Relations among multiple zeta values

KOSMOS Summer University 2013
Multiple Zeta Values in Mathematics and Physics
October 1st–5th, Humboldt-Universität zu Berlin

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October 1st, 2013

1 Single zeta values

Theorem 1.1 (Euler). For any $n \in \mathbb{N}$, the even zeta values are

$$\zeta(2n) = \frac{(-1)^{n-1}B_{2n}(2\pi)^{2n}}{2(2n)!} = \frac{(2\pi i)^{2n}B_{2n}}{2(2n)!} \in \mathbb{Q} \cdot \pi^{2n} = \mathbb{Q} \cdot \zeta^n(2)$$

(1.1)

for the Bernoulli numbers $B_n$ defined by

$$\frac{x}{e^x - 1} = \sum_{0 \leq n} B_n \frac{x^n}{k!} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} + \cdots$$

Hence, as rational multiples of powers of $\pi$ the even zetas are transcendental.

For the odd zeta values we do not expect such relations at all:

Conjecture 1.2. The elements $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.

In particular, we expect all zeta values to be transcendental. But we only know very few results on irrationality:

Theorem 1.3 ([1]). $\zeta(3)$ is irrational.

Theorem 1.4 ([13]). Infinitely many of the odd zetas $\zeta(3), \zeta(5), \zeta(7), \ldots$ are irrational. In fact, for any $\varepsilon > 0$ exists some $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$,

$$\dim_{\mathbb{Q}} \text{lin}_{\mathbb{Q}} \{1, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)\} \geq \frac{1 - \varepsilon}{1 + \log 2} \log n.$$  

(1.2)

Theorem 1.5 ([18]). At least one of $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\}$ is irrational.

Theorem 1.6 ([19]). For any odd $n \in \mathbb{N}$, at least one of $\{\zeta(n+2), \zeta(n+4), \ldots, \zeta(8n-1)\}$ is irrational.

\footnotesize{1}For details on this proof see [12] and its recently updated version.
2 Multiple zeta values

Definition 2.1. To \(d \in \mathbb{N}\) integers \(n_1, \ldots, n_d \in \mathbb{N}\) with \(n_d > 1\), the multiple zeta value

\[
\zeta(n_1, \ldots, n_d) := \sum_{0 < k_1 < \ldots < k_d} \frac{1}{k_1^{n_1} \ldots k_d^{n_d}} \in \mathbb{R}_+
\]  

(2.1)

assigns a positive real number. We call \(d\) its depth and \(n_1 + \ldots + n_d\) its weight. Be aware of the also common reverse convention [7].

Already Euler [5] knew the case \(d = 1\) of

Theorem 2.2 (Sum theorem [7]). For any depth \(d \in \mathbb{N}\) and weight \(w > 1\),

\[
\sum_{n_1 + \ldots + n_d = w, n_d > 1} \zeta(n_1, \ldots, n_d) = \zeta(w).
\]  

(2.2)

Example 2.3. In weight three, \(\zeta(3) = \zeta(1, 2)\) and \(\zeta(4) = \zeta(1, 3) + \zeta(2, 2) = \zeta(1, 1, 2)\) in weight four.

Symmetric MZVs are sums of products of single zeta values, as the stuffle identity supplies

Theorem 2.4 ([9]). Let \(n_1, \ldots, n_d \geq 2\) then

\[
\sum_{\sigma \in S_d} \zeta(n_{\sigma(1)}, \ldots, n_{\sigma(d)}) = \sum_{\text{partitions } M \text{ of } \{1, \ldots, d\}} (-1)^{d-|M|} \prod_{P \in M} (|P| - 1)! \cdot \zeta\left(\sum_{n \in P} n\right).
\]  

(2.3)

Corollary 2.5. Combining (2.3) with (1.1), we deduce

\[
E(2n, d) := \sum_{n_1 + \ldots + n_d = n} \zeta(2n_1, \ldots, 2n_d) \in \mathbb{Q} \cdot \zeta(2n).
\]  

(2.4)

In fact their generating function is computed in [8] to

\[
F(t, s) := 1 + \sum_{1 \leq k \leq n} E(2n, k) t^n s^k = \frac{\sin \left(\pi \sqrt{t(1-s)}\right)}{\sqrt{1-s} \sin \left(\pi \sqrt{t}\right)}.
\]  

(2.5)

