Groups of Autoequivalences of Derived Categories of Smooth Projective Varieties

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Introduction

This study focuses on automorphism groups of derived categories of smooth projective varieties. Part of the motivation comes from considering the derived category of such varieties as something like the ultimate cohomological invariant. In particular, one can compare the derived level with more traditional invariants like K-theory or topological cohomology.

It was Mukai who first gave an explicit example of a significant equivalence between derived categories coming from geometry [61]. He showed that the derived categories of an abelian variety and its dual are equivalent. The technique introduced by him to prove this has been generalised since then and nowadays goes by the name of Fourier-Mukai transforms. In a further step, Bondal and Orlov gave another link connecting geometry and derived categories [11], [12]: they showed how varieties of Fano type or with ample canonical sheaf can be reconstructed from their derived categories. Thus attention was drawn to the case of varieties with trivial canonical bundle. Orlov proved a derived version of the Torelli theorem for K3 surfaces [71] and also a structure theorem for derived categories of abelian varieties [72]. Meanwhile, Kawamata provided some evidence to the conjecture that two birational smooth projective varieties with trivial canonical sheaves have equivalent derived categories [49]. Bridgeland proved this in dimension three [16].

One should point out why groups of equivalences are important and interesting. Let us just note for the moment that all the fundamental results mentioned above bring forth expressions in terms of these groups. For example, the reconstruction theorem of Bondal and Orlov also implies that the groups of autoequivalences on varieties of Fano type or with ample canonical sheaf as small as possible [12]. In concrete terms, this means that every autoequivalence is a composition of an automorphism of the variety, the twist by some line bundle, and a shift of complexes. On the other hand, consider a principally polarised abelian variety. Then Mukai's initial observation can be reformulated saying that the Poincaré bundle yields a genuinely different autoequivalence on this abelian variety.

There is a certain connection between the topics addressed here and some aspects of theoretical physics [26]. Going a little bit more into detail, it is now generally accepted by string theorists that D-branes form a triangulated category. In the case of the IIB theory, which lives on a Calabi-Yau manifold, the D-branes are sheaves or, more generally, objects of the derived category. Kontsevich's homological mirror symmetry conjecture states that a symplectic manifold is the mirror of some Calabi-Yau variety precisely if a certain triangulated category constructed from the symplectic data is equivalent to the derived category on the algebraic side [52]. While not even the final definitions are completely clear on the symplectic side, this conjecture has nonetheless given impetus to research in both symplectic and algebraic geometry.

This study consists of four chapters, and we now summarise their main topics. In the first chapter, we review facts about derived categories and Fourier-Mukai transforms from the general theory. This exposition mainly collects results which are standard by now. A point worthy of notice might be the section on spherical objects as it contains a simple proof that the functors derived from them are equivalences. An auxiliary fact we could not find in the literature is that Fourier-Mukai kernels of equivalences are always simple.

Having covered the basics, we turn to K3 surfaces. The main result is that every other Hodge isometry of the full cohomology lattice can be lifted to an autoequivalence of the derived category. Explicitly, we can compare the group of autoequivalences with the group of integral Hodge isometries using a homomorphism from the former to the latter. The theorem states that the image is a subgroup of index at most two. Whether the homomorphism is actually surjective or not is not known at present. Further on in this section, we rephrase and reconsider this problem in some ways.

In the third part, we consider finite group actions on smooth projective varieties and develop a theory of groups of equivariant autoequivalences. The aim is to compare invariant Fourier-Mukai kernels and equivariant ones. How close they are depends on the group action of course but, nonetheless, there are quite general results. As an immediate application, we can show that birational Hilbert schemes of points on a K3 surface have equivalent derived categories. From this, we deduce that there are only finitely many Hilbert schemes of points on K3 surfaces up to birational isomorphism. This ties in with Kawamata's philosophy alluded to above.

The final part deals with Kummer surfaces. Since these are formed from abelian surfaces by a quotient construction with the cyclic group of order two, our results on equivariant autoequivalence groups apply particularly well. We explain what portion of the group of Kummer autoequivalences actually comes from the abelian surface. There are also certain remarks on generalised Kummer varieties.

Notation and conventions

Varieties and morphisms

If not mentioned explicitly otherwise, by a variety we mean a smooth projective scheme over the field $\mathbb{C}$. Nonetheless, we write $k(y)$ for the skyscraper sheaf of some point $y \in Y$; points are always closed. Canonical projections are denoted throughout by $p_Y : Y \times Z \to Y$ or $p_1, p_2 : Y \times Y \to Y$. Because this notation is so easy to understand, the maps will sometimes appear without further introduction.

Complexes

Complexes are written in the form $[\cdots \to A^i \to A^{i+1} \to \ldots]$ and denoted by A or $A^\bullet$, depending on emphasis. The homology of a complex A in degree i is written as $\mathrm{h}^i(A)$. The reason for this is to avoid any confusion with sheaf cohomology. Accordingly, the dimension of sheaf cohomology spaces will always be denoted by $\dim(\mathrm{H}^i(\cdot))$. The complex $A[1]$ denotes A shifted one place to the left, so that $A[1]^i = A^{i+1}$.

Derived categories

We write $\mathrm{D}(Y) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(Y))$ for the bounded category of coherent sheaves on a variety Y. The category of coherent sheaves is throughout considered as the full subcategory of complexes having only homology in degree 0. Derived functors will appear without special symbols to denote derivation. For example, considering a morphism $f : Y \to Z$, the functor $f_* : \mathrm{D}(Y) \to \mathrm{D}(Z)$ *is* exact. Taking a sheaf $A \in \mathrm{Coh}(Y)$, its direct image $f_*(A) \in \mathrm{D}(Z)$ is generally a genuine complex, and the homology sheaves $\mathrm{h}^i(f_*(A)) = R^i f_*(A) \in \mathrm{Coh}(Z)$ are the usual higher direct images.

As usual, we set $\mathrm{Hom}^i_{\mathrm{D}(Y)}(A, B) = \mathrm{Hom}_{\mathrm{D}(Y)}(A, B[i])$ for complexes $A, B \in \mathrm{D}(Y)$. For sheaves $A, B \in \mathrm{Coh}(Y)$ we have $\mathrm{Hom}_{\mathrm{D}(Y)}(A, B[i]) = \mathrm{Ext}^i_Y(A, B)$ by [30, §III.5]. We distinguish between the total homomorphism space $\mathrm{Hom}^*(A, B) := \oplus_{i \in \mathbb{Z}} \mathrm{Hom}^i(A, B)$ and the complex $\mathrm{Hom}^\bullet(A, B) := \oplus_{i \in \mathbb{Z}} \mathrm{Hom}^i(A, B)[-i]$ with zero differentials.

If a functor F is left adjoint to a functor G, we will sometimes write this as $F \dashv G$.

Remark 0. If C is a smooth curve, then $A \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{h}^i(A)[-i]$ for all $A \in \mathrm{D}(C)$ by [31, Proposition 6.3] or [42, Corollary 3.14]. Thus we are mainly interested in dimension >1.

Cohomology and intersection rings

Let Y be a variety and A be a coherent sheaf (or complex) on Y. As is common by now, we write $v(A) := \mathrm{ch}(A)\sqrt{\mathrm{td}_Y} \in \mathrm{H}^*(Y, \mathbb{Q})$ for the *Mukai vector* of A. The cycle class of a subvariety $Z \subset Y$ of codimension c is written $[Z] \in \mathrm{H}^{2c}(Y, \mathbb{Z})$. In particular, we have $1 = [Y] \in \mathrm{H}^0(Y, \mathbb{Z})$ and $[pt] \in \mathrm{H}^{2\dim(Y)}(Y, \mathbb{Z})$, the latter not depending on a particular point of Y. The same notation will apply to the intersection (or Chow) rings $\mathrm{CH}^*(Y)$, so that $1 = [Y] \in \mathrm{CH}^0(Y)$ and $[p] \in \mathrm{CH}^{\dim(Y)}(Y)$. Note that the latter cycle class in the Chow ring depends on the (rational equivalence class of the) point $p \in Y$. The product of two classes or cycles z_1, z_2 is sometimes denoted $z_1.z_2$.

1 Generalities on Fourier-Mukai transforms

1.1 Derived categories and Fourier-Mukai kernels

In this section, we will not try to reproduce the general theory but rather assemble the important facts from a concrete point of view. Let Y be a smooth projective variety over $\mathbb{C}$. Then, an interesting homological invariant of Y is $\mathrm{D}(Y) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(Y))$, the bounded derived category of coherent sheaves. While relatives of $\mathrm{D}(Y)$, using quasi-coherent sheaves or unbounded complexes, are important, they will not appear in this study. A virtue of insisting on 'smooth, projective' varieties and 'bounded' complexes of 'coherent' sheaves is that $\mathrm{D}(Y)$ turns out to be a finite $\mathbb{C}$-linear triangulated category, i.e. all Hom's are finite dimensional $\mathbb{C}$-vector spaces. This follows basically from Serre's theorem that sheaf cohomology is finite dimensional under these circumstances [36, Theorem III.5.2]. We are interested in the automorphisms of $\mathrm{D}(Y)$, thus

$$\mathrm{Aut}(\mathrm{D}(Y)) := \{\varphi : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Y) \text{ exact equivalence}\}.$$

It is obvious that the composition of exact functors is again exact; furthermore the quasi-inverse of an exact equivalence is exact. Hence $\mathrm{Aut}(\mathrm{D}(Y))$ is indeed a group. In general, there are three types of autoequivalences that we always have at our disposal, namely shifts of complexes, automorphisms of the variety and line bundle twists:

$$\begin{aligned} \mathbb{Z} &\to \mathrm{Aut}(\mathrm{D}(Y)), & n &\mapsto [n], \\ \mathrm{Aut}(Y) &\to \mathrm{Aut}(\mathrm{D}(Y)), & f &\mapsto f_* = (f^{-1})^*, \\ \mathrm{Pic}(Y) &\to \mathrm{Aut}(\mathrm{D}(Y)), & L &\mapsto \mathsf{M}_L, \quad \mathsf{M}_L(E) := L \otimes E. \end{aligned}$$

All three representations are injective. They combine to a semidirect product subgroup $\mathrm{Aut}(Y) \ltimes \mathrm{Pic}(Y) \times \mathbb{Z} \subset \mathrm{Aut}(\mathrm{D}(Y))$, as $(\mathsf{M}_{L_1} \circ f_{1*}) \circ (\mathsf{M}_{L_2} \circ f_{2*}) = \mathsf{M}_{L_1 \otimes f_{1*} L_2} \circ (f_1 f_2)_*$. Actually, this is the distinguished subgroup containing all autoequivalences which map skyscraper sheaves of points to themselves, up to shift; see Remark 1.9.

We turn to Serre functors and adjoints. The notion of Serre functor has been axiomatically defined for triangulated categories in [10] but we stick to the concrete setting.

(1) The Serre functor for $\mathrm{D}(Y)$ is given by $S_Y : \mathrm{D}(Y) \to \mathrm{D}(Y), A \mapsto A \otimes \omega_Y[\dim(Y)]$ (and similar on morphisms). It is obviously an exact equivalence.
(2) By Serre duality in the form described in [35], S_Y is an equivalence with functorial isomorphisms $\mathrm{Hom}(A, B) \xrightarrow{\sim} \mathrm{Hom}(B, S(A))^*$ for all $A, B \in \mathrm{D}(Y)$. These properties characterise S_Y uniquely up to isomorphism; see [42, §1.1].
(3) The Serre functor S_Y commutes with exact equivalences. This follows from the uniqueness in (2) and the usual Yoneda argument.
(4) If $F : \mathrm{D}(Y) \to \mathrm{D}(Z)$ is a functor (with Z another smooth projective variety) which has a right adjoint $G : \mathrm{D}(Z) \to \mathrm{D}(Y)$, then the functor $S_Y^{-1} \circ G \circ S_Z$ is a left adjoint for F. This is again an immediate consequence of (2).

Before we present a couple of facts about adjoint functors, we introduce some general notions. A contravariant cohomological functor $H : \mathrm{D}(Y)^{\mathrm{opp}} \to \mathrm{Mod}(\mathbb{C})$ to the abelian category of $\mathbb{C}$-vector spaces is of *finite type* if $\sum_i \dim H(A[i]) < \infty$ for all $A \in \mathrm{D}(Y)$. A triangulated category is called *saturated* if every contravariant cohomological functor of finite type is representable. The next theorem is crucial for the existence of adjoint functors:

Theorem 1.1 (Bondal, Van den Bergh). *For a smooth projective variety Y, the bounded derived category of coherent sheaves $\mathrm{D}(Y)$ is saturated.*

Proof. The theorem is proven in [9, Appendix A] together with vast generalisations to the singular as well as to the noncommutative case. □

The following remarks settle the question of adjoints in our setting completely.

(1) An adjoint functor of an exact functor is again exact [71, Lemma 1.2].

(2) Any exact functor $F : \mathrm{D}(Y) \to \mathrm{D}(Z)$ has a right adjoint. To see this, consider for any $B \in \mathrm{D}(Z)$ the cohomological functor $H := \mathrm{Hom}(F(\cdot), B) : \mathrm{D}(Y) \to \mathrm{Mod}(\mathbb{C})$; it is of finite type because $\mathrm{D}(Y)$ has finite dimensional total Hom's. As $\mathrm{D}(Y)$ is saturated, there is a unique object $A \in \mathrm{D}(Y)$ representing H. Setting $G(B) := A$ defines a functor $G : \mathrm{D}(Z) \to \mathrm{D}(Y)$ by the Yoneda lemma. We conclude $F \dashv G$ from $\mathrm{Hom}(F(\cdot), B) = H(\cdot) = \mathrm{Hom}(\cdot, A) = \mathrm{Hom}(\cdot, G(B))$.

(3) Finally, the last remark concerning Serre functors applies and we find that every exact functor $F : \mathrm{D}(Y) \to \mathrm{D}(Z)$ possesses both left and right adjoints.

Once we recognise the derived category $\mathrm{D}(Y)$ as an interesting invariant of the variety Y, the relations among those categories will be of importance. As has been hinted at previously, we regard $\mathrm{D}(Y)$ as a finer relative of the K-group $\mathrm{K}(Y)$, Chow ring $\mathrm{CH}^*(Y)$ or topological cohomology $\mathrm{H}^*(Y, \mathbb{Q})$. Most notably for Chow rings and cohomology, these structures have been classically studied by the means of correspondences. The basic idea is that an object on the product gives rise to a map, e.g. a cohomology class $\eta \in \mathrm{H}^*(Y \times Z, \mathbb{Q})$ yields the map $\mathrm{H}^*(Y, \mathbb{Q}) \to \mathrm{H}^*(Z, \mathbb{Q})$, $\alpha \mapsto p_{Z*}(\eta . p_Y^* \alpha)$. Here, pull-back, push-forward and product are those from topology. Maps of this sort provide all kinds of interesting examples. Since we also have the three basic operations in the derived categories $\mathrm{D}(Y)$ at our disposal, we can establish a similar notion:

Definition 1.2. Let Y and Z be smooth projective varieties and $P \in \mathrm{D}(Y \times Z)$ any object. Then the corresponding *Fourier-Mukai transform* with *kernel* P is the exact functor

$$\mathsf{FM}_P : \mathrm{D}(Y) \to \mathrm{D}(Z), \quad A \mapsto p_{Z*}(P \otimes p_Y^* A).$$

The functor FM_P is called a *Fourier-Mukai equivalence* if it is an exact equivalence between triangulated categories.

Remark 1.3. Mukai was the first to employ Fourier-Mukai transforms in [61]. There he used the Poincaré bundle as kernel to show that the derived categories of an abelian variety and of the dual abelian variety are equivalent. His use of the name Fourier stems from the comparison with Volterra operators in functional analysis which are formally correspondences between appropriate L^2-spaces; see [62].

Remark 1.4. The calculus of correspondences [29, §14] carries over from the classical versions. To quote just one instance, the composite transform of $\mathsf{FM}_Q : \mathrm{D}(X) \to \mathrm{D}(Y)$ and $\mathsf{FM}_P : \mathrm{D}(Y) \to \mathrm{D}(Z)$ has kernel $P \star Q := p_{XZ*}(p_{YZ}^* P \otimes p_{XY}^* Q) \in \mathrm{D}(X \times Z)$, where p_{XY}, p_{XZ}, p_{YZ} are the canonical projections from $X \times Y \times Z$ to the indicated factors. So we have $\mathsf{FM}_P \circ \mathsf{FM}_Q = \mathsf{FM}_{P\star Q} : \mathrm{D}(X) \to \mathrm{D}(Z)$ for the corresponding functors.

A nice feature of these transforms is that the adjoints of $\mathsf{FM}_P : \mathrm{D}(Y) \to \mathrm{D}(Z)$ are again Fourier-Mukai transforms. Their kernels are $P^L := P^\vee \otimes p_Z^* \omega_Z[\dim(Z)]$ and $P^R := P^\vee \otimes p_Y^* \omega_Y[\dim(Y)]$, respectively. This follows from $f^* \dashv f_* \dashv f^! := S_{Y_2} \circ f^* \circ S_{Y_1}^{-1}$ for a morphism $f : Y_1 \to Y_2$ where S_{Y_1} and S_{Y_2} are the Serre functors. In particular, if FM_P is fully faithful and both adjoints coincide, then FM_P is a Fourier-Mukai equivalence with quasi-inverse $\mathsf{FM}_P^{-1} \cong \mathsf{FM}_{P^L} : \mathrm{D}(Z) \xrightarrow{\sim} \mathrm{D}(Y)$.

We will use the familiar facts about Fourier-Mukai transforms without much ado whenever they are just consequences of the transforms being derived correspondences.

Remark 1.5. Note that we could also have used P to produce a functor in the direction $\mathrm{D}(Z) \to \mathrm{D}(Y)$. In order to keep the notation light, we will refrain from specifying the direction in the symbol FM_P. The functor in the reverse direction will, however, appear. In those cases, it will be denoted by FM_P^t. The assignment $\mathsf{FM}_P \mapsto \mathsf{FM}_P^t$ is a direction reversing involution although FM_P^t is not the transpose on the level of cohomology. We will be careful to introduce a kernel P always together with a prescribed direction for $\mathsf{FM}_{\mathcal{P}}$, so that the notation makes sense.

Examples 1.6. We give some examples for Fourier-Mukai equivalences. The proofs for the first three instances are straightforward.

(1) The structure sheaf of the diagonal $\Delta \subset Y \times Y$ is the kernel for the identity, i.e. $\mathsf{FM}_{\mathcal{O}_\Delta} = \mathrm{id} : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Y)$. Analogously, we get the shift functors: $\mathsf{FM}_{\mathcal{O}_\Delta[n]} = [n]$ for all $n \in \mathbb{Z}$. In the same vein, let $\Delta : Y \to Y \times Y$ be the diagonal embedding and L a line bundle on Y. Then, the kernel $L_\Delta := \Delta_* L$ gives $\mathsf{FM}_{L_\Delta} = \mathsf{M}_L : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Y), A \mapsto A \otimes L$.
(2) For a morphism $f : Y \to Z$, the structure sheaf of the graph $\Gamma(f) := (f, \mathrm{id}_Z)^{-1}(\Delta_Z)$ yields $\mathsf{FM}_{\mathcal{O}_{\Gamma(f)}} = f_*$. Note that $\mathsf{FM}^t_{\mathcal{O}_{\Gamma(f)}} = f^*$.
(3) Consider an elliptic curve E with a fixed origin p_0 and identify E and $\hat{E} = \mathrm{Pic}^0(E)$ via $\varphi : E \xrightarrow{\sim} \hat{E}, x \mapsto \mathcal{O}_E(x - p_0)$ (subtraction of divisors). The normalised Poincaré line bundle $\mathcal{P} := \mathcal{O}_{E\times E}(\Delta - \{p_0\} \times E - E \times \{p_0\})$ has a universal property which can be expressed by $\mathsf{FM}_{\mathcal{P}}(k(x)) = \varphi(x)$. It is easy to see, using Cohomology and Base Change, that $\mathsf{FM}_{\mathcal{P}} \circ \mathsf{FM}_{\mathcal{P}} = (-1)^*[-1]$. In particular, $\mathsf{FM}_{\mathcal{P}} \in \mathrm{Aut}(\mathrm{D}(E))$.

(4) Let us consider two Fourier-Mukai equivalences $F = \mathsf{FM}_{\mathcal{F}} : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Y')$ and $G = \mathsf{FM}_{\mathcal{G}} : \mathrm{D}(Z) \xrightarrow{\sim} \mathrm{D}(Z')$. Then their product gives a further equivalence $F \times G = \mathsf{FM}_{\mathcal{F}\boxtimes\mathcal{G}} : \mathrm{D}(Y \times Z) \xrightarrow{\sim} \mathrm{D}(Y' \times Z')$ whose quasi-inverse has kernel $\mathcal{F}^L \boxtimes \mathcal{G}^L$. Taking another Fourier-Mukai transform $\mathsf{FM}_P : \mathrm{D}(Y) \to \mathrm{D}(Z)$, we get the relation $\mathsf{FM}_{(F\times G)(P)} = G \circ \mathsf{FM}_P \circ F^t$ with $F^t := \mathsf{FM}^t_{\mathcal{F}}$, i.e. a commutative diagram

$$\begin{array}{ccc} \mathrm{D}(Y) & \xrightarrow{\mathsf{FM}_P} & \mathrm{D}(Z) \\ {\scriptstyle F^t=\mathsf{FM}^t_{\mathcal{F}}}\uparrow & & \downarrow{\scriptstyle G} \\ \mathrm{D}(Y') & \xrightarrow[\mathsf{FM}_{(F\times G)(P)}]{} & \mathrm{D}(Z') \end{array}$$

which is just a restatement of the following isomorphism on the kernel level:

$$(F \times G)(P) = p_{Y'Z',*}(\mathcal{F} \boxtimes \mathcal{G} \otimes p^*_{YZ}P) = p_{Y'Z',*}(p^*_{ZZ'}\mathcal{G} \otimes p^*_{YZ}P \otimes p^*_{Y'Y}\mathcal{F}),$$

the last term being the kernel of $\mathsf{FM}_{\mathcal{G}} \circ \mathsf{FM}_P \circ \mathsf{FM}^t_{\mathcal{F}}$. As a special case, consider two automorphisms $f : Y \xrightarrow{\sim} Y$, $g : Z \xrightarrow{\sim} Z$. Then, the new kernel $(f,g)^*P$ gives $\mathsf{FM}_{(f,g)^*P} = g^* \circ \mathsf{FM}_P \circ f_*$.

The notion of Fourier-Mukai transform can also be regarded, in a more fancy manner, as a functor

$$\mathsf{FM} : \mathrm{D}(Y \times Z) \to \mathrm{Fun}(\mathrm{D}(Y), \mathrm{D}(Z)), \quad P \mapsto \mathsf{FM}_P$$

where $\mathrm{Fun}(\mathrm{D}(Y), \mathrm{D}(Z))$ is the category consisting of exact functors from $\mathrm{D}(Y)$ to $\mathrm{D}(Z)$ with natural transformations as morphisms. This category can be made graded by defining the shift of $F \in \mathrm{Fun}(\mathrm{D}(Y), \mathrm{D}(Z))$ as $F \circ [1]_{\mathrm{D}(Y)} \cong [1]_{\mathrm{D}(Z)} \circ F$. Yet declaring a triangle $F' \to F \to F'' \to F'[1]$ to be distinguished in $\mathrm{Fun}(\mathrm{D}(Y), \mathrm{D}(Z))$ if the triangles $F'(A) \to F(A) \to F''(A) \to F'(A)[1]$ in $\mathrm{D}(Z)$ are distinguished for all $A \in \mathrm{D}(Y)$ does not produce a triangulated category—since cones are not functorial. See [21] for a similar, but more sophisticated view on this topic.

We will mainly be interested in equivalences between derived categories. A basic, but highly non-trivial, theorem of Orlov states that all such equivalences actually are Fourier-Mukai transforms, and thus can be described using kernels. The precise statement follows:

Theorem 1.7 (Orlov). *Let Y and Z be two smooth projective varieties and $F : \mathrm{D}(Y) \to \mathrm{D}(Z)$ an exact fully faithful functor. Then there exists an object $P \in \mathrm{D}(Y \times Z)$ such that $F \cong \mathsf{FM}_P$ are isomorphic functors. Moreover, P is unique up to isomorphism.*

Proof. For the proof see the original [71, Theorem 2.2] or Kawamata's generalisation to orbifold-type stacks, [50, Theorem 1.1]. The original statement included the additional assumption that F should possess adjoint functors. This is however automatically fulfilled in view of Theorem 1.1 and its corollary. □

Remark 1.8. This theorem allows to extend the transposition introduced in Remark 1.5 to all equivalences: for $F : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Z)$ we have $F \cong \mathsf{FM}_P$ with unique kernel $P \in \mathrm{D}(Y \times Z)$ and we set $F^t := \mathsf{FM}_P^t : \mathrm{D}(Z) \xrightarrow{\sim} \mathrm{D}(Y)$. Note that the usual transposition rules apply. In particular, we get an involution $\mathrm{Aut}(\mathrm{D}(Y)) \xrightarrow{\sim} \mathrm{Aut}(\mathrm{D}(Y)), F \mapsto (F^t)^{-1}$.

