladder only by the leading index of the subgraph associated with it, i. e.



(4.1.3)

Let Λ_{r_Γ} be a sum of r_Γ flags $\Lambda^{(i)}$

$$\Lambda_{r_\Gamma} = \sum_{i=1}^{r_\Gamma} \Lambda^{(i)}. \tag{4.1.4}$$

This sum is called a symmetric flag $\Lambda^+_{r_\Gamma}$ if

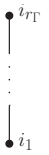
$$\tilde{\Delta}^{r_\Gamma - 1} \Lambda^+_{r_\Gamma} = \sum_\sigma \gamma_{\sigma(1)} \otimes \cdots \otimes \gamma_{\sigma(r_\Gamma)} \tag{4.1.5}$$

where the sum is over all $r_\Gamma!$ permutations of the primitive graphs γ_i. Accordingly, we say that $\Lambda^-_{r_\Gamma}$ is a total antisymmetric flag if

$$\tilde{\Delta}^{r_\Gamma - 1} \Lambda^-_{r_\Gamma} = \sum_{i_1, \dots, i_{r_\Gamma} = 1}^{r_\Gamma} \varepsilon_{i_1 \dots i_{r_\Gamma}} \gamma_{i_1} \otimes \cdots \otimes \gamma_{i_{r_\Gamma}} \tag{4.1.6}$$

for a sequence of primitive graphs γ_i with $1 \leq i \leq r_\Gamma$. In terms of ladders, $\Lambda^-_{r_\Gamma}$ can then be written as

$$\Lambda^-_{r_\Gamma} = \sum_{i_1, \dots, i_{r_\Gamma} = 1}^{r_\Gamma} \varepsilon_{i_1 \dots i_{r_\Gamma}} \lambda^{(i_{r_\Gamma}, \dots, i_1)}_{r_\Gamma} = \sum_{i_1, \dots, i_{r_\Gamma} = 1}^{r_\Gamma} \varepsilon_{i_1 \dots i_{r_\Gamma}}$$



(4.1.7)

and analogously $\Lambda^+_{r_\Gamma}$.

4.2 Angle-independence of $\Phi^{(1)}_R \left(\Lambda^-_{r_\Gamma} \right)$

Finally, we come to the central proposition of this thesis. It was already shown in [15] that the coefficient $\Phi^{(1)}_R \left(\Lambda^+_{r_\Gamma} \right)$ of the linear term is angle-independent if the renormalization point preserves scattering angles, i.e. $\Theta \equiv \Theta_0$. What we want to assert is:

Proposition 1.
Let $\Lambda^-_{r_\Gamma}$ be a total antisymmetric flag as defined in (4.1.6). Then, the L-linear term $\Phi^{(1)}_R \left(\Lambda^-_{r_\Gamma} \right)$ of the renormalized Feynman rules is independent of the second Symanzik polynomial and, therefore, angle-independent under the assumption that the renormalization point preserves scattering angles.

In order to prove the upper proposition, we will proceed as follows: At first, we will show explicitly that the angle-dependence drops out for total antisymmetric flags of co-radical degree three, four, and five. Afterwards, we give a brief explanation why

these terms cancel each other or add up to unity, such that the second Symanzik polynomial do not show up in the result. It will then be quite natural to assume that this angle-independence also holds for arbitrary co-radical degrees because there is no evidence that this is not the case.
Subsequently, we want to present a recipe how to determine the terms (and their coefficients) surviving in $\Phi_R^{(1)}\left(\Lambda_{r_\Gamma}^-\right)$ for arbitrary co-radical degree. This procedure uses a more pictorial approach and was deduced from the notion of Ferrers diagram (see appendix A).

4.2.1 (i) $r_\Gamma = 3$

We are looking at the graph $\gamma_{ijk} = \gamma_i \leftarrow \underbrace{(\gamma_j \leftarrow \gamma_k)}_{\gamma_{jk}}$ with $1 \leq i, j, k \leq 3$ and $i \neq j \neq k$.

The decorated rooted tree associated to the graph is given by 

, and the forest-set deduced from it turns out to be

$$\mathcal{F}(\gamma_{ijk}) = \{\emptyset, \gamma_{jk}, \gamma_k, \gamma_{jk} \cup \gamma_k\}\,. \tag{4.2.1}$$

The coefficient $\Phi_R^{(1)}$ of the L-linear term is (see equation (3.3.8))

$$\Phi_R^{(1)}(\gamma_{ijk}) = \int_{\mathbb{P}_\gamma} \Omega_\gamma \cdot I(\gamma_{ijk}) \qquad \text{with} \tag{4.2.2}$$

$$\begin{aligned} I(\gamma_{ijk}) = \frac{1}{\psi_{ijk}^2} - \frac{1}{\psi_{ij}^2\psi_k^2}\frac{\phi_{ij}\psi_k}{\phi_{ij}\psi_k + \phi_k\psi_{ij}} - \frac{1}{\psi_i^2\psi_{jk}^2}\frac{\phi_i\psi_{jk}}{\phi_i\psi_{jk} + \phi_{jk}\psi_i} \\ + \frac{1}{\psi_i^2\psi_j^2\psi_k^2}\frac{\phi_i\psi_j\psi_k}{\phi_i\psi_j\psi_k + (\phi_j\psi_k + \phi_k\psi_j)\,\psi_i} \end{aligned} \tag{4.2.3}$$

if we assume that the renormalization point preserves scattering angles. To make it more compact, we used the shorthand notation $\psi_{i_1\ldots i_n} \equiv \psi(\gamma_{i_1\ldots i_n})$ and analogue for ϕ. In the last term of $I(\gamma_{ijk})$, which corresponds to the forest $\gamma_{jk} \cup \gamma_k$, we used the decomposition rules (2.2.10) for products of graphs to rewrite the Symanzik polynomials. Now we intend to show the angle-independence of the L-linear term for a total antisymmetric flag of co-radical degree three. For that purpose we use

$$\varepsilon_{ijk} = \frac{1}{3}\left\{\varepsilon_{ijk} + \varepsilon_{jki} + \varepsilon_{kij}\right\} \tag{4.2.4}$$

to get

$$\begin{aligned} &\varepsilon_{ijk} I(\gamma_{ijk}) \\ &\quad = \frac{1}{3}\varepsilon_{ijk}\left\{\frac{1}{\psi_{ijk}^2} + \frac{1}{\psi_{jki}^2} + \frac{1}{\psi_{kij}^2} - \frac{1}{\psi_{ij}^2\psi_k^2} - \frac{1}{\psi_{jk}^2\psi_i^2} - \frac{1}{\psi_{ki}^2\psi_j^2} + \frac{1}{\psi_i^2\psi_j^2\psi_k^2}\right\}. \end{aligned} \tag{4.2.5}$$

A more detailed computation of the upper integrand is given in appendix B. As one can see, all ϕ-dependent terms summed up to unity such that the whole expression is independent of the scattering angles. The first and the second three terms in the upper equation are just cyclic permutations of each other. Therefore, they add up to one term if we sum over all indices i, j, and k. The last term in equation (4.2.5) cancels in the

sum because of the sign change due to the Levi-Cevita-tensor. As a consequence, the total antisymmetric sum of the L-linear terms of all graphs γ_{ijk} yields

$$\Phi_R^{(1)}\left(\Lambda_3^-\right) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\Phi_R^{(1)}\left(\gamma_{ijk}\right) = \int_{\mathbb{P}_\gamma} \Omega_\gamma \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\left\{\frac{1}{\psi_{ijk}^2} - \frac{1}{\psi_{ij}^2\psi_k^2}\right\}. \tag{4.2.6}$$

This expression only depends on the first Symanzik polynomial and is thus independent of the renormalization point.