Example 2.6. \(E(2n, 2) = \sum_{k=1}^{n-1} \zeta(2n - 2k, 2k) = \frac{3}{4} \zeta(2n)\) is already due to Euler [7]. Writing \(\{2\}^n\) for the sequence of \(n\) consecutive twos, note further

\[
E(2n, n) = \zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}.
\]  

(2.6)

Theorem 2.7 ([10]). MZV with \(n_1 = \ldots = n_{d-1} = 1\) are sums of products of single zetas,

\[
\sum_{n, m \geq 1} \zeta(\{1\}^{m-1}, n+1) s^t m^n = 1 - \frac{\Gamma(1-s)\Gamma(1-t)}{\Gamma(1-s-t)} = 1 - \exp\left(\sum_{n \geq 2} \frac{\zeta(n) \cdot t^n + s^n - (t+s)^n}{n}\right).
\]

Again, the case \(m = 2\) was known to Euler [5] already.

Theorem 2.8 ([2]). As was originally conjectured by D. Zagier, we have the identity

\[
\zeta(\{1, 3\}^n) = \frac{2 \pi^{4n}}{(4n+2)!}.
\]
3 Shuffles, stuffles and regularization

Definition 3.1 (Hoffman): The Hoffman algebra $\mathfrak{h}$ are the non-commutative polynomials

$$\mathfrak{h} := \mathbb{Q} \langle x, y \rangle \quad (3.1)$$

in two letters $x$ and $y$. It is graded by the weight (length of a word = number of letters) and depth (number of letters $y$). On the subspace

$$\mathfrak{h}^0 := \mathbb{Q} \oplus y h x \quad (3.2)$$

of admissible words words (beginning with $y$ and ending in $x$) we define the period map by

$$\zeta : \mathfrak{h}^0 \to \mathbb{R}, \quad y_{n_1} \cdots y_{n_d} \mapsto \zeta (n_1, \ldots, n_d), \quad (3.3)$$

where $y_n := y x^{n-1}$ for any $n \in \mathbb{N}$. This is extended linearly with $\zeta (1) = 1$ for the empty word. The shuffle product $\shuffle$ and stuffle (also called harmonic) product $\star$ are recursively defined by

$$av \shuffle bw := a (v \shuffle bw) + b (av \shuffle w) \quad \text{and} \quad y_n v \star y_m w := y_n (v \star y_m w) + y_m (y_n v \star w) + y_{n+m} (v \star w) \quad (3.4)$$

for any letters $a, b \in \{x, y\}$ and words $v, w \in \mathfrak{h}^0$. Both of these turn $\mathfrak{h}^0$ into a commutative, associative and free algebra.

As was shown in Olivers’ lectures, the sum representation (2.1) and the iterated integral representation provide

Lemma 3.2 (Double-shuffle relations). $\zeta$ is an algebra morphism with respect to both products $\shuffle$ and $\star$:

$$\zeta (v \shuffle w) = \zeta (v) \cdot \zeta (w) = \zeta (v \star w) \quad \text{for any} \quad v, w \in \mathfrak{h}^0. \quad (3.5)$$

Example 3.3. From $y_2 \shuffle y_2 = y x y x = 4 y x y x + 2 y x y x = 4 y_1 y_3 + 2 y_2 y_2$ and $y_2 \star y_2 = 2 y_2 y_2 + y_4$ we deduce $4 \zeta (1, 3) + 2 \zeta (2, 2) = \zeta^2 (2) = 2 \zeta (2, 2) + \zeta (4)$ and therefore $\zeta (4) = 4 \zeta (1, 3)$.

Theorem 3.4 (Hoffman relation [9]). For any $w \in \mathfrak{h}^0$, $w \shuffle y - w \star y \in \mathfrak{h}^0$ (does end in $x$) and

$$\zeta (w \shuffle y - w \star y) = 0. \quad (3.6)$$

Example 3.5. We find $y_2 \star y - y_2 \shuffle y = y_3 + y_1 y_2 + y_2 y - (2 y_1 y_2 + y_2 y) = y_3 - y_1 y_2 \in \mathfrak{h}^0$, thus $\zeta (3) = \zeta (1, 2)$.

Conjecture 3.6. All algebraic relations over $\mathbb{Q}$ among MZV are consequences of the so-called regularized double-shuffle relations meaning (3.5) and (3.6).