Remark 1.9. Every autoequivalence $F : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Y)$ which takes skyscraper sheaves $k(y)$ to skyscraper sheaves $k(f(y))$, must be of the form $\mathsf{M}_N \circ f_*$ for some line bundle $N \in \mathrm{Pic}(Y)$ and an isomorphism $f : Y \xrightarrow{\sim} Y$. To see this, choose first a Fourier-Mukai kernel P for F. Then P is actually quasi-isomorphic to a sheaf, because we can detect homology $\mathrm{h}^i(P)$ using $\mathsf{FM}_{\mathcal{P}}(k(y))$: replacing P by a locally free resolution $Q^\bullet$, we find $\mathsf{FM}_P(k(y)) \cong Q^\bullet|_{\{y\}\times Y}$ and $\mathrm{h}^i(Q^\bullet) \neq 0$ implies $\mathrm{h}^i(Q^\bullet)|_{\{y\}\times Y} = \mathrm{h}^i(Q^\bullet|_{\{y\}\times Y}) \neq 0$ for some $y \in Y$. By assumption, $\mathrm{supp}(P)$ is the graph of $f : Y \xrightarrow{\sim} Y$. Finally, P has obviously rank one on $\mathrm{supp}(P)$ and hence $\mathsf{FM}_P = \mathsf{FM}_{i_*L} = f_* \circ \mathsf{M}_L = \mathsf{M}_{f_*(L)} \circ f_*$ where $i : Y \to Y \times Y, y \mapsto (y, f(y))$ and $L \in \mathrm{Pic}(Y)$.

Remark 1.10. Consider the full subcategories of $\mathrm{D}(Y \times Z)$ and $\mathrm{Fun}(\mathrm{D}(Y), \mathrm{D}(Z))$ which consist of equivalence kernels and equivalences, respectively. Orlov's theorem states that FM induces a bijection on their skeletons (which are the sets of isomorphism classes of objects). However, the following example of Căldăraru [21, Appendix B] shows that FM is not faithful.

Example 1.11. Let E be an elliptic curve and consider the two kernels $P = \mathcal{O}_\Delta$ and $Q = \mathcal{O}_\Delta[2]$. The corresponding functors are $\mathsf{FM}_P = \mathrm{id}$ and $\mathsf{FM}_Q = [2]$. Now we have by Serre duality $\mathrm{Hom}_{\mathrm{D}(E\times E)}(P, Q) = \mathrm{Ext}^2_{E\times E}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \mathrm{Hom}_{E\times E}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)^* = \mathbb{C}$. But, on the other hand, there are no nonzero natural transformations $T : \mathrm{id} \to [2]$ in view of the Remark 0: every complex is quasi-isomorphic to its homology. Hence $\mathrm{Mor}_{\mathrm{Fun}}(\mathsf{FM}_P, \mathsf{FM}_Q) = 0$ and this shows that $\mathsf{FM} : \mathrm{D}(E \times E) \to \mathrm{Fun}(\mathrm{D}(E), \mathrm{D}(E))$ is not faithful.

Despite this example, at least $\mathrm{Hom}_{\mathrm{D}(Y\times Z)}(P, P) \to \mathrm{Hom}_{\mathrm{Fun}(\mathrm{D}(Y),\mathrm{D}(Z))}(\mathsf{FM}_P, \mathsf{FM}_P)$ is an injective map for Fourier-Mukai equivalences FM_P:

Lemma 1.12. *Let* $\mathsf{FM}_P : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Z)$ *be a Fourier-Mukai equivalence. Then the kernel P is a simple object of* $\mathrm{D}(Y \times Z)$, *i.e.* $\mathrm{Hom}_{\mathrm{D}(Y\times Z)}(P, P) = \mathbb{C}$.

Proof. Denote by Q the kernel of the quasi-inverse equivalence so that $P \star Q \cong \mathcal{O}_{\Delta_Z}$. This type of composition also works on the level of homomorphisms: endomorphisms $f : P \to P$ and $\mathrm{id}_Q : Q \to Q$ give rise to $f \star \mathrm{id}_Q : P \star Q \to P \star Q$, i.e. an endomorphism of the diagonal $\mathcal{O}_{\Delta_Z}$. From $\mathrm{Hom}_{\mathrm{D}(Z\times Z)}(\mathcal{O}_{\Delta_Z}, \mathcal{O}_{\Delta_Z}) = \mathrm{Hom}_{Z\times Z}(\mathcal{O}_{\Delta_Z}, \mathcal{O}_{\Delta_Z}) = \mathbb{C}$ now follows $f \star \mathrm{id}_Q = \lambda \cdot \mathrm{id}_{\mathcal{O}_{\Delta_Z}}$ for a $\lambda \in \mathbb{C}$. Composing again with $\mathrm{id}_P : P \to P$ gives then $f = f \star \mathrm{id}_{\mathcal{O}_{\Delta_Y}} = f \star (\mathrm{id}_Q \star \mathrm{id}_P) = (f \star \mathrm{id}_Q) \star \mathrm{id}_P = (\lambda \cdot \mathrm{id}_{\mathcal{O}_{\Delta_Z}}) \star \mathrm{id}_P = \lambda \cdot \mathrm{id}_P$. ☐

As we are primarily interested in equivalences, it proves to be very useful to have some handy criterion. Before the statement, we recall the notion of a spanning class.

Definition 1.13. A subset of objects $\mathcal{C} \subset \mathrm{D}(Y)$ is called a *spanning class* if for all objects $A \in \mathrm{D}(Y)$ the following two implications hold:

$$\begin{aligned} \mathrm{Hom}^i(A, C) = 0 \quad \forall C \in \mathcal{C}, i \in \mathbb{Z} \quad &\Longrightarrow \quad A = 0, \\ \mathrm{Hom}^i(C, A) = 0 \quad \forall C \in \mathcal{C}, i \in \mathbb{Z} \quad &\Longrightarrow \quad A = 0. \end{aligned}$$

This definition can be rephrased in terms of left and right orthogonal complements: $\mathcal{C}$ is a spanning class if and only if $\mathcal{C}^\perp = {}^\perp\mathcal{C} = 0$. Here the right orthogonal complement of $\mathcal{C}$ is $\mathcal{C}^\perp := \{A \in \mathrm{D}(Y) \ : \ \mathrm{Hom}^*(C, A) = 0 \, \forall C \in \mathcal{C}\}$ and similarly for ${}^\perp\mathcal{C}$.

Examples 1.14. The first two are the primary examples of spanning classes.

(1) $\mathcal{C} = \{k(y) \ : \ y \in Y\}$ is the spanning class of skyscraper sheaves [15, Example 2.2].

(2) Given an ample line bundle $L \in \mathrm{Pic}(Y)$, the set $\mathcal{C} = \{L^{\otimes i} \ : \ i \in \mathbb{Z}\}$ of all tensor powers is spanning by [71, Lemmas 2.13, 2.14].

(3) For the occurrence of another spanning class, see the proof of Theorem 1.27.

Theorem 1.15 (Bondal, Orlov, Bridgeland). *Let $F : \mathrm{D}(Y) \to \mathrm{D}(Z)$ be an exact functor and fix a spanning class $\mathcal{C} \subset \mathrm{D}(Y)$.*

(i) F is fully faithful if and only if $\mathrm{Hom}^i_{\mathrm{D}(Y)}(C, C') \xrightarrow{\sim} \mathrm{Hom}^i_{\mathrm{D}(Z)}(F(C), F(C'))$ holds for all $C, C' \in \mathcal{C}$ and $i \in \mathbb{Z}$.

(ii) F is an equivalence if and only in addition to the condition of (i) hold $\dim(Y) = \dim(Z)$ and $F(C \otimes \omega_Y) \cong F(C) \otimes \omega_Z$ for all $C \in \mathcal{C}$.

Proof. Introducing temporarily the shift closure of $\mathcal{C}$ by $\mathcal{C}[*] := \{C[i] \ : \ i \in \mathbb{Z}, C \in \mathcal{C}\}$, the criteria of the theorem can be rephrased by the following concise statements:

(i) F is fully faithful $\iff$ $F|_{\mathcal{C}[*]} : \mathcal{C}[*] \to \mathrm{D}(Z)$ is fully faithful;

(ii) F is equivalence $\iff$ $F|_{\mathcal{C}[*]}$ is fully faithful and commutes with the Serre functors.

Recall that by Theorem 1.7 all fully faithful exact functors are actually Fourier-Mukai transforms. Hence the above criterion could also be expressed as a condition in terms of Fourier-Mukai kernels. This is indeed what happened in the evolution of the theory: Bondal and Orlov proved in [11, Theorem 1.1] that FM_P is fully faithful if and only if it is fully faithful on shifted skyscraper sheaves of points. Actually, their statement is stronger in that the conditions $\mathrm{Ext}^i(k(y), k(y)) \xrightarrow{\sim} \mathrm{Hom}^i(\mathsf{FM}_P(k(y)), \mathsf{FM}_P(k(y)))$ do not have to be checked for $i = 1, \ldots, \dim(Y)$—a simplification peculiar to this spanning class. The fully faithfulness criterion with the spanning class of tensor powers of an ample line bundle appears in [71, Lemma 2.15]. Bridgeland extended this in [15] to arbitrary spanning classes. Furthermore, he proved that compatibility with the canonical sheaves entails the surjectivity of F in the case of skyscraper sheaves of points. For the full generality as stated, see [13, §2.6] or [42]. ☐

To conclude this section, we introduce a notion that has become of much interest to both mathematicians and physicists.

Definition 1.16. Let Y be a smooth projective variety. Then another smooth projective variety Z with $\mathrm{D}(Y) \cong \mathrm{D}(Z)$ is called a *Fourier-Mukai partner* of Y. In this case, the varieties Y and Z are also called D-*equivalent*. The set of isomorphism classes of all Fourier-Mukai partners of Y is denoted by $\mathsf{FM}(Y)$.

Examples 1.17.

(1) The reconstruction theorem of Bondal and Orlov [12] states that, for a variety Y with either ample or anti-ample canonical sheaf ω_Y, the derived category $\mathrm{D}(Y)$ determines the variety up to isomorphism. In particular, $\mathsf{FM}(Y) = \{Y\}$ for such varieties.

(2) For smooth projective curves C, we also have $\mathsf{FM}(C) = \{C\}$. This follows from the previous point if the genus is different from one. For the remaining case, one can use that D-equivalent elliptic curves have identical Hodge structures and consequently are isomorphic [42, §4].

(3) Orlov showed in [72, Corollary 2.8] that $\mathsf{FM}(A)$ is a finite set for an abelian variety A using that an abelian variety has only finitely many abelian subvarieties. If A has no principal polarisation, then its dual abelian variety $\hat{A}$ is not isomorphic to A. On the other hand, Mukai established in [61] the Poincaré line bundle as a kernel to prove $\mathrm{D}(A) \cong \mathrm{D}(\hat{A})$. In particular, we get $\#\mathsf{FM}(A) \geq 2$ for A not principally polarised. For any number N, there is an abelian surface which has at least N Fourier-Mukai partners [40].

(4) Bridgeland and Maciocia proved in [14, Proposition 5.3] that K3 surfaces and, more generally, surfaces without (-1)-curves have finitely many Fourier-Mukai partners. A closed formula for the number of Fourier-Mukai partners has been given by Hosono et al. in [39]. For a K3 surface X such that $\mathrm{Pic}(X) = \langle L \rangle$ is generated by a single ample line bundle L, Oguiso showed in [68] that $\#\mathsf{FM}(X) = 2^{p(L^2/2)-1}$ where $p(n)$ is the number of prime divisors of $n > 1$ and $p(1) = 1$. See [82] for another proof that the number of Fourier-Mukai partners of K3 surface is finite, but unbounded.

(5) Kawamata extended the techniques of Bridgeland and Maciocia in [49, Theorem 3.2] to surfaces containing (-1)-curves. Altogether, we thus find that smooth projective varieties of dimension ≤ 2 have finitely many Fourier-Mukai partners. It is conjectured that this finiteness is true in all dimensions [49, Conjecture 1.5].

1.2 Comparison with cohomology

An important feature of the derived category $\mathrm{D}(Y)$ is that we can compare complexes on other, somewhat easier levels:

$$\mathrm{D}(Y) \xrightarrow{[\cdot]} \mathrm{K}(Y) \xrightarrow{\mathrm{ch}} \mathrm{CH}^*(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{z} \mathrm{H}^*(Y, \mathbb{Q})$$

Here the map $\mathrm{D}(Y) \to \mathrm{K}(Y)$ sends a complex $A^\bullet \in \mathrm{D}(Y)$ to its natural class in the K-group, $[A^\bullet] = \sum_i (-1)^i A^i = \sum_i (-1)^i \mathrm{h}^i(A^\bullet)$. Next, we have the Chern character $\mathrm{ch} : \mathrm{K}(Y) \to \mathrm{CH}^*(Y) \otimes \mathbb{Q}$. Finally, $z : \mathrm{CH}^*(Y) \otimes \mathbb{Q} \to \mathrm{H}^*(Y, \mathbb{Q})$ is the cycle map. This last assignment uses that Y is defined over the complex numbers.

Some remarks are in order: It sometimes happens that the Chern character takes image in the integral Chow ring $\mathrm{CH}^*(Y)$. If this is the case, then the same comment applies to the cycle map having target $\mathrm{H}^*(Y, \mathbb{Z})$.

We will work mainly with the topological cohomology. By abuse of notation, we write $\mathrm{ch} : \mathrm{D}(Y) \to \mathrm{H}^*(Y, \mathbb{Q})$. Furthermore, we often substitute the Mukai vector $v(A) := \mathrm{ch}(A)\sqrt{\mathrm{td}_Y}$ for the Chern character $\mathrm{ch}(A)$ of an object $A \in \mathrm{D}(Y)$. Note that the power series defining the Todd class starts with constant term 1 and hence has a square root. See below why this replacement is desirable.

Much as the objects of $\mathrm{D}(Y)$ give elements in the K-group $\mathrm{K}(Y)$, the Chow ring $\mathrm{CH}^*(Y)$ and rational cohomology $\mathrm{H}^*(Y, \mathbb{Q})$, a Fourier-Mukai kernel $P \in \mathrm{D}(Y \times Z)$ gives maps between them. We will mainly consider the cohomology groups in this presentation. The kernel P brings forth a cohomology class $v(P) \in \mathrm{H}^*(Y \times Z, \mathbb{Q})$. The correspondence obtained from this class is written either as $\mathsf{FM}_{v(P)}$ or as $\mathsf{FM}_P^{\mathrm{H}}$. To be precise,

$$\mathsf{FM}_{v(P)} = \mathsf{FM}_P^{\mathrm{H}} : \mathrm{H}^*(Y, \mathbb{Q}) \to \mathrm{H}^*(Z, \mathbb{Q}), \quad \alpha \mapsto p_{Z*}(v(P).p_Y^*\alpha).$$

Then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{D}(Y) & \xrightarrow{\mathsf{FM}_P} & \mathrm{D}(Z) \\ \downarrow v & & \downarrow v \\ \mathrm{H}^*(Y, \mathbb{Q}) & \xrightarrow{\mathsf{FM}_P^{\mathrm{H}}} & \mathrm{H}^*(Z, \mathbb{Q}). \end{array}$$

The proof uses essentially Grothendieck-Riemann-Roch [71, Proposition 3.5]; note that here the square root of the Todd class enters. It is a basic fact that going to cohomology commutes with composition, i.e. $\mathsf{FM}_{P \star Q}^{\mathrm{H}} = (\mathsf{FM}_P \circ \mathsf{FM}_Q)^{\mathrm{H}} = \mathsf{FM}_P^{\mathrm{H}} \circ \mathsf{FM}_Q^{\mathrm{H}}$ where $Q \in \mathrm{D}(X \times Y)$ is another Fourier-Mukai kernel; see [42, §4]. Finally, $\mathsf{FM}_{\mathcal{O}_{\Delta_Y}}^{\mathrm{H}} = \mathrm{id}_{\mathrm{H}^*(Y,\mathbb{Q})}$; this follows from Riemann-Roch for the closed embedding $\Delta_Y \hookrightarrow Y \times Y$. An immediate consequence is the group homomorphism

$$(\cdot)^{\mathrm{H}} : \mathrm{Aut}(\mathrm{D}(Y)) \to \mathrm{Aut}(\mathrm{H}^*(Y, \mathbb{Q})), \quad \mathsf{FM}_P \mapsto \mathsf{FM}_P^{\mathrm{H}}.$$

Rational cohomology has several more structures: grading, cup product, and Hodge structure; in addition there is the integral lattice. Using that $v(P)$ is an algebraic class, it is easy to see that $\mathsf{FM}_P^{\mathrm{H}}$ preserves even and odd cohomology, transcendental and algebraic classes, and the Hodge diagonals $\oplus_{p-q=k} \mathrm{H}^{p,q}$ for all k. However, Fourier-Mukai transforms in general do not respect grading or multiplication. Now the product on cohomology can be used to introduce a natural quadratic form which turns $\mathsf{FM}_P^{\mathrm{H}}$

into isometries for equivalence kernels P at the expense of using complex coefficients. Explicitly, the Mukai-Căldăraru pairing is given by

$$\langle v, w\rangle := \int e^{c_1(Y)/2}.v^\vee.w$$

where $v = \sum_k v_k$ with $v_k \in H^k(Y, \mathbb{C})$ and $v^\vee := \sum_k i^k v_k$. Căldăraru shows in [21] that any Fourier-Mukai equivalence $\mathsf{FM}_P : D(Y) \xrightarrow{\sim} D(Z)$ fulfills $\langle v, w\rangle_Y = \langle \mathsf{FM}^H_P(v), \mathsf{FM}^H_P(w)\rangle_Z$ for all $v, w \in H^*(Y, \mathbb{C})$. The behaviour of Mukai vectors under dualising is given by $v(E^\vee) = v(E)^\vee\, e^{c_1(Y)/2}$ using a computation with Chern roots [42, §4]. Finally, we mention the compatibility

$$\chi(A, B) = \int \mathrm{ch}(A^\vee)\ \mathrm{ch}(B)\ \mathrm{td}_Y = \int v(A)^\vee\, e^{c_1(Y)/2}\, v(B) = \langle v(A), v(B)\rangle,$$

the Euler characteristic being defined by $\chi(A, B) := \sum_i (-1)^i \dim \mathrm{Hom}^i_{D(Y)}(A, B)$. The first step is Hirzebruch-Riemann-Roch, replacing A or B by locally free resolutions. The imaginary summands do not appear because characteristic classes have even degrees.

Remark 1.18. Let us look closer at the case of a K3 surface X. The Betti numbers of X are $b_0 = b_4 = 1$, $b_1 = b_3 = 0$ and $b_2 = 22$. In particular, there is no odd cohomology and, furthermore, $\mathrm{ch}(A) \in H^*(X, \mathbb{Z})$ as the product is an even form. Hence, we can work with integer coefficients throughout. Now let us consider the classical Mukai pairing. For $v = (v_0, v_2, v_4)$, $w = (w_0, w_2, w_4) \in H^*(X, \mathbb{Z})$, it is given by $(v, w) := v_2.w_2 - v_0 w_4 - v_4 w_0 \in \mathbb{Z}$. Note $(v, w) = -\int v^\vee.w$, and by virtue of $c_1(X) = 0$ this is up to the sign just the general construction above. Hence, for two objects $A, B \in D(X)$, Hirzebruch-Riemann-Roch yields here $(v(A), v(B)) = -\chi(A, B)$.

Examples 1.19. We will give here the cohomological versions of some easy Fourier-Mukai transforms.

(1) For a morphism $f : Y \to Z$, we denote by $i : \Gamma \hookrightarrow Y \times Z$ the inclusion of its graph. Then we get $i^*\mathrm{td}_{Y\times Z} = i^* p_Y^* \mathrm{td}_Y .i^* p_Z^* \mathrm{td}_Z = j^* \mathrm{td}_Y .j^* f^* \mathrm{td}_Z = \mathrm{td}_\Gamma^2$ using the isomorphism $j : \Gamma \xrightarrow{\sim} Y$. Now Riemann-Roch and projection formula yield $v(i_*\mathcal{O}_\Gamma) = \mathrm{ch}(i_*\mathcal{O}_\Gamma)\sqrt{\mathrm{td}_{Y\times Z}} = (i_*\,\mathrm{td}_\Gamma)/\sqrt{\mathrm{td}_{Y\times Z}} = i_*(\mathrm{td}_\Gamma / i^* \sqrt{\mathrm{td}_{Y\times Z}}) = i_*(1) = [\Gamma]$. Hence $\mathsf{FM}^H_{\mathcal{O}_\Gamma} = f_*$ by the usual calculus of correspondences. In particular, we get $\mathsf{FM}^H_{\mathcal{O}_\Delta} = \mathrm{id}_H$.

(2) For a line bundle $L \in \mathrm{Pic}(Y)$, we proceed similarly, this time using the closed embedding $\Delta : Y \hookrightarrow Y^2$ and $\Delta^*\,\mathrm{td}_{Y^2} = \mathrm{td}_Y^2$. Then, $v(\Delta_* L) = \mathrm{ch}(\Delta_* L)\,\mathrm{td}_{Y^2} / \sqrt{\mathrm{td}_{Y^2}} = \Delta_*(\mathrm{ch}(L)\ \mathrm{td}_Y)/\sqrt{\mathrm{td}_{Y^2}} = \Delta_*(\mathrm{ch}(L)\,\mathrm{td}_Y .\Delta^* \sqrt{\mathrm{td}_{Y^2}}) = \Delta_*\,\mathrm{ch}(L)$, and, accordingly, $\mathsf{M}^H_L(v) = \mathsf{FM}^H_{\Delta_* L}(v) = \mathrm{ch}(L)\cdot v$.