4.2.2 (ii) $r_\Gamma = 4$

The case $r_\Gamma = 4$ corresponds to the rooted tree  given by the graph $\gamma_{ijkl} = \gamma_i \leftarrow \underbrace{(\gamma_j \leftarrow \overbrace{(\gamma_k \leftarrow \gamma_l)}^{\gamma_{kl}})}_{\gamma_{jkl}}$ with forest set

$$\mathcal{F}\left(\gamma_{ijkl}\right) = \left\{\emptyset, \gamma_{jkl}, \gamma_{kl}, \gamma_l, \gamma_{jkl}\cup\gamma_{kl}, \gamma_{jkl}\cup\gamma_l, \gamma_{kl}\cup\gamma_l, \gamma_{jkl}\cup\gamma_{kl}\cup\gamma_l\right\}. \tag{4.2.7}$$

The integrand of the L-linear term in the renormalized Feynman rules takes the form

$$\begin{aligned}\varepsilon_{ijkl} I\left(\gamma_{ijkl}\right) = \frac{1}{4}\varepsilon_{ijkl}\Bigg\{ &\frac{1}{\psi_{ijkl}^2} + \frac{1}{\psi_{jkli}^2} + \frac{1}{\psi_{klij}^2} + \frac{1}{\psi_{lijk}^2} - \frac{1}{\psi_{ijk}^2\psi_l^2} - \frac{1}{\psi_{ij}^2\psi_{kl}^2} - \frac{1}{\psi_i^2\psi_{jkl}^2} \\ &- \frac{1}{\psi_{jk}^2\psi_{li}^2} - \frac{1}{\psi_j^2\psi_{kli}^2} - \frac{1}{\psi_k^2\psi_{lij}^2} + \frac{1}{\psi_i^2\psi_j^2\psi_{kl}^2} + \frac{1}{\psi_k^2\psi_l^2\psi_{ij}^2} \\ &+ \frac{1}{\psi_j^2\psi_k^2\psi_{li}^2} + \frac{1}{\psi_l^2\psi_i^2\psi_{jk}^2} - \frac{1}{\psi_i^2\psi_j^2\psi_k^2\psi_l^2}\Bigg\}\end{aligned} \tag{4.2.8}$$

in which we used the same trick as before to rewrite the Levi-Cevita-tensor. Taking the antisymmetric sum over all $i,j,k,l \in \{1,2,3,4\}$ reduces the coefficient of the L-linear term to

$$\begin{aligned}\Phi_R^{(1)}\left(\Lambda_4^-\right) &= \sum_{i,j,k,l=1}^{4} \varepsilon_{ijkl}\Phi_R^{(1)}\left(\gamma_{ijkl}\right) \\ &= \int_\gamma \Omega_\gamma \sum_{i,j,k,l=1}^{4} \varepsilon_{ijkl}\left\{\frac{1}{\psi_{ijkl}^2} - \frac{1}{\psi_i^2\psi_{jkl}^2} - \frac{1}{2}\frac{1}{\psi_{ij}^2\psi_{kl}^2}\right\}.\end{aligned} \tag{4.2.9}$$

4.2.3 (iii) $r_\Gamma = 5$

Now we have the graph γ_{ijklm} whose subgraph structure is given by the ladder  .

The set of forests of the graph can directly be read off from the rooted tree, yielding

$$\begin{aligned}\mathcal{F}\left(\gamma_{ijklm}\right) = \{&\emptyset, \gamma_{jklm}, \gamma_{klm}, \gamma_{lm}, \gamma_m, \gamma_{jklm}\cup\gamma_{klm}, \gamma_{jklm}\cup\gamma_{lm}, \gamma_{jklm}\cup\gamma_m, \\ &\gamma_{klm}\cup\gamma_{lm}, \gamma_{klm}\cup\gamma_m, \gamma_{lm}\cup\gamma_m, \gamma_{jklm}\cup\gamma_{klm}\cup\gamma_{lm}, \\ &\gamma_{jklm}\cup\gamma_{klm}\cup\gamma_m, \gamma_{klm}\cup\gamma_{lm}\cup\gamma_m, \gamma_{jklm}\cup\gamma_{klm}\cup\gamma_{lm}\cup\gamma_m\}.\end{aligned} \tag{4.2.10}$$

Following the same procedure as before, the coefficient of the linear term in the renormalized Feynman rules adds up to

$$\begin{aligned}\Phi_R^{(1)}\left(\Lambda_5^-\right) &= \sum_{i,j,k,l,m=1}^{5} \varepsilon_{ijklm}\Phi_R^{(1)}\left(\gamma_{ijklm}\right) \\ &= \int_\gamma \Omega_\gamma \sum_{i,j,k,l,m=1}^{5} \varepsilon_{ijklm}\left\{\frac{1}{\psi^2_{ijklm}} - \frac{1}{\psi^2_i\psi^2_{jklm}} - \frac{1}{\psi^2_{ij}\psi^2_{klm}} + \frac{1}{\psi^2_i\psi^2_{jk}\psi^2_{lm}}\right\}. \end{aligned} \tag{4.2.11}$$

4.2.4 A general argument

So far, we have seen that the L-linear term $\Phi_R^{(1)}$ is indeed angle-independent for total antisymmetric flags of co-radical degree three, four, and five. The reason for that is the following: Taking the forest $f = \left\{\gamma_{i_j\dots i_1} \cup \gamma_{i_k\dots i_1} \cup \gamma_{i_l\dots i_1} \cup \gamma_{i_m\dots i_1}\right\}$ with $r_\Gamma > j > k > l > m > 1$ that generates the term

$$\begin{gathered}\frac{1}{\prod_{d\in D}\psi_d^2}\frac{\phi_{i_{r_\gamma}\dots i_{j+1}}\psi_{i_j\dots i_{k+1}}\psi_{i_k\dots i_{l+1}}\psi_{i_l\dots i_{m+1}}\psi_{i_m\dots i_1}}{\sum_{d\in D}\phi_d\prod_{\substack{d'\in D,\\ d'\neq d}}\psi_{d'}} \\ \text{with } D = \left\{i_{r_\gamma}\dots i_{j+1}, i_j\dots i_{k+1}, i_k\dots i_{l+1}, i_l\dots i_{m+1}, i_m\dots i_1\right\}\end{gathered} \tag{4.2.12}$$

in $\Phi_R^{(1)}$. Now we have to distinguish between two cases:

(1) There are at least two $\psi_{i_s\dots i_{t+1}}$ with $s = t+1$.

Without loss of generality we assume that $k = l+1$ and $l = m+1$. Therefore, the numerator in (4.2.12) takes the form

$$\phi_{i_{r_\gamma}\dots i_{j+1}}\psi_{i_j\dots i_{k+1}}\psi_{i_k}\psi_{i_l}\psi_{i_m\dots i_1}. \tag{4.2.13}$$

When changing i_k and i_l, we get a sign flip from the Levi-Cevita-tensor but the whole fraction stays invariant under that permutation and thus these terms cancel in the sum.

(2) There is at most one $\psi_{i_s\dots i_{t+1}}$ with $s = t+1$.

