Type-definable groups in $C$-minimal structures

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Model theory

Abstract:

This paper studies type-definable groups in $C$-minimal structures. We show first for some of these groups, that they contain a cone which is a subgroup. This result will be applied to show that in any geometric locally modular non trivial $C$-minimal structure, there is a definable infinite $C$-minimal group.

Résumé:

Cet article traite des groupes type-définissables dans les structures $C$-minimales. On démontre d’abord pour certains de ces groupes, qu’ils contiennent un cône qui est un sous-groupe. Ce résultat sera appliqué pour montrer que dans toute structure géométrique $C$-minimale non triviale et localement modulaire, il y a un groupe $C$-minimal définissable infini.

1 Introduction

We prove first under certain conditions, that a type-definable group in a $C$-minimal structure $M$ contains a cone which is a definable subgroup of $M$. Similar results are already known in other contexts. In [3], Hrushovski shows that in a stable structure, a type-definable group is the intersections of definable groups. And in the case where the structure is totally transcendental, then the group is in fact definable. More recently, Milliet shows in [2] similar results for small theories. A theory is small, if for each natural number $n$, it has countably many $n$-types over the empty set, and a structure is small if its theory is. It is proved in [2] that a $\emptyset$-type-definable group of finite arity in a small structure is the intersection of definable groups, and that for any type-definable group $G$ in a simple small structure, and any finite subset $A$ of $G$, there is a definable group containing $A$.

In section 2 we prove the following:

Theorem 1 Let $M = (M, C, \ldots)$ be a $C$-minimal structure and $G = (G, , 1, C)$ an infinite type-definable $C$-group in $M$ such that $G$ is an intersection of cones of $M$. Then $G$ contains a cone which is a subgroup. In particular, $G$ contains a definable infinite $C$-group.

Theorem 1 as well as its proof, are very similar to results which can be found in [6]. In order to be self-contained, we will reproduce here most of the necessary arguments for the proof.

The next result follows from Theorem 1. It can be already found in [6], though it is not stated there as a separate result.

Theorem 2 Let $G = (G, , C, 1, \ldots)$ be a $C$-minimal group. Then $G$ contains a cone which is a subgroup.

Theorem 1 will be used to strengthen a result from [4]. We show the following

Theorem 3 Let $M = (M, C, \ldots)$ be a geometric locally modular non trivial $C$-minimal structure. Then there is a definable infinite $C$-minimal group in $M$. 

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Notations: We use $\mathcal{M}, \mathcal{N}, \ldots$ to denote structures and $M, N, \ldots$ for their underlying sets.

We start with a few definitions and preliminary results. $C$-structures have been introduced and studied in [5] and [6]. We remind in what follows their definition and principal properties. A $C$-structure is a structure $M = (M, C, \ldots)$, where $C$ is a ternary predicate satisfying the following axioms:

- $\forall x, y, z, C(x, y, z) \rightarrow C(x, z, y)$
- $\forall x, y, z, C(x, y, z) \rightarrow \neg C(y, x, z)$
- $\forall x, y, z, w, C(x, y, z) \rightarrow C(x, w, z) \lor C(w, y, z)$
- $\forall x, y, x \neq y \exists z \neq y, C(x, y, z)$

Let $M$ be a $C$-structure. We call cone any subset of $M$ of the form $\{x; M \models C(a, x, b)\}$, where $a$ and $b$ are two distinct elements of $M$. It follows from the first three axioms of $C$-relations that the cones of $M$ form a basis of a completely disconnected topology on $M$. The last axiom guarantees that all cones are infinite.

Let $(T, \leq)$ be a partially ordered set. We say that $(T, \leq)$ is a tree if the set of elements of $T$ below any fixed element is totally ordered by $\leq$, and if any two elements of $T$ have a greatest lower bound. A branch of $T$ is a maximal totally ordered subset of $T$. It is easy to check that if $a$ and $b$ are two distinct branches of $T$, then $\text{sup}(a \cap b)$ exists. On the set of branches of $T$, we define a ternary relation $C$ in the following way: we say that $C(a, b, c)$ is true if and only if $b = c$ or $a, b,$ and $c$ are all distinct and $\text{sup}(a \cap b) < \text{sup}(b \cap c)$. It is easy to check that this relation on the set of branches satisfies the first three axioms of a $C$-relation.