4 Weight filtration

Definition 4.1. Let $Z_N := \text{lin}_\mathbb{Q} \{ \zeta (n_1, \ldots, n_d) : n_1 + \ldots + n_d = N \}$ denote the space of MZV of weight $N$ and $Z := \sum_{N \geq 0} Z_N$ their sum (the $\mathbb{Q}$-space spanned by all MZV).

\[\text{In the literature the words are written in the reversed way: } y_n := x^{n-1} y \text{ and } \zeta (y_{n_d} \ldots y_1) := \zeta (n_d, \ldots, n_1).\]

Here we used this order to be consistent with the first talk of Oliver Schnetz.
From the previous examples, we know that \( \zeta(3) = \zeta(1,3) \), so \( \mathbb{Z}_3 = \mathbb{Q} \cdot \zeta(3) \) is one-dimensional.

\[
\zeta(4) = \zeta(1,1,2) = 4 \zeta(1,3) = \frac{4}{3} \zeta(2,2) = \frac{2}{5} \zeta^2(2)
\]

shows that the four MZV of weight four span only a one-dimensional space \( \mathbb{Z}_4 = \mathbb{Q} \cdot \zeta(4) \) as well. Recall the following table of Oliver’s talk obtained by using all available relations:

<table>
<thead>
<tr>
<th>weight ( N )</th>
<th>conjectured basis of ( \mathbb{Z}_N )</th>
<th>( d_N )</th>
<th>MZV of weight ( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>( \zeta(2) )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>( \zeta(3) )</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>( \zeta(2)^2 )</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>( \zeta(5), \zeta(2) \cdot \zeta(3) )</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>( \zeta(2)^3, \zeta(3)^2 )</td>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>7</td>
<td>( \zeta(7), \zeta(2) \cdot \zeta(5), \zeta(2)^2 \cdot \zeta(3) )</td>
<td>3</td>
<td>32</td>
</tr>
<tr>
<td>8</td>
<td>( \zeta(2)^4, \zeta(2) \cdot \zeta(3)^2, \zeta(3) \cdot \zeta(5), \zeta(3,5) )</td>
<td>4</td>
<td>64</td>
</tr>
</tbody>
</table>

**Conjecture 4.2** (D. Zagier, [16]). \( \mathbb{Z} \) is graded by the weight with Hilbert-Poincaré series

\[
\sum_{k \geq 0} d_k t^k = \frac{1}{1 - t^2 - t^3}.
\] (4.1)

Equivalently, the following two statements hold:

1. All relations are homogeneous in weight: \( \mathbb{Z} = \bigoplus_{N \geq 0} \mathbb{Z}_N \)

2. \( \dim_{\mathbb{Q}} \mathbb{Z}_N = d_N \) where \( d_0 = d_2 = 1 \), \( d_1 = 0 \) and then \( d_N = d_{N-2} + d_{N-3} \).

Note that this is a very strong claim as it implies conjecture 1.2 and thus transcendence of all odd zeta values. However, the results of F. Brown on motivic multiple zeta values (which will be featured in his upcoming lectures) imply

**Theorem 4.3** ([4]). *The Hoffman-elements span \( \mathbb{Z} \) in each weight, that is*

\[
\mathbb{Z}_N = \text{lin}_\mathbb{Q} \{ \zeta(n_1, \ldots, n_r) : n_1, \ldots, n_r \in \{2,3\} \quad \text{and} \quad n_1 + \ldots + n_r = N \}.
\] (4.2)

In particular this implies (as was also proved independently in [17])

\[
\dim_{\mathbb{Q}} \mathbb{Z}_N \leq d_n.
\] (4.3)

In fact the results of [4] prove the existence of a surjective algebra morphism

\[
\phi : \mathbb{Q}[f_2] \otimes_{\mathbb{Q}} (\mathbb{Q}(f_3, f_5, f_7, \ldots)_{\text{MU}}) \rightarrow \mathbb{Z}
\] (4.4)

which preserves the weight filtrations. In this picture, conjecture 4.2 is equivalent to \( \phi \) being an isomorphism.
5 Depth filtration

Definition 5.1. Let $\mathcal{Z}^{(d)}_N := \text{lin}_Q \{ \zeta(n_1, \ldots, n_r): n_1 + \ldots + n_r = N \text{ and } r \leq d \}$ denote the span of MZV of weight $N$ and depth $\leq d$.