(3) Consider an elliptic curve E with origin p_0 and the associated Poincaré bundle $\mathcal{P}$. The even cohomology of E is spanned by $[E]$ and $[pt]$. Using $\mathrm{td}_E = 1$, we see from $\mathsf{FM}_\mathcal{P}(k(p_0)) = \mathcal{O}_E$ that $\mathsf{FM}^H_\mathcal{P}([pt]) = [E]$. On the other hand, Cohomology and Base

Change easily yields $h^1\mathsf{FM}_{\mathcal{P}}(\mathcal{O}_E) = R^1 p_{2*}\mathcal{P} = k(p_0)$. Plugging this into the Base Change Theorem for $R^0 p_{2*}\mathcal{P}$, together with the fact that the zeroth direct image must be torsion free, gives $h^0\mathsf{FM}_{\mathcal{P}}(\mathcal{O}_E) = 0$. Hence, $\mathsf{FM}_{\mathcal{P}}(\mathcal{O}_E) = k(p_0)[-1]$. Using the basis $([E], [pt])$ of $H^{2*}(E, \mathbb{Z})$, we get $\mathsf{FM}^{\mathrm{H}}_{\mathcal{P}} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mathsf{M}^{\mathrm{H}}_{\mathcal{O}_E(p_0)} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

1.3 Reflections along spherical objects

There is an important class of Fourier-Mukai equivalences introduced by Seidel and Thomas in [79]. For any object $E \in \mathrm{D}(Y)$, they consider the canonical trace morphism $E^\vee \boxtimes E \to \mathcal{O}_\Delta$. It exists because $E \in \mathrm{D}(Y)$ is quasi-isomorphic to a bounded complex F of locally free sheaves; writing out the total complex $F^\vee \boxtimes F$ exhibits the trace map in degree 0. Using the cone of this complex morphism, one can define an object of $\mathrm{D}(Y \times Y)$ by $I_E := \mathrm{cone}(E^\vee \boxtimes E \to \mathcal{O}_\Delta)$. So there is an exact triangle $I_E[-1] \to E^\vee \boxtimes E \to \mathcal{O}_\Delta \to I_E$. Now consider the functor $\mathsf{T}_E := \mathsf{FM}_{I_E} : \mathrm{D}(Y) \to \mathrm{D}(Y)$ which usually goes under the name *twist* or *reflection functor*. We will mainly stick here to the latter label because, in the cases of interest to us, the cohomology operators $\mathsf{T}^{\mathrm{H}}_E$ will really be reflections, and also to distinguish functors T_E and M_L. When the term 'reflection' is not appropriate, we will use 'spherical twist'—as opposed to 'line bundle twist'. Before studying the properties of T_E, we give another description: For any $A \in \mathrm{D}(Y)$,

$$\begin{aligned} \mathsf{T}_E(A) &= p_{2*}(\mathrm{cone}(E^\vee \boxtimes E \to \mathcal{O}_\Delta) \otimes p_1^* A) \\ &= p_{2*}(\mathrm{cone}((A \otimes E^\vee) \boxtimes E \to \Delta_* A)) \\ &= \mathrm{cone}(p_{2*}((A \otimes E^\vee) \boxtimes E) \to A) \\ &= \mathrm{cone}(E \otimes p_{2*} p_1^*(A \otimes E^\vee) \to A) \\ &= \mathrm{cone}(E \otimes \pi^* \pi_*(\mathcal{H}om(E, A)) \to A) \end{aligned}$$

using that exact functors commute with cones and in the end also $p_{2*}p_1^* = \pi^*\pi_*$ where $\pi : Y \to \mathrm{Spec}(\mathbb{C})$ is the structural morphism. Now quite generally for any $B \in \mathrm{D}(Y)$, we have $\pi^*\pi_* B \cong \pi_* B \otimes_{\mathbb{C}} \mathcal{O}_Y$. In the special case $B := \mathcal{H}om(E, A)$ we use the chain rule for derived functors to get $\mathbb{R}\pi_*(\mathbb{R}\mathcal{H}om(E, A)) = \mathbb{R}\,\mathrm{Hom}(E, A)$, where for emphasis we explicitly denoted derivation. Note that the functor $\mathcal{H}om(E, \cdot) : \mathrm{K}(\mathrm{Coh}(Y)) \to \mathrm{K}(\mathrm{Coh}(Y))$ is only defined on the level of (homotopy classes of) complexes. But $\mathbb{R}\,\mathrm{Hom}(E, A)$ is a complex of vector spaces, hence it decomposes in the direct sum of its homology: $\mathbb{R}\,\mathrm{Hom}(E, A) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^i(E, A)[-i] = \mathrm{Hom}^\bullet(E, A)$. With these considerations, we can present the functor T_E alternatively as

$$\mathsf{T}_E : \mathrm{D}(Y) \to \mathrm{D}(Y), \quad A \mapsto \mathrm{cone}(\mathrm{Hom}^\bullet_{\mathrm{D}(Y)}(E, A) \otimes_{\mathbb{C}} E \to A);$$

note that $\mathrm{Hom}^\bullet(E, A) \otimes_{\mathbb{C}} E$ is a tensor product of complexes. Nevertheless, the remark above shows $\mathrm{Hom}^\bullet(E, A) \otimes_{\mathbb{C}} E = \bigoplus_{k \in \mathbb{Z}} E^{\oplus h_k}[k]$ where $h_k := \dim \mathrm{Hom}^{-k}(E, A)$. The latter description of T_E is ambiguous, because there is no way to specify these cones functorially. The point is that we can circumvent this by choosing *one* cone at the level of Fourier-Mukai kernels.

Definition 1.20. An object $E \in \mathrm{D}(Y)$ is *spherical* if $E \otimes \omega_Y \cong E$ and $\mathrm{Hom}^\bullet_{\mathrm{D}(Y)}(E, E) = \mathbb{C} \oplus \mathbb{C}[-d]$ where $d := \dim(Y)$, i.e.

$$\dim \mathrm{Hom}^i(E, E) = \begin{cases} 1 & i = 0, d \\ 0 & \text{else} \end{cases}$$

Examples 1.21. We give a couple of simple examples of spherical objects. There is a relation between exceptional and spherical objects; see [79, §3c].

(1) Skyscraper sheaves $k(p)$ are spherical on any curve C. Note $k(p)^\vee = k(p)[-1]$, and thus $\mathrm{cone}(k(p) \boxtimes k(p)[-1] \to \mathcal{O}_\Delta) = \Delta_* \mathcal{O}_C(p)$ in view of the exact sequence $0 \to \mathcal{O}_\Delta \to \mathcal{O}_\Delta(p, p) = \Delta_* \mathcal{O}_C(p) \to k(p, p) \to 0$. Hence, $\mathsf{T}_{k(p)} = \mathsf{M}_{\mathcal{O}_C(p)}$.

(2) $\mathcal{O}_Y$ is spherical if and only if Y is a Calabi-Yau variety in the strict sense, so that $\omega_Y \cong \mathcal{O}_Y$ and $\mathrm{H}^{0,p}(Y) = \mathrm{H}^p(\mathcal{O}_Y) = 0$ for $0 < p < d$. In any case, $\mathsf{T}_{\mathcal{O}_Y} = \mathsf{FM}_{\mathcal{I}_\Delta}[1]$ where $\mathcal{I}_\Delta \subset \mathcal{O}_{Y \times Y}$ is the ideal sheaf of Δ_Y.
On the other hand, if Y is not a strict Calabi-Yau variety, then there are no spherical vector bundles. This follows from $E^\vee \otimes E \cong \mathcal{O}_Y \oplus \mathcal{E}nd_0(E)$ for a locally free sheaf E, where $\mathcal{E}nd_0(E)$ is the bundle of trace-less endomorphisms. The existence of the splitting uses characteristic 0. In particular, $\mathrm{Ext}^i(E, E) = \mathrm{H}^i(\mathcal{O}_Y) \oplus \mathrm{H}^i(\mathcal{E}nd_0(E))$. The same reasoning is valid for objects $E^\bullet \in \mathrm{D}(Y)$ with $\mathrm{rk}(E^\bullet) \neq 0$.

(3) Consider a smooth rational (-2)-curve C on a surface S given by an embedding $i : \mathbb{P}^1 \hookrightarrow S$. Then for all $k \in \mathbb{Z}$, the sheaves $\mathcal{O}_C(k) := i_* \mathcal{O}_{\mathbb{P}^1}(k) \in \mathrm{Coh}(S)$ are spherical. This follows from the adjunction formula: $-2 = 2g_C - 2 = \deg(K_C) = \deg((K_S \otimes \mathcal{O}_S(C))|_C) = \deg(K_S|_C) - 2$, so $\deg(K_S|_C) = 0$ and $\omega_S|_C \cong \mathcal{O}_C$. We have $1 = \dim \mathrm{Hom}_S(\mathcal{O}_C(k), \mathcal{O}_C(k)) = \dim \mathrm{Ext}^2_S(\mathcal{O}_C(k), \mathcal{O}_C(k))$ by Serre duality; finally $\dim \mathrm{Ext}^1_S(\mathcal{O}_C(k), \mathcal{O}_C(k)) = 0$ as C is exceptional and hence rigid.

(4) If X is a K3 surface with a smooth rational curve C, then $K_X = 0$ automatically implies $C^2 = -2$ and so $\mathcal{O}_C(k)$ is spherical as in the last example. In particular, later on the case $k = -1$ will appear because of the relation $v(\mathcal{O}_C(-1)) = [C]$: $v(i_* \mathcal{O}_{\mathbb{P}^1}(k)) = i_*(\mathrm{ch}(\mathcal{O}_{\mathbb{P}^1}(k))\ \mathrm{td}_{\mathbb{P}^1}) = i_*((1 + k[pt])(1 + [pt])) = [C] + (1 + k)[pt]$.

(5) On an elliptic curve E with origin p, we have spherical objects $\mathcal{O}_E$ and $k(p)$. This seemingly innocuous example displays two interesting features: Firstly, they form an A_2-configuration in $\mathrm{D}(E)$, i.e. $\dim \mathrm{Hom}^*_{\mathrm{D}(E)}(\mathcal{O}_E, k(p)) = 1$. A consequence of this is the braid relation $\mathsf{T}_{k(p)} \mathsf{T}_{\mathcal{O}_E} \mathsf{T}_{k(p)} = \mathsf{T}_{\mathcal{O}_E} \mathsf{T}_{k(p)} \mathsf{T}_{\mathcal{O}_E}$; see [79] or Section 2.4.
Secondly, the Fourier-Mukai equivalence $\mathsf{FM}_{\mathcal{P}} : \mathrm{D}(E) \xrightarrow{\sim} \mathrm{D}(E)$, given by the normalised Poincaré bundle $\mathcal{P} \in \mathrm{Pic}(E \times E)$, can be described in terms of the reflection functors: $\mathsf{FM}_P^{-1} = \mathsf{T}_{k(p)} \mathsf{T}_{\mathcal{O}_E} \mathsf{T}_{k(p)}$. Burban and Kreussler used this in [20] to extend the homological mirror symmetry picture to nodal Weierstrass cubics.

(6) If $E \in \mathrm{D}(Y)$ is spherical, then so is $F(E)$ for any autoequivalence $F \in \mathrm{Aut}(\mathrm{D}(Y))$. Note $\mathsf{T}_{F(E)}(A) \cong F(\mathsf{T}_E(F^{-1}(A)))$ and this even extends to a functor isomorphism $\mathsf{T}_{F(E)} \cong F \circ \mathsf{T}_E \circ F^{-1}$; see [42, §8.2]. In particular, $\mathsf{T}_{f^* E} \cong f^* \circ \mathsf{T}_E \circ f_*$ for $f \in \mathrm{Aut}(Y)$, $\mathsf{T}_{E \otimes L} \cong \mathsf{M}_L \circ \mathsf{T}_E \circ \mathsf{M}_L^{-1}$ for $L \in \mathrm{Pic}(Y)$, and $\mathsf{T}_{E[k]} \cong \mathsf{T}_E$ for $k \in \mathbb{Z}$.

(7) Serre duality shows that $E^\vee$ is spherical for $E \in \mathrm{D}(Y)$ spherical; we have $\mathsf{T}_{E^\vee} \cong \mathsf{T}^t_E$.

Lemma 1.22. *Let $E \in \mathrm{D}(Y)$ be a spherical object and introduce its right orthogonal complement $E^\perp := \{A \in \mathrm{D}(Y) : \mathrm{Hom}^*(E, A) = 0\}$. Let $\langle E \rangle$ be the smallest triangulated subcategory of $\mathrm{D}(Y)$ containing E. Then, for any object $A \in \mathrm{D}(Y)$, we have*

(i) $A \in E^\perp \implies \mathsf{T}_E(A) \cong A$,

(ii) $\mathsf{T}_E(E) \cong E[1-d]$,

(iii) $A \in \langle E \rangle \implies \mathsf{T}_E(A) \cong A[1-d]$.

If the dimension $d > 1$, then

(iv) $A \in E^\perp \iff \mathsf{T}_E(A) \cong A$.

Question 1.23. Is it also true that $\mathsf{T}_E(A) \cong A[1-d] \implies A \in \langle E \rangle$ if $d > 1$?

Proof. For (i) just note $\mathsf{T}_E(A) \cong \mathrm{cone}(\mathrm{Hom}^\bullet(E, A) \otimes E \to A) = A$ if $\mathrm{Hom}^*(E, A) = 0$. For (ii) we compute $\mathsf{T}_E(E)$: The triangle defining T_E yields

$$\begin{array}{ccccccc} \mathsf{T}_E(E)[-1] & \longrightarrow & \mathrm{Hom}^\bullet_{\mathrm{D}(Y)}(E,E) \otimes_{\mathbb{C}} E & \longrightarrow & E & \longrightarrow & \mathsf{T}_E(E) \\ & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \\ E[-d] & \longrightarrow & E \oplus E[-d] & \longrightarrow & E & \longrightarrow & E[1-d] \end{array}$$

This shows $\mathsf{T}_E(E) \cong E[1-d]$ because the morphism $E \oplus E[-d] \to E$ in the lower row induces an isomorphism on E. Since $\mathrm{Hom}(E, E[1-d]) = 0$, the triangle actually splits. Note however, that the morphisms are not canonical inclusion or projection of the direct sum.

In order to demonstrate (iii), we will use the fact that T_E is an equivalence. The proof of Theorem 1.27 below depends only on parts (i) and (ii) of this lemma, however. The non-trivial extension corresponding to $\mathrm{Hom}^d(E, E) = \mathbb{C}$ is given by a non-split triangle $E \to E[d] \to P \to E[1]$. Applying T_E and using (ii) yields $E[1-d] \to E[1] \to \mathsf{T}_E(P) \to E[2-d]$. As E is spherical and T_E an equivalence, this latter triangle is the former one shifted by $[1-d]$; hence $\mathsf{T}_E(P) \cong P[1-d]$. Similarly, any further object of $\langle E \rangle$ is obtained by consecutive extensions and hence subject to the same argument.

Finally, to evidence (iv), we introduce temporary notation $H^i := \mathrm{Hom}^i(E, A)$. Note that $\mathrm{Hom}^i(E, \mathrm{Hom}^*(E, A) \otimes E) = \mathrm{Hom}^i(E, A) \oplus \mathrm{Hom}^{i-d}(E, A) = H^i \oplus H^{i-d}$ as E is spherical. Now assume that $\mathsf{T}_E(A) \cong A$ but $A \notin E^\perp$. Applying $\mathrm{Hom}(E, \cdot)$ to the triangle $\mathrm{Hom}^*(E, A) \otimes E \to A \to A$ gives an exact sequence $H^{i-1} \to H^{i-1} \to H^i \oplus H^{i-d} \to H^i \to H^i$. Let $j \in \mathbb{Z}$ be maximal with $H^j \neq 0$. Then, the middle part of the long exact sequence at $i = j + d$ gives $H^{j+d-1} \to H^{j+d} \oplus H^j \to H^{j+d}$. Now by assumption and $d > 1$ all terms except H^j vanish, hence $H^j = 0$, too—contradiction. □

Example 1.24. In general, it is not easy to present concrete elements of $E^\perp$ for some spherical object $E \in \mathrm{D}(Y)$. Simple examples are provided by a skyscraper sheaf $k(p)$ on a curve (then $k(q) \in k(p)^\perp$ for $q \neq p$) and by the structure sheaf $\mathcal{O}_E$ of an elliptic curve E (then $L \in \mathcal{O}_E^\perp$ for all nontrivial $L \in \mathrm{Pic}^0(E)$). We can however also present a specimen in a more interesting setting: consider a K3 surface X with fixed polarisation H. Note first that a vector bundle E on X is spherical if and only if it is simple (i.e. $\mathrm{Hom}_X(E,E) = \mathbb{C}$) and rigid (i.e. $\mathrm{Ext}^1_X(E,E) = 0$). The Mukai vector $v(E) \in \mathrm{H}^*(X,\mathbb{Z})$ of such a spherical bundle fulfills $v(E)^2 = -\chi(E^\vee \otimes E) = -2$ as in Remark 1.18. On the other hand, given an algebraic class $v \in \mathrm{H}^*(X,\mathbb{Z})$ such that $v^2 = -2$ and $v_0 > 0$, there is a μ_H-semistable vector bundle E with $v(E) = v$ by Kuleshov's Theorem 2.5. Hein proved in [37] the following criterion: a sheaf E on X is μ_H-semistable if and only if for a general curve $C \in |mH|$ and $m \gg 0$ there is a vector bundle V on C with $\mathrm{H}^*(E \otimes V) = 0$. As E is locally free, this is equivalent to $\mathrm{Hom}^*(E^\vee, V) = 0$ in $\mathrm{D}(X)$. Using that E is μ_H-semistable if and only if $E^\vee$ is μ_H-semistable, we see that for every μ_H-semistable spherical bundle E (e.g. E stable with $v(E)^2 = -2$) on X there is a torsion sheaf $V \in E^\perp$.

Question 1.25. This raises the following general question: Does every spherical object $E \in \mathrm{D}(Y)$ have a nonempty orthogonal complement?

Remark 1.26. Despite this uncertainty, the spherical twist can never be just a shift. For suppose $\mathsf{T}_E \cong [1-d]$, then the kernel would have to be $I_E \cong \mathcal{O}_\Delta[1-d]$. In the triangle $E^\vee \boxtimes E \to \mathcal{O}_\Delta \to \Delta[1-d]$ the last morphism is zero for $d > 1$ because of $\mathrm{Hom}_{\mathrm{D}(Y\times Y)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[1-d]) = \mathrm{Ext}^{1-d}_Y(\mathcal{O}_Y, \mathcal{O}_Y) = 0$ (the case of curves can be explicitly dealt with using Examples 1.21(1, 2) and Remark 0). Thus, $E^\vee \boxtimes E \cong \mathcal{O}_\Delta \oplus \mathcal{O}_\Delta[-d]$, or $\mathsf{FM}_{E^\vee \boxtimes E} = \mathrm{id} \oplus \mathrm{id}[-d]$ which is absurd in view of $\mathsf{FM}_{E^\vee\boxtimes E}(A) = E \otimes_{\mathbb{C}} \mathrm{Hom}^\bullet(E,A)$.

Theorem 1.27 (Seidel, Thomas). *For a spherical object $E \in \mathrm{D}(Y)$ the reflection functor $\mathsf{T}_E : \mathrm{D}(Y) \to \mathrm{D}(Y)$ is an equivalence.*

Proof. For the proof of the theorem, see the original paper [79]. We give a short new proof invoking Theorem 1.15 by checking conditions

(a) $\mathcal{C} := \{E\} \cup E^\perp$ is a spanning class;

(b) $\mathrm{Hom}^i(C,C') \to \mathrm{Hom}^i(\mathsf{T}_E(C), \mathsf{T}_E(C'))$ is bijective for all $C, C' \in \mathcal{C}$ and $i \in \mathbb{Z}$;

(c) $\mathsf{T}_E(C \otimes \omega_Y) \cong \mathsf{T}_E(C) \otimes \omega_Y$ for all $C \in \mathcal{C}$.

Proof of (a): First take an $A \in \mathrm{D}(Y)$ with $\mathrm{Hom}^*(C,A) = 0$ for all $C \in \mathcal{C}$. In particular, $\mathrm{Hom}^*(E,A) = 0$ and thus $A \in E^\perp \subset \mathcal{C}$. But then $\mathrm{id}_A \in \mathrm{Hom}(A,A) = 0$ and so $A = 0$. Now assume $\mathrm{Hom}^*(A,C) = 0$ for all $C \in \mathcal{C}$. Especially for $C := E \otimes \omega_Y$, we obtain by Serre duality $0 = \mathrm{Hom}^i(A, E \otimes \omega_Y) = \mathrm{Hom}^{d-i}(E,A)$ for all i and hence $A \in E^\perp$. Thus, for $C := A$ and $i = 0$, we get $\mathrm{Hom}(A,A) = 0$, and again $A = 0$.

Proof of (b): Consider $C = C' = E$. For a non-trivial element $f \in \mathrm{Hom}^d(E,E)$ we compute the morphism $\mathsf{T}_E(f)$. The middle square in the diagram

$$\begin{array}{ccccccc}
E[-d] & \xrightarrow{f[-d],-id} & E \oplus E[-d] & \xrightarrow{\mathrm{id}\oplus f[-d]} & E & \xrightarrow{0} & E[1-d] = \mathsf{T}_E E \\
\Big\downarrow{\scriptstyle \mathsf{T}_E(f)[-1]} & & \Big\downarrow{\scriptstyle f\oplus f[-d]} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathsf{T}_E(f)} \\
E & \xrightarrow{f,-\mathrm{id}} & E[d] \oplus E & \xrightarrow{\mathrm{id}\oplus f} & E[d] & \xrightarrow{0} & E[1] = \mathsf{T}_E E[d]
\end{array}$$

is given; it implies $\mathsf{T}_E(f) = f[1-d]$ and hence $\mathrm{Hom}^i(E,E) \to \mathrm{Hom}^i(\mathsf{T}_E(E), \mathsf{T}_E(E))$ is bijective for $i = d$. The bijectivity for the other $i \in \mathbb{Z}$ is easy as are the cases where $C \in E^\perp$ or $C' \in E^\perp$.

Proof of (c): The claim is trivial for $C = E$ in view of $\mathsf{T}_E(E) = E[1-d]$ and $E \otimes \omega_Y \cong E$. For $C = A \in E^\perp$ just note that also $A \in (E \otimes \omega_Y^\vee)^\perp$ and hence $A \otimes \omega_Y \in E^\perp$. ☐

Finally, we turn to the traces of spherical reflections on K-theory and cohomology. Fix a spherical object $E \in \mathrm{D}(Y)$ and any object $A \in \mathrm{D}(Y)$. By definition of T_E we have the relation $[\mathsf{T}_E(A)] = [A] - \chi(E,A)[E]$ in $\mathrm{K}(Y)$ and thus

$$\begin{aligned}[\mathsf{T}_E^2(A)] &= [A] - \chi(E,A)[E] - \chi(E, \mathsf{T}_E A)[E] \\ &= [A] - \chi(E,A)[E] - (\chi(E,A) - \chi(E,A)\chi(E,E))[E] \\ &= [A] + \chi(E,A)(\chi(E,E) - 2)[E].\end{aligned}$$

Assume that $\dim(Y)$ is even; then $\chi(E,E) = 2$ and the formula implies $(\mathsf{T}_E^2)^{\mathrm{K}} = \mathrm{id}_{\mathrm{K}}$, i.e. $\mathsf{T}_E^{\mathrm{K}}$ is an involution. Let us denote the Mukai vectors by $v := v(E) \in \mathrm{H}^{2*}(Y)$ and $v' := v(E^\vee) = v(E)^\vee \cdot \mathrm{e}^{c_1(Y)/2}$, using notion and notation from Section 1.2. Then, the triangle defining I_E gives $v(I_E) = v(\mathcal{O}_\Delta) - v(E^\vee \boxtimes E) = [\Delta_Y] - v' \boxtimes v$. So we see that on cohomology

$$\mathsf{T}_E^{\mathrm{H}}(\alpha) = \alpha - p_{2*}(v' \boxtimes v.\alpha \boxtimes [Y]) = \alpha - (\textstyle\int v'.\alpha)\, v = \alpha - \langle v, \alpha \rangle\, v$$

where the last step invokes the Mukai-Căldăraru pairing. Suppose again that $\dim(Y)$ is even. Then we find $\langle v, v \rangle = \chi(E,E) = 2$ and the above formula exhibits $\mathsf{T}_E^{\mathrm{H}}$ as the reflection given by v. We sum this part up:

Lemma 1.28. *Let $E \in \mathrm{D}(Y)$ be a spherical object on a smooth projective variety Y of even dimension, and let $v(E) \in \mathrm{H}^{2*}(Y, \mathbb{Z})$ be its Mukai vector. Then,*

$$\mathsf{T}_E^{\mathrm{H}} = s_{v(E)} : \mathrm{H}^*(Y, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y, \mathbb{Q}),$$

where $s_{v(E)}$ is the reflection in the cohomology lattice on the $(+2)$-class $v(E)$.

Remark 1.29. We will primarily be interested in the case of a K3 surface X. Here the above lemma clearly applies. However, due to the use of the Mukai's original pairing as explained in Remark 1.18, we see that the reflection functors from spherical objects correspond then to (-2)-reflections.

Remark 1.30. Under the assumptions of the lemma, we have $(\mathsf{T}_E^2)^{\mathrm{H}} = \mathrm{id}$, i.e. T_E^2 is cohomologically trivial. On the other hand, it cannot be an even shift (which obviously is also cohomologically trivial) by Remark 1.26; or also because $\mathsf{T}_E(E) \cong E[1-d]$ with d even. See Section 2.4 for more details about cohomologically trivial autoequivalences.

Remark 1.31. Ishii and Uehara consider in [46] a smooth surface X containing a chain Z of (-2)-curves and the derived category $\mathrm{D}_Z(X)$ of coherent sheaves supported on Z. Thus, they work essentially with $Y = \mathrm{Spec}(\mathbb{C}[[x,y,z]]/(x^2+y^2+z^{n+1}))$, the local model of an A_n-singularity. Their result is that the group of Fourier-Mukai autoequivalences of $\mathrm{D}_Z(Y)$ is generated by spherical twists, multiplications with line bundles, surface automorphisms and shifts [46, Theorem 1.3].

1.4 On the condition $K_Y = 0$

From now on, we will consider only varieties Y (mostly surfaces) with $K_Y = 0$, or equivalently, $\omega_Y \cong \mathcal{O}_Y$. There are several reasons for doing so.

1. Bondal and Orlov proved in [12] that, on varieties with either ample or anti-ample canonical sheaf, the derived category determines the variety itself. On the other hand, varieties with $\omega_Y \cong \mathcal{O}_Y$ provide plenty of examples where non-isomorphic varieties have the same derived categories.

2. In a similar vein, the group of autoequivalences is also explicitly described in [12] for varieties with ample or anti-ample canonical sheaf. Roughly speaking, only the expected autoequivalences occur: automorphisms of the variety, tensor products by line bundles and shifts of complexes. On the contrary again, varieties with $\omega_Y \cong \mathcal{O}_Y$ have much more interesting groups of autoequivalences.

