

String – inspired representations of photon/gluon amplitudes

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Abstract

The string-based Bern-Kosower rules provide an efficient way for obtaining parameter integral representations of the one-loop N - photon/gluon amplitudes involving a scalar, spinor or gluon loop, starting from a master formula and using a certain integration-by-parts (“IBP”) procedure. Strassler observed that this algorithm also relates to gauge invariance, since it leads to the absorption of polarization vectors into field strength tensors. Here we present a systematic IBP algorithm that works for arbitrary N and leads to an integrand that is not only suitable for the application of the Bern-Kosower rules but also optimized with respect to gauge invariance. In the photon case this means manifest transversality at the integrand level, in the gluon case that a form factor decomposition of the amplitude into transversal and longitudinal parts is generated naturally by the IBP, without the necessity to consider the nonabelian Ward identities. Our algorithm is valid off-shell, and provides an extremely efficient way of calculating the one-loop one-particle-irreducible off-shell Green’s functions (“vertices”) in QCD. It can also be applied essentially unchanged to the one-loop gauge boson amplitudes in open string theory. In the abelian case, we study the systematics of the IBP also for the practically important case of the one-loop N - photon amplitudes in a constant field.

1 Introduction

It is by now well-known that techniques originally developed for the computation of amplitudes in string theory can be used also for the simplification of calculations in ordinary quantum field theory. Already in 1972 Gervais and Neveu observed that the field theory limit of string theory generates Feynman rules for Yang-Mills theory in a special gauge that has certain calculational advantages [1]. Actual calculations along these lines were done, however, only much later [2, 3, 4, 5]. A systematic investigation of the field theory limit at the tree and one-loop level was undertaken by Bern and Kosower, and led to the establishment of a new set of rules for the construction of the one-loop gluon amplitudes in QCD [6, 7, 8]. These ‘‘Bern-Kosower rules’’ were used for a first calculation of the five – gluon amplitudes [9]. Their relation to the usual Feynman rules was clarified in [10].

Shortly afterwards, a simpler approach to the derivation of Bern-Kosower type formulas was initiated by Strassler [11] based on the representation of one-loop amplitudes in terms of first-quantized path integrals. For the case of QED representations of this type had been known for a long time [12] although they had rarely been considered as a tool for state-of-the-art calculations. Subsequently such representations were derived also for Yukawa and axial couplings [13, 14, 15], and generalized to higher loop orders [16, 17, 18, 19] as well as to the inclusion of constant external fields [20, 21, 22], gravitation [23, 24, 25, 26, 27, 28] and finite temperature [29, 30, 31]. For a review see [32].

The central formula in the Bern-Kosower formalism is the following ‘master formula’:

$$\begin{aligned} \Gamma_{\text{scal}}[k_1, \varepsilon_1; \dots; k_N, \varepsilon_N] &= (-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \\ &\times \exp \left\{ \sum_{i,j=1}^N \left[\frac{1}{2} G_{Bij} k_i \cdot k_j - i \dot{G}_{Bij} \varepsilon_i \cdot k_j + \frac{1}{2} \ddot{G}_{Bij} \varepsilon_i \cdot \varepsilon_j \right] \right\} \Big|_{\text{lin}(\varepsilon_1, \dots, \varepsilon_N)} \end{aligned} \quad (1.1)$$

As it stands, this formula represents the one-loop N – photon amplitude in scalar QED, with photon momenta k_i and polarisation vectors ε_i . m denotes the mass, e the charge and T the total proper time of the scalar loop particle ¹.

Each of the integrals $\int d\tau_i$ represents one photon leg moving around the loop. The integrand is written in terms of the ‘bosonic’ worldline Green’s function G_B and its derivatives,

$$\begin{aligned} G_B(\tau_1, \tau_2) &= |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} \\ \dot{G}_B(\tau_1, \tau_2) &= \text{sign}(\tau_1 - \tau_2) - 2 \frac{(\tau_1 - \tau_2)}{T} \\ \ddot{G}_B(\tau_1, \tau_2) &= 2\delta(\tau_1 - \tau_2) - \frac{2}{T} \end{aligned} \quad (1.2)$$

¹ We work in the Euclidean throughout. With our conventions a Wick rotation $k_i^4 \rightarrow -ik_i^0, T \rightarrow is$ yields the N – photon amplitude in the conventions of [33].

Dots generally denote a derivative acting on the first variable, $\dot{G}_B(\tau_1, \tau_2) \equiv \frac{\partial}{\partial \tau_1} G_B(\tau_1, \tau_2)$, and we abbreviate $G_{Bij} = G_B(\tau_i - \tau_j)$ etc. In deriving the master formula a formal exponentiation has been used that needs to be undone by expanding out the exponential in (1.1) in the polarization vectors, and keeping only the terms linear in each of them (in this paper our interest will be mostly in the off-shell case, but it will still be useful to keep the polarization vectors as book-keeping devices).

We will not dwell here on the derivation of this formula, which can be obtained either from the infinite string tension limit of string theory [6, 7, 8, 34] or using the worldline path integral formalism [11, 32]. Its role in the Bern-Kosower formalism is central, since it provides the input for the Bern-Kosower rules, which allow one to obtain, from the scalar QED integrand, the corresponding integrand for the photon amplitudes in fermion QED, as well as for the (on-shell) N - gluon amplitude in QCD. However, those rules do not apply to the master formula as it stands. Writing out the exponential in eq.(1.1) one obtains an integrand

$$\exp\left\{\cdot\right\}\Big|_{\text{multi-linear}} = (-i)^N P_N(\dot{G}_{Bij}, \ddot{G}_{Bij}) \exp\left[\frac{1}{2} \sum_{i,j=1}^N G_{Bij} k_i \cdot k_j\right] \quad (1.3)$$

with a certain polynomial P_N depending on the various $\dot{G}_{Bij}, \ddot{G}_{Bij}$ and on the kinematic invariants. The application of the Bern - Kosower rules requires one to now remove all second derivatives \ddot{G}_{Bij} appearing in P_N by suitable integrations by parts in the variables τ_i .

That this removal of all \ddot{G}_B 's is possible for any N was shown in appendix B of [7]. The new integrand is written in terms of the G_{Bij} 's and \dot{G}_{Bij} 's alone, and serves as the input for the Bern - Kosower rules. Those allow one to classify the various contributions to the N - photon/gluon amplitude in terms of ϕ^3 - diagrams, and moreover lead to simple relations between the integrands for the scalar, spinor and gluon loop cases. A complete formulation of the Bern-Kosower rules is lengthy, and we refer the reader to [8, 34, 32, 35]. For our present purposes, the most relevant part of the rules is that, up to a global factor of -2 correcting for the differences in degrees of freedom and statistics, the integrand for the spinor loop case can be obtained from the one for the scalar loop simply by replacing every closed cycle of \dot{G}_B 's appearing in Q_N according to the ‘‘replacement rule’’

$$\dot{G}_{Bi_1i_2} \dot{G}_{Bi_2i_3} \cdots \dot{G}_{Bi_ni_1} \rightarrow \dot{G}_{Bi_1i_2} \dot{G}_{Bi_2i_3} \cdots \dot{G}_{Bi_ni_1} - G_{Fi_1i_2} G_{Fi_2i_3} \cdots G_{Fi_ni_1} \quad (1.4)$$

where $G_{F12} \equiv \text{sign}(\tau_1 - \tau_2)$ denotes the ‘fermionic’ worldline Green’s function. Note that an expression is considered a cycle already if it can be put into cycle form using the antisymmetry of \dot{G}_B (e.g. $\dot{G}_{B12} \dot{G}_{B12} = -\dot{G}_{B12} \dot{G}_{B21}$). A similar ‘‘cycle replacement rule’’ holds for the gluon loop case. Of course, in the nonabelian case there will also be many other modifications.

As was discussed already in [7], the IBP procedure is generally ambiguous. However, this does not constitute an impediment to the application of the Bern-Kosower rules, whose application requires only that all second derivatives \ddot{G}_{Bij} have been removed. For this reason, in the application to the computation of gluon amplitudes in [8, 9] the partial integration had been performed in an essentially arbitrary way.

A closer look at the IBP was taken by Strassler in [36], who noted that this procedure bears an interesting relation to gauge invariance. For each photon leg, define the corresponding field

strength tensor,

$$F_i^{\mu\nu} \equiv k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu \quad (1.5)$$

Remove all \ddot{G}_{Bi_j} 's and combine all terms contributing to a given ‘ τ -cycle’ $\dot{G}_{Bi_1i_2} \dot{G}_{Bi_2i_3} \cdots \dot{G}_{Bi_ni_1}$. Then the sum of their Lorentz factors can be written as a ‘Lorentz cycle’ $Z_n(i_1i_2 \dots i_n)$, defined by

$$\begin{aligned} Z_2(ij) &\equiv \frac{1}{2} \text{tr} \left(F_i F_j \right) = \varepsilon_i \cdot k_j \varepsilon_j \cdot k_i - \varepsilon_i \cdot \varepsilon_j k_i \cdot k_j \\ Z_n(i_1i_2 \dots i_n) &\equiv \text{tr} \left(\prod_{j=1}^n F_{i_j} \right) \quad (n \geq 3) \end{aligned} \quad (1.6)$$

Thus Z_n generalizes the familiar transversal projector. However, in [36] no systematic way was found to perform the partial integrations at arbitrary N , and also the absorption of polarisation vectors into field strength tensors (a process to be called ‘covariantization’ in the following) worked only partially; after the IBP and the sorting of the resulting integrand in terms of ‘cycle content’ some terms are just cycles or products of cycles, but, starting from the three-point case, there are also terms with left-overs, called ‘tails’ in [36], and the polarisation vectors in them were not absorbed yet into field strength tensors.

The IBP procedure was further studied in [37], where a definite partial integration prescription was given which works for any N , preserves the full permutation symmetry and is suitable for computerization. This algorithm is completely satisfactory as far as the application of the Bern-Kosower rules is concerned. However, it is obviously an interesting question whether some IBP algorithm exists which leads to an integrand where *all* polarisation vectors would be contained in field strength tensors, thus making gauge invariance, i.e. transversality, manifest at the integrand level. It is the purpose of the present work to present such an algorithm ².