In this case, we can find cyclic permutation $i_1\dots i_{r_\Gamma} \to i_{j+1}\dots i_{r_\Gamma}i_1\dots i_j \to i_{k+1}\dots i_{r_\Gamma}$ $i_1\dots i_k \to i_{l+1}\dots i_{r_\Gamma}i_1\dots i_l \to i_{m+1}\dots i_{r_\Gamma}i_1\dots i_m$ that leave the denominator of (4.2.12) invariant. If we assume $j = k+1$ and sum over all cyclic permutations of the indices, we get an expression of the form

$$\frac{1}{\prod_{d\in D}\psi_d^2}\frac{\phi_{i_{r_\gamma}\dots i_{k+2}}\psi_{i_{k+1}}\psi_{i_k\dots i_{l+1}}\psi_{i_l\dots i_{m+1}}\psi_{i_m\dots i_1} + (\text{cyclic permutations})}{\sum_{d\in D}\phi_d\prod_{\substack{d'\in D,\\ d'\neq d}}\psi_{d'}}. \tag{4.2.14}$$

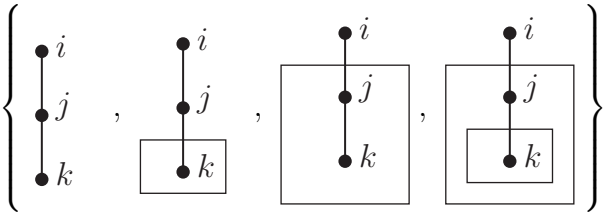
Since the numerator and denominator of the second factor are equal, the whole term turns to unity, and what we get is just the prefactor

$$\frac{1}{\prod_{d\in D}\psi_d^2} \tag{4.2.15}$$

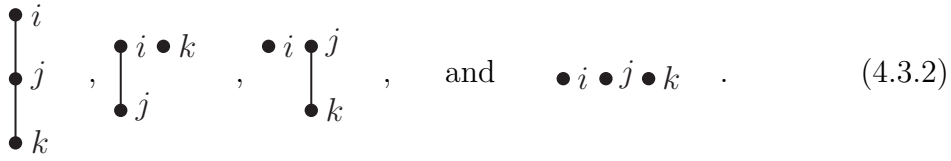
that only depends on the first Symanzik polynomial and no longer on the scattering angles.

4.3 A general formula

The method we want to present now is in principle based on the idea to figure out all possible combinations of subgraphs building a forest and, afterwards, discard those forests that cancel in the sum. Let us look again at the graph γ_{ijk} we already considered in section 4.2.1. The forest-set of the graph is given in (4.2.1) and we will write it pictorially as

$$\mathcal{F}(\gamma_{ijk}) = \Bigg\{$$  $$\Bigg\}$$ (4.3.1)

where each box stands for a forest and the nested ones for a union of trees building up that forest (see example 2.1). For reasons that will become clear later, we cut the trees at each edge that is crossed by a box, yielding the graphs



(4.3.2)

From this, it seems like that the forest-set is equal to the set of possibilities to cut the rooted tree into different parts, using 0, 1, and 2 cuts. For 0 and 2 cuts, there is obviously only one possibility, assuming that it does not matter in which order the cuts are performed, and for 1 cut we have two possibilities.
Comparing the integrand (4.2.3) of $\Phi_R^{(1)}(\gamma_{ijk})$ with the upper representation of the forests in terms of dissected rooted trees, there are two things that should attract our attention: For one thing, the products that show up in the integrand have just as many factors as dissected trees that build up the corresponding forest. For another thing, the polynomials in the products are subscripted by the decorations of the dissected trees. For example, the forest denoted by

corresponds to $$-\frac{1}{\psi_i^2\psi_{jk}^2}\frac{\phi_i\psi_{jk}}{\phi_i\psi_{jk}+\phi_{jk}\psi_i} \tag{4.3.3}$$

with the sign given by -1 to the power of $\#f$, which equals the number of cuts we performed to build the forest out of the ladder.
The reason for the close connection between the dissected trees and the integrand lies in the decomposition rule (2.2.10) of the second Symanzik polynomial. For a union of trees $\gamma_{ijk} \cup \gamma_{jk}$ we can write $\phi_{\gamma_{ijk}\cup\gamma_{jk}} = \phi_{\gamma_{ijk}/\gamma_{jk}}\psi_{\gamma_{jk}} + \phi_{\gamma_{jk}}\psi_{\gamma_{ijk}/\gamma_{jk}} = \phi_i\psi_{jk} + \phi_{jk}\psi_i$. Thus, cutting the tree can be understood as a contraction, splitting the graph γ into a product of graphs $(\gamma/\gamma_d)\ \gamma_d = \gamma_r\gamma_d$ where the rooted part γ_r is generated by contracting the original graph γ with the dissected graph γ_d. Therefore, the upper set of dissected trees (4.3.2) in terms of graphs is given by

$$\gamma_{ijk}, \quad \gamma_{ij}\gamma_k, \quad \gamma_i\gamma_{jk}, \quad \text{and} \quad \gamma_i\gamma_j\gamma_k \tag{4.3.4}$$

generating the integrand (4.2.3).

Remark 4.1. *Note that the graphs in (4.3.4) have nothing to do with the true forests of the graph and do not show up in the forest set. These graphs only arise from cutting the rooted tree in all possible ways and can be identified with the resulting dissected trees.*

Like in section 4.2.1, we assume angle preservation such that the integrand of the antisymmetric flag of co-radical degree three takes the form (cf. equation (4.2.5))

$$\varepsilon_{ijk} I\left(\gamma_{ijk}\right) = \varepsilon_{ijk}\left\{\frac{1}{\psi_{ijk}^2} - \frac{1}{2}\frac{1}{\psi_{ij}^2\psi_k^2} - \frac{1}{2}\frac{1}{\psi_i^2\psi_{jk}^2} + \frac{1}{3}\frac{1}{\psi_i^2\psi_j^2\psi_k^2}\right\}. \tag{4.3.5}$$

If we bear in mind that we have to sum over all $1 \leq i, j, k \leq 3$, the last term cancels in the sum since interchanging any of the three indices changes the sign (due to ε_{ijk}) but not the expression itself and, therefore, they sum up to zero. Furthermore, the second and third term in the integrand add up to one term because they can be converted into each other via cyclic permutation of the indices. Altogether, the only forests that show up in the result (4.2.6) are the empty one $\emptyset$ and γ_k or, figuratively speaking,  and  .

Remark 4.2. *Indeed, the choice of the second forest is not unique since we sum over all permutations. Therefore, it is also possible to write* $\frac{1}{\psi_i^2\psi_{jk}^2}$ *instead of* $\frac{1}{\psi_{ij}^2\psi_k^2}$ *and choose the forest*  *together with the uncut tree as the contributory elements of the forest-set.*

The question we want to face know is, how far the upper representation (4.3.2) of the forests can be linked to the integrand of $\Phi_R^{(1)}\left(\Lambda_{r_\Gamma}^-\right)$ or more precisely: Is it possible to deduce the terms in the resulting integrand by just looking at the possible cuts or dissections of the corresponding tree? The answer is yes. To see how this can be done, we first want to reformulate the L-linear term of the renormalized Feynman rules (3.3.8) for ladder graphs.
Let $\mathcal{P}\left(\lambda_{r_\Gamma}^{(i_1\dots i_{r_\Gamma})}\right)$ be the set of all partitions of the ladder $\lambda_{r_\Gamma}^{(i_1\dots i_{r_\Gamma})}$ into 1 up to r_Γ ladders $\lambda_{k_i}^{(d_i)}$ with weight k_i and decoration d_i. Of course we have $|d_i| = k_i$, otherwise this would not make sense. The set $(D, \prec)$ of all decorations $D = \{i_1, \dots, i_{r_\Gamma}\}$ has a strict total ordering $i_k \prec i_j,\ \forall j, k \in \mathbb{R} : j < k$ such that the Hasse diagram of D is given by the decorated rooted tree belonging to it.