3. On the technical side, there are certain advantages of the condition $\omega_Y \cong \mathcal{O}_Y$. For any Fourier-Mukai kernel $P \in \mathrm{D}(Y \times Y)$, the left and right adjoints of FM_P are equal; their common kernel is $P^\vee[\dim(Y)]$. In particular, FM_P fully faithful implies that FM_P is an equivalence. Serre functors appear at several places in the general theory. Under the condition $\omega_Y \cong \mathcal{O}_Y$, they reduce to mere shifts. A reflection of this is the simple form Serre duality takes, making the bilinear form $\chi(A,B) := \sum_i (-1)^i \dim \mathrm{Hom}^i_{\mathrm{D}(Y)}(A,B)$ on $\mathrm{K}(Y) \times \mathrm{K}(Y)$ symmetric if $\dim(Y)$ is even and alternating if $\dim(Y)$ is odd.

Accordingly, a lot of research has been devoted to the case of trivial canonical bundle. By the decomposition theorem of Beauville [6] and Bogomolov [8], a variety Y with $\omega_Y \cong \mathcal{O}_Y$ is up to a finite unramified covering $f\colon Y' \to Y$ a product $Y' = C_Y \times A_Y \times H_Y$ where the factors are: C_Y a product of Calabi-Yau manifolds, A_Y an abelian variety and H_Y a product of irreducible symplectic manifolds. Note that elliptic curves are at the junction of abelian varieties and Calabi-Yau manifolds (provided we define a variety Y to be Calabi-Yau if $\omega_Y \cong \mathcal{O}_Y$ and $\mathrm{H}^i(Y, \mathcal{O}_Y) = 0$ for all $0 < i < \dim(Y)$). Similarly, K3 surfaces occupy the intersection of irreducible symplectic manifolds and Calabi-Yau's. In this study, Chapter 2 is devoted to K3 surfaces and in Chapter 4 we consider abelian varieties. Hyperkähler varieties appear at some odds and ends; see Sections 3.4 and 4.4.

Three mutually connected conjectures stand out as yardsticks for the development:

(C)	Manifolds with $K = 0$ have only finitely many birational $K = 0$ manifolds.
(DK)	Birational manifolds with $K = 0$ have equivalent derived categories.
(FMP)	Manifolds with $K = 0$ have only finitely many Fourier-Mukai partners.

Assertion (C) means that the number of isomorphism classes of minimal models of Y is finite. It is expected to hold without the $K = 0$ assumption, perhaps with the exception of the (rational ruled) Hirzebruch surfaces, which are mutually birational to each other; confer [55, §12.3]. A proof for threefolds of general type is given in [51]. An example of a Calabi-Yau threefold has been worked out, namely the fibre product of two rational elliptic surfaces over $\mathbb{P}^1$; see [32] and [59, §5]. The statement above is connected to Morrison's original 'Cone Conjecture' [58] as follows. The Kähler cone $\mathcal{K}(Y)$ of a manifold Y with $K = 0$ is an open cone inside $\mathrm{H}^{1,1}(Y, \mathbb{R})$ if $\mathrm{H}^{2,0}(Y) = 0$, which, however, need not be a finite polyhedral cone. Morrison's claim is that there exists a finite rational polyhedral cone $\mathcal{P}(Y)$ with $\mathrm{Aut}(Y) \cdot \mathcal{P}(Y) = \overline{\mathcal{K}(Y)}$, i.e. $\mathcal{P}(Y)$ is a fundamental domain for the action of $\mathrm{Aut}(Y)$ on $\mathcal{K}(Y)$. Now there is a generalisation of this, the 'Birational Cone Conjecture' [59, §5], where one replaces $\mathrm{Aut}(Y)$ and $\mathcal{K}(Y)$ by $\mathrm{Bir}(Y)$ and $\mathcal{M}(Y)$, respectively. Here, $\mathrm{Bir}(Y)$ is the group of all birational automorphisms and $\mathcal{M}(Y)$ is Kawamata's movable cone consisting of divisor classes with at least 2-codimensional base loci. See [28] for a very explicit description of all this, where Y is a Horrocks-Mumford quintic threefold.

The next statement is a special case of Kawamata's Conjecture 1.2 in [49] that two birationally equivalent smooth projective varieties are D-equivalent if and only if they are K-equivalent. The latter means that the canonical divisors are linearly equivalent on some common birational pull-back. Birational equivalence of $K = 0$ manifolds implies derived equivalence in dimension up to three, as has been proven by Bridgeland [16] using moduli spaces of perverse sheaves. Concerning the general conjecture, an example of Uehara [84] shows though, that a further assumption like $K = 0$ or general type is necessary.

The last conjecture about Fourier-Mukai partners is confirmed for abelian varieties [72] and for K3 surfaces [14, Proposition 5.3]. This conjecture is expected to hold true without the $K = 0$ assumption; see the examples after Definition 1.16 for references.

We propose here a tool to single out Calabi-Yau manifolds among the three building blocks for varieties with trivial canonical bundles. As has been mentioned already in Example1.21(2), Calabi-Yau manifolds always carry lots of spherical objects: every line bundle is an example and depending on the geometry there might be many more. The next proposition—which is also the content of Bridgeland's [18, Lemma 14.1]—shows that the situation differs drastically for abelian varieties. From this point of view, elliptic curves behave rather like Calabi-Yau manifolds and unlike abelian varieties.

Proposition 1.32. *Let A be an abelian variety of dimension $d > 1$. Then there are no spherical objects in* $\mathrm{D}(A)$.

Proof. We note first that there can be no spherical sheaves of positive rank on A in view of $\mathrm{H}^1(\mathcal{O}_A) \neq 0$, confer Example 1.21(2). Now for a torsion sheaf $T \in \mathrm{Coh}(A)$, we consider the global deformation m^*T on $A \times A$ where $m : A \times A \to A$ is the addition; the members are the translated sheaves $t_{a*}T$. This deformation is not trivial as it moves the support; thus it gives a non-trivial infinitesimal deformation and so $\mathrm{Ext}^1(T,T) \neq 0$.

Now take a complex $E \in \mathrm{D}(A)$ and let l be maximal with $\mathrm{h}^l(E) \neq 0$. The spectral sequence $E_1^{pq} = \bigoplus_{i-j=p} \mathrm{Ext}^q_A(\mathrm{h}^i(E), \mathrm{h}^j(E)) \Rightarrow \mathrm{Hom}^{p+q}_{\mathrm{D}(A)}(E,E)$ of [30, Exercise IV.2.2] has a surviving element in $\mathrm{Hom}^1(E,E)$ for $q = 1$, $p = 0$, $i = j = l$. Thus, no object of $\mathrm{D}(A)$ can be spherical. □

This immediately raises the question how a projective irreducible symplectic manifold H behaves in this regard. We are not able to present a definite answer. However, it seems that there are only slim chances for the existence of spherical objects in $\mathrm{D}(H)$ when $\dim(H) =: 2n > 4$. There can be no spherical objects $E \in \mathrm{D}(H)$ with $\mathrm{rk}(E) \neq 0$ in view of Example 1.21(2) and $\mathrm{H}^2(\mathcal{O}_H) \cong \mathrm{H}^{2,0}(H) = \mathbb{C}$. Among torsion sheaves on H, one can check that neither $\mathcal{O}_L$ with $L \subset H$ Lagrangian nor $\mathcal{O}_Y$ with $Y \subset H$ a symplectic subvariety might be spherical. This leads to the

Conjecture 1.33. *If H is a projective irreducible symplectic variety H of dimension at least 4, then* $\mathrm{D}(H)$ *contains no spherical objects.*

Recent work of Huybrechts and Thomas [45] suggests that instead of spherical objects one should look for objects $E \in \mathrm{D}(H)$ with $\mathrm{Hom}^*(E,E) \cong \mathrm{H}^*(\mathbb{P}^{2n}, \mathbb{Z})$ (as graded rings). They call such E $\mathbb{P}$-object and show that it gives rise to an equivalence $P_E \in \mathrm{Aut}(\mathrm{D}(H))$ in a manner reminiscent of the construction of the spherical twists.

For the greater picture, consider Orlov's Theorem 4.2 on Fourier-Mukai kernels on abelian varieties. It states that all kernels of autoequivalences of abelian varieties are shifted sheaves. The same statement is true for varieties Z which have either ample or anti-ample canonical sheaf. This follows from the relation $\mathrm{Aut}(\mathrm{D}(Z)) = \mathrm{Aut}(Z) \ltimes \mathrm{Pic}(Z) \times \mathbb{Z}$ of Bondal and Orlov [12, Theorem 3.1]. Similarly, $\mathrm{D}(Z)$ can carry lots of spherical objects. Yet the point we want to make concerns varieties with trivial canonical class. It seems that abelian varieties are characterised as the $K = 0$ varieties where all equivalence kernels are sheaves in the standard t-structure, i.e. have exactly one non-vanishing homology. On the other hand, accepting the conjecture, Calabi-Yau varieties would be characterised as $K = 0$ varieties whose derived categories contain spherical objects. A derived category description for irreducible symplectic varieties would be nice to have; perhaps using the $\mathbb{P}$-objects of Huybrechts and Thomas.

2 K3 surfaces

2.1 Generalities

By definition, a K3 surface is a smooth proper surface X over $\mathbb{C}$ such that $\omega_X \cong \mathcal{O}_X$ and $\mathrm{H}^1(\mathcal{O}_X) = 0$. In other words, it is a strict Calabi-Yau manifold of dimension 2. We have $c_1(X) = 0$ by definition and $c_2(X) = 24[pt] \in \mathrm{H}^4(X,\mathbb{Z})$ from Noether's formula. This immediately implies $\mathrm{td}_X = [X] + 2[pt]$ and $\sqrt{\mathrm{td}_X} = (1,0,1) \in \mathrm{H}^0(X) \oplus \mathrm{H}^2(X) \oplus \mathrm{H}^4(X)$. All facts used here about K3 surfaces can be found in [5] or [3]. They form a 20-dimensional family, among which the projective K3 surfaces constitute a 19-dimensional subfamily that is dense in the analytical topology. As we are interested in derived categories, we will always work under the projectivity assumption.

Examples 2.1. The following types of K3 surfaces will be referred to later on.

(1) The classical example is given by a smooth quartic $X \subset \mathbb{P}^3$. It follows from the Lefschetz theorem [57] that X is simply-connected. Eventually, the adjunction formula shows $\omega_X = \omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(X)|_X = \omega_{\mathbb{P}^3} \otimes \mathcal{N}_{X/\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_X(4) = \mathcal{O}_X$.

(2) We will appeal to Kummer construction a great deal. Let A be a abelian surface. Then, the involution $A \to A$, $a \mapsto -a$ acts on A with 16 fixed points, and the quotient $Y := A/\pm 1$ has 16 nodal singularities. Blowing up the fixed locus, we obtain a K3 surface $K_1(A)$ which is called the Kummer surface corresponding to A. Kummer surfaces are dense in the moduli space of K3 surfaces. The notation $K_1(A)$ refers to Beauville's generalisation of this construction; see Section 4.4.

(3) Another important class are elliptic K3 surfaces. These are given by a morphism $X \to \mathbb{P}^1$ such that almost all fibres are smooth curves of genus 1. In order for the topological Euler number to be 24, there must be 24 singular fibres, if counted with multiplicities. A projective elliptic fibration always has a multisection, i.e. a morphism $\sigma : \mathbb{P}^1 \to X$ such that $\sigma.f = n$ for some $n \geq 1$ where σ and f denote the cycle classes of section and fibre, respectively. Elliptic K3 surfaces are also dense in the moduli space of K3 surfaces.

(4) A further way to construct K3 surfaces is as double covering $\pi : X \to \mathbb{P}^2$, branched over a curve $C \subset \mathbb{P}^2$ of degree 6. It is given by the line bundle $L := \mathcal{O}_{\mathbb{P}^2}(3)$ in view of $L^{\otimes 2} \cong \mathcal{O}_{\mathbb{P}^2}(C)$; see [3, §17]. Obviously, X is simply connected. The Hurwitz formula yields $\omega_X \cong \pi^*(\omega_{\mathbb{P}^2} \otimes L) = \mathcal{O}_X$.

Singular cohomology $\mathrm{H}^*(X,\mathbb{Z})$ is free of rank 24 with Betti numbers $b_0 = b_4 = 1$, $b_1 = b_3 = 0$ and $b_2 = 22$. The intersection pairing (cup product) on $\mathrm{H}^2(X,\mathbb{R})$ has signature $(3,19)$ and turns $\mathrm{H}^2(X,\mathbb{Z})$ into an even unimodular lattice. Its Hodge decomposition over $\mathbb{C}$ is $\mathrm{H}^2(X,\mathbb{C}) = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2}$, and from the K3 condition it follows that $\mathrm{H}^{2,0} = \mathrm{H}^0(\omega_X) = \mathbb{C}$. Further, we have

$$\mathrm{Pic}(X) = \mathrm{CH}^1(X) = \mathrm{NS}(X) = \mathrm{H}^2(X,\mathbb{Z}) \cap \mathrm{H}^{1,1}(X,\mathbb{R})$$

using firstly $\mathrm{Pic}(X) = \mathrm{H}^1(\mathcal{O}_X^*) = \mathrm{Im}(c_1) = \mathrm{NS}(X)$ as the map $c_1 : \mathrm{H}^1(\mathcal{O}_X^*) \to \mathrm{H}^2(X,\mathbb{Z})$ is injective due to $\mathrm{H}^1(\mathcal{O}_X) = 0$ and secondly $\mathrm{NS}(X) = \mathrm{H}^2(X,\mathbb{Z}) \cap \mathrm{H}^{1,1}(X,\mathbb{R})$ because $\mathrm{H}^*(X,\mathbb{Z})$ has no torsion. We will use labels depending on emphasis.

The plane $P := (\mathrm{H}^{2,0} \oplus \mathrm{H}^{0,2}) \cap \mathrm{H}^2(X,\mathbb{R})$ is positive. It is called the period of X and determines X up to isomorphism by the Torelli theorem. Now its orthogonal complement $P^\perp = \mathrm{H}^{1,1} \subset \mathrm{H}^2(X,\mathbb{R})$ has signature $(1,19)$ and is a hyperbolic space. Thus, the cone $\{v \in \mathrm{H}^{1,1}(X,\mathbb{R}) : v^2 > 0\}$ has two connected components which we denote by $\mathcal{C}_X$ and $-\mathcal{C}_X$. Let $\mathcal{C}_X$ be the component containing the ample divisors or, equivalently, the Kähler classes. It is called the positive cone of X.

The set $D_X := \{v \in \mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X,\mathbb{Z}) : v^2 = -2\}$ of (-2)-classes is used to generate the Weyl group in the sense $W_X := \langle s_v \in \mathrm{Aut}(\mathrm{H}^2(X,\mathbb{Z})) : v \in D_X \rangle$ where $s_v(x) := x + (v,x)v$ is the reflection on $v^\perp$. With this notation, we can quote a part of the strong Torelli theorem for K3 surfaces which describes the group $\mathrm{Aut}(\mathrm{H}^2(X,\mathbb{Z}))$ of integral Hodge isometries as a semidirect product:

Theorem 2.2 (Strong Torelli theorem for K3 surfaces).

$$\mathrm{Aut}(\mathrm{H}^2(X,\mathbb{Z})) = W(X) \ltimes \mathrm{Aut}(X) \times \langle \iota \rangle,$$

where $\iota := -\,\mathrm{id}_{\mathrm{H}^2(X)}$ *swaps the two components* $\mathcal{C}$ *and* $-\mathcal{C}$*.*

Proof. Confer [3, §VIII.11] or [5, §VII]. □

Note that a Hodge isometry $\varphi : \mathrm{H}^2(X) \xrightarrow{\sim} \mathrm{H}^2(X)$ preserves the positive cone if and only if for all $v \in \mathrm{H}^{1,1}(X)$ with $v^2 > 0$ we have $v.\varphi(v) > 0$. Equivalently by [3, Proposition VIII.11.3], φ preserves the positive cone if and only if $v.\varphi(v) > 0$ for some $v \in \mathrm{H}^{1,1}(X)$ with $v^2 > 0$.

Following Mukai [60, §2], it is possible to generalise a great part of the above discussion to the whole cohomology $\mathrm{H}^*(X)$. Firstly, as in Remark 1.18 we extend the intersection pairing by setting

$$(\cdot,\cdot) : \mathrm{H}^*(X,\mathbb{Z}) \times \mathrm{H}^*(X,\mathbb{Z}) \to \mathbb{Z}, \quad (v,w) := v_2.w_2 - v_0 w_4 - v_4 w_0.$$

Note that this is up to sign Căldăraru's pairing from Section 1.2. Furthermore, define a Hodge structure $\tilde{\mathrm{H}}$ on $\mathrm{H}^2(X,\mathbb{C})$ of weight 2 by $\tilde{\mathrm{H}}^{2,0} := \mathrm{H}^{2,0}(X)$, $\tilde{\mathrm{H}}^{0,2} := \mathrm{H}^{0,2}(X)$ and $\tilde{\mathrm{H}}^{1,1} := \mathrm{H}^0(X,\mathbb{C}) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^4(X,\mathbb{C})$. We will denote by $\tilde{\mathrm{H}}(X)$ the total cohomology group $\mathrm{H}^*(X,\mathbb{C})$ with its pairing, Hodge and lattice structure, and call it the Mukai lattice of X. Accordingly, $\mathrm{Aut}(\tilde{\mathrm{H}}(X))$ denotes the group of all integral Hodge isometries.

Note that the group homomorphism $\mathrm{Aut}(\mathrm{D}(X)) \to \mathrm{Aut}(\mathrm{H}^*(X,\mathbb{Q}))$ from Section 1.2 factors over $\mathrm{Aut}(\tilde{\mathrm{H}}(X))$: It is obvious that a Fourier-Mukai transform respects the new Hodge structure on cohomology since the Chern characters of equivalence kernels are algebraic. That it preserves the Mukai pairing is shown in [60] and [71, Proposition

3.5]. In order to see that the linear map $\mathsf{FM}_P^{\mathrm{H}}$ is defined on the integral lattice, we need $\mathrm{ch}(P) \in \mathrm{H}^*(X \times X, \mathbb{Z})$. This can also be found in [60]. Altogether we obtain

$$\mathrm{Aut}(\mathrm{D}(X)) \to \mathrm{Aut}(\tilde{\mathrm{H}}(X)), \quad \mathsf{FM}_P \mapsto \mathsf{FM}_P^{\mathrm{H}}.$$

Of fundamental importance is the Torelli theorem: by means of their cohomology lattices, it classifies K3 surfaces up to isomorphism. This classical statement is given below together with a derived analogue by Orlov which classifies K3 surfaces up to D-equivalence via their full Mukai lattices.

Theorem 2.3 (Global Torelli theorem for K3 surfaces).

(i) Two (not necessarily projective) K3 surfaces X and X' are isomorphic if and only if there is a Hodge isometry $\mathrm{H}^2(X, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(X', \mathbb{Z})$.

(ii) Two projective K3 surfaces X and X' are D*-equivalent, i.e.* $\mathrm{D}(X) \cong \mathrm{D}(X')$ *if and only if there is an integral Hodge isometry $\tilde{\mathrm{H}}(X) \xrightarrow{\sim} \tilde{\mathrm{H}}(X')$.*

Proof. The classical Torelli theorem for K3 surfaces has a long history; the statement given is the so-called weak version. See the concluding remark in chapter VIII of [3] for details and references.

The derived K3 Torelli theorem originates from Orlov [71]. Instead of the full Mukai lattices, one could actually use the transcendental lattices: It is obvious that a Hodge isometry $\varphi : \tilde{\mathrm{H}}(X) \xrightarrow{\sim} \tilde{\mathrm{H}}(X')$ induces an isometry of the transcendental sublattices as $T(X)$ is the smallest primitive sublattice of $\mathrm{H}^2(X, \mathbb{R})$ containing $\mathrm{H}^{2,0}(X)$ after complexification. In the other direction, one uses a theorem of Nikulin [67, Theorem 1.14.1]: it implies (see below for details) that an isometry of the primitive even sublattice $T(X) \subset \tilde{\mathrm{H}}(X)$ can be extended to an isometry of $\tilde{\mathrm{H}}(X)$ since $T(X)^{\perp}$ contains the hyperbolic plane $\mathrm{H}^0(X, \mathbb{Z}) \oplus \mathrm{H}^4(X, \mathbb{Z})$.

Nikulin's setting is this: Let L be an even unimodular lattice of signature (p_L, n_L) and let S be an even lattice of signature (p_S, n_S). Using the inclusion $S \hookrightarrow S^* := \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Z})$, $x \mapsto (x, \cdot)$, the quotient $A_S := S^*/S$ is a finite abelian group of order $|\det(S)|$ which inherits a bilinear form $b_S : A_S \times A_S \to \mathbb{Q}/\mathbb{Z}$ from the natural extension $(\cdot, \cdot) : S^* \times S^* \to \mathbb{Q}$. As S is even, we also get a quadratic form $q_S : A_S \to \mathbb{Q}/2\mathbb{Z}$, called the discriminant form of L. Consider the following two conditions: 1) The lattice K with signature $(p_L - p_S, n_L - n_S)$ and discriminant form $(A_K, q_K) \cong (A_S, -q_S)$ is unique. 3) The homomorphism $\mathrm{O}(K) \to \mathrm{O}(A_K, q_K)$ is surjective. Then, the analog of Witt's theorem in the strong form holds for L and S (this is [67, Theorem 1.14.1]). This means that all primitive embeddings $S \hookrightarrow L$ (i.e. L/S is free) are isomorphic, so that two such embeddings differ by an isometry of L.

Furthermore, Theorem [67, Theorem 1.14.2] gives a criterion when an even, indefinite lattice K satisfies 1) and 3). We refrain from restating it here, and instead just introduce $l(A_K)$ which by definition is the minimal number of generators of A_K. A consequence of [67, Theorem 1.14.2] is that an even, indefinite lattice K with $\mathrm{rk}(K) = p_K + n_K \geq l(A_K) + 2$ fulfills conditions 1) and 3).

Finally, if $S \hookrightarrow L$ is a primitive, even sublattice of an even, unimodular lattice, then $K := S^\perp$ has signature $(p_S - p_L, n_S - n_L)$ and discriminant form $(A_K, q_K) = (A_S, -q_S)$. If, in addition, we have a decomposition $K = K' \oplus U$ (where U denotes a hyperbolic plane), then $\mathrm{rk}(K) = \mathrm{rk}(K') + 2$ and $A_K = A_{K'} \oplus A_U = A_{K'}$ since U is unimodular. The inequality $\mathrm{rk}(K) \geq l(A_K) + 2$ now follows from the trivial $\mathrm{rk}(K') \geq l(A_{K'})$. Thus, we can apply both theorems alluded to above and find that the primitive embedding $S \hookrightarrow L$ is unique up to isometries of L. □

Remark 2.4. Note that if X is a K3 surface and Y some smooth projective variety such that $\mathrm{D}(X) \cong \mathrm{D}(Y)$, then Y must necessarily be a K3 surface, too: Firstly, uniqueness of Serre functors and their compatibility with equivalences show that Y is a surface with trivial canonical bundle, hence either a K3 surface or an abelian surface. Now the latter case is excluded because $\mathrm{D}(X)$ contains the spherical object $\mathcal{O}_X$, whereas by Proposition 1.32 $\mathrm{D}(Y)$ does not. Thus Y must also be a K3 surface, indeed.

Finally, we will repeatedly make use of the assertion that algebraic (-2)-classes of the Mukai lattice $\tilde{\mathrm{H}}(X)$ can up to sign be realised by sheaves.

Theorem 2.5 (Kuleshov). *Let X be a K3 surface and $H \in \mathrm{Pic}(X)$ a polarisation. For any algebraic class $v = (v_0, v_2, v_4) \in \tilde{\mathrm{H}}(X)$ with $v^2 = -2$ and $v_0 > 0$, there exists a μ_H-semistable vector bundle E on X such that $v(E) = v$.*

Proof. For the proof see [53]. There is also an argument by Yoshioka using moduli spaces in [86, §8]. To apply it, one has to deform the polarisation in such a way that μ_H-semistable sheaves are actually μ_H-stable. If $v_0 < 0$, then $-v$ is a (-2)-class to which the theorem applies.

Note that a (-2)-class $v \in \tilde{\mathrm{H}}(X)$ with $v_0 = 0$ is of the form $v = (0, D, v_4)$, where $D = \sum a_i C_i$ is a divisor containing at least one (-2)-curve. For a (-2)-curve $C \subset X$ and any $v_4 \in \mathbb{Z}$, we deduce $v(\mathcal{O}_C(v_4 - 1)) = (0, [C], v_4)$ from Example 1.21(4). The proof of Theorem 2.2 will show that for a general (-2)-class $v = (0, v_2, v_4)$ there is a spherical object $E \in \mathrm{D}(X)$ with $v(E) = v$. □

2.2 Autoequivalences for K3 surfaces

In this section, we demonstrate the following theorem [74]. The proof is a consequence of results from Mukai on moduli spaces [60] and from Orlov on derived categories on K3 surfaces [71]. The same method also appears in [38].