In detail, we will do the following: In chapters 2 and 3 we find a surprisingly simple way of using IBPs to covariantize the Bern-Kosower master formula itself; the resulting representation will be called the R-representation. In 4 we summarize the ‘symmetric IBP procedure’ of [37], leading to the Q - and Q' -representations. The next two sections 5 and 6 define our new algorithm, which combines elements of both the R - and the Q' -representation, and results in what we will call the S-representation of the N photon/gluon amplitudes. As an aside, in section 7 we present a further improvement of the ‘two-tail’ which leads to a particularly compact integrand at the four-point level (but does not seem to generalize to the N -point case). In section 8 we shortly comment on a direct treatment of the spinor QED case in the worldline super formalism. Section 9 is devoted to our main application, which is the calculation of the one-loop off-shell one-particle-irreducible N -gluon amplitudes (or ‘ N -vertices’). While in the abelian case there are never any boundary terms in our IBPs, since all integrations run over the full loop and the integrand is written in terms of worldline Green’s functions with the appropriate boundary conditions, in the nonabelian case the color ordering of the gluon legs leads to the restriction of the multiple parameter integrals to ordered sectors, and to the appearance of such boundary

²An IBP algorithm for the general worldline integrand was also developed in [38], but for the unrelated purpose of tensor reduction.

terms [11, 36]. Those generally can be combined into color commutators and, in x -space, would in principle allow one to achieve a complete nonabelian extension of the covariantization, namely to rewrite the final integrand in terms of full nonabelian field strength tensors, and to complete all derivatives to covariant ones. This is not possible in momentum space, but here instead the IBP procedure generates a natural form factor decomposition of the N -vertices, where the bulk terms are manifestly transversal and all non-transversality has been pushed into boundary terms. Finding such a decomposition by standard methods usually involves a tedious analysis of the nonabelian Ward identities, and so far has been completed only for the three-point case [39].

In section 10 we return to the abelian case, but now with a constant external field added. Here the replacement rule (1.4) applies as well [21, 22], however the IBP procedure is complicated by the fact that the worldline Green's functions become nontrivial Lorentz matrices. We will point out here the necessary modifications, and also find a way of using the IBP procedure to effectively eliminate the nonvanishing coincidence limits of the generalized Green's functions $\dot{\mathcal{G}}_B$, \mathcal{G}_F (see (10.2),(10.5) below) which otherwise would lead to a proliferation of terms as compared to the vacuum case.

In the conclusions section we summarize the properties of our new IBP algorithm, and discuss various applications, some of which have already been published or are actually in progress. We shortly comment on possible generalizations to gravity and string theory.

2 The P - representation

We will call ‘‘P-representation’’ the integrand obtained directly from the expansion of the Bern-Kosower master formula,

$$\begin{aligned} \Gamma_{\text{scal}}[k_1, \varepsilon_1; \dots; k_N, \varepsilon_N] &= (-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \\ &\times \prod_{i=1}^N \int_0^T d\tau_i P_N(\dot{G}_{Bij}, \ddot{G}_{Bij}) \exp\left\{ \sum_{i,j=1}^N \frac{1}{2} G_{Bij} k_i \cdot k_j \right\} \end{aligned} \quad (2.1)$$

Explicitly, the polynomial P_N is given by

$$\begin{aligned} P_N &= \dot{G}_{B1i_1 \varepsilon_1} \cdot k_{i_1} \dot{G}_{B2i_2 \varepsilon_2} \cdot k_{i_2} \cdots \dot{G}_{BNi_N \varepsilon_N} \cdot k_{i_N} \\ &- \sum_{\substack{a,b=1 \\ a < b}}^N \ddot{G}_{Bab \varepsilon_a \cdot \varepsilon_b} \dot{G}_{B1i_1 \varepsilon_1} \cdot k_{i_1} \cdots \widehat{\dot{G}_{Bai_a \varepsilon_a} \cdot k_{i_a}} \cdots \widehat{\dot{G}_{Bbi_b \varepsilon_b} \cdot k_{i_b}} \cdots \dot{G}_{BNi_N \varepsilon_N} \cdot k_{i_N} \\ &+ \sum_{\substack{a,b,c,d=1 \\ a < b < c < d}}^N (\ddot{G}_{Bab \varepsilon_a \cdot \varepsilon_b} \ddot{G}_{Bcd \varepsilon_c \cdot \varepsilon_d} + \ddot{G}_{Bac \varepsilon_a \cdot \varepsilon_c} \ddot{G}_{Bbd \varepsilon_b \cdot \varepsilon_d} + \ddot{G}_{Bad \varepsilon_a \cdot \varepsilon_d} \ddot{G}_{Bbc \varepsilon_b \cdot \varepsilon_c}) \\ &\times \widehat{\dot{G}_{B1i_1 \varepsilon_1} \cdot k_{i_1}} \cdots \widehat{\dot{G}_{Bai_a \varepsilon_a} \cdot k_{i_a}} \cdots \widehat{\dot{G}_{Bbi_b \varepsilon_b} \cdot k_{i_b}} \cdots \widehat{\dot{G}_{Bci_c \varepsilon_c} \cdot k_{i_c}} \cdots \widehat{\dot{G}_{Bdi_d \varepsilon_d} \cdot k_{i_d}} \cdots \dot{G}_{BNi_N \varepsilon_N} \cdot k_{i_N} \\ &- \dots \end{aligned} \quad (2.2)$$

Here and in the following the dummy indices i_1, i_2, \dots should be summed over from 1 to N , and a ‘hat’ denotes omission. Note that all terms in P_N are obtained from the first one by a simultaneous replacement of pairs of $\dot{G}_{B\hat{r}i_r \varepsilon_r} \cdot k_{i_r} \dot{G}_{B\hat{s}i_s \varepsilon_s} \cdot k_{i_s}$ by $-\ddot{G}_{Brs} \varepsilon_r \cdot \varepsilon_s$, which has to be done in all possible ways. Note also that $\dot{G}_{Bii} = 0$ by antisymmetry.

These P-representation integrals are still directly related to the ones arising in a standard Feynman or Schwinger parameter calculation of the N photon amplitude [10, 11]. The exponential factor will, after a multiple rescaling and performance of the global T - integration, turn into the standard one-loop N - point Feynman denominator polynomial. The δ - function contained in \ddot{G}_{Bij} will bring together the photons i and j , corresponding to a quartic vertex, and the contributions of such terms match the ones from the seagull vertex of scalar QED.

3 The R - representation

Before coming to the ‘old’ IBP procedure of [11, 37] and its intended improvement, it will be useful to solve a simpler problem, namely how to covariantize the Bern-Kosower master formula itself. In the following we will often abbreviate

$$e^{\frac{1}{2} \sum G_{Bij} k_i \cdot k_j} \equiv e^{(\cdot)} \quad (3.1)$$

Consider first the case of $N = 2$, where

$$P_2 = \dot{G}_{B12} \varepsilon_1 \cdot k_2 \dot{G}_{B21} \varepsilon_2 \cdot k_1 - \ddot{G}_{B12} \varepsilon_1 \cdot \varepsilon_2 \quad (3.2)$$

We choose two vectors r_1, r_2 that fulfill

$$r_i \cdot k_i \neq 0 \quad (3.3)$$

but are arbitrary otherwise. Adding to the integrand $P_2 e^{(\cdot)}$ the following sum of total derivative terms,

$$-\frac{r_1 \cdot \varepsilon_1}{r_1 \cdot k_1} \partial_1 \left(\dot{G}_{B21} \varepsilon_2 \cdot k_1 e^{(\cdot)} \right) - \frac{r_2 \cdot \varepsilon_2}{r_2 \cdot k_2} \partial_2 \left(\dot{G}_{B12} \varepsilon_1 \cdot k_2 e^{(\cdot)} \right) + \frac{r_1 \cdot \varepsilon_1}{r_1 \cdot k_1} \frac{r_2 \cdot \varepsilon_2}{r_2 \cdot k_2} \partial_1 \partial_2 e^{(\cdot)} \quad (3.4)$$

($\partial_i \equiv \frac{\partial}{\partial \tau_i}$) the total result is a change of P_2 into R_2 ,

$$R_2 := \dot{G}_{B12} \frac{r_1 \cdot F_1 \cdot k_2}{r_1 \cdot k_1} \dot{G}_{B21} \frac{r_2 \cdot F_2 \cdot k_1}{r_2 \cdot k_2} + \ddot{G}_{B12} \frac{r_1 \cdot F_1 \cdot F_2 \cdot r_2}{r_1 \cdot k_1 r_2 \cdot k_2} \quad (3.5)$$

Thus we have managed to absorb the polarization vectors into field strength tensors. And this procedure can be immediately generalized to the N -point case: let us abbreviate

$$\rho_i := \frac{r_i \cdot \varepsilon_i}{r_i \cdot k_i} \quad (3.6)$$

$$T_r(i) := \sum_j \dot{G}_{Bij} \frac{r_i \cdot F_i \cdot k_j}{r_i \cdot k_j} \quad (3.7)$$

$$W_r(ij) := \ddot{G}_{Bij} \frac{r_i \cdot F_i \cdot F_j \cdot r_j}{r_i \cdot k_i k_j \cdot r_j} \quad (3.8)$$

and choose vectors r_1, \dots, r_N fulfilling (3.3). Then it is a matter of simple combinatorics to verify that

$$P_N e^{(\cdot)} + \left[\prod_{a=1}^N (1 - \rho_a \partial_a \Delta_a) - 1 \right] \left[P_N e^{(\cdot)} \right] = P_N \left(\dot{G}_{Bai_a} \varepsilon_a \cdot k_{i_a} \rightarrow T_r(a), -\ddot{G}_{Bab} \varepsilon_a \cdot \varepsilon_b \rightarrow W_r(ab) \right) e^{(\cdot)} \quad (3.9)$$

where the operator Δ_a is defined as follows: each term in $P_N e^{(\cdot)}$ either involves the index a in a second derivative factor \ddot{G}_B , or it carries a factor of $\dot{G}_{Bai_a} \varepsilon_a \cdot k_{i_a}$. In the former case the term will be annihilated by Δ_a , in the latter case the action of Δ_a is to replace the factor of $\dot{G}_{Bai_a} \varepsilon_a \cdot k_{i_a}$ by 1.