Remark 4.3. *To visualize the ordering of a set S one can draw the corresponding Hasse diagram as follows (see [21]): At first, we identify each element in S with a vertex decorated by the element itself. Afterwards, we start by drawing the vertex, decorated by the largest element a on top and connect it to the vertex b by an edge if and only if $b \prec a$, i.e. $\nexists c \in S$ with $b \prec c \prec a$. From b we draw an edge to the vertex decorated by c, if c is the next lower element, and so on until we end up with drawing the vertex associated to the smallest element at the bottom. Indeed, there can be more than one minimal element and hence the set S has no minimum. The same holds for the maximal element. In both cases the Hasse diagram is no longer a ladder but a branched tree diagram.*

We decompose the set $\mathcal{P}$ into subsets $\mathcal{P}^{(n)}$ fulfilling

$$\mathcal{P} = \bigcup_{n=1}^{r_\Gamma} \mathcal{P}^{(n)} \quad \text{and} \quad \mathcal{P}^{(n)} = \left\{ p \in \mathcal{P} : p = \bigcup_{i=1}^{n} \lambda_{k_i}^{(d_i)} \wedge \sum_{i=1}^{n} k_i = r_\Gamma \wedge d_{i+1} \prec d_i \right\} \tag{4.3.6}$$

with subsets $d_i \subseteq D$ such that $D = \bigcup_{i=1}^{n} d_i$. The ordering condition $d_{i+1} \prec d_i$ means that $i_k \prec i_j,\ \forall i_j \in d_i, i_k \in d_{i+1}$ and of course $i_l \prec i_j,\ \forall i_j, i_l \in d_i : j < l$. Thus, the order of the decorations always has to stay the same no matter how many dissections were performed. For example, the set (4.3.2) can be written as

$$\mathcal{P}\left(\lambda_3^{(ijk)}\right) = \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)} \cup \mathcal{P}^{(3)} \qquad \text{with}$$
$$\mathcal{P}^{(1)} = \left\{\lambda_3^{(ijk)}\right\}, \quad \mathcal{P}^{(2)} = \left\{\lambda_2^{(ij)}\lambda_1^{(k)}, \lambda_1^{(i)}\lambda_2^{(jk)}\right\}, \quad \text{and} \quad \mathcal{P}^{(3)} = \left\{\lambda_1^{(i)}\lambda_1^{(j)}\lambda_1^{(k)}\right\}. \tag{4.3.7}$$

Since each element of $\mathcal{P}\left(\lambda_{r_\Gamma}^{(D)}\right)$ can be linked to a term in $\Phi_R^{(1)}(\gamma_D)$ of the graph γ_D, whose subgraph structure is visualized through $\lambda_{r_\Gamma}^{(D)}$, we can reformulate $\Phi_R^{(1)}\left(\lambda_{r_\Gamma}^{(D)}\right)$ in terms of ladders as follows

$$\begin{aligned}\Phi_R^{(1)}\left(\gamma_{i_1\dots i_{r_\Gamma}}\right) &\equiv \Phi_R^{(1)}\left(\lambda_{r_\Gamma}^{(i_1\dots i_{r_\Gamma})}\right) \\ &= \sum_{n=1}^{r_\Gamma} (-1)^{n+1} \sum_{\substack{p \in \mathcal{P}^{(n)}(\lambda_{r_\Gamma}^{(D)}) \\ p = \cup_{i=1}^{n} \lambda_{k_i}^{(d_i)}}} \frac{1}{\prod_{i=1}^{n} \psi_{d_i}^2} \frac{\phi_{d_1} \prod_{i=2}^{n} \psi_{d_i}}{\sum_{i=1}^{n} \phi_{d_i} \prod_{\substack{j=1, \\ j\neq i}}^{n} \psi_{d_j}}.\end{aligned} \tag{4.3.8}$$

For convenience, we make use of the notation $\psi_{d_i} = \psi\left(\lambda_{k_i}^{(d_i)}\right) = \psi(\gamma_{d_i})$ and analogously ϕ_{d_i}.

Now we go one step further and ask for $\Phi_R^{(1)}\left(\Lambda_{r_\Gamma}^{-}\right)$ in terms of the possible partitions $p \in \mathcal{P}$. Therefore, we have to recall the cases (1) and (2) from section 4.2.4 for the integrand of $\Phi_R^{(1)}\left(\Lambda_{r_\Gamma}^{-}\right)$. From case (1) it follows in terms of partitions that all elements $p \in \mathcal{P}$ containing more than one ladder of weight 1 do not contribute to the integrand, as we have seen for the fourth forest in (4.3.2). The second case tells us that the sum of all partitions of a graph consisting of the same number of ladders with fixed weight, contributes only one single term to the integrand (cf. the second and third forest in (4.3.2)).

To get a more adapted formulation of those two cases, we define a multiplicity $m_p(k)$ for each element p in (4.3.6), given by the number of ladders $\lambda_{k_i=k}^{(d_i)}$ of weight k contained in p. Let m_p be the r_Γ-tuple of all multiplicities of p, i.e. $m_p = (m_p(k))_{k=1,\dots,r_\Gamma} = (m_p(1), \dots, m_p(r_\Gamma))$. Then, we claim that two elements p and p' of $\mathcal{P}$ are independent of each other if and only if there exists at least one weight k such that $m_p(k) \neq m_{p'}(k)$. That is to say that p and p' have not the same tuple of multiplicities. Therefore, it follows that the two forests  and  are not independent of each other since they have the same multiplicity tuple given by $(m(1) = 1, m(2) = 1, m(3) = 0)$.

Based on the invented notion of the multiplicity tuple, we define

$$\mathcal{P}_{\text{ind}}\left(\lambda_{r_\Gamma}^{(D)}\right) \coloneqq \left\{ p \in \mathcal{P}\left(\lambda_{r_\Gamma}^{(D)}\right) : m_p \neq m_{p'}\, \forall\, p, p' \in \mathcal{P}_{\text{ind}}\left(\lambda_{r_\Gamma}^{(D)}\right) \right\} \tag{4.3.9}$$

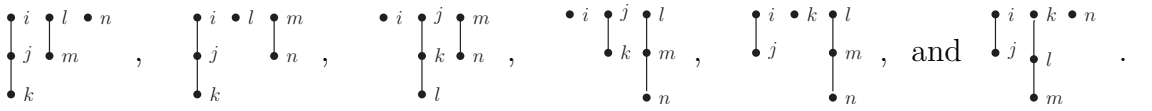
to be the set of all independent partitions of $\lambda_{r_\Gamma}^{(D)}$, which means that all elements of $\mathcal{P}_{\text{ind}}$ are pairwise independent of each other. Clearly, this set is not unique because out of all partitions in $\mathcal{P}\left(\lambda_{r_\Gamma}^{(D)}\right)$ with the same multiplicity tuple we have to choose only

one to be contained in $\mathcal{P}_{\text{ind}}$. Nevertheless, this will not bother us since our formula is completely independent of the choice of the specific partition as we will see. [16] Consequently, without loss of generality, we will always choose the partition consisting of ladders with equal or decreasing weight for a fixed multiplicity tuple, meaning that $p = \cup_i \lambda_{k_i}^{(d_i)}$ with $k_i \geq k_{i+1} \geq 1$. This selection rule allows us to write the set of independent partitions of a ladder $\lambda_{r_\Gamma}^{(D)}$ as

$$\mathcal{P}_{\text{ind}} = \bigcup_{n=1}^{r_\Gamma} \mathcal{P}_{\text{ind}}^{(n)} \quad \text{with}$$

$$\mathcal{P}_{\text{ind}}^{(n)} = \left\{ p \in \mathcal{P}_{\text{ind}} : p = \bigcup_{i=1}^{n} \lambda_{k_i}^{(d_i)} \wedge \sum_{i=1}^{n} k_i = r_\Gamma \wedge k_i \geq k_{i+1} \geq 1 \wedge d_{i+1} \prec d_i \right\}. \quad (4.3.10)$$

