The goal of this exercise is for you to provide a working computer code in your language of preference with implementations of:

i) the Gradient Descent Method with a fixed step size,
ii) the Gradient Descent Method with Armijo step sizes,
iii) the Semi-smooth Newton Method.

If this is the first time you code something, it is recommended to use MatLab. You may use the finite difference discretization of the differential operators seen in class. However, this is not a restriction, you may use finite elements if you have already done so.

For each of the following problems you are asked to

a) Discretize the underlying PDE with mesh sizes $h = 2^{-k}$ with $k = 4, 5, 6, 7, 8, 9$ on the unit square $\Omega = (0, 1)^2$.
b) Choose the data $y_d \in L^2(\Omega)$, $\lambda > 0$, etc. for each problem.
c) Iteratively solve the problem until the appropriate residual drops below a small number $\epsilon \le 10^{-7}$ or the number of outer iterations n reaches 500.
d) Count the number of outer iterations for each mesh size and plot the behaviour of the residuals.
e) For methods (i) and (ii), plot the numbers μ_n, and for method (iii), plot the numbers ν_n against n. Here,
$$\mu_n = \frac{|u_{n+1} - u_n|_{L^2(\Omega)}}{|u_n - u_{n-1}|_{L^2(\Omega)}}, \quad \text{and} \quad \nu_n = \frac{|u_{n+1} - u^*|_{L^2(\Omega)}}{|u_n - u^*|_{L^2(\Omega)}},$$
where u_n is the control variable at iteration n and u^* is a computed solution with a residual that is much smaller than ϵ.
f) Compare between methods and mesh-sizes. Choose different values for λ and for the parameter ϵ and compare again the convergence behaviour.

More details given in each specific problem.

Exercise 8.1. Consider the following distributed control problem:
$$\begin{aligned} \text{Minimize } & \frac{1}{2}\int_\Omega |y - y_d|^2 dx + \frac{\lambda}{2}\int_\Omega |u|^2 dx \\ \text{s.t. } & -\Delta y = u, \quad \text{in } \Omega \\ & y = 0 \quad \text{on } \partial\Omega \\ & u \in U_{\text{ad}} \end{aligned}$$
where $U_{\text{ad}} := \{w \in L^2(\Omega) : w(x) \le \overline{u} \text{ a.e. for } x \in \Omega\}$ with $\overline{u} \in H^1(\Omega)$ and $y_d \in L^2(\Omega)$.

a) Choose $\lambda > 0, y_d \in L^2(\Omega)$ and $\overline{u} \in H^1(\Omega)$ so that the optimal control is not inactive, that is, an optimal control u^* should satisfy $u^*(x) = \overline{u}(x)$ on a set of positive measure.
b) As residual for termination use $r(u) = |u - P_{U_{ad}}(-p/\lambda)|_{L^2(\Omega)}$ where p is the associated adjoint state to $y(u)$.

Exercise 8.2. Consider the following boundary control problem:
$$\begin{aligned} \text{Minimize } & \frac{1}{2}\int_\Omega |y - y_d|^2 dx + \frac{\lambda}{2}\int_{\partial\Omega} |u|^2 dS \\ \text{s.t. } & -\Delta y = f, \quad \text{in } \Omega \\ & \frac{\partial y}{\partial \nu} = u - y \quad \text{on } \partial\Omega \\ & u \in U_{\text{ad}} \end{aligned}$$
where $U_{\text{ad}} := \{w \in L^2(\partial\Omega) : w(s) \ge \underline{u} \text{ a.e. for } s \in \partial\Omega\}$ with $\underline{u} \in \mathbb{R}$ and $f, y_d \in L^2(\Omega)$.
Proceed analogously as with the previous problem.

Please send your working code via email to carlos.rautenberg@math.hu-berlin.de and to loebhard@math.hu-berlin.de by January, 1 2016.