We can then reexponentiate the new integrand, and arrive at the following covariantized version of the Bern-Kosower master formula (1.1):

$$\begin{aligned} \Gamma_{\text{scal}}[k_1, \varepsilon_1; \dots; k_N, \varepsilon_N] &= (-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \\ &\times \exp \left\{ \sum_{i,j=1}^N \left[\frac{1}{2} G_{Bij} k_i \cdot k_j - i \dot{G}_{Bij} \frac{r_i \cdot F_i \cdot k_j}{r_i \cdot k_i} - \frac{1}{2} \ddot{G}_{Bij} \frac{r_i \cdot F_i \cdot F_j \cdot r_j}{r_i \cdot k_i r_j \cdot k_j} \right] \right\} \Big|_{\text{lin}(F_1, \dots, F_N)} \end{aligned} \quad (3.10)$$

Thus we have achieved manifest gauge invariance at the integrand level, with a large freedom of choosing the vectors r_1, \dots, r_N . We will call this the ‘‘R-representation’’ of the N - photon amplitudes. Note that it reduces to the original master formula (1.1) if $r_i \cdot \varepsilon_i = 0$ for all i .

4 The Q and Q' - representations

Next, we review the IBP procedure motivated by the Bern-Kosower rules, whose primary purpose is to get rid of all second derivatives \ddot{G}_B [7, 36, 37].

For the two-point case P_2 has been written down already in (3.2). After an IBP of the second term in either τ_1 or τ_2 , and using $\dot{G}_{B12} = -\dot{G}_{B21}$, it turns into

$$Q_2 = \dot{G}_{B12} \dot{G}_{B21} (\varepsilon_1 \cdot k_2 \varepsilon_2 \cdot k_1 - \varepsilon_1 \cdot \varepsilon_2 k_1 \cdot k_2) = \dot{G}_{B12} \dot{G}_{B21} Z_2(12) \quad (4.1)$$

Proceeding to the three-point case, here (2.2) becomes

$$\begin{aligned}
P_3 &= \dot{G}_{B1i}\varepsilon_1 \cdot k_i \dot{G}_{B2j}\varepsilon_2 \cdot k_j \dot{G}_{B3k}\varepsilon_3 \cdot k_k \\
&\quad - \left[\ddot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \dot{G}_{B3i}\varepsilon_3 \cdot k_i + (1 \rightarrow 2 \rightarrow 3) + (1 \rightarrow 3 \rightarrow 2) \right]
\end{aligned} \tag{4.2}$$

In this three-point case it is still possible to remove all \ddot{G}_B 's in a single step. To remove, e. g., the term involving $\ddot{G}_{B12}\dot{G}_{B31}$ in the second term of P_3 , we can add the total derivative term

$$- \partial_2 \left(\dot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \dot{G}_{B31}\varepsilon_3 \cdot k_1 e^{(\cdot)} \right) \tag{4.3}$$

This term together with five similar ones removes all the \ddot{G}_B 's. Decomposing the new integrand according to its ‘‘cycle content’’, P_3 gets replaced by $Q_3 = Q_3^3 + Q_3^2$, where

$$\begin{aligned}
Q_3^3 &= \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123), \\
Q_3^2 &= \dot{G}_{B12}\dot{G}_{B21}Z_2(12)T(3) + \dot{G}_{B13}\dot{G}_{B31}Z_2(13)T(2) + \dot{G}_{B23}\dot{G}_{B32}Z_2(23)T(1)
\end{aligned} \tag{4.4}$$

Note that Q_3^3 contains a cycle of length three and Q_3^2 a cycle of length two, as indicated by the upper indices, and that each τ -cycle appears together with the corresponding ‘‘Lorentz-cycle’’. This motivates the further definition of a ‘‘bicycle’’ as the product of the two:

$$\dot{G}(i_1 i_2 \cdots i_n) := \dot{G}_{B i_1 i_2} \dot{G}_{B i_2 i_3} \cdots \dot{G}_{B i_n i_1} Z_n(i_1 i_2 \cdots i_n) \tag{4.5}$$

But the terms of Q_3^2 have, apart from the cycle, also a ‘‘one-tail’’, defined by

$$T(a) := \dot{G}_{B a i} \varepsilon_a \cdot k_i \tag{4.6}$$

This tail still has a polarisation vector that is not absorbed into a field strength tensor. It is easy to see that, nonetheless, each term in Q_3^2 is individually gauge invariant; if one replaces in, e.g., the term

$$\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\dot{G}_{B3k}\varepsilon_3 \cdot k_k e^{(\cdot)}$$

ε_3 by k_3 , then it becomes proportional to

$$\partial_3 \left(\dot{G}_{B12}\dot{G}_{B21}Z_2(12) e^{(\cdot)} \right)$$

However, our aim here is to make gauge invariance manifest even at the integrand level. Now in the three-point case there are already various chains of IBP that can be used to remove all the \ddot{G}_B 's, but if one assumes that the corresponding total derivative terms are added with constant

coefficients (i.e., they involve no dependences on momentum or polarization other than the ones already present in the term which one wishes to modify), then it is easy to convince oneself that they all lead to the same Q_3 of (4.4). Thus we have to look for a more general type of IBP. We will now essentially apply the procedure of the previous section to the tails. Consider again the first term in Q_3^2 above, eq. (4.4). Choose a momentum vector r_3 such that $r_3 \cdot k_3 \neq 0$, and add the total derivative

$$-\frac{r_3 \cdot \varepsilon_3}{r_3 \cdot k_3} Z_2(12) \partial_3 \left(\dot{G}_{B12} \dot{G}_{B21} e^{(\cdot)} \right) \quad (4.7)$$

The addition of this term to the first term in Q_3^2 , and of similar terms to the second and third one, transforms Q_3^2 into

$$\begin{aligned} R_3^2 := & \dot{G}_{B12} \dot{G}_{B21} Z_2(12) \dot{G}_{B3k} \frac{r_3 \cdot F_3 \cdot k_k}{r_3 \cdot k_3} + \dot{G}_{B13} \dot{G}_{B31} Z_2(13) \dot{G}_{B2j} \frac{r_2 \cdot F_2 \cdot k_j}{r_2 \cdot k_2} \\ & + \dot{G}_{B23} \dot{G}_{B32} Z_2(23) \dot{G}_{B1i} \frac{r_1 \cdot F_1 \cdot k_i}{r_1 \cdot k_1} \end{aligned} \quad (4.8)$$

Thus we have completed the covariantization of the integrand.

In the abelian case the 3-point amplitude must, of course, vanish, which we can see by noting that the integrand is odd under the orientation-reversing transformation of variables $\tau_i = T - \tau'_i$, $i = 1, 2, 3$.

Proceeding to the four-point case, here even using only total derivative terms with constant coefficients (in the above sense) there are already many ways to remove the \ddot{G}_B 's by IBP, with a large ambiguity for the final integrand, and it is not obvious how one should proceed. But certainly one would like to preserve the manifest permutation symmetry of the N -photon amplitudes, and in [37] this was used as a guiding principle to develop the following algorithm:

1. In every step, partially integrate away *all* the second derivative factors \ddot{G}_{Bij} 's appearing in the term under inspection simultaneously. This is possible since different \ddot{G}_{Bij} 's never share variables.
2. In the first step, for every factor of \ddot{G}_{Bij} present use both τ_i and τ_j for the IBP, and take the mean of the results.
3. At every following step, any \ddot{G}_{Bij} appearing must have been created in the previous step. Therefore either both variables τ_i and τ_j were used in the previous step, or just one of them. If both were used, then both should be used again in the actual IBP step, and the mean of the results be taken. If only one of the variables was used in the previous step, then the other variable should be used in the actual step.

In the four-point case, applying this algorithm to P_4 and decomposing the resulting integrand according to cycle content, leads to the following version of the numerator polynomial Q_4 [37]:

$$\begin{aligned}
Q_4 &= Q_4^4 + Q_4^3 + Q_4^2 - Q_4^{22} \\
Q_4^4 &= \dot{G}(1234) + \dot{G}(1243) + \dot{G}(1324) \\
Q_4^3 &= \dot{G}(123)T(4) + \dot{G}(234)T(1) + \dot{G}(341)T(2) + \dot{G}(412)T(3) \\
Q_4^2 &= \dot{G}(12)T(34) + \dot{G}(13)T(24) + \dot{G}(14)T(23) + \dot{G}(23)T(14) + \dot{G}(24)T(13) + \dot{G}(34)T(12) \\
Q_4^{22} &= \dot{G}(12)\dot{G}(34) + \dot{G}(13)\dot{G}(24) + \dot{G}(14)\dot{G}(23)
\end{aligned} \tag{4.9}$$

Here we have now further introduced the two-tail,

$$T(ij) := \sum_{r,s} \left\{ \dot{G}_{Bir}\varepsilon_i \cdot k_r \dot{G}_{Bjs}\varepsilon_j \cdot k_s + \frac{1}{2} \dot{G}_{Bij}\varepsilon_i \cdot \varepsilon_j \left[\dot{G}_{Bir}k_i \cdot k_r - \dot{G}_{Bjr}k_j \cdot k_r \right] \right\} \tag{4.10}$$

Thus the final representation of the (still off-shell) four-photon amplitude in scalar QED becomes

$$\begin{aligned}
\Gamma_{\text{scal}}[k_1, \varepsilon_1; \dots; k_4, \varepsilon_4] &= \frac{e^4}{(4\pi)^{\frac{D}{2}}} (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} T^{-\frac{D}{2}} e^{-m^2 T} \\
&\times \int_0^T d\tau_1 \cdots d\tau_4 Q_4(\dot{G}_{Bij}) \exp \left\{ \frac{T}{2} \sum_{i,j=1}^4 G_{Bij} k_i \cdot k_j \right\}
\end{aligned} \tag{4.11}$$

We shortly summarize the advantages of this representation compared to a standard Feynman/Schwinger parameter integral representation (see [32] for details):

First, the rhs of (4.11) represents already the complete amplitude, with no need to add ‘‘crossed’’ terms. The summation over ‘‘crossed’’ diagrams which would have to be done in a standard field theory calculation here is implicit in the integration over the various ordered sectors.