A theorem from [1] shows that $C$-structures can be looked at as a set of branches of a tree. We will then associate to any $C$-structure $M$ a tree $T$. We will call it the underlying tree of $M$, and the elements of $T$ will be called nodes. To any $x, y \in M$, $x \neq y$, we associate the node $t := \text{sup}(x \cap y)$, where $x$ and $y$ are seen as subsets of $T$. This operation is well defined, and we say then that $x$ and $y$ branch at $t$. If $a$ and $b$ are two elements of $M$ branching at a node $t$, and if $D$ is the cone $D := \{x \in M; C(a, x, b)\}$, we say then that $D$ is the cone at the node $t$ containing $b$. If $t$ and $t'$ are two nodes, we denote by $t||t'$ the property that $t$ and $t'$ are not comparable in $T$ with respect to the relation $\leq$. If $A$ and $B$ are two sets of nodes, we denote by $A||B$ the property that, for any $t \in A$ and $t' \in B$, $t||t'$.

**Definition 4** Let $\mathcal{M} = (M, C, \ldots)$ be a $C$-structure. We say that $\mathcal{M}$ is $C$-minimal if and only if for any structure $\mathcal{M}' = (M', C, \ldots)$ elementarily equivalent to $\mathcal{M}$, any definable subset of $M'$ can be defined without quantifiers using only the relations $C$ and $=$.

2 A cone of a $C$-minimal group is a subgroup

**Definition 5** We say that $G = (G, 1, C)$ is a $C$-group if and only if $G$ is a $C$-structure, $(G, 1)$ is a group and for all $x, y, z, a, b \in G, G \models C(x, y, z) \rightarrow C(a, x, b a, y, b, a z, b)$.

Let $G = (G, 1, C)$ be a $C$-group and $T$ its underlying tree. Let $t \in T$, and $x, y, x', y', z \in G$ be such that, $x$ and $y$, as well as $x'$ and $y'$ branch at $t$ (recall that the elements of a $C$-structure are looked at as branches of the underlying tree). It is easy to check that $z, x$ and $z, y$, as well as $z, x'$ and $z, y'$, all branch again at the same node, which we denote by $t^2$. We can then define a left action of $G$ on $T$, $(z, t) \mapsto t^z$, and check that this action preserves $\leq$. We will speak then of orbits of $G$ on $T$. Similarly one can check that if $D$ is a cone of $G$ at a node $t$, then $z, D := \{z, x; x \in D\}$ is a cone at the node $t^z$.

**Proposition 6** Let $G = (G, 1, C)$ be a $C$-group and $T$ its underlying tree. Suppose that some orbit $\Omega$ of $G$ on $T$ is an antichain. Then there is a cone in $G$ which is a subgroup.

**Proof:** Let $s \in \Omega$ and $g \in G$ be such that $s \in g$. Since $g^{-1}g = 1$, then $t := s g^{-1}$ is an element of $\Omega \cap 1$ (here we see $1$ as a branch of $T$). Let $X$ be the cone at the node $t$ containing $1$. We want to show that $X$ is a subgroup of $G$. Take $h \in X$. Then $h X$ is a cone at the node $t^h$. But $t^h \in \Omega$, and
if \( t^h \neq t \), \( t^h \) is incomparable with \( t \) (\( \Omega \) is an antichain). In this case, since a chain contains no two incomparable elements, \( h.X \cap X = \emptyset \). But this cannot happen since \( 1 \) and \( h \) are two elements of \( X \), and thus \( h \in h.X \cap X \). We have shown that \( h.X \) is the cone at the node \( t \) containing \( h \), and then \( h.X = X \). And since \( 1 \in X \), \( h^{-1} \in X \). Since this is true for any \( h \in X \), \( X \) is a subgroup of \( G \). \( \square \)

We now will show in what follows that the same result holds for the \( C \)-groups of the statement of Theorem 1 in the case where there is an orbit which is not an antichain.

**Lemma 7** Let \( \mathcal{M} = (M,C,\ldots) \) and \( \mathcal{G} = (G,.,1,C) \) be as in the statement of Theorem 1. There is a definable subset \( V \) of \( M \) and an \( M \)-definable function \( F : V \times V \mapsto M \) such that \( G \in V, F|G \times G = . \) and \( F \) is a \( C \)-isomorphism in each variable.