Now we are on the verge of giving a compact formulation of the final integrand $\Phi_R^{(1)}\left(\Lambda_{r_\Gamma}^-\right)$. Calling to mind that all partitions containing more than one ladder of weight 1 do not contribute to the integrand (cf. case (1) in section 4.2.4), $\mathcal{P}_{\text{ind}}\left(\lambda_{r_\Gamma}^{(D)}\right)$ gives the full set of partitions we need to build up the integrand of $\Lambda_{r_\Gamma}^-$ if we discard all elements p with $m_p(1) > 1$. The only thing we need for the final expression is the numerical prefactor of the terms contributing to the integrand. Let us see how this can be done by giving an example. Consider the case $r_\Gamma = 6$. The set of partitions p with multiplicity tuple $m_p = (1, 1, 1)$ is given by



(4.3.11)

As we already know, those terms will sum up to the single one $\frac{1}{\psi_{ijk}^2 \psi_{lm}^2 \psi_n^2}$ times a numerical prefactor which is composed as follows. The number of partitions with the same multiplicity for r_Γ fixed is just the number of possibilities to arrange the ladders in the respective partition. If we forget the decorations of the trees for a moment, this number would be just the number of ladders in p factorial. Since the decorations of the ladders are strictly ordered we do not get a new partition if we interchange ladders of the same weight in the partition. Therefore, the factorial of the number of ladders in p has to be divided by the multiplicity factorial for each weight. In our example we get $\frac{3!}{1!1!1!} = 6$ for the number of partitions with multiplicity $m_p = (1, 1, 1)$. The integrand associated to the first partition in the upper example is (cf. equation (4.3.8)) $\frac{1}{\psi_{ijk}^2 \psi_{lm}^2 \psi_n^2} \frac{\phi_{ijk}\psi_{lm}\psi_n}{\phi_{ijk}\psi_{lm}\psi_n + \phi_{lm}\psi_{ijk}\psi_n + \phi_n \psi_{ijk}\psi_{lm}}$. As one can see, each three of the partitions add up in such a way that the angle dependent term turns to unity. More generally, if n is the number of ladders in the partition, each n of the partitions will add up to one term in the sum. Therefore, dividing the number of partitions with the same multiplicity by the number of ladders in each partition gives the prefactor in the integrand we are looking for. In the present example this factor is $\frac{3!}{1!1!1!3} = 2$. In general, the prefactor can be defined as

$$\frac{\frac{n_p!}{\prod_{i=1}^{r_\Gamma} m_p(k_i)}}{n_p} = \frac{(n_p - 1)!}{\prod_{i=1}^{r_\Gamma} m_p(k_i)!} = \frac{\left(\sum_{j=1}^{r_\Gamma} m_p(k_j) - 1\right)!}{\prod_{i=1}^{r_\Gamma} m_p(k_i)!} \quad (4.3.12)$$

with n_p the number of ladders in the partition p. For an example of the prefactor for some partitions see table 4.1, in which we omitted the factors 0! in the denominators and

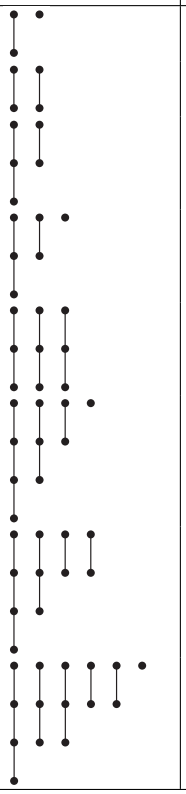
Partition p	Prefactor $\frac{(n_p-1)!}{\prod_i m_p(k_i)!}$
	$\frac{1!}{1!1!} = 1$
	$\frac{1!}{2!} = \frac{1}{2}$
	$\frac{1!}{1!1!} = 1$
	$\frac{2!}{1!1!1!} = 2$
	$\frac{2!}{3!} = \frac{1}{3}$
	$\frac{3!}{1!1!1!1!} = 6$
	$\frac{3!}{2!1!1!} = 3$
	$\frac{5!}{1!2!2!1!} = 30$

Table 4.1: The prefactor from equation (4.3.12) for a variety of partitions.

the decorations of the trees for convenience. Finally, the linear term of the renormalized Feynman rules for an antisymmetric flag takes the compact form

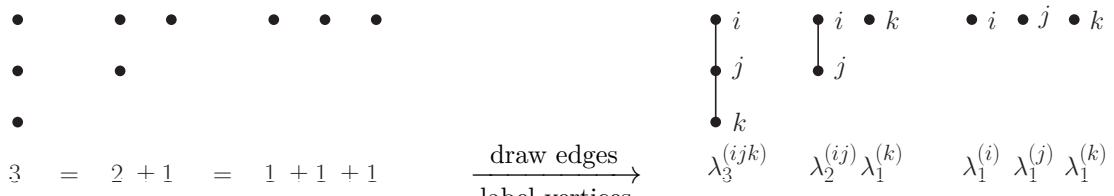
$$\Phi_R^{(1)}\left(\Lambda^-_{r_\Gamma}\right) = \int_\gamma \Omega_\gamma \sum_{n=1}^{r_\Gamma} (-1)^{n+1} \sum_{d\in D} \sum_{d=1}^{r_\Gamma} \varepsilon_{(D)} \sum_{\substack{p\in \mathcal{P}^{(n)}_{\text{ind}}(\lambda^{(D)}_{r_\Gamma}) \\ p=\cup_{i=1}^n \lambda^{(d_i)}_{k_i} \\ m_p(1)\leq 1}} \frac{(n-1)!}{\prod_{j=1}^{r_\Gamma} m_p(k_j)!} \frac{1}{\prod_{i=1}^n \psi_{d_i}^2} \tag{4.3.13}$$

where $\varepsilon_{(D)}$ is the Levi-Cevita-tensor indexed by the full decoration set of the corresponding flag and the sum $\sum_{d\in D}\sum_{d=1}^{r_\Gamma}$ runs over all elements of the decoration set, each of them taking on values from 1 up to r_Γ. Note, that d denotes an element and d_i an ordered subset of D so they should not be confused with each other.
The upper result is very striking since the problem of finding all the forests of a graph is boiled down to the task of finding all partitions of the corresponding ladder graph which is straightforward. A very easy and pictorial way to cope with this task will be given in the following section.

4.4 A pictorial approach using flag diagrams

In the last section we presented a formula that allows us to calculate the linear term of the renormalized Feynman rules of an antisymmetric flag by looking at the possible (independent) partitions of the corresponding ladder. There is also a pictorial way to deduce the set $\mathcal{P}_{\text{ind}}\left(\lambda^{(D)}_{r_\Gamma}\right)$ for a given ladder $\lambda^{(D)}_{r_\Gamma}$, which is based on the idea of Ferrers

[16] To some extent, this was already discussed in remark 4.2.



(a) Ferrers diagrams for $r_\Gamma = 3$ (b) Flag diagram for $r_\Gamma = 3$

Figure 4.1: Ferrers diagrams of all possible partitions of $r_\Gamma = 3$ into positive integers and the corresponding flag diagram constructed out of them.

diagrams (see appendix A). Such a pictorial representation of the set of independent partitions of a ladder graph, we will refer to as a flag diagram. To see how these diagrams can be constructed, we first think of the partition of the co-radical degree r_Γ into n positive integers k_i such that (cf. equation (A.1))

$$r_\Gamma = \sum_{i=1}^{n} k_i \qquad \text{and} \qquad k_i \geq k_{i+1} \geq 1. \tag{4.4.1}$$

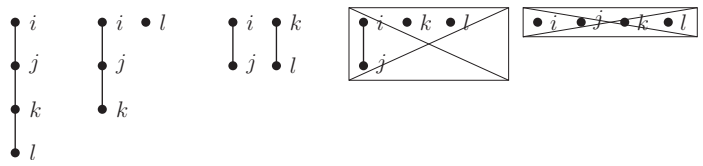
This partition can be illustrated by drawing the corresponding Ferrers diagram (see for example figure A.1). Alternatively we can write

$$\lambda_{r_\Gamma}^{(D)} = \bigcup_{i=1}^{n} \lambda_{k_i}^{(d_i)} \qquad \text{with}$$