Second, the IBP procedure has homogenized the integrand; every term in Q_N has N factors of \dot{G}_{Bij} and N factors of external momentum. In the four-point case this has the additional advantage of making the UV finiteness of the photon-photon scattering amplitude manifest before integration. While the original numerator P_4 contains terms involving products of two \ddot{G}_{Bij} ’s which lead to spurious divergences in the T - integration, after the IBP the integrand is finite term by term.

Third, It allows one to obtain the corresponding spinor QED amplitude by the application of the replacement rule (1.4). In applying the rules it must be observed, though, that the form of the integrand given above still contains, apart from the explicit cycle factors, additional cycles from the tail factors for certain values of the dummy indices. In the four – point case this occurs for Q_4^2 only: The two – tails contained in Q_4^2 as given in (4.9) above each contain a two-cycle, since the content of the braces on the rhs of (4.10) for $r = j, s = i$ turns into

$$\dot{G}_{Bij}\dot{G}_{Bji} \left(\varepsilon_i \cdot k_j \varepsilon_j \cdot k_i - \varepsilon_i \cdot \varepsilon_j k_i \cdot k_i \right) = \dot{G}(ij) \tag{4.12}$$

For the application of the “replacement rules” it is therefore convenient to decompose Q_4 in a slightly different way [37]. Namely, note that

$$Q_4^2 = Q_4'^2 + 2Q_4^{22} \quad (4.13)$$

where $Q_4'^2$ is obtained from Q_4^2 by eliminating the term with $r = j, s = i$ from the sum over dummy indices. With this definition, and setting $Q_4^{(\cdot)} = Q_4^{(\cdot)}$ for the remaining components, we can write

$$Q_4 = Q_4'^4 + Q_4'^3 + Q_4'^2 + Q_4'^{22} \quad (4.14)$$

In this form all cycle factors are explicit, and moreover all the coefficients in the decomposition turn out to be unity. Generally, we will denote by $T'(i_1 \dots i_n)$ a tail whose cycles have been removed by the appropriate restrictions on the multiple dummy sums appearing in it.

Thus the four-photon amplitude for the spinor loop case can now be obtained from the scalar loop formula (4.11) simply by multiplying with a global factor of -2 from spin and statistics, and by replacing, simultaneously, each bicycle $\dot{G}(i_1 \dots i_n)$ by the corresponding “super-bicycle”

$$\dot{G}_S(i_1 \dots i_n) := (\dot{G}_{Bi_1i_2} \dot{G}_{Bi_2i_3} \dots \dot{G}_{Bi_ni_1} - G_{Fi_1i_2} G_{Fi_2i_3} \dots G_{Fi_ni_1}) Z_n(i_1 \dots i_n) \quad (4.15)$$

(the notation refers to the worldline supersymmetry underlying the replacement rule (1.4), see [32]). This “symmetric partial integration” procedure has been worked out explicitly for up to the six-photon case; see [32, 37] for the explicit formulas.

5 The QR representation

So far we have established two seemingly unrelated IBP procedures, the first one leading to the manifestly gauge invariant R-representation, the second one to the Q' - representation that is suitable for the application of the Bern-Kosower rules, but manifestly gauge invariant only in the cycle factors, not in the tails. We will now combine the two IBP strategies, using the following three simple observations:

First, it had been noted in the appendix C of [32] that the Q - representation is recursive, in the following sense: Each term in the cycle decomposition of Q_N contains at least one cycle [32], so that any tail appearing in the N - point amplitude has at most $N - 2$ arguments. And a tail of length, say, M , is related to the (undecomposed) lower-order Q_M simply by writing Q_M in the tail variables, and then extending the range of all dummy variables occurring in it to run over the full set of indices $1, \dots, N$. For example, writing out (4.10) for the two-tail $T(12)$ in the last term of Q_4^2 in (4.9) gives

$$T(12) = \sum_{r,s=1}^4 \dot{G}_{B1r\varepsilon_1} \cdot k_r \dot{G}_{B2s\varepsilon_2} \cdot k_s + \frac{1}{2} \dot{G}_{B12\varepsilon_1} \cdot \varepsilon_2 \sum_{r=1}^4 \left[\dot{G}_{B1r} k_1 \cdot k_r - \dot{G}_{B2r} k_2 \cdot k_r \right] \quad (5.1)$$

which can also be obtained by writing Q_2 , defined in (4.1), as

$$Q_2(12) \equiv \dot{G}_{B12}\varepsilon_1 \cdot k_2 \dot{G}_{B21}\varepsilon_2 \cdot k_1 + \frac{1}{2}\dot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \left[\dot{G}_{B12}k_1 \cdot k_2 - \dot{G}_{B21}k_2 \cdot k_1 \right] \quad (5.2)$$

and introducing appropriate dummy index summations. That this property holds in general can be easily seen by considering those terms in the cycle factors of the decomposition of Q_M that do not contain factors of $\varepsilon_i \cdot \varepsilon_j$, and thus can have involved IBPs only in the tail and not in the cycle variables; see the appendix C of [32] for more details.

Second, for each N the symmetric IBP procedure defines a unique Q_N and thus a total derivative term

$$S_N e^{(\cdot)} \equiv Q_N e^{(\cdot)} - P_N e^{(\cdot)} \quad (5.3)$$

Consider now an arbitrary term in the cycle decomposition of Q_N . It will have the form $C(i_1 \dots i_L)T(j_1 \dots j_M)$, where $M \leq N-2$, $C(i_1 \dots i_L)$ is a bicycle or product of bicycles involving the variables $\tau_{i_1}, \dots, \tau_{i_L}$, and $T(j_1 \dots j_M)$ is the unique (in the symmetric IBP scheme) tail of M variables, written in the remaining variables $\tau_{j_1}, \dots, \tau_{j_M}$. Consider $C(i_1 \dots i_L)S_M(j_1 \dots j_M) e^{(\cdot)}$, where possible dummy variable summations in S_M are extended to run over the full range of variables τ_1, \dots, τ_N as above. This is still a total derivative term (involving only derivatives in the tail variables), and the above simple relation between the M - tail and Q_M implies, that

$$C(i_1 \dots i_L)T(j_1 \dots j_M) e^{(\cdot)} - C(i_1 \dots i_L)S_M(j_1 \dots j_M) e^{(\cdot)} = C(i_1 \dots i_L)T_p(j_1 \dots j_M) e^{(\cdot)} \quad (5.4)$$

with a new version T_p of the M - tail which relates to P_M in the same way as the standard tail T to Q_M , i.e. by an extension of the dummy index sums.

Continuing with our example above, here we have

$$\begin{aligned} S_2 e^{G_{B12}k_1 \cdot k_2} &\equiv (Q_2 - P_2) e^{G_{B12}k_1 \cdot k_2} \\ &= \left\{ \frac{1}{2}\dot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \left[\dot{G}_{B12}k_1 \cdot k_2 - \dot{G}_{B21}k_2 \cdot k_1 \right] + \ddot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \right\} e^{G_{B12}k_1 \cdot k_2} \\ &= \frac{1}{2}(\partial_1 - \partial_2) \left(\dot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 e^{G_{B12}k_1 \cdot k_2} \right) \end{aligned} \quad (5.5)$$

and (5.4) becomes

$$\dot{G}(34)T(12) e^{(\cdot)} - \frac{1}{2}(\partial_1 - \partial_2) \left(\dot{G}(34)\dot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 e^{(\cdot)} \right) = \dot{G}(34)T_p(12) e^{(\cdot)} \quad (5.6)$$

where $T(12)$ was given in (5.1) and $T_p(12)$ is given by

$$T_p(12) = \sum_{r,s=1}^4 \dot{G}_{B1r}\varepsilon_1 \cdot k_r \dot{G}_{B2s}\varepsilon_2 \cdot k_s - \ddot{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \quad (5.7)$$

Finally, the new tail $T_p(\dots)$ can now be covariantized by a simple extension of (3.9):

$$\prod_{a=1}^M (1 - \rho_{j_a} \partial_{j_a} \Delta_{j_a}) \left(C(\cdot) T_p(j_1 \cdots j_M) e^{(\cdot)} \right) = C(\cdot) T_p \left(\dot{G}_{Bai_a} \varepsilon_a \cdot k_{i_a} \rightarrow T_r(a), -\ddot{G}_{Bab} \varepsilon_a \cdot \varepsilon_b \rightarrow W_r(ab) \right) e^{(\cdot)} \quad (5.8)$$

In our two-tail example, this lead to the following result:

$$T_r(12) = \dot{G}_{B1r} \frac{r_1 \cdot F_1 \cdot k_r}{r_1 \cdot k_1} \dot{G}_{B2s} \frac{r_2 \cdot F_2 \cdot k_s}{r_2 \cdot k_2} + \ddot{G}_{B12} \frac{r_1 \cdot F_1 \cdot F_2 \cdot r_2}{r_1 \cdot k_1 k_2 \cdot r_2} \quad (5.9)$$

where the sums over r and s run from 1 to 4. This should be compared with R_2 of (3.5).

6 The S representation

Thus in the QR - representation we have the usual decomposition into cycle and tail factors, with the tails already covariantized, and in a form that generalizes the (lower order) R - representation by an extension of the dummy index sums to run over all N variables, including those belonging to the cycle factors of the term under consideration. To finish our quest for an integrand that would be both covariant and suitable for the application of the Bern-Kosower rules, two more steps are needed: First, we need to remove the remaining \dot{G}_B 's; this can be done by reapplying the symmetric IBP procedure of chapter 4, without any modifications. And finally all cycles still contained in the tails have to be separated out. We will call this final, in some sense optimized result for the integrand of the N photon/gluon amplitudes, the ‘‘S - representation’’, and denote the corresponding tails by $T'_s(\cdot)$.