**Proof:** By compactness we know that the group operation of \( \mathcal{G} \) is definable in \( \mathcal{M} \). Denote then by \( F \) an \( M \)-definable ternary relation which restriction to \( G \) is \( .\), the group operation of \( \mathcal{G} \). For an element \( x \) of \( M \), denote by \( C_x := \{ y \in M; \neg C(y,x,1) \} \). Let \( V \) be the set of elements \( x \) of \( M \) such that, \( F \) defines on \( C_x \times C_x \) a binary function which is a \( C \)-isomorphism in each variable. So \( V \) is definable, and using the fact that \( G \) is an intersection of cones of \( M \), we see easily that \( V \) and \( F \) do the job. \( \square \)

**Notations:** Let from now on \( \mathcal{M} = (M,C,\ldots) \) and \( \mathcal{G} = (G,.,1,C) \) be as in the statement of Theorem 1, and let \( V \) and \( F \) be as in the statement of Lemma 7. \( (T,<) \) will be the underlying tree of \( \mathcal{G} \), and \( (T',<) \) will be the underlying tree of \( V \) (we have that \( T \subset T' \)).

Now doing as above, but using \( F \) instead of \( . \), we can define the left action of \( V \) on \( T' \). We will speak then of orbits of \( V \) on \( T' \) via \( F \).

**Lemma 8** Let \( t \in T \) and \( x \in V \setminus G \). Then \( t^x \notin T \).

**Proof:** Since \( G \) is an intersection of cones of \( M \), it is enough to show that for all \( y \in G \), \( F(x,y) \notin G \).

Suppose not. If \( F(x,y) = z \in G \), by the fact that \( F \) is one-to-one in each variable and the fact that the restriction of \( F \) to \( G \times G \) is the operation \( . \) of \( G \), we get that \( x = y^{-1}.z \in G \). Contradiction. \( \square \)

**Lemma 9** Let \( \Omega \) be an orbit of \( G \) on \( T \) and \( z \in G \). Then in \( T \), \( \Omega \cap z \) is a finite union of intervals and points.

**Proof:** Let \( t \in \Omega \), and \( \Omega' \) be the orbit of \( V \) on \( T' \) via \( F \) containing \( t \). By \( C \)-minimality \( \Omega' \cap z \) is a finite union of intervals and points. By Lemma 8, \( \Omega = \Omega' \cap T \), so \( \Omega \cap z = \Omega' \cap z \cap T \). And then in \( T \), \( \Omega \cap z \) is a finite union of intervals and points.

The two following results can be found in [6]. But we will restate them here in a slightly more general context.

**Proposition 10** Let \( \Omega \) be an orbit of \( G \) on \( T \). Then there are no elements \( r, s, t \in \Omega \) such that \( r < s, r < t \) and \( s||t \).

**Proof:** This is exactly Lemma 4.6 of [6], except for the fact that in our case \( \mathcal{G} \) is not necessarily \( C \)-minimal: \( \mathcal{G} \) satisfies only the hypothesis of Theorem 1, namely that it is a type-definable \( C \)-group in a \( C \)-minimal structure \( \mathcal{M} \) and its universe \( G \) is an intersection of cones of \( \mathcal{M} \). The same proof of Lemma 4.6 of [6] works as well in our case, except for replacing the centralizer of an element \( h \) of \( \mathcal{G} \) by the set \( C^*_L(h) := \{ x \in V, F(x,h) = F(h,x) \} \). Lemma 7 is used only at this step. \( \square \)

Let \( \Omega \) be an orbit of \( G \) on \( T \). For all \( t \in \Omega \), let \( L_t := \{ t' \in \Omega; t \leq t' \lor t \geq t' \} \). Using Proposition 10, it is easy to check that the relation \( \sim \) defined on \( \Omega \) by \( t \sim t' \iff t' \in L_t \) is an equivalence relation. For all \( t, t' \in \Omega \), we denote by \( \bar{t} \) the class of \( t \) modulo \( \sim \). Note that \( \bar{t} = \bar{t'} \) if and only if \( L_t = L_{t'} \). Set \( L \equiv \Omega \setminus \Omega/\sim \). It is obvious that \( \{ L_t; t \in \Omega/\sim \} \) is a partition of \( \Omega \) and that if \( t \neq t' \in \Omega/\sim \), \( L_t||L_{t'} \) (for the notations see the introduction). Now if \( \Omega \) is not an antichain, at least one of the \( L_t \) is not a singleton. And since \( G \) acts transitively on the set \( \{ L_t; t \in \Omega/\sim \} \), none of the \( L_t \) is a singleton.
Proposition 11 Let \( \Omega \) be an orbit of \( G \) on \( T \) which is not an antichain. As above, we write \( \Omega := \bigcup_{\epsilon \in \Omega/\sim} L_{\epsilon} \). Then for all \( \epsilon \), there is no \( g \in G \) such that \( L_{\epsilon} \subset g \).