Theorem 2.6. *The image of the group homomorphism $\mathrm{Aut}(\mathrm{D}(X)) \to \mathrm{Aut}(\tilde{\mathrm{H}}(X))$ is a subgroup of index at most 2. More precisely, given a Hodge isometry $\varphi \in \mathrm{Aut}(\tilde{\mathrm{H}}(X))$, either φ or $\varphi \circ \iota$ has a preimage in $\mathrm{Aut}(\mathrm{D}(X))$ where $\iota := \mathrm{id}_{\mathrm{H}^0} \oplus (-\mathrm{id}_{\mathrm{H}^2}) \oplus \mathrm{id}_{\mathrm{H}^4}$.*

Proof. Let $\varphi : \tilde{\mathrm{H}}(X) \to \tilde{\mathrm{H}}(X)$ be a Hodge isometry. We prove the theorem in three steps, improving φ in the process with Hodge isometries that lift to the derived level. The first goal is to modify φ into φ_1 such that, putting $\varphi_1(0,0,1) =: (r, \alpha, s)$, we have

(a) $r > 1$,

(b) α is the class of an ample divisor,

(c) the greatest common divisor of the set $\{r, s, \ell.\alpha\}$ is 1 where ℓ ranges through ample classes in $\mathrm{NS}(X)$.

For this, we need three kinds of Hodge isometries which lift to $\mathrm{Aut}(\mathrm{D}(X))$:

Shift: $-\mathrm{id}_{\mathrm{H}^*(X)}$ which comes from the shift $[1] \in \mathrm{Aut}(D(X))$.

Swap: $v = (v_0, v_2, v_4) \mapsto (-v_4, v_2, -v_0)$. Note that this is the reflection given by the (-2)-class $(1,0,1) = v(\mathcal{O}_X)$. By Lemma 1.28, this Hodge isometry lifts to the reflection functor $\mathsf{T}_{\mathcal{O}_X}$ as $\mathcal{O}_X$ is spherical.

Twist: For a line bundle $L \in \mathrm{Pic}(X)$ the Hodge isometry $v \mapsto \mathrm{ch}(L) \cdot v$ lifts to the autoequivalence $\mathsf{M}_L : D(X) \xrightarrow{\sim} D(X), E \mapsto L \otimes E$ by Example 1.19(2).

We start by examining the effect of $\mathsf{M}_L^{\mathrm{H}}$ on (r, α, s):

$$\mathsf{M}_L^{\mathrm{H}}(r, \alpha, s) = \mathrm{ch}(L) \cdot (r, \alpha, s) = (1, \ell, \ell^2/2) \cdot (r, \alpha, s) = (r, r\ell + \alpha, s + \ell.\alpha + r\ell^2/2)$$

where we denote $\ell := c_1(L) \in \mathrm{H}^2(X, \mathbb{Z})$ for the line bundle $L \in \mathrm{Pic}(X)$. As $\alpha \in \mathrm{NS}(X)$, we can represent this class uniquely by a line bundle $A \in \mathrm{Pic}(X)$, i.e. $c_1(A) = \alpha$. Using $\mathsf{M}_A^{\mathrm{H}}$ followed possibly by swap or shift, we can assume that $r > 1$. Once we have φ with $r > 1$, choose an ample line bundle L. Inspecting the above formula for $\mathsf{M}_L^{\mathrm{H}}(r, \alpha, s)$, we see that the resulting divisor class is ample after some twisting with L. Obviously, we keep the condition $r > 1$. Thus, at this stage we work with $\pm \mathsf{T}_{\mathcal{O}_X}^{\epsilon} \mathsf{M}_L^{\mathrm{H}} \varphi$ where ϵ is 0 or 1. Now $-1 = \langle (0,0,1), (1,0,0) \rangle = \langle \varphi(0,0,1), \varphi(1,0,0) \rangle = \langle (r, \alpha, s), \varphi(1,0,0) \rangle$, and examining this numerical property, we see that (c) is fulfilled after enough twisting with L from the right side. It is actually possible to achieve $(r,s) = 1$ as well using the same methods; see [38] for this. Write φ_1 for the composition of φ with the various twists, and possibly swap and shift isometries.

Consider the moduli space $M(v)$ of stable vector bundles on X with Mukai vector $v := \varphi_1(0,0,1)$. We denote by $[E]$ the closed point of $M(v)$ corresponding to a stable bundle E with $v(E) = v$. The three conditions discussed above ensure that $M(v)$ is a non-empty, proper and fine moduli space, confer [60, Proposition 4.1] or [43, Remark 4.6.8]. It is a general fact that the tangent spaces are given by infinitesimal deformations: $\mathsf{T}_{[E]}M(v) = \mathrm{Ext}^1(E, E)$. For all $[E] \in M(v)$ we have, by stability, $\dim \mathrm{Hom}(E, E) = 1$ and thus $\dim \mathrm{Ext}^2(E, E) = 1$ by Serre duality. Hence $\dim \mathrm{Ext}^1(E, E) = 2 - \chi(E, E) = 2 - v^2 = 2$, independently of E. Therefore, $M(v)$ is a smooth surface. Results of Mukai now show that $M(v)$ is indeed a K3 surface; see [60, Theorem 1.4]. As the moduli space is fine, we have a universal bundle $\mathcal{E}$ on $M(v) \times X$ at our disposal. The Fourier-Mukai transform $\mathsf{FM}_{\mathcal{E}} : D(M(v)) \to D(X)$ is an equivalence. This follows immediately from Theorem 1.15 and the remarks concerning the spanning class of points, using that

$\mathsf{FM}_{\mathcal{E}}(k([E])) = E$. The general theory of Section 1.2 produces an isomorphism $\mathsf{FM}^{\mathrm{H}}_{\mathcal{E}} : \mathrm{H}^*(M(v), \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^*(X, \mathbb{Z})$ and, as explained in Section 2.1, this isomorphism is actually a Hodge isometry. Furthermore, $v(\mathsf{FM}_{\mathcal{E}}(k([E]))) = v(E) = v$ implies $\mathsf{FM}^{\mathrm{H}}_{\mathcal{E}}(0,0,1) = v$. Thus, $\varphi_2 := \varphi_1^{-1} \circ \mathsf{FM}^{\mathrm{H}}_{\mathcal{E}} : \mathrm{H}^*(M(v), \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^*(X, \mathbb{Z})$ is a Hodge isometry that sends $(0,0,1)$ to $(0,0,1)$. Using that $\mathrm{H}^2(X, \mathbb{Z}) = (0,0,1)^\perp / \mathbb{Z}(0,0,1)$, we infer that there is some isomorphism $\mathrm{H}^2(M(v), \mathbb{Z}) \cong \mathrm{H}^2(X, \mathbb{Z})$ between second cohomologies.

However, it is not necessarily true that φ_2 already restricts to an isomorphism of H^2's. In order to rectify this, we consider also $\varphi_2(1,0,0) =: (v_0, v_2, v_4)$. The isometry φ_2 yields $-1 = ((1,0,0),(0,0,1)) = (\varphi_2(1,0,0), \varphi_2(0,0,1)) = ((v_0,v_2,v_4),(0,0,1)) = -v_0$ and thus $0 = (1,0,0)^2 = (v_0,v_2,v_4)^2 = v_2^2 - 2v_0v_4 = v_2^2 - 2v_4$. Hence, we arrive at $\varphi_2(1,0,0) = (1, v_2, v_2^2/2)$. Since the class $v_2 \in \mathrm{H}^2(X)$ is algebraic as the image of the algebraic class $(1,0,0)$ under some Hodge isometry, there is a unique line bundle L_2 such that $c_1(L_2) = v_2$. In view of $(1, v_2, v_2^2/2) = \mathrm{ch}(L_2)$, we find that $\varphi_3 := \mathsf{M}^{\mathrm{H}}_{L_2^\vee} \circ \varphi_2$ maps $\varphi_3 : (1,0,0) \mapsto (1,0,0)$ and $\varphi_3 : (0,0,1) \mapsto \mathrm{ch}(L_2^\vee) \cdot (0,0,1) = (0,0,1)$.

Thus, we have produced a genuine Hodge isometry φ_3 of H^2-lattices. By the weak Torelli Theorem 2.3(i) the K3 surfaces X and $M(v)$ must be isomorphic. Identifying them, we consider φ_3 as an automorphism of $\mathrm{H}^2(X, \mathbb{Z})$. According to the strong Torelli Theorem 2.2, it can be written uniquely as a composition of reflections on (-2)-curves and a surface automorphism, possibly followed by the orientation reversing ι. The Hodge isometry $f^* \in \mathrm{Aut}(\mathrm{H}^2(X))$ coming from a surface automorphism $f : X \xrightarrow{\sim} X$ lifts to the autoequivalence $f^* \in \mathrm{Aut}(\mathrm{D}(X))$. The (-2)-reflection $s_C \in \mathrm{Aut}(\mathrm{H}^2(X, \mathbb{Z}))$ along a smooth rational curve $C \subset X$ lifts to the reflection functor $\mathsf{T}_{\mathcal{O}_C(-1)}$ obtained from the spherical sheaf $\mathcal{O}_C(-1)$. Altogether, at each step we have modified our Hodge isometry using autoequivalences of $\mathrm{D}(X)$—except possibly for the single appearance of ι at the very end. Thus the image of $\mathrm{Aut}(\mathrm{D}(X))$ in $\mathrm{Aut}(\tilde{\mathrm{H}}(X))$ has index 2, if ι does not lift to an autoequivalence; else the map $\mathrm{Aut}(\mathrm{D}(X)) \to \mathrm{Aut}(\tilde{\mathrm{H}}(X))$ is surjective. ☐

Problem 2.7. It is not known if ι actually lifts to the derived level; one generally assumes this not to be true. The first instance of this appears in Szendrői's [83, Conjecture 5.4].

2.3 The Orientation Problem

It is instructive to look at the question of the surjectivity of $\mathrm{Aut}(\mathrm{D}(X)) \to \mathrm{Aut}(\tilde{\mathrm{H}}(X))$ from another angle. Let M be the differentiable 4-manifold underlying a K3 surface. Giving a K3 surface X is now equivalent to specifying a complex structure I on M. From the differential geometric point of view, an automorphism of such a K3 pair (M, I) is a diffeomorphism $f : M \xrightarrow{\sim} M$ such that $f^*I = I$. Denoting the K3 cohomology lattice by $\Lambda := \mathrm{H}^2(M, \mathbb{Z})$, we see that every automorphism $f \in \mathrm{Aut}(M, I)$ gives rise to an integral isometry f^*. Hence, we find a group homomorphism $\mathrm{Aut}(M, I) \to \mathrm{O}(\Lambda)$. The lattice Λ has signature $(3, 19)$, we fix an orientation of a positive 3-space $P_\Lambda \subset \Lambda_{\mathbb{R}}$. Then we can speak of the group $O_+(\Lambda)$ consisting of all orientation-preserving isometries. Here, we say that an isometry $\alpha : \Lambda \xrightarrow{\sim} \Lambda$ preserves the orientation precisely if the composition $P_\Lambda \hookrightarrow \Lambda \xrightarrow{\alpha} \Lambda \twoheadrightarrow P_\Lambda$ does. Donaldson proved in [25] that the image of the homomorphism $\mathrm{Aut}(M, I) \to O(\Lambda)$ is exactly $O_+(\Lambda)$. Note that $O_+(\Lambda) \subset O(\Lambda)$ has index 2 and that $-\mathrm{id}_\Lambda$ generates the coset of orientation-reversing isometries.

Recall the definition of the positive cone $\mathcal{C} \subset \mathrm{NS}(X)$ as a certain cone in the hyperbolic space $H^{1,1}(X, \mathbb{R})$. Since a Hodge isometry of $H^2(X)$ has to fix the positive plane $H^{2,0}(X) \oplus H^{0,2}(X) \cap H^2(X, \mathbb{R})$, it is an orientation-preserving lattice automorphism if and only if it maps $\mathcal{C} \to \mathcal{C}$ in the notation of Theorem 2.2.

Tying in with the philosophy of extending notions from $H^2(X)$ to $H^*(X)$, we follow [44] (see also [83, §5]) in order to define an orientation of the positive 4-space in $\tilde{H}(X)$: Let $\sigma \in H^{2,0}$ be the cohomology class of a symplectic form and let $\omega \in H^{1,1}(X, \mathbb{R})$ be an ample class. Then the real and imaginary parts $\mathrm{Re}(\sigma)$ and $\mathrm{Im}(\sigma)$ are positive classes in $H^2(X, \mathbb{R})$. Similarly, $1 - \omega^2/2$ and ω span a positive plane P' orthogonal to $P = \langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma) \rangle = (H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R})$. We define an orientation by ordering the basis as in $\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1 - \omega^2/2, \omega$. As above, there is the index two subgroup $\mathrm{Aut}_+(\tilde{H}(X)) \subset \mathrm{Aut}(\tilde{H}(X))$ of orientation-preserving Hodge isometries. Note that the Hodge isometry ι is orientation-reversing. Hence, Problem 2.7 can be phrased thus: Does $\mathsf{FM}_P^H \in \mathrm{Aut}_+(\tilde{H}(X))$ hold for all Fourier-Mukai equivalences $\mathsf{FM}_P \in \mathrm{Aut}(D(X))$? This is true for many instances [44, §5], e.g. line bundle twists and transforms given by fine moduli spaces as used in the proof of Theorem 2.6.

Reflections

Let V be a $\mathbb{R}$-vector space with a pairing $(\cdot, \cdot)$, i.e. a non-degenerate symmetric bilinear form. For any non-isotropic vector $w \in V$, the formula for the reflection along the orthogonal complement $w^\perp := \{v \in V : (v, w) = 0\}$ from linear algebra is

$$s_w : V \xrightarrow{\sim} V, \quad v \mapsto v - 2\frac{(w, v)}{(w, w)} w.$$

Furthermore, s_w is both isometry and involution: $s_w \in O(V, (\cdot, \cdot))$ and $s_w^2 = \mathrm{id}_V$. Now suppose that $\Lambda \subset V$ is a lattice such that $(\cdot, \cdot)|_\Lambda$ is integer valued. We obtain isometries $s_\lambda : \Lambda \xrightarrow{\sim} \Lambda$, using the same formula, for all $\lambda \in \Lambda$ such that $(\lambda, \lambda) = \pm 1$ or $(\lambda, \lambda) = \pm 2$.

The cases of interest here are $V = H^2(X, \mathbb{R})$, $\Lambda = H^2(X, \mathbb{Z})$ with the cup product and $V = \tilde{H}(X, \mathbb{R})$, $\Lambda = \tilde{H}(X, \mathbb{Z})$ with the Mukai pairing. Because this is even for K3 surfaces, we look for classes $\lambda \in H^2(X, \mathbb{Z})$ with $\lambda^2 = \pm 2$ to produce reflections. All reflections coming from (-2)-classes $\sigma \in \tilde{H}(X)$ can be lifted to the derived level using reflection functors as stated in Theorem 2.5.

On the contrary, reflections coming from $\tau \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R})$ with $\tau^2 = +2$ are less geometric in nature. The principal difference between them is that (-2)-reflections preserve the positive cones whereas $(+2)$-reflections do not: for $v \in \mathrm{NS}(X)$ with $v^2 > 0$

we have

$$\sigma^2 = -2 \quad \Longrightarrow \quad v.s_\sigma(v) = v^2 + (\sigma.v)^2 > 0,$$
$$\tau^2 = 2 \quad \Longrightarrow \quad v.s_\tau(v) = v^2 - (\tau.v)^2 < 0.$$

The latter inequality follows from the fact that it is enough to check some class in $\mathrm{NS}(X)$ of positive square and using $v := \tau$ yields $\tau^2 - (\tau^2)^2 = -2$. Next come some examples for $(+2)$-classes in $\mathrm{H}^2(X,\mathbb{Z})$ and their reflections in a somewhat geometric setting:

Examples 2.8.

(1) Let $\pi : X \to \mathbb{P}^1$ be a generic elliptic surface possessing a section $\sigma : \mathbb{P}^1 \hookrightarrow X$. By generic, we mean that $\mathrm{NS}(X)$ is of rank 2 and thus spanned by section and fibre: $\mathrm{NS}(X) = \langle \sigma, f \rangle$. The intersection product is determined by $\sigma^2 = -2$, $f^2 = 0$ and $\sigma.f = 1$. We first have the usual Picard-Lefschetz reflection $s_\sigma : v \mapsto v + (v.\sigma)\sigma$ given by the (-2)-class σ. However, $\sigma + 2f$ is a $(+2)$-class and hence gives rise to a reflection $s_{\sigma+2f} : v \mapsto v - (v.\sigma + 2v.f) \cdot (\sigma + 2f)$ as well. Note $s_\sigma(\sigma) = -\sigma$, $s_\sigma(f) = f + \sigma$, $s_{\sigma+2f}(\sigma) = \sigma$, $s_{\sigma+2f}(f) = -\sigma - f$ and so $s_\sigma = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $s_{\sigma+2f} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$. In particular, $s_{\sigma+2f} \circ s_\sigma^{-1} = \iota$ because the reflections do not act on the transcendental cycles at all. So, $s_{\sigma+2f}$ comes from $\mathrm{Aut}(\mathrm{D}(X))$ if and only if ι comes from $\mathrm{Aut}(\mathrm{D}(X))$.

(2) Let $\pi : X \to \mathbb{P}^2$ be a K3 surface which is branched over a sextic $C \subset \mathbb{P}^2$ as in Example 2.1(4). Let $L \subset \mathbb{P}^2$ be any line meeting C properly and let $D := \pi^* L$ be the corresponding divisor on X. Since $\deg(\pi) = 2$, we find $[D]^2 = 2[L]^2 = 2$, so that $[D]$ is a $(+2)$-class in $\mathrm{H}^2(X,\mathbb{Z})$. Obviously, the reflection $s_D : \mathrm{H}^*(X,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^*(X,\mathbb{Z})$ sends $[D] \mapsto -[D]$, $[X] \mapsto [X]$, $[pt] \mapsto [pt]$ and fixes $[D]^\perp \subset \mathrm{H}^2(X,\mathbb{Z})$. Now, let us also take into account the deck transformation $\varphi : X \xrightarrow{\sim} X$, which interchanges the two sheets. As an involution, φ^* has eigenvalues 1 and -1 only. We have $\varphi^*[D] = \varphi^*\pi^*[L] = \pi^*[L] = [D]$ and also $\varphi^*[X] = [X]$, $\varphi^*[pt] = [pt]$. Note that, up to multiples, $[D] \in \mathrm{H}^2(X,\mathbb{Z})$ is the only cohomology class coming from $\mathbb{P}^2$. As a class $c \in \mathrm{H}^2(X,\mathbb{Z})$ is φ^*-invariant if and only if it descends to $\mathbb{P}^2$, we find $\varphi^*(c) = -c$ whenever $c.[D] = 0$. Summing up, this shows $\iota = s_{[D]} \circ \varphi^*$. In particular, ι comes from $\mathrm{Aut}(\mathrm{D}(X))$ if and only if the $(+2)$-reflection $s_{[D]}$ comes from $\mathrm{Aut}(\mathrm{D}(X))$.

(3) The vector $v := (1,0,-1) \in \tilde{\mathrm{H}}(X,\mathbb{Z})$ has $v^2 = 2$. Note that $v = \mathrm{ch}(\mathcal{I}_p) = v(\mathcal{I}_{p,q})$ where $\mathcal{I}_p$ and $\mathcal{I}_{p,q}$ are ideal sheaves of a point p and two points p, q, respectively. The induced reflection is $s_v : x = (x_0, x_2, x_4) \mapsto (x_4, x_2, x_0)$. Similarly, the vector $w = (1,0,1) = v(\mathcal{O}_X)$ has $w^2 = -2$. Its reflection is $s_w : (x_0, x_2, x_4) \mapsto (-x_4, x_2, -x_0)$ and it lifts to $\mathsf{T}_{\mathcal{O}_X} \in \mathrm{Aut}(\mathrm{D}(X))$. From $\iota = s_v \circ s_w$, we see again that ι lifts to the derived level if and only if s_v does.

One might hope for a better description of reflections in $\tilde{\mathrm{H}}(X)$ along the following lines:

Conjecture 2.9. *Let* $\tilde{D}_X := \{v \in \tilde{\mathrm{H}}^{1,1} \cap \mathrm{H}^*(X,\mathbb{Z}) \,:\, v^2 = -2\}$ *be the set of integral algebraic* (-2)*-classes in the Mukai lattice and* $\tilde{W}(X) := \langle s_v \,:\, v \in \tilde{D}_X\rangle \subset \mathrm{Aut}(\tilde{\mathrm{H}}(X))$ *the 'big' Weyl group generated by them. Then, any* $(+2)$*-reflection* s_v *along* $v \in \mathrm{NS}(X)$*,* $v^2 = 2$ *is in the group generated by* (-2)*-reflections* $\mathsf{T}^{\mathrm{H}}_E$*, line bundle twists* $\mathsf{M}^{\mathrm{H}}_L$*, surface automorphisms* f^* *and* ι *and* $-\,\mathrm{id}$*, i.e.*

$$s_v \in \langle \tilde{W}(X), \mathrm{Pic}(X), \mathrm{Aut}(X), \mathrm{Aut}(U)\rangle$$

where $U := \mathrm{H}^0(X) \oplus \mathrm{H}^4(X)$ *with* $\mathrm{Aut}(U) = \{\pm\,\mathrm{id}, \pm\iota\}$*.*

If we ignore the Hodge condition for a moment, i.e. we consider the group $\mathrm{O}(\mathrm{H}^*(X))$ of all isometries of $\mathrm{H}^*(X,\mathbb{Z})$, then the analogue of the conjecture holds true. This is Wall's result on automorphism groups of the sum of an unimodular lattice with a hyperbolic plane [85, §5]. Going more into detail, Wall shows $\mathrm{O}(\mathrm{H}^2(X) \oplus U) = \langle \mathrm{O}(\mathrm{H}^2(X)), \mathrm{O}(U), \mathrm{H}^2(X)\rangle$ where $v \in \mathrm{H}^2(X)$ acts by multiplication with $\exp(v)$.

The conjecture aims at a better understanding of $\mathrm{Aut}(\tilde{\mathrm{H}}(X))$. However, there seems to be a flaw: Example 2.8(3) indicates that the truth of this conjecture would imply $\mathrm{Aut}(\tilde{\mathrm{H}}(X)) = \langle \tilde{W}(X), \mathrm{Pic}(X), \mathrm{Aut}(X), \mathrm{Aut}(U)\rangle$. This looks like an analogue of the strong Torelli Theorem 2.2, but note the following deficiency: among the isometries listed, only surface automorphisms act non-trivially on the transcendental lattice of $\tilde{\mathrm{H}}(X)$. So perhaps it would be more prudent to restrict the conjecture to K3 surfaces with big Picard rank.

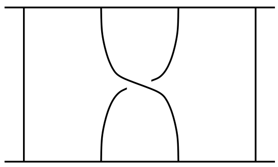
2.4 Cohomologically trivial autoequivalences

Of interest is the kernel of the homomorphism $\mathrm{Aut}(\mathrm{D}(X)) \to \mathrm{Aut}(\tilde{\mathrm{H}}(X))$ which we denote by $\mathrm{Aut}_0(\mathrm{D}(X))$. It consists of autoequivalences which are trivial on the level of cohomology. The general philosophy is to get knowledge about $\mathrm{Aut}(\mathrm{D}(X))$ by studying $\mathrm{Aut}_0(\mathrm{D}(X))$ and $\mathrm{Aut}(\tilde{\mathrm{H}}(X))$ individually. But not much is known at present about $\mathrm{Aut}_0(\mathrm{D}(X))$ and we just present concrete examples. However, Bridgeland's Conjecture 2.15 predicts what $\mathrm{Aut}_0(\mathrm{D}(X))$ should look like; it is described at the end of this section.

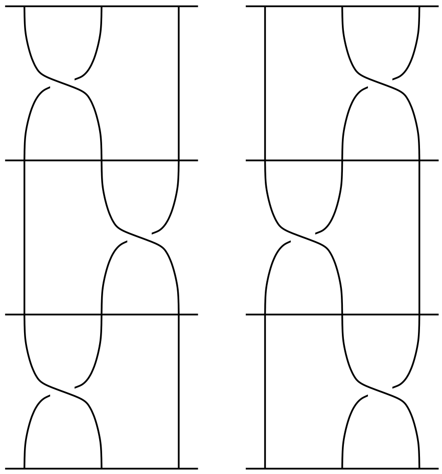
Pure braid subgroups of $\mathrm{Aut}_0(\mathrm{D}(X))$

Obviously, all even shifts $[2k]$ belong to $\mathrm{Aut}_0(\mathrm{D}(X))$. By Remark 1.30, we see also that $\mathsf{T}^2_E \in \mathrm{Aut}_0(\mathrm{D}(X))$ for a spherical object $E \in \mathrm{D}(X)$. Every realisation of an algebraic (-2)-class in $\tilde{\mathrm{H}}(X)$ by a sheaf gives rise to such a spherical reflection. Furthermore, for any collection of spherical objects $\{E_i\}_{i\in I}$, the cohomologically trivial autoequivalences $[2k]$, $\mathsf{T}^2_{E_i}$ with $k \in \mathbb{Z}$, $i \in I$ are all distinct by Lemma 1.22, at least if all $E_i^{\perp}$ are non-trivial; see also Remark 2.10. In order to find subgroups of $\mathrm{Aut}_0(\mathrm{D}(X))$, it is convenient to use certain configurations of spherical objects which are related to braid group actions.