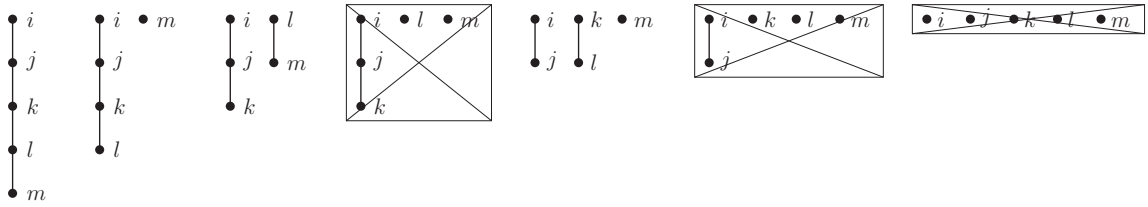
$$\sum_{i=1}^{n} k_i = r_\Gamma, \qquad k_i \geq k_{i+1} \geq 1, \qquad \text{and} \qquad d_{i+1} \prec d_i \tag{4.4.2}$$

in terms of ladders. The set of all possible partitions of $\lambda_{r_\Gamma}^{(D)}$ into $n = 1$ up to $n = r_\Gamma$ ladders fulfilling equation (4.4.2), equals the set (4.3.10) of all independent partitions of the ladder. But the problem of finding all those partitions can be reduced to the problem of finding all possible decompositions of the co-radical degree r_Γ into $n = 1, \ldots, r_\Gamma$ positive integers as in (4.4.1).
To see how this can be done, we consider the case $r_\Gamma = 3$. The possible decompositions of r_Γ are $r_\Gamma = 3 = 2 + 1 = 1 + 1 + 1$ with the corresponding Ferrers diagrams given in figure 4.1(a). Now, the set of independent partitions for $r_\Gamma = 3$ can directly be constructed out of Ferrers diagrams in figure 4.1(a) by simply drawing edges between the dots in the same column. Afterwards, we can label the dots in each diagram by the elements in the decoration set, going from top to bottom and from left to right, yielding the flag diagram in figure 4.1(b). The generalization of this construction is straightforward.
To determine the partitions in the flag diagram that contribute to the integrand of $\Phi_R^{(1)}\left(\Lambda_{r_\Gamma}^-\right)$, we have to discard all partitions with more than one ladder of weight one, e.g. the third one in figure 4.1(b). Or, in terms of Ferrers diagrams, we discard all diagrams containing more than one column with only one entry. Figure 4.2 shows the flag diagram for co-radical degree four and five, in which we already crossed out those partitions that do not show up in the integrand. At the end of this section, we want to give an explicit example for $r_\Gamma = 6$ in which we first draw the flag diagram and afterwards calculate the integrand using equation (4.3.13).

Example 4.1.
We



(a) Flag diagram for $r_\Gamma = 4$



(b) Flag diagram for $r_\Gamma = 5$

Figure 4.2: Flag diagrams for co-radical degree four and five with all partitions that do not contribute to integrand crossed out.

sum of positive integers, namely

$$\begin{aligned} 6 &= 5+1 = 4+2 = 4+1+1 = 3+3 = 3+2+1 = 3+1+1+1 \\ &= 2+2+2 = 2+2+1+1 = 2+1+1+1+1 = 1+1+1+1+1+1. \end{aligned} \tag{4.4.3}$$

Each of those decompositions can be illustrated by a Ferrers diagram, and from the set of diagrams we can deduce the corresponding flag diagram given in figure 4.3. Our goal

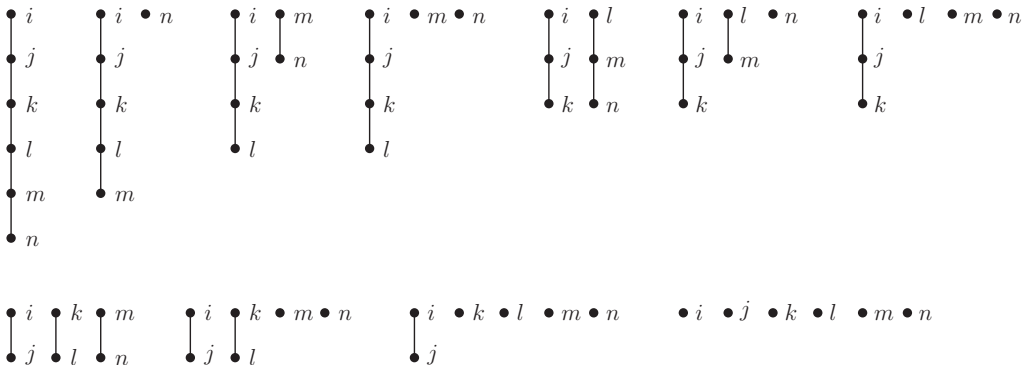


Figure 4.3: Flag diagram for co-radical degree $r_\Gamma = 6$.

is to calculate the integrand of the antisymmetric flag. Therefore, we can discard the fourth and seventh partition in the first, and all but the first partition in the second line of the flag diagram. The remaining partitions that contribute to the integrand are

$$\begin{gathered} \lambda_6^{(ijklmn)}, \quad \lambda_5^{(ijklm)}\lambda_1^{(n)}, \quad \lambda_4^{(ijkl)}\lambda_2^{(mn)}, \quad \lambda_3^{(ijk)}\lambda_3^{(lmn)}, \quad \lambda_3^{(ijk)}\lambda_2^{(lm)}\lambda_1^{(n)}, \\ \textit{and} \quad \lambda_2^{(ij)}\lambda_2^{(kl)}\lambda_2^{(mn)} \end{gathered} \tag{4.4.4}$$

with prefactors (see equation (4.3.12))

$$1, \quad 1, \quad 1, \quad \frac{1}{2}, \quad 2, \quad \textit{and} \quad \frac{1}{3}. \tag{4.4.5}$$

Thus, the integrand from equation (4.3.13) evaluates to

$$\Phi_R^{(1)}\left(\Lambda_6^-\right) = \int \Omega \sum_{i,j,k,l,m,n=1}^{6} \varepsilon_{ijklmn} \left[\frac{1}{\psi_{ijklmn}^2} - \frac{1}{\psi_{ijklm}^2 \psi_n^2} - \frac{1}{\psi_{ijkl}^2 \psi_{mn}^2} \right.$$
$$\left. -\frac{1}{2}\frac{1}{\psi_{ijk}^2 \psi_{lmn}^2} + \frac{2}{\psi_{ijk}^2 \psi_{lm}^2 \psi_n^2} + \frac{1}{3}\frac{1}{\psi_{ij}^2 \psi_{kl}^2 \psi_{mn}^2} \right]. \quad (4.4.6)$$

This calculation was done without any great effort and nearly took a half page. In contrast, we can think of the explicit calculation as it was outlined in sections 4.2.1 - 4.2.3. For the ladder of co-radical degree 6 *we would get a total number of* 32 *forests, each giving us one term in the integrand corresponding to the forest formula in equation (3.3.8). To see how the angle-dependence cancels out, we can rewrite the integrand using* $\varepsilon_{ijklmn} = \frac{1}{6}\left[\varepsilon_{ijklmn} + \varepsilon_{jklmni} + \dots\right]$*, giving us a total amount of* $32 \times 6 = 192$ *terms that have to be combined until we end up with the expression (4.4.6). Thus, our formula not only saves a lot of time and paperwork but also is more elegant in a combinatorial sense.*

