Exercise 9.1. Let Ω be bounded in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $1 < p < \infty$. Let $a(\xi) = |\xi|^{p-1}\operatorname{sign}(\xi)$ for $\xi \in \mathbb{R}$, $b \in L^\infty(\Omega)$, $c \in L^\infty(\partial\Omega)$ with both b and c non-negative. Define A as

$$\langle A(u), v\rangle := \int_\Omega \sum_{j=1}^n a(\partial_j u(x))\partial_j v(x) + b(x)a(u(x))v(x)\mathrm{d}x + \int_{\partial\Omega} c(s)a(u(s))v(s)\mathrm{d}s,$$

for $u, v \in W^{1,p}(\Omega)$. Let $f \in L^{p'}(\Omega)$ and $f_0 \in L^{p'}(\partial\Omega)$ and define F as

$$F(v) := \int_\Omega f(x)v(x)\mathrm{d}x + \int_{\partial\Omega} f_0(s)v(s)\mathrm{d}s,$$

for $v \in W^{1,p}(\Omega)$.

The idea of these problems is to put in practice the theorems seen in class and also walk the inverse path of finding which PDE is behind the problem $A(y) = F$. You may guess the PDE and then derive its weak form (this is acceptable); however, it would be better if you could try to work directly from the variational equality $\langle A(y), v\rangle = F(v)$, for all $v \in V$ (Hint: assume higher regularity of y, consider test functions $v \in C_0^\infty(\Omega)$ to derive behaviour on Ω. With this information and the variational equality, you should be able to derive the boundary conditions by use of the divergence theorem and density results).

a) Let $V = W^{1,p}(\Omega)$, provide conditions on b so that the problem

$$\text{Find } y \in V : \langle A(y), v\rangle = F(v), \qquad \forall v \in V, \tag{1}$$

has a unique solution. Characterize the PDE and the boundary conditions from which this problem is a weak form (you may have to proceed formally or assume higher regularity of y).

b) Let Γ_1 be a strict subset of $\partial\Omega$ and $\Gamma_2 = \partial\Omega \setminus \Gamma_1$. Choose $V = \{u \in W^{1,p}(\Omega) : \tau u = 0 \quad \text{on } \Gamma_1\}$ where $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the trace map. Do analogously as in the previous item.

c) The theorems provided for existence of solutions are strong enough to tackle problems that have non-standard boundary conditions. The purpose of this exercise is to address one of these boundary conditions which are *non-local*. Let V be defined as

$$V = \{v \in W^{1,p}(\Omega) : (\tau v)(s) = r \quad \text{for some } r \in \mathbb{R} \text{ and almost all } s \in \partial\Omega\},$$

that is, V is the set of functions $W^{1,p}(\Omega)$ whose traces are constant on $\partial\Omega$ (note that this means any constant and not some fixed constant). Prove that V is a closed subspace of $W^{1,p}(\Omega)$. Provide conditions on b so that (1) has a unique solution. Provide the PDE+boundary conditions from which this problem arises (this is not trivial!).