**Proof:** Suppose for a contradiction that for some \( \epsilon, g \), \( L_{\epsilon} \subset g \). Let \( s < t \in L_{\epsilon} \), and \( h \in G \) be such that \( h \) branches with \( g \) in \( s \). The image \( L_{\epsilon}^{g^{-1}.h} \) of \( L_{\epsilon} \) under the left action of \( g^{-1}.h \) is a subset of \( h \). But from Proposition 10 and the fact that \( \Omega^{g^{-1}.h} = \Omega \), \( L_{\epsilon}^{g^{-1}.h} \) contains no elements of \( h \) above \( s \), so we get that \( L_{\epsilon}^{g^{-1}.h} \subset g \cap h \). On the other hand, we can find \( t_{1} \in \Omega \) such that \( t_{1}^{g^{-1}.h} = t \). But then \( t_{1} \notin L_{\epsilon} \) and \( t_{1} \) is not comparable with \( t \). But \( t_{1}^{g^{-1}.h} \in L_{\epsilon} \) is comparable with \( t^{g^{-1}.h} \in h \cap g \). Contradiction. \( \square \)

Proposition 12 Suppose that some orbit \( \Omega \) of \( G \) on \( T \) is not an antichain. Then there is a cone of \( G \) which is a \( C \)-subgroup.

**Proof:** We use the notations of Proposition 11. Let \( L_{\epsilon} \) be such that \( 1 \cap L_{\epsilon} \neq \emptyset \). Let \( x \) be an element of \( G \) containing a node of \( L_{\epsilon} \setminus 1 \) (such an element exists by Proposition 11). Let \( t \) be the node of \( T \) at which \( x \) branches with \( 1 \). We want to show that the cone containing \( 1 \) at the node \( t \) is a subgroup of \( G \). Denote this cone by \( D \), and let \( h \in D \). Since \( 1 \in D \), \( h.D \) is a cone containing \( h \) at the node \( t^{h} \). Note first that \( t, t^{h} \in h \), then either \( t \leq t^{h} \) or \( t^{h} \leq t \). Note also that \( t^{h} \in L_{\epsilon} \). We want to show that \( h.D = D \). Since \( h \in D \cap h.D \), it is enough to show that \( t^{h} = t \). Suppose not. If \( t^{h} > t \), then by the definitions of \( t \) and \( D \) and the fact that \( t^{h} \in L_{\epsilon}, t^{h} \notin h \). But this is impossible. And if \( t^{h} < t \), then \( t^{h-1} > t \), and for the same reason as above, \( 1 \notin h^{-1}.D \), which is impossible. So for all \( h \in D \), \( h.D = D \), and since \( 1 \in D \), \( D \) is a \( C \)-subgroup of \( G \). \( \square \)

Theorem 1 follows directly from Propositions 6 and 12. \( \square \)

**Proof of Theorem 3:** Let \( M = (M, C, \ldots) \) be a non trivial locally modular geometric \( C \)-minimal structure, and let \( M^{\prime} \) be an \( \omega \)-saturated structure elementarily equivalent to \( M \). We show in [4] that in \( M^{\prime} \) there is an infinite type-definable \( C \)-group \( G^{\prime} = (G^{\prime}, 1, C) \), and moreover, \( G^{\prime} \) is an intersection of cones of \( M \). Thus \( G^{\prime} \) satisfies the hypothesis of Theorem 1, and there is a cone \( D \) of \( G^{\prime} \) which is an infinite \( C \)-group definable in \( M^{\prime} \). Since \( M \equiv M^{\prime} \), there is an infinite \( C \)-group \( G \) definable in \( M \). And \( G \) is \( C \)-minimal because \( M \) is \( C \)-minimal. \( \square \)

**References**