Continuing with our example of the two-tail in Q_4^2 , the first step transforms $T_r(12)$ of (5.9) into

$$\begin{aligned} T_s(12) &\equiv \sum_{r,s=1}^4 \dot{G}_{B1r} \frac{r_1 \cdot F_1 \cdot k_r}{r_1 \cdot k_1} \dot{G}_{B2s} \frac{r_2 \cdot F_2 \cdot k_s}{r_2 \cdot k_2} \\ &\quad - \frac{1}{2} \dot{G}_{B12} \left(\sum_{r=1}^4 \dot{G}_{B1r} k_1 \cdot k_r - \sum_{s=1}^4 \dot{G}_{B2s} k_2 \cdot k_s \right) \frac{r_1 \cdot F_1 \cdot F_2 \cdot r_2}{r_1 \cdot k_1 k_2 \cdot r_2} \end{aligned} \quad (6.1)$$

This form of the two-tail, like the original two-tail $T(12)$ of (4.10), still contains a cycle - the terms on the rhs with $r = 2$ and $s = 1$ combine to form a $\dot{G}(12)$, as in (4.12). Eliminating these terms from the tail one arrives at the final form,

$$\begin{aligned}
T'_s(12) \equiv & \sum_{\substack{r,s=1 \\ r,s \neq (2,1)}}^4 \dot{G}_{B1r} \frac{r_1 \cdot F_1 \cdot k_r}{r_1 \cdot k_1} \dot{G}_{B2s} \frac{r_2 \cdot F_2 \cdot k_s}{r_2 \cdot k_2} \\
& - \frac{1}{2} \dot{G}_{B12} \left(\sum_{\substack{r=1 \\ r \neq 2}}^4 \dot{G}_{B1r} k_1 \cdot k_r - \sum_{\substack{s=1 \\ s \neq 1}}^4 \dot{G}_{B2s} k_1 \cdot k_s \right) \frac{r_1 \cdot F_1 \cdot F_2 \cdot r_2}{r_1 \cdot k_1 k_2 \cdot r_2}
\end{aligned} \tag{6.2}$$

For the one-tail there is no difference between $T_r(i)$, $T_s(i)$ and $T'_s(i)$, being all given by (3.7).

We can now write down a covariantized version of the four photon amplitude, simply by taking over (4.9), (4.11), and (4.14) and replacing all one-tails by (3.7) and all two-tails by (6.2). Explicit formulas for higher point amplitudes will be given elsewhere.

7 Alternative version of the two-tail

For the two-tail, there is actually yet another form which is covariant, free of \ddot{G}_B 's and at the same time more compact than (6.1). Starting again with $T_p(12)$ of (5.7) we add the following total derivative term to $T_p(12)$ (omitting now the inert cycle factors $C(\cdot)$)

$$\frac{1}{(k_1 \cdot k_2)^2} \text{tr}(F_1 F_2) \partial_1 \partial_2 e^{(\cdot)} + \frac{1}{k_1 \cdot k_2} \left[\varepsilon_1 \cdot \varepsilon_2 \partial_1 \partial_2 e^{(\cdot)} - \varepsilon_1 \cdot k_2 \varepsilon_2 \cdot k_j \partial_1 \left(\dot{G}_{B2j} e^{(\cdot)} \right) - \varepsilon_2 \cdot k_1 \varepsilon_1 \cdot k_i \partial_2 \left(\dot{G}_{B1i} e^{(\cdot)} \right) \right] \tag{7.1}$$

One obtains the new two-tail

$$T_H(12) \equiv \dot{G}_{B1i} \dot{G}_{B2j} k_i \cdot H_{12} \cdot k_j \tag{7.2}$$

where we have introduced the tensor

$$H_{12}^{\mu\nu} \equiv \frac{(F_1 F_2)^{\mu\nu} k_1 \cdot k_2 - k_1^\mu k_2^\nu \text{tr}(F_1 F_2)}{(k_1 \cdot k_2)^2} \tag{7.3}$$

Note that $\text{tr} H_{12} = 0$ and $H_{12}^T = H_{21}$. Note also that the term with $i = 2, j = 1$ in (7.2) as before produces a $\dot{G}(12)$, since

$$k_2 \cdot H_{12} \cdot k_1 = \frac{1}{2} \text{tr}(F_1 F_2) - \text{tr}(F_1 F_2) = -\frac{1}{2} \text{tr}(F_1 F_2) = -Z_2(12) \tag{7.4}$$

Thus $T_H(ij)$ can be used as well as $T(ij)$ and $T_s(ij)$ in the construction of the four-point amplitudes, including the application of the replacement rule (1.4) and the simple sign change in passing from (4.9) to (4.14). However, contrary to $T_s(ij)$ it appears that $T_H(ij)$ has no natural generalization to the higher-point tails.

8 The case of spinor QED

One of the main purposes of the IBP procedure is to trivialize the transition to the spinor QED case though the replacement rule (1.4). Still, it is interesting to note (and of possible practical relevance for the generalization to the case of open fermion lines, where the IBP is less attractive due to the existence of boundary terms) that in the worldline formalism there is also a more direct treatment of the spin $\frac{1}{2}$ case using an approach based on explicit worldline supersymmetry [40, 17, 32, 38, 41]. It allows one to write down a master formula for N – photon scattering [40] which is formally analogous to the one for the scalar loop, eq.(1.1):

$$\begin{aligned} \Gamma_{\text{spin}}[k_1, \varepsilon_1; \dots; k_N, \varepsilon_N] &= -2(-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \\ &\times \prod_{i=1}^N \int_0^T d\tau_i \int d\theta_i \exp \left\{ \sum_{i,j=1}^N \left[\frac{1}{2} \hat{G}_{ij} k_i \cdot k_j + i D_i \hat{G}_{ij} \varepsilon_i \cdot k_j + \frac{1}{2} D_i D_j \hat{G}_{ij} \varepsilon_i \cdot \varepsilon_j \right] \right\} \Bigg|_{\text{lin}(\varepsilon_1, \dots, \varepsilon_N)} \end{aligned} \quad (8.1)$$

Here we have further introduced integrals over the Grassmann variables $\theta_1, \dots, \theta_N$, such that $\int d\theta_i \theta_i = 1$, and the super derivative

$$D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial \tau} \quad (8.2)$$

The two worldline Green's functions $G_{B,F}$ now appear combined in the super Green's function

$$\hat{G}(\tau_1, \theta_1; \tau_2, \theta_2) \equiv G_B(\tau_1, \tau_2) + \theta_1 \theta_2 G_F(\tau_1, \tau_2) \quad (8.3)$$

Now also the polarization vectors $\varepsilon_1, \dots, \varepsilon_N$ are to be treated as Grassmann variables. The overall sign of the master formula refers to the standard ordering of the polarization vectors $\varepsilon_1 \varepsilon_2 \dots \varepsilon_N$.

Starting with this master formula, all the manipulations which we have applied in the previous chapters to the scalar loop integrands can, *mutatis mutandis*, also be used in the spinor loop case starting from (8.1). Here we will be satisfied with pointing out that the covariantized Bern-Kosower master formula (3.10) generalizes to the spinor QED case as follows:

$$\begin{aligned} \Gamma_{\text{spin}}[k_1, \varepsilon_1; \dots; k_N, \varepsilon_N] &= -2(-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \\ &\times \prod_{i=1}^N \int_0^T d\tau_i \int d\theta_i \exp \left\{ \sum_{i,j=1}^N \left[\frac{1}{2} \hat{G}_{ij} k_i \cdot k_j + i D_i \hat{G}_{ij} \frac{r_i \cdot F_i \cdot k_j}{r_i \cdot k_i} - \frac{1}{2} D_i D_j \hat{G}_{ij} \frac{r_i \cdot F_i \cdot F_j \cdot r_j}{r_i \cdot k_i r_j \cdot k_j} \right] \right\} \Bigg|_{\text{lin}(F_1, \dots, F_N)} \end{aligned} \quad (8.4)$$

where now F_1, \dots, F_N have to be treated as Grassmann variables.

9 The nonabelian case

As far as concerns the calculation of the on-shell QED photon amplitudes, or of the on-shell gluon S-matrix elements via the Bern-Kosower rules, the availability of a representation that is manifestly transversal at the integrand level is satisfying, but the significance of this fact for practical calculations is not obvious. To the contrary, it is easy to recognize the advantages of such a representation when it comes to the gluon amplitudes off-shell. Here the natural objects to consider in QCD are the “N-vertices”, that is the one-particle-irreducible N - point functions, and for applications of those it is often essential to decompose them into a basis of transversal and longitudinal tensor structures (see, e.g., [39, 42, 43, 44]). Such a “transversality-based form factor decomposition” in the present approach emerges essentially automatically in the IBP procedure through the appearance of field strength tensors. We have seen how this happens for the abelian case, but it is true also for the nonabelian case; here in principle one would like to see the full nonabelian field strength tensor emerging,

$$F_{\mu\nu} \equiv F_{\mu\nu}^a T^a = F_{\mu\nu}^0 + ig[A_\mu^b T^b, A_\nu^c T^c] \quad (9.1)$$

where by

$$F_{\mu\nu}^0 \equiv (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) T^a \quad (9.2)$$

we now denote its “abelian part”; and indeed Strassler demonstrated already for some simple cases how this happens [11, 36]: When the external particles are gluons, the various ordered sectors of the integral $\int d\tau_1 \cdots d\tau_N$ need to be considered separately, since they carry different color factors. Therefore boundary terms now arise in the IBP procedure, and the commutator terms are generated as differences of boundary terms between adjacent sectors that in the abelian case would cancel, but cannot do so any more in the presence of color since two of the color matrices appear in different orders. In an x -space calculation of the effective action, those commutator terms could then be combined with the “abelian” parts of the field strength tensor, but this is not possible in a momentum space calculation of the N - point function at fixed N , since any term in the nonabelian effective action after Fourier transformation contributes to amplitudes with various numbers of external particles; e.g., the term $\text{tr}(D_\mu F_{\alpha\beta} D^\mu F^{\alpha\beta})$ will contribute to the N - point functions with N between two and six. Generally, each term in the nonabelian effective Lagrangian has a “core” term, which has a counterpart already in the abelian case (in the example this would be $\partial_\mu F_{\alpha\beta}^0 \partial^\mu F^{0\alpha\beta}$) and a number of “covariantizing” terms that all involve commutators, and belong to amplitudes with more legs than the core term. In this section, we will explain the essentials of how to calculate the scalar, spinor and gluon loop contributions to the one-loop N - gluon vertex. The details and a full recalculation of the three-gluon-vertex will be left to a separate paper [45].