Chapter 5

Conclusion

Within the scope of the present thesis, we initially considered scalar Feynman integrals in parametric representation. After introducing dimensionless scattering angles and carrying out one of the integrations, it becomes apparent that the renormalized Feynman rules can be written as a polynomial in the scaling parameter $L = \ln(S/S_0)$. Assuming that L is very small, the dominant contributions of Φ_R arise from the low-order terms in the polynomial. Therefore, we focus on the L-linear term of the renormalized Feynman rules. In quest of finding combinations of graphs such that the linear term is significantly simplified, we end up at so-called flags. It turned out that in the case of total antisymmetric flags (as well as in the case of symmetric flags) the ϕ-dependence and thus the angle-dependence drops out in the linear term of Φ_R if we assume that scattering angles are preserved. Based on our discovery, we thought about a concept how to compute $\Phi_R^{(1)}$ for antisymmetric flags of arbitrary order and developed a formula whereby the calculations are facilitated and shortened. In a concluding example, we showed how advantageous this formula can be. Indeed, the formula can also be used to compute $\Phi_R^{(1)}$ for symmetric flags. The sole difference is that all ladders in the flag-diagram contribute to the integrand. Consequently, the claim $m_p(1) \leq 1$ is dropped (cf. equation (4.3.13)). Apart from this, the formula is unchanged such as the computation of the prefactor (cf. equation (4.3.12)). Within the context of this thesis, we also considered combinations of ladder graphs with branched rooted trees for r_Γ fixed. However, investigations of those combinations of graphs gave no reason to expect the angle-independence of the L-linear term so far.
In a future work, one could examine if the angle-independence of $\Phi_R^{(1)}\left(\Lambda_{r_\Gamma}^{\pm}\right)$ also shows up in the parametric representation of other theories such as QED, where the PhD thesis of Matthias Sars [20] could serve as a basis. Additionally, further studies on this topic, based on the paper of Dirk Kreimer and Olaf Krüger [15], are possible by looking at the Lie-brackets of different graphs.

Appendix A

Ferrers diagram

Based upon [5] and [21], we briefly give an overview on what is called Ferrers diagram, named after the mathematician N. M. Ferrers (1829 - 1903), which can also be connected to Young diagrams.
Consider a partition of an integer n into k parts, given by the k-tuple $n = (y_1, \ldots, y_k)$ of positive integers y_i with

$$y_1 + y_2 + \cdots + y_k = n \quad \text{and} \quad y_1 \geq y_2 \geq \cdots \geq y_k \geq 1. \tag{A.1}$$

Alternatively, we can write $n = x_1^{m_1} \ldots x_l^{m_l}$ in terms of the multiplicity m_i of the different integers x_i showing up in the partition, such that $x_1 > x_2 > \cdots > x_l \geq 1$ and $\sum_{i=1}^{l} m_i = k$. For example, one possibility to decompose $n = 15$ into $k = 7$ parts is $15 = (4, 3, 2, 2, 2, 1, 1)$ or $15 = 4^1 3^1 2^3 1^2$. Note that the ladder one is not a product but a listing of the different elements in the partition and their multiplicity.
A useful way to represent such a partition pictorially as an array of points is Ferrers diagram. This diagram consists of k rows and $y_1 = x_1$ columns, where the first row contains y_1 points, the second one y_2 points, and so on, such that the number of points in the columns and rows decreases when going from left to right and top to bottom, respectively. For example, the Ferrers diagram of the partition of 15 into 7 parts, we mentioned above, is shown in figure A.1(a).

There is also a conjugate partition if we consider the columns (and not the rows)

(a) Ferrers diagram

(b) Transposed Ferrers diagram

Figure A.1: Pictorial representation (a) of the partition $15 = 4^1 3^1 2^3 1^2$ via Ferrers diagram and (b) its conjugate partition $15 = 7^1 5^1 2^1 1^1$ as the transposed of the original diagram.

of the diagram from left to right and link the number of points contained in them to

positive integers z_i, such that $n = \sum_i z_i$. Indeed, this partition is obtained from the diagram by interchanging the rows and columns, that is to say taking the transpose of the diagram. For example, the transpose of figure A.1(a) leads to the partition $15 = 7 + 5 + 2 + 1 = 7^1 5^1 2^1 1^1$ (see figure A.1(b)). The map from n to the conjugate n' gives the following theorem whose proof is also given in [21]:

Theorem A.1. ([21, theorem 3.1])
The number of partitions of n with k parts is equal to the number of partitions of n whose largest part is k.

Appendix B

Computation of $\Phi_R^{(1)}\left(\Lambda_3^-\right)$

We consider the graph γ_{ijk} of co-radical degree three, whose forest set is given in (4.2.1). The linear term of the renormalized Feynman rules already stated in equations (4.2.2) and (4.2.3) can be used to calculate $\Phi_R^{(1)}\left(\Lambda_3^-\right)$. Therefore, we rewrite the Levi-Cevita tensor as in equation (4.2.4), such that the integrand takes the form

$$\begin{aligned}
&\varepsilon_{ijk} I\left(\gamma_{ijk}\right) \\
&\quad = \frac{1}{3}\Bigg\{ \varepsilon_{ijk}\left[\frac{1}{\psi_{ijk}^2} - \frac{1}{\psi_{ij}^2\psi_k^2}\frac{\phi_{ij}\psi_k}{\phi_{ij}\psi_k+\phi_k\psi_{ij}} - \frac{1}{\psi_i^2\psi_{jk}^2}\frac{\phi_i\psi_{jk}}{\phi_i\psi_{jk}+\phi_{jk}\psi_i}\right. \\
&\qquad\qquad \left. + \frac{1}{\psi_i^2\psi_j^2\psi_k^2}\frac{\phi_i\psi_j\psi_k}{\phi_i\psi_j\psi_k+\left(\phi_j\psi_k+\phi_k\psi_j\right)\psi_i}\right] \\
&\qquad + \varepsilon_{jki}\left[\frac{1}{\psi_{jki}^2} - \frac{1}{\psi_{jk}^2\psi_i^2}\frac{\phi_{jk}\psi_i}{\phi_{jk}\psi_i+\phi_i\psi_{jk}} - \frac{1}{\psi_j^2\psi_{ki}^2}\frac{\phi_j\psi_{ki}}{\phi_j\psi_{ki}+\phi_{ki}\psi_j}\right. \\
&\qquad\qquad \left. + \frac{1}{\psi_j^2\psi_k^2\psi_i^2}\frac{\phi_j\psi_k\psi_i}{\phi_j\psi_k\psi_i+\left(\phi_k\psi_i+\phi_i\psi_k\right)\psi_j}\right] \\
&\qquad + \varepsilon_{kij}\left[\frac{1}{\psi_{kij}^2} - \frac{1}{\psi_{ki}^2\psi_j^2}\frac{\phi_{ki}\psi_j}{\phi_{ki}\psi_j+\phi_j\psi_{ki}} - \frac{1}{\psi_k^2\psi_{ij}^2}\frac{\phi_k\psi_{ij}}{\phi_k\psi_{ij}+\phi_{ij}\psi_k}\right. \\
&\qquad\qquad \left. + \frac{1}{\psi_k^2\psi_i^2\psi_j^2}\frac{\phi_k\psi_i\psi_j}{\phi_k\psi_i\psi_j+\left(\phi_i\psi_j+\phi_j\psi_i\right)\psi_k}\right]\Bigg\}
\end{aligned} \tag{B.1}$$