Let us recall the braid groups. The braid group on m strands is denoted by $\mathcal{B}_m$ and explicitly defined using $m-1$ generators $\beta_1, \ldots, \beta_{m-1}$ satisfying the relations $\beta_i\beta_j = \beta_j\beta_i$ if $|i-j| > 1$ and $\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}$ for all $i = 1, \ldots, m-2$. There is a kind of geometric realisation for this group, in virtue of $\mathcal{B}_m = \pi_1(F_m(\mathbb{R}^2))$ with $F_m(\mathbb{R}^2) = \{(v_1, \ldots, v_m) \in (\mathbb{R}^2)^m \ : \ v_i \neq v_j \ \forall i \neq j\}$. The generators and relations then have the form shown in the figure.

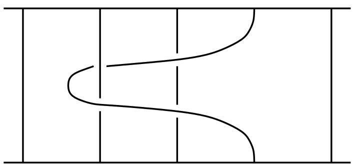


A generator $\beta_i \in \mathcal{B}_m$.



The relation $\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}$.

There is a natural surjective group homomorphism $\mathcal{B}_m \to S_m$, $\beta_i \mapsto (i,\ i+1)$, the latter term denoting transpositions in the symmetric group. The pure braid group on m strands is defined as the kernel $\mathcal{P}_m := \ker(\mathcal{B}_m \to S_m)$. It is generated by the elements $\alpha_{i,j} := \beta_{j-1}\beta_{j-2}\ldots\beta_{i+1}\beta_i^2\beta_{i+1}^{-1}\beta_{i+2}^{-1}\ldots\beta_{j-1}^{-1} \in \mathcal{P}_m$ for $1 \leq i < j \leq m$; see [1] for this and also for the relations. The geometrical representation of $\alpha_{i,j}$ is depicted below.



A generator $\alpha_{i,j} \in \mathcal{P}_m$.

In their study of spherical reflections [79], Seidel and Thomas also introduced the following notion of an A_m-configuration in $\mathrm{D}(Y)$ for a smooth projective variety Y. It is given by an ordered collection $(E_1, \ldots, E_m)$ of spherical objects in $\mathrm{D}(Y)$ such that $\dim(\mathrm{Hom}^*(E_i, E_{i+1})) = 1$ for $i = 1, \ldots, m-1$ and $\mathrm{Hom}^*(E_i, E_j) = 0$ if $|i-j| > 1$. Considering $\dim(\mathrm{Hom}^*(E_i, E_j))$ as a kind of intersection number, these conditions correspond up to sign to the Dynkin diagram of type A_m as $\dim(\mathrm{Hom}^*(E_i, E_i)) = 2$.

Now the condition $\mathrm{Hom}^*(E_i, E_j) = 0$ for $|i-j| > 1$ just means $E_i \in E_j^{\perp}$ and implies $\mathsf{T}_{E_j}(E_i) = E_i$ by Lemma 1.22. In view of $\mathsf{T}_{F(E_i)} = F \circ \mathsf{T}_{E_i} \circ F^{-1}$ for any $F \in \mathrm{Aut}(\mathrm{D}(Y))$, we get $\mathsf{T}_{E_i} = \mathsf{T}_{\mathsf{T}_{E_j}(E_i)} = \mathsf{T}_{E_j}\mathsf{T}_{E_i}\mathsf{T}_{E_j}^{-1}$ and thus $\mathsf{T}_{E_i}\mathsf{T}_{E_j} = \mathsf{T}_{E_j}\mathsf{T}_{E_i}$. Next consider the objects $A := E_i$ and $B := E_{i+1}$ which sit side by side in the A_m-configuration. By assumption we have $\mathrm{Hom}^l(A, B) \neq 0$ for a unique $l \in \mathbb{Z}$; hence, $\mathrm{Hom}^\bullet(A, B) \otimes A = A[-l]$, and, using $B \otimes \omega_Y \cong B$ and Serre duality, $\mathrm{Hom}^\bullet(B, A) \otimes B = B[l-d]$ where $d := \dim(Y)$. Applying T_B to the triangle $A[-l] \to B \to \mathsf{T}_A B$ yields $\mathsf{T}_B A[-l] \to B[1-d] \to \mathsf{T}_B \mathsf{T}_A B$ where $\mathsf{T}_B B \cong B[1-d]$ according to Lemma 1.22. Shifting by $l-1$ gives $B[l-d] \to \mathsf{T}_B \mathsf{T}_A B[l-1] \to \mathsf{T}_B A$ which is just the triangle defining $\mathsf{T}_B A$. Since all morphisms are non-trivial and unique up to scalars, we find $\mathsf{T}_B \mathsf{T}_A B \cong A[1-l]$ which in turn shows $\mathsf{T}_A \mathsf{T}_B \mathsf{T}_A = \mathsf{T}_B \mathsf{T}_{\mathsf{T}_B^{-1} A} \mathsf{T}_A = \mathsf{T}_B \mathsf{T}_{\mathsf{T}_A B[l-1]} \mathsf{T}_A = \mathsf{T}_B \mathsf{T}_{\mathsf{T}_A B} \mathsf{T}_A = \mathsf{T}_B \mathsf{T}_A \mathsf{T}_B$. To conclude, we arrive at $\mathsf{T}_{E_i} \mathsf{T}_{E_j} = \mathsf{T}_{E_j} \mathsf{T}_{E_i}$ for $|i-j| > 1$ and $\mathsf{T}_{E_i} \mathsf{T}_{E_{i+1}} \mathsf{T}_{E_i} = \mathsf{T}_{E_{i+1}} \mathsf{T}_{E_i} \mathsf{T}_{E_{i+1}}$.

Therefore, the T_{E_i} fulfill the braid relations and thus induce a group homomorphism $\mathcal{B}_{m+1} \to \mathrm{Aut}(\mathrm{D}(Y))$. The main theorem in [79] states that this representation is faithful if $\dim(Y) \geq 2$ which we will assume from now on. We denote the braid subgroup of $\mathrm{Aut}(\mathrm{D}(Y))$ coming from the A_m-configuration $E_\bullet$ by $\mathcal{B}(E_\bullet)$, and analogously for the pure braid subgroup by $\mathcal{P}(E_\bullet)$. Suppose in addition that $\dim(Y)$ is even. Then, the generators $A_{i,j} := \mathsf{T}_{E_{j-1}} \dots \mathsf{T}_{E_{i+1}} \mathsf{T}_{E_i}^2 \mathsf{T}_{E_{i+1}}^{-1} \dots T_{E_{j-1}}^{-1}$ of $\mathcal{P}(E_\bullet)$ act trivially on $\mathrm{H}^*(Y, \mathbb{Q})$ by Lemma 1.28 and we obtain an induced faithful pure braid group representation $\mathcal{P}(E_\bullet) \hookrightarrow \mathrm{Aut}_0(\mathrm{D}(Y))$. This is valid in particular for a K3 surface X.

Remark 2.10. Fix two spherical objects $A, B \in \mathrm{D}(Y)$ with $\mathrm{Hom}^\bullet(A, B) = \mathbb{C}[0]$. Then $\mathsf{T}_A^2 \neq \mathsf{T}_B^2$ follows from [79], but can be seen directly, too: we have $A[1] \cong \mathsf{T}_B \mathsf{T}_A B$ as above (with $l = 0$); analogously $B[1-d] \cong \mathsf{T}_A \mathsf{T}_B A$. If $\mathsf{T}_A^2 = \mathsf{T}_B^2$, we get a contradiction: $A[2-d] \cong \mathsf{T}_B \mathsf{T}_A (B[1-d]) \cong \mathsf{T}_B \mathsf{T}_A (\mathsf{T}_A \mathsf{T}_B A) \cong \mathsf{T}_B^4 A \cong \mathsf{T}_A^4 A \cong A[4-4d]$.

Example 2.11. The second cohomology lattice of a K3 surface has type $(-E_8)^{\oplus 2} \oplus U^{\oplus 3}$ with $U \cong (\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ denoting the hyperbolic plane. Oguiso and Zhang construct in [69, Proposition 1.7(2)] a K3 surface X as the desingularisation of a surface such that the configuration of exceptional divisors corresponds to a Dynkin diagram of type A_{19}. So X possesses an A_{19}-configuration $C_1, \dots, C_{19}$ of smooth (-2)-curves. The torsion sheaves $\mathcal{O}_{C_i}$ are spherical by Example 1.21(4). Also the intersection conditions $C_i.C_j = 0$ for $|i-j| > 1$ and $1 = C_i.C_{i+1} = \langle v(\mathcal{O}_{C_i}(-1)), v(\mathcal{O}_{C_{i+1}}(-1)) \rangle = -\chi(\mathcal{O}_{C_i}(-1), \mathcal{O}_{C_{i+1}}(-1))$ translate into $(\mathcal{O}_{C_1}, \dots, \mathcal{O}_{C_{19}})$ being an A_{19}-configuration in $\mathrm{D}(X)$. In particular, we find a pure braid group $\mathcal{P}_{20} \cong \mathcal{P}(\mathcal{O}_{C_\bullet})$ inside $\mathrm{Aut}_0(\mathrm{D}(X))$.

Example 2.12. Despite T_E^2 being trivial on the levels of cohomology and K-theory, the autoequivalence is far from being simple. Consider $E := \mathcal{O}_X$ and let us evaluate T_E^2 on $k(x)$, the skyscraper sheaf of some point $x \in X$. From $\mathrm{Hom}^\bullet(\mathcal{O}_X, k(x)) = \mathbb{C}[0]$, we get $\mathsf{T}_{\mathcal{O}_X}(k(x)) = \mathrm{cone}(\mathcal{O}_X \to k(x)) = \mathcal{I}_x[1]$ in view of the exact ideal sheaf sequence $0 \to \mathcal{I}_x \to \mathcal{O}_X \to k(x) \to 0$. Now from $\mathrm{Hom}^i(\mathcal{O}_X, \mathcal{I}_x[1]) = \mathrm{Ext}^{i+1}(\mathcal{O}_X, \mathcal{I}_x) = \mathrm{H}^{i+1}(\mathcal{I}_x)$ and the same ideal sheaf sequence, we find $\mathrm{Hom}^\bullet(\mathcal{O}_X, \mathcal{I}_x[1]) = \mathbb{C}[-1]$. Hence the cone of the morphism $\mathcal{O}_X[-1] \to \mathcal{I}_x[1]$ corresponds to a non-trivial extension $0 \to \mathcal{I}_x \to A^{-1} \to A^0 \to \mathcal{O}_X \to 0$. Both the morphism and the extension are unique up to scalars. We thus find $\mathsf{T}_{\mathcal{O}_X}^2(k(x)) = A^\bullet$ with a complex $A^\bullet \in \mathrm{D}(X)$ having cohomology in degrees both 0 and -1. Note that $[A^\bullet] = [k(x)]$ as classes in K-theory, as required.

Bridgeland's conjecture on $\mathrm{Aut}_0(\mathrm{D}(X))$

Bridgeland introduced an entirely new invariant of triangulated categories in [17]. He was guided by Douglas' work in string theory [26] related to Π-stability. The basic idea is to associate to $\mathrm{D}(Y)$ (or more generally, any triangulated category) a manifold $\mathrm{Stab}(Y)$ consisting of stability conditions on $\mathrm{D}(Y)$. One hope is that 'ungeometrical' (or 'unphysical') complexes having homology in various places might be better understood as being stable with respect to some generalised stability condition. Furthermore, the group $\mathrm{Aut}(\mathrm{D}(Y))$ naturally acts on $\mathrm{Stab}(Y)$ and thus gives another point of view on autoequivalences. It should be remarked that there is another notion of stability on triangulated categories proposed in [31]. However, this approach completely neglects the aspect of the manifold $\mathrm{Stab}(Y)$ and is thus not suited for the description below.

For the smooth projective variety Y, let $\mathrm{K}(Y)$ be its K-group; this is also the K-group of the triangulated category $\mathrm{D}(Y)$. Now $\mathrm{K}(Y)$ is equipped with the bilinear Euler form $\chi(A_1, A_2) := \sum_i (-1)^i \dim \mathrm{Hom}^i(A_1, A_2)$. We define $\mathcal{N}(Y) := \mathrm{K}(Y)/\mathrm{K}(Y)^\perp$ where $\mathrm{K}(Y)^\perp$ is the right χ-orthogonal subgroup. By the Hirzebruch-Riemann-Roch theorem, $\mathcal{N}(Y)$ is a free abelian group of finite rank. More precisely, $\mathcal{N}(Y)$ is isomorphic to the group $\mathrm{CH}^*(Y)/num$ of cycles modulo numerical equivalence and $\mathcal{N}(Y)$ is also isomorphic to the algebraic lattice inside $\mathrm{H}^*(Y, \mathbb{Z})$.

A *stability condition* on $\mathrm{D}(Y)$ consists of a linear map $Z : \mathcal{N}(Y) \to \mathbb{C}$ and a collection of full subcategories $\mathcal{P}(t) \subset \mathrm{D}(Y)$ for all $t \in \mathbb{R}$, subject to the following conditions

(a) $A \in \mathcal{P}(t)$ implies $Z(A) = m(A) \cdot \mathrm{e}^{i\pi t}$ with $m(A) \in \mathbb{R}_{>0}$.

(b) $\mathcal{P}(t+1) = \mathcal{P}(t)[1]$ for all $t \in \mathbb{R}$.

(c) $\mathrm{Hom}(A_1, A_2) = 0$ for all $A_1 \in \mathcal{P}(t_1)$, $A_2 \in \mathcal{P}(t_2)$ with $t_1 > t_2$.

(d) For $0 \neq A \in \mathrm{D}(Y)$ there are $t_1 > t_2 > \cdots > t_n$ and objects $Q_j \in \mathcal{P}(t_j)$, $A_j \in \mathrm{D}(Y)$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 \longrightarrow \cdots \longrightarrow A_{n-1} & \longrightarrow & A_n = A \, . \\ & {\scriptstyle [1]}\nwarrow \;\; \swarrow & & {\scriptstyle [1]}\nwarrow \;\; \swarrow & & {\scriptstyle [1]}\nwarrow \;\; \swarrow & \\ & Q_1 & & Q_2 & & Q_n & \end{array}$$

The last axiom captures the Harder-Narashiman property of the usual stability notions: objects $A \in \mathcal{P}(t)$ are said to be *semistable* of phase t with respect to $(Z, \mathcal{P})$. The definition above contains two further axioms imposed in [17]; what we have described are 'locally finite' and 'numerical' stability conditions. Let $\mathrm{Stab}(Y)$ be the set of all such stability conditions. Bridgeland provided $\mathrm{Stab}(Y)$ with a metric and proved [18]:

Theorem 2.13 (Bridgeland). *For every connected component $\Sigma \subset \mathrm{Stab}(Y)$ there is a linear subspace $V(\Sigma) \subset (\mathcal{N}(Y) \otimes \mathbb{C})^*$, topologised by a linear topology, such that the continuous map $\Sigma \to V(\Sigma)$, $(Z, \mathcal{P}) \mapsto Z$ is a local homeomorphism.*

This implies in particular, that all connected components of $\mathrm{Stab}(Y)$ are complex manifolds. A priori, the dimensions of different components might vary. However, no instance of such behaviour is known. In all the examples discussed below, one actually has $V(\Sigma) = (\mathcal{N}(Y) \otimes \mathbb{C})^*$.

The group $\mathrm{Aut}(\mathrm{D}(Y))$ acts naturally on $\mathrm{Stab}(Y)$: for a stability condition (Z, P) and an autoequivalence $F : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Y)$, we define a new stability condition (Z', P') by $Z' := Z \circ (F^{\mathcal{N}})^{-1}$ and $P'(t) := F(P(t))$ where $F^{\mathcal{N}}$ denotes the obvious automorphism of $\mathcal{N}(Y)$ induced by F. The quotient $\mathrm{Stab}(Y)/\mathrm{Aut}(\mathrm{D}(Y))$ turns out to be particularly interesting. Both $\mathrm{Stab}(Y)$ and this quotient have been computed in some cases.

Example 2.14. A smooth projective curve has C has $\mathcal{N}(C) \cong \mathbb{Z}^2$, the isomorphism given by rank and determinant. Because of this, for any automorphism $f : C \xrightarrow{\sim} C$, the action of f^* on $\mathcal{N}(C)$ is trivial, i.e. $(f^*)^{\mathcal{N}} = \mathrm{id}$. Similarly, for a line bundle $L \in \mathrm{Pic}^0(C)$ of degree 0 we have $\mathsf{M}_L^{\mathcal{N}} = \mathrm{id}$.

For the projective line, Okada [70] computed $\mathrm{Stab}(\mathbb{P}^1) \cong \mathbb{A}^2$. Since $\mathrm{Aut}(\mathrm{D}(\mathbb{P}^1)) = \mathrm{Aut}(\mathbb{P}^1) \ltimes \mathrm{Pic}(\mathbb{P}^1) \times \mathbb{Z}$ by [12, §3], we derive $\mathrm{Stab}(\mathbb{P}^1)/\mathrm{Aut}(\mathrm{D}(\mathbb{P}^1)) \cong \mathbb{A}^2/\mathbb{Z}^2$, the quotient by shifts and line bundle twists. However, the $\mathbb{Z}^2$-action is not made explicit in [70], and we cannot give a more precise description of the quotient.

Now let C be a smooth projective curve of genus at least three. Bridgeland [17, §7] proved that there is a connected component $\mathrm{Stab}^0(C) \subset \mathrm{Stab}(C)$ which is isomorphic to $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. The latter is the universal covering space of the set $\mathrm{GL}^+(2, \mathbb{R})$ of matrices with positive determinant. This shows $\mathrm{Stab}^0(C)/\mathrm{Aut}(\mathrm{D}(C)) \cong \mathrm{GL}^+(2, \mathbb{R})/\langle \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \rangle$, using that shifts act such that $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})/\langle [1] \rangle \cong \mathrm{GL}^+(2, \mathbb{R})$ and that both $\mathrm{Aut}(C)$ and $\mathrm{Pic}^0(C)$ act trivially on $\mathcal{N}(C)$.

For an elliptic curve E, one has $\mathrm{Stab}(E) \cong \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$; confer again [17, §7]. It follows from results of Mukai and Orlov that $\mathrm{Aut}(\mathrm{D}(E))$ is generated by $\mathrm{Aut}(E) \ltimes \mathrm{Pic}(E) \times \mathbb{Z}$ and the autoequivalence $\mathsf{FM}_{\mathcal{P}}$ given by the Poincaré bundle $\mathcal{P}$; see Section 4.1. As explained in Example 1.19(3), $\mathsf{FM}_{\mathcal{P}}$ acts on $\mathcal{N}(E)$ by $\mathsf{FM}_{\mathcal{P}}^{\mathcal{N}} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Accordingly, $\mathrm{Stab}(E)/\mathrm{Aut}(\mathrm{D}(E)) \cong \mathrm{GL}^+(2, \mathbb{R})/\langle \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \rangle \cong \mathrm{GL}^+(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{R})$.

For a surface S, we have $\mathcal{N}(S) = \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \mathbb{Z}$, the three summands corresponding to rank, c_1 and c_2. Let us now turn to the case of an algebraic K3 surface X. Then, $\mathcal{N}(X)$ is the algebraic lattice inside $\mathrm{H}^*(X, \mathbb{Z})$ and we equip $\mathcal{N}(X)$ with the Mukai pairing as in Remark 1.18. Let $\rho := \mathrm{rk}(\mathrm{NS}(X))$ be the Picard rank; since X is supposed to be algebraic, we have $1 \leq \rho \leq 20$. By the Hodge index theorem, $\mathrm{NS}(X) \otimes \mathbb{R}$ has index $(1, \rho - 1)$ and thus $\mathcal{N}(X) \otimes \mathbb{R}$ has signature $(2, \rho)$.

Consider the open subset $\mathcal{P}^\pm \subset \mathcal{N}(X) \otimes \mathbb{C}$ consisting of vectors such that their real and imaginary parts span a positive 2-plane of $\mathcal{N}(X) \otimes \mathbb{R}$. For example, if $\omega \in \mathrm{NS}(X)$ is the class of an ample line bundle, then $(1, i\omega, -\omega^2/2) \in \mathcal{P}^\pm$. In analogy with real hyperbolic spaces, $\mathcal{P}^\pm$ has two connected components which get interchanged by complex conjugation on $\mathcal{N}(X) \otimes \mathbb{C}$. We fix one component $\mathcal{P}^+$ by the condition $(1, i\omega, -\omega^2/2) \in \mathcal{P}^+$ for one (and hence all) ample classes $\omega \in \mathrm{NS}(X) \otimes \mathbb{R}$. Let $\tilde{D}_X := \{\delta \in \mathcal{N}(X) : \delta^2 = -2\}$ be the set of algebraic (-2)-classes and denote by

$\delta^\perp := \{v \in \mathcal{N}(X) \otimes \mathbb{C} \,:\, (v, \delta) = 0\}$ the complex hyperplane induced by δ. Finally, we need another open subset $\mathcal{P}_0^+ := \mathcal{P}^+ \setminus \bigcup_{\delta \in \tilde{D}_X} \delta^\perp$ by removing the walls corresponding to (-2)-classes. Note that $\mathcal{P}^+$ is contractible but obviously $\mathcal{P}_0^+$ possesses non-trivial loops. The main result of [18] states that there is a distinguished connected component $\Sigma(X) \subset \mathrm{Stab}(X)$ such that the local homeomorphism $\pi : \Sigma(X) \to (\mathcal{N}(X) \otimes \mathbb{C})^*$ is a surjection onto $\mathcal{P}_0^+$ and the map $\pi : \Sigma(X) \to \mathcal{P}_0^+$ is a topological covering. The group of deck transformations is $\mathrm{Aut}_{0,\Sigma}(\mathrm{D}(X))$, consisting of cohomologically trivial autoequivalences $F : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(X)$ such that the induced isomorphism on $\mathrm{Stab}(X)$ keeps the component $\Sigma(X)$ fixed. Bridgeland has formulated the following prediction about this covering [18, Conjecture 1.2].

Conjecture 2.15 (Bridgeland). $\Sigma(X)$ *is simply-connected and* $\mathrm{Aut}(\mathrm{D}(X))$ *fixes* Σ*.*

This would imply that $\mathrm{Aut}_0(\mathrm{D}(X)) = \mathrm{Aut}_{0,\Sigma}(\mathrm{D}(X)) = \pi_1(\mathcal{P}_0^+)$ is the group of deck transformations of the covering $\pi : \Sigma(X) \to \mathcal{P}_0^+$. Since an autoequivalence of $\mathrm{D}(X)$ which is orientation-reversing swaps the components $\mathcal{P}^+$ and $\mathcal{P}^-$, it could not possibly fix Σ. Hence, the truth of the conjecture would lead to an exact sequence

$$1 \to \pi_1(\mathcal{P}_0^+) \to \mathrm{Aut}(\mathrm{D}(X)) \to \mathrm{Aut}_+(\tilde{\mathrm{H}}(X)) \to 1.$$

In particular, this entails the statements that $\mathrm{Aut}_0(\mathrm{D}(X)) \cong \pi_1(\mathcal{P}_0^+)$ and that the image of $\mathrm{Aut}(\mathrm{D}(X)) \to \mathrm{Aut}(\tilde{\mathrm{H}}(X))$ has index 2. The latter statement is a reformulation of Problem 2.7, that $\iota \in \mathrm{Aut}(\tilde{\mathrm{H}}(X))$ does not lift to an autoequivalence.

3 Finite group actions and derived categories

3.1 Linearisations for finite groups

Let Y be a smooth projective variety with an action by a finite group G. Among the coherent sheaves, one might consider the full additive subcategory of G-invariant sheaves

$$\mathrm{Coh}(Y)^G := \{F \in \mathrm{Coh}(Y) : g^*F \cong F\ \forall g \in G\} \subset \mathrm{Coh}(Y).$$

However, the isomorphisms $g^*F \xrightarrow{\sim} F$ are by no means canonical. Hence $\mathrm{Coh}(Y)^G$ is not abelian and is thus not suitable for forming derived categories. As suggested from Geometric Invariant Theory, we will employ G-linearisations. The general definition, valid for any algebraic group G, is given first. The morphisms $\sigma : G \times Y \to Y$ and $\mu : G \times G \to G$ are action and multiplication of G, respectively; $p_2 : G \times Y \to Y$ and $p_{23} : G \times G \times Y \to G \times Y$ are the projections.

