Exercise 2.1.

a) Let $J : X \to Y$ be Fréchet differentiable at $x \in X$ where X, Y are Banach spaces. Show that J is Gâteaux differentiable and that both derivatives are equal.

b) Let $J : C([0,1]) \to \mathbb{R}$, be defined as

$$J(u) := \int_0^1 xu(x) + (u(x))^2 \mathrm{d}x.$$

Show that J is Gâteaux differentiable at every point $u \in C([0,1])$ and find the derivative. Is it also Fréchet differentiable? (As usual, considered $C([0,1])$ endowed with norm $|u|_{C([0,1])} = \sup_{x\in[0,1]} |u(x)|$)

Exercise 2.2. We have proven that in a Hilbert space, the projection map $P_{U_{\mathrm{ad}}} : H \to U_{\mathrm{ad}}$ for U_{ad} closed, convex and non-empty, is continuous in the strong topology. We now prove that this is not the case for weak convergence! Let $\{\phi_n\}$ be an orthonormal basis for H and $U_{\mathrm{ad}} := \{f \in H : |f|_H \leq 1\}$. Suppose $u_n := \phi_1 + \phi_n$. Find $P_{U_{\mathrm{ad}}}(u_n)$, show that $u_n \rightharpoonup u^*$, $P_{U_{\mathrm{ad}}}(u_n) \rightharpoonup y^*$ for some $u^*, y^* \in H$ but $P_{U_{\mathrm{ad}}}(u^*) \neq y^*$.
[Hint: in order to find $P_{U_{\mathrm{ad}}}(u_n)$ think in terms of two dimensions (ϕ_1 and ϕ_n) and then later prove that this is actually the projection via the variational inequality associated with $P_{U_{\mathrm{ad}}}(u_n)$.]

Exercise 2.3. Let $a_\eta : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ be defined as

$$a_\eta(u, v) := \int_\Omega \eta(x)u(x)v(x)\mathrm{d}x,$$

for some $\eta : \Omega \to \mathbb{R}$ in some Banach space X such that the integral above is well-defined. Define

$$J_\eta(u) = \frac{1}{2}a_\eta(u, u) + \int_\Omega f(x)u(x)\mathrm{d}x,$$

for some $f \in L^2(\Omega)$ and let $U_{\mathrm{ad}} \subset L^2(\Omega)$ be closed, convex and non-empty.

a) Specify X and find sufficient conditions on η so that the problem $\min_{u\in U_{\mathrm{ad}}} J_\eta(u)$ has a unique solution.

b) Consider the map $\eta \mapsto u(\eta)$, where η is of the type found in the above item and $u(\eta) \in L^2(\Omega)$ solves the problem $\min_{u\in U_{\mathrm{ad}}} J_\eta(u)$. Suppose U_{ad} is **bounded** and prove that the map $\eta \mapsto u(\eta)$ is Lipschitz continuous if η is restricted to some subset of X.

[Hint: Check what we did in class to prove that the projection mapping was Lipschitz continuous. Analogously, you should be able to obtain a similar result for the map $\eta \mapsto u(\eta)$]

c) Can we drop the boundedness assumption on U_{ad} and still claim the previous result as true?