Starting with the scalar loop case, here the master formula (1.1) generalizes to the nonabelian case simply by supplying a global color factor, and keeping the gluons in a fixed order:

$$\begin{aligned}
\Gamma_{\text{1PI,scal}}^{a_1 \dots a_N}[k_1, \varepsilon_1; \dots; k_N, \varepsilon_N] &= (-ig)^N \text{tr}(T^{a_1} \dots T^{a_N}) (2\pi)^D i\delta(\sum k_i) \int_0^\infty dT (4\pi T)^{-D/2} e^{-m^2 T} \\
&\times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{N-2}} d\tau_{N-1} \exp \left\{ \sum_{i,j=1}^N \left[\frac{1}{2} G_{Bij} k_i \cdot k_j - i\dot{G}_{Bij} \varepsilon_i \cdot k_j + \frac{1}{2} \ddot{G}_{Bij} \varepsilon_i \cdot \varepsilon_j \right] \right\} \Big|_{\text{lin}(\varepsilon_1, \dots, \varepsilon_N)}
\end{aligned} \tag{9.3}$$

Here the T^a are the generators of the gauge group in the representation of the loop scalar. This treatment of color corresponds to a “color-ordered” representation (although not necessarily in the usual sense, where the T^a would be in the fundamental representation, see, e.g., [46, 47]). Note that we have not only fixed the ordering of the gluons along the loop but also used the translation invariance in τ to set $\tau_N = 0$. Summation over all $(N-1)!$ inequivalent orderings of the N vertex operators is implied. Also, Γ has been given an index “1PI” to indicate that the rhs gives only the one-particle-irreducible part of the N - gluon amplitude, not including the reducible part that now also exists, differently from the abelian case. Starting from (9.3) one can apply the IBP procedure leading from the P - representation to the S - representation as before, the only novelty being the boundary terms. Since for the bulk terms all the polarization vectors ε_i get absorbed into the transversal structures (1.5) (which now, however, represent only the “abelian part” of the field strength tensor), in the final representation the non-transversal part of the N - vertex must be entirely in the boundary terms, given by lower-point integrals. For those one still has to choose the vectors r_i , preferably in a way that is consistent with the cyclic invariance of the nonabelian amplitudes. In the three-point case a convenient cyclic choice is $r_1 = k_2 - k_3$, $r_2 = k_3 - k_1$, $r_3 = k_1 - k_2$, and indeed it turns out [45] that, with this choice, the resulting form factor decomposition matches precisely the standard Ball-Chiu decomposition [39] of the three-gluon-vertex.

Coming to the spinor loop case, here the only issue is whether the replacement rule (1.4) can be applied also to all the boundary terms now arising in the IBP procedure. This is indeed the case, as we can see as follows: it suffices to show the corresponding statement for the effective action, rather than the momentum space Green’s functions. Now, the effective Lagrangian can in principle be written as an infinite series of terms that are Lorentz scalars formed using any number of field strength tensors and covariant derivatives. As was already mentioned, each such term has a core term, whose calculation is not different from the abelian case for either the scalar or spinor loop, such that the replacement rule applies to it. All covariantizing terms of a core term must share its coefficient, and low-order calculations show that, as one would expect, the way this works is that they all involve the same parameter integral [36, 45]. And for the whole structure to continue to be gauge invariant for the spinor loop case it is necessary that the same replacement rule applies to all the covariantizing terms as well as for the core term.

Similarly one can convince oneself that also the replacement rules that connect the scalar with the gluon loop cases [8, 34, 32, 11, 21] can be extended from the core terms to the ones involving boundary contributions [45]. An additional issue with the gluon loop contribution to the N - vertex is that one has to choose a gauge for the gluon propagator. The application of the gluonic replacement rules gives the N - vertex corresponding to the use of the background field method with Feynman gauge for the quantum part [11, 21], which is also known to coincide with the result of the application of the pinch technique [48, 49, 50]. This version of the gluonic vertex is also the one that leads to SUSY sum rules [51].

10 The constant external field case

The Bern-Kosower master formula (1.1) has the following straightforward extension to the N -photon amplitude in a constant external field [20, 21, 22]:

$$\begin{aligned} \Gamma_{\text{scal}}[k_1, \epsilon_1; \dots; k_N, \epsilon_N] &= (-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty [4\pi T]^{-\frac{D}{2}} e^{-m^2 T} \det^{-\frac{1}{2}} \left[\frac{\sin \mathcal{Z}}{\mathcal{Z}} \right] \prod_{i=1}^N \int_0^T d\tau_i \\ &\times \exp \left\{ \sum_{i,j=1}^N \left[\frac{1}{2} k_i \cdot \mathcal{G}_{Bij} \cdot k_j - i \epsilon_i \cdot \dot{\mathcal{G}}_{Bij} \cdot k_j + \frac{1}{2} \epsilon_i \cdot \ddot{\mathcal{G}}_{Bij} \cdot \epsilon_j \right] \right\} \Big|_{\text{lin}(\epsilon_1, \dots, \epsilon_N)} \end{aligned} \quad (10.1)$$

Here we have introduced the matrix $\mathcal{Z}_{\mu\nu} \equiv eTF_{\mu\nu}$, $\mathcal{G}_{Bij} \equiv \mathcal{G}_B(\tau_i, \tau_j)$ denotes the ‘‘bosonic’’ worldline Green’s function in the constant field background, and $\dot{\mathcal{G}}_{Bij}, \ddot{\mathcal{G}}_{Bij}$ its first and second derivatives with respect to τ_i . Explicitly, they are given by [21, 22]

$$\begin{aligned} \mathcal{G}_{B12} &= \frac{T}{2\mathcal{Z}^2} \left(\frac{\mathcal{Z}}{\sin \mathcal{Z}} e^{-i\mathcal{Z}\dot{\mathcal{G}}_{B12}} + i\mathcal{Z}\dot{\mathcal{G}}_{B12} - 1 \right) \\ \dot{\mathcal{G}}_{B12} &= \frac{i}{\mathcal{Z}} \left(\frac{\mathcal{Z}}{\sin \mathcal{Z}} e^{-i\mathcal{Z}\dot{\mathcal{G}}_{B12}} - 1 \right) \\ \ddot{\mathcal{G}}_{B12} &= 2\delta(\tau_1 - \tau_2) - \frac{2}{T} \frac{\mathcal{Z}}{\sin \mathcal{Z}} e^{-i\mathcal{Z}\dot{\mathcal{G}}_{B12}} \end{aligned} \quad (10.2)$$

while the fermionic Green’s function \mathcal{G}_{Fij} generalizes to

$$\mathcal{G}_{F12} = G_{F12} \frac{e^{-i\mathcal{Z}\dot{\mathcal{G}}_{B12}}}{\cos \mathcal{Z}} \quad (10.3)$$

These expressions are to be understood as power series in the matrix \mathcal{Z} . We note the symmetry properties

$$\mathcal{G}_{Bji} = \mathcal{G}_{Bij}^T, \quad \dot{\mathcal{G}}_{Bji} = -\dot{\mathcal{G}}_{Bij}^T, \quad \ddot{\mathcal{G}}_{Bji} = \ddot{\mathcal{G}}_{Bij}^T, \quad \mathcal{G}_{Fji} = -\mathcal{G}_{Fij}^T \quad (10.4)$$

and the coincidence limits

$$\begin{aligned} \mathcal{G}_B(\tau, \tau) &= \frac{T}{2\mathcal{Z}^2} (\mathcal{Z} \cot \mathcal{Z} - 1) \\ \dot{\mathcal{G}}_B(\tau, \tau) &= i \cot \mathcal{Z} - \frac{i}{\mathcal{Z}} \\ \mathcal{G}_F(\tau, \tau) &= -i \tan \mathcal{Z} \end{aligned} \quad (10.5)$$

Note that those are independent of τ . For the following, it will also be convenient to introduce the “subtracted” Green’s function

$$\bar{\mathcal{G}}_B(\tau_1, \tau_2) := \mathcal{G}_B(\tau_1, \tau_2) - \mathcal{G}_B(\tau, \tau) = \frac{T}{2\mathcal{Z}} \left(\frac{e^{-i\mathcal{Z}\dot{\mathcal{G}}_{B12}} - \cos \mathcal{Z}}{\sin \mathcal{Z}} + i\dot{\mathcal{G}}_{B12} \right) \quad (10.6)$$

The corresponding spinor QED amplitude is obtained from this master formula in the same way as in the vacuum case, with the following minor modifications: First, note from (10.2),(10.5) that the worldline Green’s functions for the constant field are generally nontrivial Lorentz matrices. Thus the definition of a ‘cycle’ must now be slightly generalized; for example, a term

$$\varepsilon_1 \cdot \dot{\mathcal{G}}_{B12} \cdot k_2 \varepsilon_2 \cdot \dot{\mathcal{G}}_{B23} \cdot \varepsilon_3 k_3 \cdot \dot{\mathcal{G}}_{B31} \cdot k_1$$

would have to be replaced by

$$\varepsilon_1 \cdot \dot{\mathcal{G}}_{B12} \cdot k_2 \varepsilon_2 \cdot \dot{\mathcal{G}}_{B23} \cdot \varepsilon_3 k_3 \cdot \dot{\mathcal{G}}_{B31} \cdot k_1 - \varepsilon_1 \cdot \mathcal{G}_{F12} \cdot k_2 \varepsilon_2 \cdot \mathcal{G}_{F23} \cdot \varepsilon_3 k_3 \cdot \mathcal{G}_{F31} \cdot k_1$$

Second, the above nonvanishing coincidence limits of $\dot{\mathcal{G}}_B$ and \mathcal{G}_F must be considered as cycles of length one, and included in the replacement rule (1.4):

$$\dot{\mathcal{G}}_B(\tau_i, \tau_i) \rightarrow \dot{\mathcal{G}}_B(\tau_i, \tau_i) - \mathcal{G}_F(\tau_i, \tau_i) \quad (10.7)$$

Third, there is now also a replacement rule for the determinant factor:

$$\det^{-\frac{1}{2}} \left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}} \right] \rightarrow \det^{-\frac{1}{2}} \left[\frac{\tan(\mathcal{Z})}{\mathcal{Z}} \right] \quad (10.8)$$

This algorithm yields the one-loop N - photon amplitudes in a constant field for spinor QED.