Definition 3.1. A *G-linearisation* of a coherent sheaf F is an isomorphism $\lambda : \sigma^*F \xrightarrow{\sim} p_2^*F$ of $\mathcal{O}_{G\times Y}$-modules subject to the cocycle condition $(\mu \times \mathrm{id}_Y)^*\lambda = p_{23}^*\lambda \circ (\sigma \times \mathrm{id}_G)^*\lambda$.

Remark 3.2. In the case of a finite group G, there is an obvious reformulation (with the minor modification of reversing the direction of λ): A G-linearisation of F is given by isomorphisms $\lambda_g : F \xrightarrow{\sim} g^*F$ for all $g \in G$ subject to $\lambda_1 = \mathrm{id}_F$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$, i.e.

$$F \xrightarrow{\lambda_h} h^*F \xrightarrow{h^*\lambda_g} h^*g^*F, \qquad \lambda_{gh} : F \to h^*g^*F.$$

Example 3.3. If G acts on Y as above, the identities $g^*\mathcal{O}_Y = \mathcal{O}_Y$ show that the structure sheaf is canonically G-linearised. The canonical sheaf ω_Y is G-linearisable, as well: the morphism $g : Y \to Y$ induces a morphism of cotangent bundles $g_* : g^*\Omega_Y \to \Omega_Y$. Going to determinants and using adjunction yields $\lambda_g := \det(g_*^{-1}) : \omega_Y \xrightarrow{\sim} g^*\omega_Y$.

If (F, λ) and (F', λ') are two G-linearised sheaves, then the vector space $\mathrm{Hom}(F, F')$ becomes a G-representation via $g \cdot f := (\lambda'_g)^{-1} \circ g^*f \circ \lambda_g$ for $f : F \to F'$. Let $\mathrm{Coh}^G(Y)$ be the category whose objects are G-linearised sheaves and whose morphisms are the G-invariant sheaf morphisms: $\mathrm{Hom}((F,\lambda),(F',\lambda')) := \mathrm{Hom}(F,F')^G$. This category is abelian: the kernel of a morphism $f : (F,\lambda) \to (F',\lambda')$ is given by $\ker(f)$, linearised by $\lambda|_{\ker(f)}$. Similarly, its cokernel is $\mathrm{Coker}(f)$ with linearisation induced from λ'. The necessary axioms follow from their validity in $\mathrm{Coh}(Y)$.

We will encounter the special case where G acts trivially on Y. Then a G-linearisation λ of a sheaf F is merely a group homomorphism $\lambda : G \to \mathrm{Aut}(F)$. As G is finite, this representation decomposes into a direct sum over the irreducible G-representations $\rho_0 = 1, \rho_1 \ldots, \rho_k$, i.e. $F \cong \bigoplus_i F_i \otimes \rho_i$ in $\mathrm{Coh}^G(Y)$ with ordinary sheaves $F_i \in \mathrm{Coh}(Y)$. There exist no homomorphisms between summands corresponding to two different representations, and hence we obtain two mutually adjoint and exact functors, the latter of which is 'taking G-invariants':

$$\cdot \otimes \rho_0 : \mathrm{Coh}(Y) \to \mathrm{Coh}^G(Y), \quad F_0 \mapsto F_0 \otimes \rho_0 = (F_0, \lambda_0) \text{ with } \lambda_0 : G \to \mathrm{Aut}(F), g \mapsto \mathrm{id}_{F_0},$$
$$[\,\cdot\,]^G : \mathrm{Coh}^G(Y) \to \mathrm{Coh}(Y), \quad (F, \lambda) \cong \bigoplus_i F_i \otimes \rho_i \mapsto [F]^G = F_0 \otimes \rho_0.$$

A G-equivariant morphism $\Phi : Y \to Z$ allows defining functors $\Phi_* : \mathrm{Coh}^G(Y) \to \mathrm{Coh}^G(Z)$ and $\Phi^* : \mathrm{Coh}^G(Z) \to \mathrm{Coh}^G(Y)$ in a straightforward manner. For example, if $(F, \lambda) \in \mathrm{Coh}^G(Y)$, then $\Phi_*(F)$ is canonically G-linearised via $\Phi_*\lambda_g : \Phi_*F \xrightarrow{\sim} \Phi_*g^*F = g^*\Phi_*F$. We have the usual adjointness $\Phi^* \dashv \Phi_*$ because the natural transformation $\Phi^*\Phi_* \to \mathrm{id}$ induces G-invariant morphisms.

By definition every linearisable sheaf is invariant, which amounts to the forgetful functor *for* $: \mathrm{Coh}^G(Y) \to \mathrm{Coh}(Y)^G$. There is also a functor that assigns to any coherent sheaf a G-linearised sheaf via *inf* $: \mathrm{Coh}(Y) \to \mathrm{Coh}^G(Y), F \mapsto \mathit{inf}(F) := \bigoplus_{g \in G} g^*F$ where the linearisation comes canonically from permuting the summands. This map is usually called inflation.

It pays off to extend much of this story to a relative setting involving two groups; our presentation follows closely [7, §8]. Let G_1 and G_2 be finite groups acting on smooth projective varieties Y_1 and Y_2, respectively. Further, let $\varphi : G_1 \to G_2$ be a group homomorphism and $\Phi : Y_1 \to Y_2$ be a φ-map, that means $\Phi \circ g_1 = \varphi(g_1) \circ \Phi$ for all $g_1 \in G_1$. Then, there is a pull-back $\Phi^* : \mathrm{Coh}^{G_2}(Y_2) \to \mathrm{Coh}^{G_1}(Y_1)$ given by $\Phi^*\lambda_{\varphi(g_1)} : \Phi^*F_2 \xrightarrow{\sim} \Phi^*\varphi(g_1)^*F_2 = g_1^*\Phi^*F_2$ for $(F_2, \lambda) \in \mathrm{Coh}^{G_2}(Y_2)$.

If $\varphi : G_1 \twoheadrightarrow G_2$ is surjective with kernel $K \subset G_1$, then there is also a push-forward $\Phi_*^K : \mathrm{Coh}^{G_1}(Y_1) \to \mathrm{Coh}^{G_2}(Y_2)$: let G_1 act naturally on Y_2 via φ; thus, the kernel K acts trivially on Y_2. For $F_1 \in \mathrm{Coh}^{G_1}(Y_1)$, the direct image $\Phi_*F_1 \in \mathrm{Coh}(Y_2)$ is canonically G_1-linearised. Taking K-invariants, we define $\Phi_*^K(F_1) := [\Phi_*F_1]^K$, and this sheaf is G_2-linearised because φ was supposed to be surjective. The adjoint property $\Phi^* \dashv \Phi_*^K$ is a consequence of the fact that for G_1-invariant sheaf morphisms $\Phi^*F_2 \to F_1$, the adjoint morphism $F_2 \to \Phi_*F_1$ has image in $\Phi_*^K F_1$.

Let us consider a subgroup $H \subset G$. Generalising the discussion of the case $H = 1$ above, there is an inflation functor $\mathrm{Inf}_H^G : \mathrm{Coh}^H(Y) \to \mathrm{Coh}^G(Y)$ as well as a restriction functor $\mathrm{Res}_G^H : \mathrm{Coh}^G(Y) \to \mathrm{Coh}^H(Y)$. The restriction being trivial, we explain the construction of $\mathrm{Inf}_H^G(F)$ for a H-linearised sheaf $F \in \mathrm{Coh}^H(Y)$. The product $G \times H$ acts on $G \times Y$ via $(g, h) \cdot (g', y) := (gg'h^{-1}, hy)$. Obviously, this turns the projection $p : G \times Y \to Y$ into a $G \times H$-equivariant map. The functors introduced above show that p^*F is canonically $G \times H$-linearised. Note that H acts freely on $G \times Y$, and we have the projection $q : G \times Y \to G \times Y/H =: G \times_H Y$. With the projection of groups $\varphi : G \times H \twoheadrightarrow G$, the morphism q becomes a φ-map. Using the equivariant push-forward, we obtain the G-linearised sheaf $q_*^H p^*F$. Finally, we need the multiplication $m : G \times_H Y \to Y, (g, y) \mapsto gy$, which is a G-map. Then, $\mathrm{Inf}_H^G(F) := m_*q_*^H p^*F$ is the sought after G-inflation of F. As ordinary sheaves, we have $\mathrm{Inf}_H^G(F) = \bigoplus_{[g]\in G/H} g^*F$. It is also possible to write down an explicit G-linearisation for the latter sum. Note that Res_G^H and Inf_H^G are exact functors with the adjoint property $\mathrm{Inf}_H^G \dashv \mathrm{Res}_G^H$. Especially, *for* $= \mathrm{Res}_G^1$ and *inf* $= \mathrm{Inf}_1^G$ for the trivial subgroup $1 \subset G$.

Example 3.4. The canonical G-linearisation of $\mathcal{O}_Y$ corresponds to $\lambda_g = 1$ for all $g \in G$. However, every group homomorphism $\chi : G \to \mathbb{C}^*$ gives rise to another linearisation. In the other direction, two different homomorphisms $G \to \mathbb{C}^*$ endow $\mathcal{O}_Y$ with different G-linearisations.

We will study the question whether a similar statement is true for simple G-invariant sheaves as well. To state the result in a concise way, we utilise group cohomology of G with values in the multiplicative abelian group $\mathbb{C}^*$. The reference for this, with the notation taken up here, is Serre's book [80]. Let us recall some basic facts: if the finite group G (written multiplicatively) acts on the abelian group M (which is written additively), then there is a sequence of cohomology groups defined in the usual way as quotients $\mathrm{H}^i(G, M) := Z^i(G, M)/B^i(G, M)$ of i-cycles modulo i-boundaries. For $i \leq 2$ these quotients explicitly look like

$i = 0$: $\mathrm{H}^0(G, M) = M^G$ is just the subgroup of G-invariants. Taking invariants is a left exact functor and the higher cohomology groups can be intrinsically introduced as the right derived functors.

$i = 1$: A 1-cycle is a map $c : G \to M$ such that $c(gh) = g \cdot c(h) + c(g)$ for all $g, h \in G$. We always have $c(1) = 0$. A 1-boundary is a map given by $g \mapsto g \cdot m - m$ for some $m \in M$. Note that $\mathrm{H}^1(G, M) = \mathrm{Hom}(G, M)$ if the G-action on M is trivial.

$i = 2$: A 2-cycle is a map $c : G^2 \to M$ satisfying $c(gh, k) + c(g, h) = g \cdot c(h, k) + c(g, hk)$ for all $g, h, k \in G$. It is a 2-boundary if there is some map $e : G \to M$ such that $c(g, h) = g \cdot e(h) + e(g) - e(gh)$ for all $g, h \in G$.

Furthermore, let us denote by $\mathrm{Lin}_G(F)$ the set of non-isomorphic G-linearisations on F. As another matter of terminology, we write $\hat{G}$ for the group of all homomorphisms $\chi : G \to \mathbb{C}^*$ and call these homomorphisms characters of G. In the following lemma and its proof, the group of coefficients $M = \mathbb{C}^*$ is written multiplicatively, of course.

Lemma 3.5. *Let Y be a smooth projective variety with an action of a finite group G and let F be a simple G-invariant sheaf on Y.*

(i) There is a class $[F] \in \mathrm{H}^2(G, \mathbb{C}^*)$ such that F is G-linearisable if and only if $[F] = 0$.

(ii) If $[F] = 0$, then $\mathrm{Lin}_G(F)$ is a $\hat{G}$-torsor; in particular $\#\mathrm{Lin}_G(F) = \#\hat{G}$.

Remark 3.6. The invariant $\mathrm{H}^2(G, \mathbb{C}^*)$ was studied by Schur in [77] long before the advent of group cohomology proper; see also [23, §53]. The classical name for $\mathrm{H}^2(G, \mathbb{C}^*)$ is 'Schur multiplier'. It is always a finite abelian group whose exponent is a divisor of $\#G$. The long exact group cohomology sequence obtained from $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$ yields $\mathrm{H}^2(G, \mathbb{C}^*) \cong \mathrm{H}^3(G, \mathbb{Z})$ because $\mathbb{C}$ is torsion free and hence G-acyclic. As group cohomology can also be expressed as singular cohomology of the classifying space BG, these invariants have been investigated by topologists. We give some examples for Schur multipliers, following [41, §25]:

$\mathrm{H}^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^*) = 0$ (cyclic groups)
$\mathrm{H}^2(D_{2n}, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$ and $\mathrm{H}^2(D_{2n+1}, \mathbb{C}^*) = 0$ (dihedral groups)
$\mathrm{H}^2(S_3, \mathbb{C}^*) = 0$ and $\mathrm{H}^2(S_n, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 4$ (symmetric groups)
$\mathrm{H}^2((\mathbb{Z}/n\mathbb{Z})^k, \mathbb{C}^*) = (\mathbb{Z}/n\mathbb{Z})^{k(k-1)/2}$
$\mathrm{H}^2(G_1 \times G_2, \mathbb{C}^*) = \mathrm{H}^2(G_1, \mathbb{C}^*) \times \mathrm{H}^2(G_2, \mathbb{C}^*) \times (\mathrm{H}^1(G_1, \mathbb{C}^*) \otimes \mathrm{H}^1(G_2, \mathbb{C}^*))$

The first relation also appears in Spanier's book [81, Theorem 9.5.6]. The last equation is the Künneth formula and its repeated application for $G_1 = G_2 = \mathbb{Z}/n\mathbb{Z}$ yields the preceding example.

Question 3.7. The lemma immediately raises questions like this: for $G = S_n$ with $n > 3$, is there some intrinsic property of G-invariant sheaves which measures whether they are linearisable?

Problem 3.8. The methods of the proof completely fail in the case of sheaves with non-trivial endomorphisms. For example, we are not even able to associate a cohomology class in $\mathrm{H}^2(G, \mathrm{Aut}(F))$ to a G-invariant sheaf F in general. Note that for non-abelian $\mathrm{Aut}(F)$, the group cohomology $\mathrm{H}^2(G, \mathrm{Aut}(F))$ is just a pointed set.

Remark 3.9. The statement of the lemma differs from the results of Geometric Invariant Theory [64, §1.3] where G usually is a connected linear algebraic group. Typically only line bundles are considered in this setting. Some basic results are: a G-invariant line bundle need not be G-linearisable. If there is no surjection $G \twoheadrightarrow \mathbb{C}^*$, then at most one linearisation can exist. Furthermore, any line bundle has a G-linearisation after an appropriate twist with an ample G-linearised line bundle. The example from [64, §1.3] is the canonical action of $G = \mathrm{PGL}(n+1)$ on $Y = \mathbb{P}^n$. Then the G-invariant line bundle $F = \mathcal{O}_{\mathbb{P}^n}(1)$ admits no G-linearisation whatsoever but $F^{\otimes n+1} = \mathcal{O}_{\mathbb{P}^n}(n+1)$ does.

Proof of the lemma. We will use the following facts throughout: $\mathrm{Aut}(F) = \mathbb{C}^*$ is abelian and the G-action on $\mathrm{Aut}(F)$ is trivial. The last statement follows from $g^* \mathrm{id}_F = \mathrm{id}_F$ together with the fact that sheaf homomorphisms are $\mathbb{C}$-linear. The proof of (i) rests on the following two facts:
(a) By assumption, there exist isomorphisms $\mu_g : F \xrightarrow{\sim} g^*F$ for all $g \in G$. As F is simple, we can define units $c_{g,h} \in \mathbb{C}^*$ by $\mu_{gh} = h^*\mu_g \circ \mu_h \cdot c_{g,h}$. The map $c : G^2 \to \mathbb{C}^*$ is a 2-cocycle of G with values in $\mathbb{C}^*$, i.e. $c \in Z^2(G, \mathbb{C}^*)$.
(b) Replacing the isomorphisms μ_g with some other μ'_g yields the map $e : G \to \mathbb{C}^*$ such that $\mu'_g = \mu_g \cdot e_g$. The two cocycles $c, c' : G^2 \to \mathbb{C}^*$ derived from μ and μ' differ by the boundary coming from e, i.e. $c/c' = d(e)$ and thus $c = c' \in \mathrm{H}^2(G, \mathbb{C}^*)$.
Taking these for granted, we see that the G-invariant sheaf F gives rise to a unique class

$[F] := c \in \mathrm{H}^2(G, \mathbb{C}^*)$. In these terms, F is G-linearisable if and only if $c \equiv 1$, i.e. $[F]$ vanishes.
Check of (a): Invoking the definition of μ in two ways, we get for $g, h, k \in G$

$$\mu_{ghk} = k^*\mu_{gh} \circ \mu_k \cdot c_{gh,k} = k^*(h^*\mu_g \circ \mu_h \cdot c_{g,h}) \circ \mu_k \cdot c_{gh,k}$$
$$\mu_{ghk} = (hk)^*\mu_g \circ \mu_{hk} \cdot c_{g,hk} = (hk)^*\mu_g \circ (k^*\mu_h \circ \mu_k \cdot c_{h,k}) \cdot c_{g,hk}.$$

Using the remarks from the beginning of the proof, we find $c_{g,h} \cdot c_{gh,k} = c_{h,k} \cdot c_{g,hk}$, so that c is a 2-cycle, indeed.
Check of (b): For $g, h \in G$, we have $\mu'_{gh} = h^*\mu'_g \circ \mu'_h \cdot c'_{g,h} = h^*(\mu_g \cdot e_g) \circ \mu_h \cdot e_h \cdot c'_{g,h}$, on the one hand, and $\mu'_{gh} = \mu_{gh} \cdot e_{gh} = h^*\mu_g \circ \mu_h \cdot e_{gh}$, on the other hand. With the aforementioned remarks, we find $c_{g,h}/c'_{g,h} = e_h \cdot e_g \cdot e_{gh}^{-1}$, as claimed.

To show (ii), we consider the $\hat{G}$-action $\hat{G} \times \mathrm{Lin}_G(F) \to \mathrm{Lin}_G(F)$, $(\chi, \lambda) \mapsto \chi \cdot \lambda$ on $\mathrm{Lin}_G(F)$. First take $\chi \in \hat{G}$ and $\lambda \in \mathrm{Lin}_G(F)$ such that $\chi \cdot \lambda = \lambda$. Then, there is an isomorphism $f : (F, \lambda) \xrightarrow{\sim} (F, \chi \cdot \lambda)$ which in turn immediately implies $\chi = 1$, again using $f \in \mathrm{Aut}(F) = \mathbb{C}^*$. Thus, the action is effective. Now take two elements $\lambda, \lambda' \in \mathrm{Lin}_G(F)$ and consider $\lambda_g^{-1} \circ \lambda'_g : F \xrightarrow{\sim} g^*F \xrightarrow{\sim} F$. As F is simple, we have $\lambda_g^{-1} \circ \lambda'_g = \chi(g) \cdot \mathrm{id}_F$. It follows from the cocycle condition for linearisations that χ is multiplicative, i.e. $\chi \in \hat{G}$. In other words, $\lambda' = \chi \cdot \lambda$ and the action is also transitive. Both properties together can be summed up by saying that the action is simply transitive. This is tantamount to statement (b). ☐

Example 3.10. Note that in (b) simplicity is necessary: We take an abelian surface A with its involution action so that $G = \{\pm \mathrm{id}_A\}$. Fix a geometrical point $a \in A$ that is not 2-torsion. The sheaf $F := k(a) \oplus k(-a)$ is invariant but not simple. Now a linearisation of F is given by a matrix

$$\begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix} : F = k(a) \oplus k(-a) \xrightarrow{\sim} k(-a) \oplus k(a) = (-1)^*F$$

with the condition $\lambda_1\lambda_2 = 1$ stemming from $\mathrm{id} = (-1)^*\lambda \circ \lambda$ (note that $(-1)^*\lambda$ is given by the same matrix as λ). Assume that we have two linearisations λ and μ for F, i.e. $\lambda_1\lambda_2 = 1$ and $\mu_1\mu_2 = 1$. Then, an endomorphism $f : F \to F$ is given by a diagonal matrix and yields an isomorphism between (F, λ) and (F, μ) if and only if the square

$$\begin{array}{ccc} F & \xrightarrow{f} & F \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} \\ (-1)^*F & \xrightarrow{(-1)^*f} & (-1)^*F \end{array} \qquad \text{i.e.} \qquad \begin{array}{ccc} k(a) \oplus k(-a) & \xrightarrow{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}} & k(a) \oplus k(-a) \\ {\scriptstyle\begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix}}\downarrow & & \downarrow{\scriptstyle\begin{pmatrix} 0 & \mu_1 \\ \mu_2 & 0 \end{pmatrix}} \\ k(-a) \oplus k(a) & \xrightarrow{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}} & k(-a) \oplus k(a) \end{array}$$

commutes, which means just $\mu_1\beta = \lambda_2\alpha$ and $\mu_2\alpha = \lambda_1\beta$. However, this shows that by setting $\alpha := \lambda_1$, $\beta := \mu_2$, we get an isomorphism $f : (F, \lambda) \xrightarrow{\sim} (F, \mu)$. Hence, all linearisations of F are actually isomorphic.

3.2 Equivariant Fourier-Mukai functors and derived McKay correspondence

First let us merge the derived with the linearised world. As above, the finite group G acts on the smooth projective variety Y. For the abelian category $\mathrm{Coh}^G(Y)$ of coherent sheaves equipped with G-linearisations, we have

$$\mathrm{Hom}_{\mathrm{Coh}^G(Y)}(\tilde{E}, \tilde{E}') := \mathrm{Hom}_{\mathrm{Coh}(Y)}(E, E')^G, \quad \mathrm{Ext}^i_{\mathrm{Coh}^G(Y)}(\tilde{E}, \tilde{E}') = \mathrm{Ext}^i_{\mathrm{Coh}(Y)}(E, E')^G$$

for $\tilde{E} = (E, \lambda), \tilde{E}' = (E', \lambda') \in \mathrm{Coh}^G(Y)$. The former relation between Hom's is the definition, where the action of G on morphisms is given by

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle \lambda_g^{-1}}\uparrow & & \downarrow{\scriptstyle \lambda'_g} \\ g^*E & \overset{g^*f}{\dashrightarrow} & g^*E' \end{array}$$

The relation between Ext's is then a consequence. Note that the two conceivable notions of equivariant Ext groups coincide by the universal property of derived functors: they are both the G-invariants of the ordinary Ext groups as well as the derived functors of equivariant Hom's.

We introduce $\mathrm{D}^G(Y) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}^G(Y))$; this is the bounded derived category of G-linearised sheaves. Using induction on the length of complexes, the above relation for equivariant Ext groups translates to

$$\mathrm{Hom}^i_{\mathrm{D}^G(Y)}(\tilde{E}^\bullet, \tilde{E}'^\bullet) = \mathrm{Hom}^i_{\mathrm{D}(Y)}(E^\bullet, E'^\bullet)^G$$

for G-linearised complexes $\tilde{E}^\bullet, \tilde{E}'^\bullet \in \mathrm{D}^G(Y)$. Note that all facts about linearisations of sheaves also apply to complexes of sheaves. In particular, Lemma 3.5 holds in $\mathrm{D}^G(Y)$. Hence, simple G-invariant complexes of coherent sheaves can be linearised as well, provided that their obstruction classes in $\mathrm{H}^2(G, \mathbb{C}^*)$ vanish.

It will be useful to look at $\mathrm{D}^G(Y)$ in another way: denote for a moment by $\mathcal{T}$ the category of G-linearised objects of $\mathrm{D}(Y)$, so that the objects of $\mathcal{T}$ are complexes $E^\bullet \in \mathrm{D}(Y)$ together with isomorphisms $\lambda_g : E^\bullet \xrightarrow{\sim} g^*E^\bullet$ in $\mathrm{D}(Y)$ as in Remark 3.2. There is a canonical functor $\mathrm{D}^G(Y) \to \mathcal{T}$ which is fully faithful in view of the above formula. Furthermore, for $(E^\bullet, \lambda) \in \mathcal{T}$ we can choose a bounded injective resolution $I^\bullet \xrightarrow{\sim} E^\bullet$. Then the morphism λ corresponds to a genuine complex map $\lambda_g : g^*I^\bullet \xrightarrow{\sim} I^\bullet$. Hence the functor $\mathrm{D}^G(Y) \to \mathcal{T}$ is essentially surjective and yields an equivalence of categories.

Remark 3.11. Everything we do in this section is tailor-made for the case of finite groups. There is a substantial theory for the general case developed in [7]. Note that for arbitrary groups G, neither the derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}^G(Y))$ of bounded complexes consisting of G-linearised sheaves nor $\mathcal{T}$ is the right category.