Using $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$, the terms in the afore-stated expression can be combined as follows

$$\begin{aligned}
&\varepsilon_{ijk} I\left(\gamma_{ijk}\right) \\
&\quad = \frac{1}{3}\varepsilon_{ijk}\Bigg\{ \frac{1}{\psi_{ijk}^2} + \frac{1}{\psi_{jki}^2} + \frac{1}{\psi_{kij}^2} - \frac{1}{\psi_{ij}^2\psi_k^2}\frac{\phi_{ij}\psi_k + \cancel{\phi_k\psi_{ij}}}{\cancel{\phi_{ij}\psi_k} + \phi_k\psi_{ij}}^{1} - \frac{1}{\psi_i^2\psi_{jk}^2}\frac{\phi_i\psi_{jk} + \cancel{\phi_{jk}\psi_i}}{\cancel{\phi_i\psi_{jk}} + \phi_{jk}\psi_i}^{1} \\
&\qquad - \frac{1}{\psi_j^2\psi_{ki}^2}\frac{\phi_j\psi_{ki} + \cancel{\phi_{ki}\psi_j}}{\cancel{\phi_j\psi_{ki}} + \phi_{ki}\psi_j}^{1} + \frac{1}{\psi_i^2\psi_j^2\psi_k^2}\frac{\phi_i\psi_j\psi_k + \phi_j\psi_k\psi_i + \cancel{\phi_k\psi_i\psi_j}}{\cancel{\phi_i\psi_j\psi_k} + \left(\phi_j\psi_k + \phi_k\psi_j\right)\psi_i}^{1}\Bigg\}
\end{aligned} \tag{B.2}$$

which is identical to equation (4.2.5). Taking the sum over i, j, and k the first and second three terms add up to one term with a factor of 3 that cancels with the prefactor and the last term drops out, giving

$$\sum_{i,j,k=1}^{3} \varepsilon_{ijk} I\left(\gamma_{ijk}\right) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\left\{\frac{1}{\psi_{ijk}^2} - \frac{1}{\psi_{ij}^2\psi_k^2}\right\}. \tag{B.3}$$

Bibliography

[1] C. Bergbauer and D. Kreimer. "Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology". In: *IRMA Lect.Math.Theor.Phys.* 10 (2006), pp. 133–164. arXiv: 0506190 [hep-th].

[2] C. Bogner. "Mathematical Aspects of Feynman Integrals". PhD thesis. Mainz, Germany: Johannes Gutenberg-Universität, Aug. 2009. URL: http://wwwthep.physik.uni-mainz.de/Publications/theses/dis-bogner.pdf (visited on 05/13/2015).

[3] C. Bogner and S. Weinzierl. "Feynman graph polynomials". In: *Int.J.Mod.Phys.* A25 (2010), pp. 2585–2618. arXiv: 1002.3458 [hep-th].

[4] F. Brown and D. Kreimer. "Angles, scales and parametric renormalization". In: *Lett.Math.Phys.* 103 (2013), pp. 933–1007. arXiv: 1112.1180 [hep-th].

[5] L. Comtet. *Advanced Combinatorics. The Art of Finite and Infinite Expansions*. Springer Netherlands, 1974. ISBN: 978-94-010-2198-2.

[6] A. Connes and D. Kreimer. "Hopf Algebras, Renormalization and Noncommutative Geometry". In: *Commun.Math.Phys.* 199 (1998), pp. 203–242. arXiv: 9808042 [hep-th].

[7] K. Ebrahimi-Fard and D. Kreimer. "Hopf algebra approach to Feynman diagram calculations". In: *J.Phys.* A38 (2005), R385–R406. arXiv: 0510202 [hep-th].

[8] L. Foissy. *An introduction to Hopf algebras of trees*. preprint. URL: http://loic.foissy.free.fr/pageperso/preprint3.pdf (visited on 05/04/2015).

[9] C. Itzykson and J.-B. Zuber. *Quantum Field Theory*. McGraw-Hill International Book Co., 1980. ISBN: 9780070320710.

[10] C. Kassel. *Quantum Groups*. Graduate Texts in Mathematics 155. Springer New York, 1995. ISBN: 9781461269007.

[11] M. Kontsevich and D. Zagier. "Periods". In: *Mathematics unlimited - 2001 and beyond*. Ed. by B. Engquist and W. Schmid. Springer-Verlag Berlin Heidelberg, 2001, pp. 771 –808. ISBN: 978-3-540-66913-5.

[12] D. Kreimer. "Anatomy of a gauge theory". In: *Annals Phys.* 321 (2006), pp. 2757–2781. arXiv: 0509135 [hep-th].

[13] D. Kreimer. "On the Hopf algebra structure of perturbative quantum field theories". In: *Adv.Theor.Math.Phys.* 2 (1998), pp. 303–334. arXiv: 9707029 [q-alg].

[14] D. Kreimer. *Renormalization & Renormalization Group*. Lecture notes by L. Klaczynski. Humboldt-Universität zu Berlin. 2012. URL: http://www2.mathematik.hu-berlin.de/~kreimer/wp-content/uploads/SkriptRGE.pdf.

[15] D. Kreimer and O. Krüger. "Filtrations in Dyson-Schwinger equations: next-to$^{\{j\}}$-leading log expansions systematically". In: (2014). arXiv: 1412.1657 [hep-th].

[16] D. Manchon. “Hopf algebras, from basics to applications to renormalization”. In: *ArXiv Mathematics e-prints* (Aug. 2004). arXiv: 0408405 [math].

[17] E. Panzer. “Feynman integrals and hyperlogarithms”. PhD thesis. Berlin, Germany: Humboldt-Universität zu Berlin, Oct. 2014. URL: http://www2.mathematik.hu-berlin.de/~kreimer/wp-content/uploads/Panzerphd.pdf (visited on 05/13/2015).

[18] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory*. Advanced book program. Westview Press Reading (Mass.), 1995. ISBN: 9780201503975.

[19] L. Rotheray. “Hopf subalgebras from Green’s functions”. Master thesis. Berlin, Germany: Humboldt-Universität zu Berlin, Dec. 2014. URL: http://www2.mathematik.hu-berlin.de/~kreimer/wp-content/uploads/Lucia.pdf (visited on 05/13/2015).

[20] M. Sars. “Parametric Representation of Feynman Amplitudes in Gauge Theories”. PhD thesis. Berlin, Germany: Humboldt-Universität zu Berlin, Jan. 2015. URL: http://www2.mathematik.hu-berlin.de/~kreimer/wp-content/uploads/SarsThesis.pdf (visited on 05/14/2015).

[21] D. Stanton and D. White. *Constructive Combinatorics*. Undergraduate Texts in Mathematics. Springer New York, 1986. ISBN: 978-0-387-96347-1.

[22] W. D. van Suijlekom. “Renormalization of gauge fields: A Hopf algebra approach”. In: *Commun.Math.Phys.* 276 (2007), pp. 773–798. arXiv: 0610137 [hep-th].

[23] D. B. West. *Introduction to Graph Theory*. 2nd ed. Prentice Hall, 2000. ISBN: 9780130144003.

Acknowledgment

First of all, I want to thank Dirk Kreimer for supervising my thesis and teaching us so much about Hopf algebra and renormalization. I am grateful to him for supporting me and giving me the chance to work in such an interesting field. Moreover, I thank Christian Bogner, not only for agreeing to advise the present thesis, but also for his hints and improvement suggestions. In addition to it, I thank everyone in the research group for the great atmosphere. I would like to say thanks to Horst Purkart, Lisa Frach, Kira Mitzlaff, Christoph Harmuth, and Mathilda Barchmann for participating in proofreading the thesis. Last but not least, I thank my family, friends, and especially my girlfriend for the mental support during the last years of my studies and for helping me to get my mind off and relax at the end of a hard day.

Selbstständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Masterarbeit

The signed permutation group on Feynman graphs

selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe. Darüber hinaus bestätige ich meinerseits die erstmalige Einreichung einer Masterarbeit in diesem Studiengang.
Die Arbeit wurde in gleicher oder ähnlicher Form bisher bei keiner anderen Institution eingereicht.

Ort, Datum

Unterschrift