The stuffle relation involves MZV of different depths, so in contrast to the weight, the depth can only be a filtration on $\mathcal{Z}$. For example recall the example

$$\zeta(\{2\}^n) = \zeta(2, \ldots, 2) = \frac{\pi^{2n}}{(2n + 1)!} \in \mathbb{Q} \cdot \zeta(2n) = \mathcal{Z}^{(1)}_{2n}$$

that is of depth one (not $n$). Similarly note

Theorem 5.2 ([17]). Setting $H(n) := \zeta(\{2\}^n)$ and $H(a, b) := \zeta(\{2\}^a, 3 \cdot \{2\}^b)$,

$$H(a, b) = 2 \sum_{r=1}^{a+b+1} (-1)^r \left( \frac{2r}{2a+2} \right) - \left( 1 - \frac{1}{2^{2r}} \right) \left( \frac{2r}{2b+1} \right) H(a+b-r+1) \zeta(2r+1). \quad (5.2)$$

In particular it implies that $H(a, b) \in \mathcal{Z}^{(2)}_{2a+2b+3}$ is of depth at most two, even though it is originally a MZV of high depth $a+b+1$.

Conjecture 5.3 (D. Broadhurst and D. Kreimer, [3]). $\mathcal{Z}$ is graded by the weight and the depth filtration has dimensions

$$d_{n,k} = \text{dim}_Q \left( \mathcal{Z}_{n}^{(k)} / \mathcal{Z}_{n}^{(k-1)} \right)$$

given by the generating series

$$\sum_{n,k} d_{n,k} x^k y^n = \frac{1 + \mathbb{E} y}{1 - \mathbb{O} y + \mathbb{S} y^2 (1 - y^2)} \quad (5.3)$$

where $\mathbb{E} := \frac{x^2}{1-x^2} = x^2 + x^4 + x^6 + \ldots$ and $\mathbb{O} := \frac{x^3}{1-x^3} = x^3 + x^5 + x^7 + \ldots$ count the even and odd single zetas in each weight, while $\mathbb{S} := \frac{x^{12}}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)}$ is the generating series of the dimensions of the spaces of cusp forms of $\text{SL}_2(\mathbb{Z})$ in each weight.

Note that this is a refinement of conjecture [12] (set $y = 1$ and use $d_n = \sum_k d_{n,k}$). Expanding (5.3) to the first orders in the depth $y$, observe

$$\sum_{n,k} d_{n,k} = 1 + (\mathbb{E} + \mathbb{O}) y + (\mathbb{E} \mathbb{O} + \mathbb{O}^2 - \mathbb{S}) + (\cdots) y^3 + \ldots \quad (5.4)$$

In particular we see in depth two, that the products $\mathbb{E} \mathbb{O}$ of even and odd single zeta values are the only generators in odd weights. This is known more generally as

Theorem 5.4 ([11] [15]). Every $\zeta(n_1, \ldots, n_d)$ with weight $n = n_1 + \ldots + n_d$ and depth $d \neq n \mod 2$ of different parity is a $\mathbb{Q}$-linear combination of products of MZV of depth smaller than $d$.

Example 5.5.

$$\zeta(4, 2, 2) = \zeta(4) \zeta(2, 2) + \zeta(2) [4\zeta(4, 2) + 6\zeta(3, 3) + 7\zeta(2, 4) + 8\zeta(1, 5)] - 8\zeta(6, 2) - 10\zeta(5, 3) - \frac{33}{2} \zeta(4, 4) - 12\zeta(3, 5) - \frac{15}{2} \zeta(2, 6)$$
In particular, every depth-two $\zeta(n_1, n_2)$ with odd $n_1 + n_2$ is a sum of products of single zetas. An explicit formula is given as

**Theorem 5.6** (Proposition 7 in [17]). For $m \geq 1$, $n \geq 2$ of odd weight $k = m + n = 2K + 1$,

$$\zeta(m, n) = (-1)^m \sum_{s=0}^{K-1} \left[ \binom{k - 2s - 1}{m - 1} + \binom{k - 2s - 1}{n - 1} \right] - \delta_{n,2s} + (-1)^m \delta_{s,0} \right] \zeta(2s) \zeta(k - 2s).$$

On the other hand, for even weights $N$, the contribution $O^2$ to (5.4) counts the $\zeta(n, m)$ with odd entries $n, m \geq 3$ and $n + m = N$. However, these are in general not independent. The first relation (up to $\mathbb{Q} \cdot \zeta(N) = 2Z^{(1)}_N$) appears at weight $N = 12$:

$$28\zeta(3, 9) + 150\zeta(5, 7) + 168\zeta(7, 5) = \frac{5197}{691}\zeta(12).$$ (5.5)

The origin of these exotic relations in depth two correspond [6] to period polynomials for cusp forms of $\text{SL}_2(\mathbb{Z})$ and are counted by $S$. These connections might be enlightened by the upcoming lectures of José I. Burgos.

**References**