Although the worldline formalism in a constant field has already found extensive applications [52, 53, 54, 55, 21, 56, 57, 58, 59] a systematic investigation of the IBP procedure for the constant field case has so far been lacking. Inspection of the symmetric partial integration algorithm of section 4 shows that it applies as well as to this case, and it leads to essentially the same decomposition of the integrand into cycles and tails, with only two modifications: First, one-cycles have to be included; and second, the definition of the “bicycle of length n ”, now to be denoted by $\dot{\mathcal{G}}(i_1 i_2 \dots i_n)$, has to be changed to

$$\begin{aligned} \dot{\mathcal{G}}(i) &:= \frac{1}{2} \text{tr}(F_i \cdot \dot{\mathcal{G}}_{Bii}) = \varepsilon_i \cdot \dot{\mathcal{G}}_{Bii} \cdot k_i \\ \dot{\mathcal{G}}(i_1 i_2) &:= \frac{1}{2} \text{tr}(F_{i_1} \cdot \dot{\mathcal{G}}_{B i_1 i_2} \cdot F_{i_2} \cdot \dot{\mathcal{G}}_{B i_2 i_1}) \\ &= \varepsilon_1 \cdot \dot{\mathcal{G}}_{B12} \cdot k_2 \varepsilon_2 \cdot \dot{\mathcal{G}}_{B21} \cdot k_1 - \varepsilon_1 \cdot \dot{\mathcal{G}}_{B12} \cdot \varepsilon_2 k_2 \cdot \dot{\mathcal{G}}_{B21} \cdot k_1 \\ \dot{\mathcal{G}}(i_1 i_2 \dots i_n) &:= \text{tr}(F_{i_1} \cdot \dot{\mathcal{G}}_{B i_1 i_2} \cdot F_{i_2} \cdot \dot{\mathcal{G}}_{B i_2 i_3} \dots F_{i_n} \cdot \dot{\mathcal{G}}_{B i_n i_1}) \quad (n \geq 3) \end{aligned} \quad (10.9)$$

These bicycles generalize the vacuum ones (4.5), and will turn into them in the limit $F \rightarrow 0$.

However, it turns out that the presence of one-cycles - which would lead to a significant proliferation of terms as compared to the vacuum case - can be altogether avoided, by the following

slight modification of the IBP: First, we observe that we have the freedom to shift both the Green's function \mathcal{G}_{Bij} and its derivatives $\dot{\mathcal{G}}_{Bij}$ by arbitrary constant matrices. For \mathcal{G}_{Bij} , which appears only in the first term in the exponent of the master formula (10.1), this is an obvious consequence of momentum conservation, $\sum_j k_j = 0$. For $\dot{\mathcal{G}}_{Bij}$, we note that it can appear in a $\varepsilon_i \cdot \dot{\mathcal{G}}_{Bij} \cdot k_j$, $k_i \cdot \dot{\mathcal{G}}_{Bij} \cdot k_j$ or $\varepsilon_i \cdot \dot{\mathcal{G}}_{Bij} \cdot \varepsilon_j$. The first type of terms comes directly from the second term in the master formula, and here again a constant matrix added to $\dot{\mathcal{G}}_{Bij}$ drops out immediately because of momentum conservation. The second type of terms arises in the integration-by-parts procedure as a

$$\partial_i \sum_{l,j=1}^N \frac{1}{2} G_{Blj} k_l \cdot k_j = \sum_{j=1}^N k_i \cdot \dot{\mathcal{G}}_{Bij} \cdot k_j$$

with j running, so that again we can modify $\dot{\mathcal{G}}_{Bij}$ by a constant. The third type arises as the integral of a $\varepsilon_i \cdot \dot{\mathcal{G}}_{Bij} \cdot \varepsilon_j$ in the IBP procedure, and here the constant matrix can be added as an integration constant.

For the scalar QED case, we can use this freedom to directly eliminate all one-cycles by subtracting from $\dot{\mathcal{G}}_{Bij}$ its (constant) coincidence limit, that is, replacing $\dot{\mathcal{G}}_{Bij}$ by $\bar{\mathcal{G}}_{Bij} \equiv \dot{\mathcal{G}}_{Bij} - \dot{\mathcal{G}}_{Bii}$ throughout. For the spinor QED case, this would not make sense, since there are still the fermionic one-cycles. Here instead one should anticipate the application of the replacement rule for one-cycles (10.8), and use the freedom of modifying $\dot{\mathcal{G}}_{Bij}$ to replace it by

$$\begin{aligned} \hat{\mathcal{G}}_{Bij} &:= \bar{\mathcal{G}}_{Bij} + \mathcal{G}_{Fii} = \dot{\mathcal{G}}_{Bij} - \dot{\mathcal{G}}_{Bii} + \mathcal{G}_{Fii} \\ &= i \left(\frac{e^{-iZ\dot{\mathcal{G}}_{B12}}}{\sin Z} - \cot Z - \tan Z \right) \end{aligned} \quad (10.10)$$

This eliminates all one-cycles, since now by construction $\hat{\mathcal{G}}_{Bii} - \mathcal{G}_{Fii} = 0$.

With these modifications, we can write down the one-loop N -photon amplitudes in a constant field for scalar and spinor QED in a way which is almost as compact as in vacuum. For example, the formula (4.11) for the four-photon amplitude in scalar QED generalizes to

$$\begin{aligned} \Gamma_{\text{scal}}[k_1, \varepsilon_1; \dots; k_4, \varepsilon_4] &= \frac{e^4}{(4\pi)^{D/2}} (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T^{1+D/2}} e^{-m^2 T} \det^{-\frac{1}{2}} \left[\frac{\sin Z}{Z} \right] \\ &\times \int_0^T d\tau_1 d\tau_2 d\tau_3 d\tau_4 \left(\mathcal{Q}_4^4 + \mathcal{Q}_4^3 + \mathcal{Q}_4^2 + \mathcal{Q}_4^{22} \right) \exp \left\{ \sum_{i,j=1}^4 \left[\frac{1}{2} k_i \cdot \bar{\mathcal{G}}_{Bij} \cdot k_j \right] \right\} \end{aligned} \quad (10.11)$$

where

$$\begin{aligned} \mathcal{Q}_4^4 &= \bar{\mathcal{G}}(1234) + \bar{\mathcal{G}}(1243) + \bar{\mathcal{G}}(1324) \\ \mathcal{Q}_4^3 &= \bar{\mathcal{G}}(123)\bar{\mathcal{T}}'(4) + \bar{\mathcal{G}}(234)\bar{\mathcal{T}}'(1) + \bar{\mathcal{G}}(341)\bar{\mathcal{T}}'(2) + \bar{\mathcal{G}}(412)\bar{\mathcal{T}}'(3) \\ \mathcal{Q}_4^2 &= \bar{\mathcal{G}}(12)\bar{\mathcal{T}}'(34) + \bar{\mathcal{G}}(13)\bar{\mathcal{T}}'(24) + \bar{\mathcal{G}}(14)\bar{\mathcal{T}}'(23) \\ &\quad + \bar{\mathcal{G}}(23)\bar{\mathcal{T}}'(14) + \bar{\mathcal{G}}(24)\bar{\mathcal{T}}'(13) + \bar{\mathcal{G}}(34)\bar{\mathcal{T}}'(12) \\ \mathcal{Q}_4^{22} &= \bar{\mathcal{G}}(12)\bar{\mathcal{G}}(34) + \bar{\mathcal{G}}(13)\bar{\mathcal{G}}(24) + \bar{\mathcal{G}}(14)\bar{\mathcal{G}}(23) \end{aligned} \quad (10.12)$$

Here the bicycles $\bar{\mathcal{G}}(i_1 \cdots i_n)$ differ from the original ones (10.9) only by the replacement of $\dot{\mathcal{G}}_{Bij}$ by $\bar{\mathcal{G}}_{Bij}$, and the tails $\bar{\mathcal{T}}(i_1 i_2 \dots i_n)$ are isomorphic to the one of the vacuum case (4.6),(4.10),

$$\begin{aligned}\bar{\mathcal{T}}'(i) &:= \sum_a' \varepsilon_i \cdot \bar{\mathcal{G}}_{Bia} \cdot k_a \\ \bar{\mathcal{T}}'(ij) &:= \sum_{a,b}' \varepsilon_i \cdot \bar{\mathcal{G}}_{Bia} \cdot k_a \varepsilon_j \cdot \bar{\mathcal{G}}_{Bjb} \cdot k_b + \frac{1}{2} \sum_a' \varepsilon_i \cdot \bar{\mathcal{G}}_{Bij} \cdot \varepsilon_j \left(k_i \cdot \bar{\mathcal{G}}_{Bia} \cdot k_a - k_j \cdot \bar{\mathcal{G}}_{Bja} \cdot k_a \right)\end{aligned}\tag{10.13}$$

As before, a ‘‘prime’’ on a tail means that its cycle have been removed; note that now this is necessary already for the one-tail.

The corresponding representation for spinor QED is obtained from (10.11), (10.12) by the change of determinants (10.8), multiplication by the global factor of -2 , and replacement of the bicycles $\bar{\mathcal{G}}(i_1 \dots i_n)$ by the ‘‘super-bicycles’’ $\hat{\mathcal{G}}_S(i_1 \dots i_n)$

$$\begin{aligned}\hat{\mathcal{G}}_S(i_1 i_2) &:= \frac{1}{2} \text{tr} \left(F_{i_1} \cdot \hat{\mathcal{G}}_{B i_1 i_2} \cdot F_{i_2} \cdot \hat{\mathcal{G}}_{B i_2 i_1} \right) \\ &\quad - \frac{1}{2} \text{tr} \left(F_{i_1} \cdot \mathcal{G}_{F i_1 i_2} \cdot F_{i_2} \cdot \mathcal{G}_{F i_2 i_1} \right) \\ \hat{\mathcal{G}}_S(i_1 i_2 \cdots i_n) &:= \text{tr} \left(F_{i_1} \cdot \hat{\mathcal{G}}_{B i_1 i_2} \cdot F_{i_2} \cdot \hat{\mathcal{G}}_{B i_2 i_3} \cdots F_{i_n} \cdot \hat{\mathcal{G}}_{B i_n i_1} \right) \\ &\quad - \text{tr} \left(F_{i_1} \cdot \mathcal{G}_{F i_1 i_2} \cdot F_{i_2} \cdot \mathcal{G}_{F i_2 i_3} \cdots F_{i_n} \cdot \mathcal{G}_{F i_n i_1} \right) \quad (n \geq 3)\end{aligned}\tag{10.14}$$

Moreover, the $\bar{\mathcal{G}}_{Bij}$ ’s in the tails (10.13) must also be replaced by $\hat{\mathcal{G}}_{Bij}$ ’s.

The covariantization of the tails does not seem to extend to the constant field case in a natural manner, except for the modification of the one-tail (3.8), which generalizes to

$$T_r(i) := \sum_a \frac{r_i \cdot F_i \cdot \bar{\mathcal{G}}_{Bia} \cdot k_a}{r_i \cdot k_i}\tag{10.15}$$

Also the optimized form of the two-tail (7.2) does not seem to generalize to the case of the general constant field. It does so, however, for the important special case of a self-dual field, which obeys $F^2 \sim \mathbb{1}$ [22, 58, 59]. Here one can generalize the total derivative (7.1) to

$$\frac{1}{(k_1 \cdot k_2)^2} \text{tr}(F_1 F_2) \partial_1 \partial_2 e^{(\cdot)} + \frac{1}{k_1 \cdot k_2} \left[\varepsilon_1 \cdot \varepsilon_2 \partial_1 \partial_2 e^{(\cdot)} - \varepsilon_1 \cdot k_2 \partial_1 \left(\varepsilon_2 \cdot \bar{\mathcal{G}}_{B2j} \cdot k_j e^{(\cdot)} \right) - \varepsilon_2 \cdot k_1 \partial_2 \left(\varepsilon_1 \cdot \bar{\mathcal{G}}_{B1i} \cdot k_i e^{(\cdot)} \right) \right]\tag{10.16}$$

and adding this to the (unsubtracted) two-tail $\bar{\mathcal{T}}(12)$ of (10.13), one arrives at

$$\mathcal{T}_H(ij) := \sum_{a,b} k_a \cdot \bar{\mathcal{G}}_{Bai} \cdot H_{ij} \cdot \bar{\mathcal{G}}_{Bjb} \cdot k_b\tag{10.17}$$

with the same tensor $H_{ij}^{\mu\nu}$ as in (7.3).

11 Conclusions

We have continued here the systematic investigation of the Bern-Kosower partial integration procedure, initiated in [36] and continued in [37]. We have presented an IBP algorithm that unambiguously leads to a form of the integrand of the one-loop N photon amplitude in scalar QED which is manifestly gauge invariant (transversal) at the integrand level, and suitable for an application of the Bern-Kosower rules. We have worked out this integrand explicitly at the four-point (two-tail) level. The N - point integrand contains N vectors r_1, \dots, r_N which are constrained only by the condition (3.3). Further study will be needed to find out what is the significance of this ambiguity, and how to make the best use of it. As far as concerns the on-shell N - photon/gluon amplitudes, one would surmise that the dependence of the integrand on the vectors r_i is related to the usual dependence on the reference vectors q_1, \dots, q_N which one would normally have in the application of the spinor helicity formalism, but does not exist any more once all polarization vectors are absorbed into field strength tensors. And indeed, there is clearly a relation: for example, consider the case of the the on-shell N -photon amplitudes with all helicities positive. When using the P representation together with the standard spinor helicity formalism (see, e.g., [46]), one can remove all terms involving a $\tilde{G}_{Bij}\varepsilon_i \cdot \varepsilon_j$ by choosing, in the spinor helicity formalism, the same reference vector $q_i = q$ for all legs, since then $\varepsilon_i^+ \cdot \varepsilon_j^+ = 0$. Similarly, in the S representation one could make disappear all the factors $r_i \cdot F_i \cdot F_j \cdot r_j$ by choosing all $r_i = r$ equal, and r on-shell, since then

$$r \cdot F_i \cdot F_j \cdot r = \frac{1}{2} r \cdot \{F_i, F_j\} \cdot r = -\frac{1}{4} [ij]^2 r^2 = 0 \quad (11.1)$$

on account of the identity [60]

$$\{F_i^+, F_j^+\}^{\mu\nu} = -\frac{1}{2} [ij]^2 \eta^{\mu\nu} \quad (11.2)$$

However, it is clear that this match between the freedom of choosing the r_i 's and the q_i 's cannot be a perfect one, since the r_i 's need not be chosen as on-shell.

All our representations are valid off-shell. This makes them relevant for state-of-the-art calculations already at the four-point level, since neither the four-photon nor the four-gluon amplitudes are presently available in the literature fully off-shell (for any spin in the loop). This fact is particularly conspicuous in the case of spinor QED, where the on-shell four-photon scattering amplitude was obtained already in 1951 by Karplus and Neumann [61], and the extension to the case of two off-shell legs in 1971 by Costantini et al. [62]. Our integral representations for the QED four-photon amplitudes are, with any of the various definitions of the tails, manifestly finite term by term and thus suitable for a numerical evaluation as they stand. For analytical purposes one would still like to reduce the various parameter integrals appearing in them to scalar box, triangle and bubble integrals. This could be done using existing tensor reduction algorithms (see [63] and refs. therein), but in a companion paper [64] we will rather perform this tensor reduction in a way that is specifically adapted to the structure of the worldline integrals.

Due to this validity off-shell our representations can also be used to construct, by sewing, all higher-loop N - photon amplitudes. From the calculation of the two-loop QED β function [17] it is clear that when calculating those multiloop amplitudes in the worldline formalism significative simplifications can be expected from a judicious application of IBP.

The manifest transversality is probably a more significant issue in the nonabelian case. Here we have in mind not so much the calculation of on-shell gluon amplitudes, for which other extremely powerful methods have been developed in recent years (see, e.g., [65, 66] and refs. therein), but rather of the one-particle-irreducible off-shell N - gluon vertices, for which there is presently still a dearth of efficient methods. For the use of these amplitudes, e.g. in Schwinger-Dyson equations, it is usually important to have them in a form that separates them into transversal and non-transversal parts, which normally requires a tedious analysis of the nonabelian Ward identities. This is one of the reasons why, using standard methods, the explicit calculation of these off-shell vertices has been completed so far only for the $N = 3$ case [39, 67, 51]. In another companion paper [45] we use the S representation to recalculate the three-gluon vertex, for the scalar, spinor, and gluon loops, achieving a drastic reduction in computational effort compared to earlier attempts (a summary of this calculation was given in [68]). The main advantages of our approach are that the gluon and spinor loop cases can be effortlessly obtained from the scalar loop one through the (off-shell extended) Bern-Kosower replacement rules, and that there is no necessity to solve the Ward identities, rather the decomposition into transversal and non-transversal pieces emerges automatically in the IBP procedure. In relation with the latter point it must also be mentioned that the boundary terms in the IBP procedure applied to the N -vertex always involve color-commutators, and are always connected to some lower-point term, even appearing with the same integral. Thus the algorithm makes it also easy to separate the genuinely new structures appearing in the N -vertex from those that, in terms of the effective action, only serve the completion of lower-point expressions to fully gauge-invariant ones; see [45].

In the abelian case, we have also extended the IBP procedure to the QED N -photon amplitudes in a constant external field. Here our main motivation is that these amplitudes can be used for the construction of higher-loop Euler-Heisenberg Lagrangians, and those as a tool to obtain insight about the asymptotic behavior of the QED N - photon amplitudes at high loop orders and large photon number [69, 58, 59, 70, 71]. In that context it would be highly desirable to calculate the three-loop Euler-Heisenberg Lagrangian in various dimensions, and indeed the IBP procedure presented in section 10 has made it possible to finally achieve this goal at least for the 1+1 dimensional case [72].

It should be possible, and very interesting, to generalize our approach to the inclusion of gravity. On-shell, the gauge theory Bern-Kosower rules were generalized to the construction of string-based representations for the one-loop N -graviton amplitudes in [73], and these gravity rules were then successfully applied at the four-point level in [74]. They also involve an IBP and “replacement rules” connecting the amplitudes with different spins in the loop. Worldline path integral representations of the one-loop effective actions for gravity have so far been constructed for spin zero [24], spin half [25, 26] and spin one [27] in the loop, and can be used to obtain parameter integral representations of the corresponding off-shell one-particle irreducible N -graviton amplitudes that are closely related to those string-based representations. They are written in terms of the same worldline Green’s functions as the ones for gauge theory, and the challenge is again to find an IBP algorithm that would allow one to apply the replacement rules and at the same time make covariance manifest, where the latter now means the emergence of full Riemann tensors in the IBP. This algorithm can, however, not be a simple extension of the one which we have presented here, as one can see from the fact that in gravity there are no boundary terms in the IBP, so that the nonlinear terms in the Riemann tensor now have to be created by δ -functions; therefore a complete removal of all \check{G}_B ’s in the IBP is not called for, and in fact also not possible, as one can easily see already from the case of the graviton propagator.

Finally, in open string theory the one-loop gauge boson amplitudes can be written in terms of a master formula that is analogous to the master formula (1.1), only that the variables τ_i parametrize positions along the boundary of the string worldsheet, and the Green's functions are worldsheet Green's functions (see, e.g., [75]). Since in all of our manipulations we have, apart from the translation invariance in proper-time, not used any specific properties of the worldline Green's functions, our IBP procedure could as well be applied at the string level to achieve a form factor decomposition based on gauge invariance.

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