Many of the results from Section 1.1 about Fourier-Mukai transforms carry over to this setting. Again, we allude to the calculus of correspondences, the only novelty being the use of equivariant push-forwards. The explicit definition of Fourier-Mukai functors in the linearised setting is spelled out thus: assume we have two smooth projective varieties Y_1 and Y_2 and two finite groups G_1 and G_2 acting on them. Then a $G_1 \times G_2$-linearised complex $P \in \mathrm{D}^{G_1 \times G_2}(Y_1 \times Y_2)$ gives a Fourier-Mukai transform

$$\mathsf{FM}_P : \mathrm{D}^{G_1}(Y_1) \to \mathrm{D}^{G_2}(Y_2), \quad (E, \lambda) \mapsto p_{2*}^{G_1}(P \otimes p_1^* E) = [p_{2*}(P \otimes p_1^* E)]^{G_1}$$

where $p_2 : Y_1 \times Y_2 \to Y_2$ is compatible with the group surjection $G_1 \times G_2 \twoheadrightarrow G_2$ and $p_{2*}^{G_1} : \mathrm{D}^{G_1 \times G_2}(Y_1 \times Y_2) \to \mathrm{D}^{G_2}(Y_2)$ is the equivariant push-forward of Section 3.1.

G-Hilbert schemes and McKay correspondence

Now we turn to the McKay correspondence which, assuming favorable circumstances, describes an equivariant category $\mathrm{D}^G(Y)$ as $\mathrm{D}(\tilde{Y})$ where $\tilde{Y}$ is a special resolution of Y/G. Let Y be a smooth quasi-projective variety with an effective finite group action, i.e. $G \subset \mathrm{Aut}(Y)$. We assume that ω_Y is locally trivial as an object of $\mathrm{Coh}^G(Y)$, i.e. there is a covering of Y by G-invariant open subsets $U \subset Y$ such that $\mathcal{O}_U \cong \omega_Y|_U$ as G-linearised sheaves. Y/G has quotient singularities and there is a resolution in form of G-$\mathrm{Hilb}(Y)$, the Nakamura-Hilbert scheme. Besides the usual definition as the scheme representing a certain functor, one can give the following description of closed points: $\xi \in G$-$\mathrm{Hilb}(Y)$ corresponds to a G-cluster Z_ξ, that is a G-invariant 0-dimensional subscheme $Z_\xi \subset Y$ such that its $\mathbb{C}$-vector space of global sections is isomorphic to the regular representation, i.e. $\mathrm{H}^0(\mathcal{O}_{Z_\xi}) \cong \mathbb{C}[G]$ as G-representations. In particular, $\mathrm{length}(Z_\xi) = \#G$. A typical example of a G-cluster is given by some free G-orbit.

Let $\tilde{Y}$ be the irreducible component of G-$\mathrm{Hilb}(Y)$ which contains the G-clusters of free orbits. There is a morphism G-$\mathrm{Hilb}(Y) \to Y/G$ (usually called Hilbert-Chow map), mapping a G-cluster to its supporting orbit. This morphism is always projective and the irreducible component $\tilde{Y} \subset G$-$\mathrm{Hilb}(Y)$ is mapped birationally onto Y/G. It is not known in general, whether G-$\mathrm{Hilb}(Y)$ is connected, let alone irreducible or smooth. For details see [65] and [47]. There is a universal subscheme $\mathcal{Z} \subset \tilde{Y} \times Y$, and from a geometrical point of view, $\mathcal{Z}$ is just the incidence subvariety. The following theorem is the culmination of a long development; check [13, §1] for the history. It is shown that structure sheaf $\mathcal{O}_\mathcal{Z}$ provides a good Fourier-Mukai kernel. Note that $\mathcal{O}_\mathcal{Z}$ is canonically $1 \times G$-linearised.

Theorem 3.12 (Bridgeland, King, Reid). *Let the finite group G act on Y via $G \hookrightarrow \mathrm{Aut}(Y)$ such that ω_Y is locally trivial in $\mathrm{Coh}^G(Y)$. If $\dim(\tilde{Y} \times_{Y/G} \tilde{Y}) \leq \dim(Y) + 1$, then*

$$\mathsf{FM}_{\mathcal{O}_\mathcal{Z}} : \mathrm{D}(\tilde{Y}) \xrightarrow{\sim} \mathrm{D}^G(Y), \quad E \mapsto p_{Y*}(\mathcal{O}_\mathcal{Z} \otimes p_{\tilde{Y}}^* E)$$
$$\mathsf{FM}_{\mathcal{O}_\mathcal{Z}^L} : \mathrm{D}^G(Y) \xrightarrow{\sim} \mathrm{D}(\tilde{Y}), \quad (F, \lambda) \mapsto [p_{\tilde{Y}*}(\mathcal{O}_\mathcal{Z}^L \otimes p_Y^*(F, \lambda)]^G$$

are mutual quasi-inverse equivalences where $\mathcal{O}_\mathcal{Z}^L = \mathcal{O}_\mathcal{Z}^\vee \otimes p_Y^* \omega_Y[\dim(Y)]$.

Proof. As in Remark 1.4, $\mathcal{O}_{\mathcal{Z}}^{L}$ is the usual description for the Fourier-Mukai kernel of the quasi-inverse of $\mathsf{FM}_{\mathcal{O}_{\mathcal{Z}}}$. In this situation, we can be more specific concerning the derived dual $\mathcal{O}_{\mathcal{Z}}^{\vee}$ by saying $\mathcal{O}_{\mathcal{Z}}^{\vee} = \mathcal{E}xt^{d}(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{\tilde{Y}\times Y})[-d]$ with $d := \dim(Y)$; see [42, Corollary 3.35].

For the proof, see the excellent exposition in [13]. Note that the theorem implies an isomorphism on the level of K-theory, which goes more classically by the name of McKay correspondence. One should at least mention, that the resolution $\tilde{Y} \to Y/G$ is crepant and that $\tilde{Y}$ is smooth if the hypothesis of the above theorem is fulfilled. □

Remark 3.13. In case $\dim(Y) \leq 3$, the dimension condition of the theorem is always fulfilled: e.g. if $\dim(Y) = 3$, then the exceptional divisors are 2-dimensional, and hence give at most 4-dimensional irreducible components of the fibre product. The proof of the theorem actually shows that the G-Hilbert scheme is irreducible if $\dim(Y) \leq 3$.
We will need another case where the dimension condition is automatically satisfied: suppose that Y is a holomorphic symplectic manifold on which G acts by symplectic automorphisms and that $\tilde{Y} \subset G\text{-Hilb}(Y)$ is a crepant resolution of Y/G. Then the resolution is semismall, and we obtain $\mathrm{D}^{G}(Y) \cong \mathrm{D}(\tilde{Y})$; see the references around [13, Corollary 1.3].

Remark 3.14. There is another description of the categories appearing in the theorem. Consider the stack $[Y/G]$ instead of the quotient variety. It is covered by one étale chart, given by the projection $Y \to Y/G$, or more explicitly, by the fibre square (intersection in the étale topology)

$$\begin{array}{ccc} G \times Y & \xrightarrow{p} & Y \\ \downarrow{\scriptstyle \sigma} & & \downarrow \\ Y & \longrightarrow & Y/G \end{array}$$

Now a sheaf on the stack $[Y/G]$ is just a sheaf E on the chart Y with $p^{*}E \cong \sigma^{*}E$, and the descend condition translates into the linearisation property. Henceforth, the abelian categories $\mathrm{Coh}([Y/G])$ and $\mathrm{Coh}^{G}(Y)$ are equivalent, and consequently they give rise to equivalent derived categories. However, the equivalence of the theorem is deeper, as it is only true on the derived level.

It is important to note that there is no completely straightforward generalisation of the cohomological picture from Section 1.2 to the equivariant setting. Let us consider a setting where $\mathrm{D}(\tilde{Y}) \cong \mathrm{D}^{G}(Y)$, as in the above theorem. One would like to compare the following items: invariants of the crepant resolution $\tilde{Y}$, equivariant invariants of Y, and invariants of the orbifold $[Y/G]$. In a topological setting, equivariant K-theory has been studied for a long time (see [78]), and there are also several notions of equivariant cohomology. However, orbifold K-theory is in the focus of interest only since the rise of global quotients in string theory [24]. Physicists also expected that orbifold Hodge numbers should coincide with the Hodge numbers of a crepant resolution. This is indeed the case, see [4]. The question of providing orbifold cohomology with a good multiplication is delicate: Chen and Ruan introduced a product [22] and conjectured that there is a ring isomorphism $\mathrm{H}^*([Y/G]) \cong \mathrm{H}^*(\tilde{Y})$ for a hyperkähler orbifold Y on which G acts by symplectic automorphisms such that the crepant resolution $\tilde{Y}$ is also hyperkähler. This has been confirmed up to minor sign changes for Hilbert schemes [27, §3] and for generalised Kummer varieties [19]. Furthermore, Fantechi and Göttsche point out [27, §2] that the conjecture of Chen and Ruan fails in general with rational coefficients in the orbifold cohomology. It can be seen, however, that one obtains a ring isomorphism using complex coefficients. In Section 4.3, we deal with Kummer surfaces; in this case everything works out nicely.

3.3 Groups of equivariant autoequivalences

Let Y and Y' be smooth projective varieties on which a finite group G acts. Then we have induced actions on $Y \times Y'$ by G^2 and, diagonally, by G. The latter will be called the G_Δ-action of G on $Y \times Y'$.

Take an object $P \in \mathrm{D}^{G_\Delta}(Y \times Y')$ so that P is a complex of sheaves on $Y \times Y'$ equipped with a G_Δ-linearisation $\lambda_g : P \xrightarrow{\sim} (g,g)^* P$. We construct a new object from this via

$$G \cdot P := \mathrm{Inf}_{G_\Delta}^{G^2}(P) = m_* q_*^{G_\Delta} p^* P$$

where $p : G^2 \times Y \times Y' \to Y \times Y'$ and $q : G^2 \times Y \times Y' \to G^2 \times_{G_\Delta} (Y \times Y')$ are the projections and $m : G^2 \times Y \times Y' \to Y \times Y'$ is the multiplication. Note that

$$\begin{aligned} p^* P &= P \boxtimes \mathcal{O}_{G^2} = \bigoplus_{\underline{g} \in G^2} (P, \underline{g}) \\ q_* p^* P &= \bigoplus_{[\underline{g}] \in G^2/G_\Delta} (P^{\oplus \# G}, [\underline{g}]) \\ q_*^{G_\Delta} p^* P &= \bigoplus_{[\underline{g}] \in G^2/G_\Delta} (P, [\underline{g}]) \\ m_* q_*^{G_\Delta} p^* P &= \bigoplus_{[\underline{g}] \in G^2/G_\Delta} [\underline{g}]^* P \end{aligned}$$

and so

$$G \cdot P = \bigoplus_{g \in G} (g, 1)^* P$$

where the last equality relies on the fact that the $(g,1)$'s for $g \in G$ represent the cosets in G^2/G_Δ. Accordingly, we will call $G \cdot P$ the *left inflation* of P. In the explicit shape of the last formula, the G^2-linearisation is given by the isomorphisms for $a, b \in G$

$$\begin{aligned}(a,b)^* G \cdot P &= \bigoplus_{g \in G} (a,b)^*(g,1)^* P \;=\; \bigoplus_{g \in G} (ga, b)^* P \\ &= \bigoplus_{g \in G} (b^{-1}ga, 1)^*(b,b)^* P \;\cong_\lambda\; \bigoplus_{g \in G} (b^{-1}ga, 1)^* P \;\cong_{a,b}\; G \cdot P\end{aligned}$$

where all isomorphisms denoted '$=$' are canonical, the one labelled '$\cong_\lambda$' comes from the given G_Δ-linearisation λ on P, and the one labelled '$\cong_{a,b}$' is the obvious permutation of summands. Hence using $G \cdot P \in \mathrm{D}^{G^2}(Y \times Y')$, we get an equivariant Fourier-Mukai transform $\mathsf{FM}_{G \cdot P} : \mathrm{D}^G(Y) \to \mathrm{D}^G(Y')$. We will also use the notation $\mathsf{FM}^G_P := \mathsf{FM}_{G \cdot P}$.

Example 3.15. The structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset Y \times Y$ has a canonical G_Δ-linearisation. The left inflation of $\mathcal{O}_\Delta$ is $G \cdot \mathcal{O}_\Delta = \bigoplus_{g \in G} \mathcal{O}_{(g,1)\Delta}$, and its G^2-linearisation $\pi_{h,k}$ is given by the permutation of summands via $G \to G$, $g \mapsto kgh^{-1}$.

For an object $(F, \lambda) \in \mathrm{D}^G(Y)$, the pull-back $p_1^* F$ is G^2-linearised by $\lambda_{h,k} = p_1^* \lambda_h : p_1^* F \xrightarrow{\sim} (h,k)^* p_1^* F$. Note that $\lambda_{h,k} = \lambda_{h,1}$ for all $k \in G$. The G^2-linearisation of the tensor product $G \cdot \mathcal{O}_\Delta \otimes p_1^* F = \bigoplus_{g \in G} (1,g)_* F$ is $\mu_{h,k} = \pi_{h,k} \otimes \lambda_{h,k}$; this acts on $h^* F$ via λ_h and permutes summands as $\pi_{h,k}$ in view of $(h,k)^* \bigoplus_{g \in G} (1,g)_* F = \bigoplus_{g \in G} (1, k^{-1}gh)_* h^* F$.

Next consider the push-forward: $p_{2*}(G \cdot \mathcal{O}_\Delta \otimes p_1^* F) = \bigoplus_{g \in G} g_* F$. It is G^2-linearised via $p_{2*}\mu_{h,k}$; the matrices for the push-forward are the *same* as for μ. The difference is just that $G \times 1$ now acts trivially.

Taking $G \times 1$-invariants: consider $\bigoplus_{g \in G} g_* F$ with its $G \times 1$-linearisation given by $p_{2*}\mu_{h,1}$; this acts by λ_h and permuting summands according to $g \mapsto gh^{-1}$. The invariant part can be written as $F \xrightarrow{\sim} [\bigoplus_{g \in G} g_* F]^{G \times 1}$, the map being the direct sum of the isomorphisms $\lambda_{g^{-1}} : F \xrightarrow{\sim} g_* F$. In words, this amounts to a kind of diagonal inside $\bigoplus_{g \in G} g_* F$. The important thing is that $[\bigoplus_{g \in G} g_* F]^{G \times 1}$ inherits a G-linearisation from the $1 \times G$-part of $p_{2*}\mu$. We see that the latter uses *only* permutations of summands. However, due to the form of $[\bigoplus_{g \in G} g_* F]^{G \times 1}$ just described, we see that this means exactly equipping F with the linearisation λ. To conclude, the Fourier-Mukai kernel $G \cdot \mathcal{O}_\Delta$ transforms the object $(F, \lambda) \in \mathrm{D}^G(Y)$ into itself. In other words, $\mathsf{FM}^G_{\mathcal{O}_\Delta} = \mathrm{id} : \mathrm{D}^G(Y) \to \mathrm{D}^G(Y)$.

Remark 3.16. G_Δ-linearised kernels are well behaved with respect to compositions. To substantiate this claim, we will associate to two objects $(P, \lambda) \in \mathrm{D}^{G_\Delta}(Y \times Y')$ and $(Q, \mu) \in \mathrm{D}^{G_\Delta}(Y' \times Y'')$ a new one $(R, \nu) \in \mathrm{D}^{G_\Delta}(Y \times Y'')$ as follows: first put $R := Q \star P$, so that $\mathsf{FM}_R = \mathsf{FM}_Q \circ \mathsf{FM}_P$ as functors $\mathrm{D}(Y) \to \mathrm{D}(Y'')$. Composing the isomorphisms $\lambda_{(g,g)} : P \xrightarrow{\sim} (g,g)^* P$ and $\mu_{(g,g)} : Q \xrightarrow{\sim} (g,g)^* Q$ in the sense of correspondences, we get

$$\nu_{(g,g)} := \mu_{(g,g)} \star \lambda_{(g,g)} : Q \star P \xrightarrow{\sim} (g,g)^* Q \star (g,g)^* P = (g,g)^*(Q \star P),$$

the latter isomorphism $(g,g)^* Q \star (g,g)^* P \xrightarrow{\sim} (g,g)^*(Q \star P)$ being completely canonical. The required compatibilities follow at once from the cocycle conditions of λ and μ.

Lemma 3.17. *For $P \in \mathrm{D}^{G_\Delta}(Y \times Y')$, a G_Δ-linearised complex on $Y \times Y'$, we have*

(i) For any $F \in \mathrm{D}^G(Y)$, the two objects $for(\mathsf{FM}^G_P(F)) \in \mathrm{D}(Y')$ (i.e. disregarding the linearisation) and $\mathsf{FM}_P(F)$ coincide. More precisely, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{D}^G(Y) & \xrightarrow{\mathsf{FM}^G_P} & \mathrm{D}^G(Y') \\ \big\downarrow{\scriptstyle for} & & \big\downarrow{\scriptstyle for} \\ \mathrm{D}(Y) & \xrightarrow{\mathsf{FM}_P} & \mathrm{D}(Y') \end{array}$$

(ii) The left inflation is compatible with composition, i.e. $\mathsf{FM}^G_Q \circ \mathsf{FM}^G_P = \mathsf{FM}^G_{Q\star P}$ for kernels $P \in \mathrm{D}^{G_\Delta}(Y \times Y')$ and $Q \in \mathrm{D}^{G_\Delta}(Y' \times Y'')$ where $Q \star P$ is G_Δ-linearised as above.

(iii) $\mathsf{FM}_P : \mathrm{D}(Y) \to \mathrm{D}(Y')$ fully faithful $\implies$ $\mathsf{FM}^G_P : \mathrm{D}^G(Y) \to \mathrm{D}^G(Y')$ fully faithful.

(iv) $\mathsf{FM}_P : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Y')$ equivalence $\implies$ $\mathsf{FM}^G_P : \mathrm{D}^G(Y) \xrightarrow{\sim} \mathrm{D}^G(Y')$ equivalence.

Proof. For (i) note that Example 3.15 demonstrates the claim for $P = \mathcal{O}_\Delta$. Now take any object $F \in \mathrm{D}^G(Y)$ and denote its G-linearisation by λ. Then, we have from the definition of equivariant Fourier-Mukai transforms

$$\mathsf{FM}_{G\cdot P} = \mathsf{FM}^G_P : \mathrm{D}^G(Y) \to \mathrm{D}^G(Y), \quad F \mapsto [p_{2*}(G \cdot P \otimes p_1^* F)]^{G\times 1}.$$

The $G \times 1$-linearisation of $G \cdot P$ is given by permutations (the G_Δ-linearisation of P does not enter). Since $p_{2*}((g,1)^* P \otimes p_1^* F) = p_{2*}(g,1)^*(P \otimes p_1^* g^{-1^*} F) = p_{2*}(P \otimes p_1^* F)$, we see that (exactly as in the case $P = \mathcal{O}_\Delta$ studied above) $p_{2*}(G \cdot P \otimes p_1^* F) \cong \bigoplus_G p_{2*}(P \otimes p_1^* F)$ and $G \times 1$ acts with permutation matrices where the 1's are replaced by $p_1^*\lambda_g$'s. Again taking $G \times 1$-invariants singles out a subobject of this sum isomorphic to one summand.

A morphism $f : F_1 \to F_2$ in $\mathrm{D}^G(Y)$ is analogously first taken to a G-fold direct sum. The final taking of $G \times 1$-invariants then leaves one copy of $\mathsf{FM}_P(f)$.

(ii) The composite $\mathsf{FM}_{G\cdot Q} \circ \mathsf{FM}_{G\cdot P}$ has the kernel

$$\begin{aligned}(G\cdot Q) \star (G \cdot P) &= [p_{13*}(p_{12}^*(G\cdot P) \otimes p_{23}^*(G \cdot Q))]^{1\times G\times 1} \\ &= [p_{13*}(p_{12}^* \bigoplus_{g\in G}(g,1)^* P \otimes p_{23}^* \bigoplus_{h\in G}(h,1)^* Q)]^{1\times G\times 1} \\ &\cong [p_{13*}(p_{12}^* \bigoplus_{g\in G}(g,1)^* P \otimes p_{23}^* \bigoplus_{h\in G}(1,h^{-1})^* Q)]^{1\times G\times 1} \\ &= [\bigoplus_{g,h\in G} p_{13*}(g,1,h^{-1})^*(p_{12}^* P \otimes p_{23}^* Q)]^{1\times G\times 1} \\ &= [\bigoplus_{g,h\in G} (g,h^{-1})^* p_{13*}(p_{12}^* P \otimes p_{23}^* Q)]^{1\times G\times 1}.\end{aligned}$$

Now note that $(1,c,1) \in 1 \times G \times 1$ acts on $(G\cdot Q) \star (G \cdot P)$ via permutations (inverse multiplications from left) and λ on P, and $(1,c,1)$ acts purely by permutations (which are multiplications from right) on Q. Plugging this into the last equation, we find that after taking invariants we end up with $\bigoplus_{d\in G}(d^{-1},d)^* p_{13*}(p_{12}^* P \otimes p_{23}^* Q)$. Since the (d^{-1},d)'s give all classes in G^2/G_Δ, we find that $(G\cdot Q)\star(G\cdot P)\cong G\cdot(Q\star P)$.

(iii) Fix two objects $(A_1,\lambda_1),(A_2,\lambda_2)\in \mathrm{D}^G(Y)$ and recall that $\mathit{for}:\mathrm{D}^G(Y)\to\mathrm{D}(Y)$ is the functor forgetting linearisations, so that $\mathit{for}(A_1),\mathit{for}(A_2)\in\mathrm{D}(Y)$ are the underlying complexes. By definition of morphisms in $\mathrm{D}^G(Y)$, we have a diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{D}(Y)}(\mathit{for}(A_1),\mathit{for}(A_2)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{D}(Y')}(\mathsf{FM}_P(\mathit{for}(A_1)),\mathsf{FM}_P(\mathit{for}(A_2))) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathrm{D}(Y)}(\mathit{for}(A_1),\mathit{for}(A_2))^G & & \mathrm{Hom}_{\mathrm{D}(Y')}(\mathit{for}(\mathsf{FM}_{G\cdot P}(A_1)),\mathit{for}(\mathsf{FM}_{G\cdot P}(A_2)))^G \\
\| & & \| \\
\mathrm{Hom}_{\mathrm{D}^G(Y)}(A_1,A_2) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}^G(Y')}(\mathsf{FM}_{G\cdot P}(A_1),\mathsf{FM}_{G\cdot P}(A_2))
\end{array}$$

and, by assumption, the topmost morphism is a bijection. Thus the lower map of the diagram is injective, and hence $\mathsf{FM}_{G\cdot P}$ is faithful. To see that $\mathsf{FM}_{G\cdot P}$ is full as well, consider the following variation of the former diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{D}(Y)}(\mathit{for}(A_1),\mathit{for}(A_2)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{D}(Y')}(\mathsf{FM}_P(\mathit{for}(A_1)),\mathsf{FM}_P(\mathit{for}(A_2))) \\
\downarrow \vartheta & & \downarrow \vartheta \\
\mathrm{Hom}_{\mathrm{D}(Y)}(\mathit{for}(A_1),\mathit{for}(A_2))^G & & \mathrm{Hom}_{\mathrm{D}(Y')}(\mathit{for}(\mathsf{FM}_{G\cdot P}(A_1)),\mathit{for}(\mathsf{FM}_{G\cdot P}(A_2)))^G \\
\| & & \| \\
\mathrm{Hom}_{\mathrm{D}^G(Y)}(A_1,A_2) & \longrightarrow & \mathrm{Hom}_{\mathrm{D}^G(Y')}(\mathsf{FM}_{G\cdot P}(A_1),\mathsf{FM}_{G\cdot P}(A_2))
\end{array}$$

using the averaging morphism (which is often called Reynolds operator)

$$\begin{aligned}
\vartheta : \mathrm{Hom}_{\mathrm{D}(Y)}(\mathit{for}(A_1),\mathit{for}(A_2)) \to \mathrm{Hom}_{\mathrm{D}(Y)}(\mathit{for}(A_1),\mathit{for}(A_2)),& \\
f \mapsto \frac{1}{\#G}\sum_{g\in G}(\lambda_{2,g})^{-1}\circ g^* f\circ\lambda_{1,g}.&
\end{aligned}$$

Using the defining property $\lambda_{gh}=h^*\lambda_g\circ\lambda_h$ of linearisations repeatedly, we find

$$\begin{aligned}
h^*\vartheta(f)\circ\lambda_{1,h} &= \frac{1}{\#G}\sum_{g\in G} h^*(\lambda_{2,g})^{-1}\circ h^*g^*f\circ h^*\lambda_{1,g}\circ\lambda_{1,h} \\
&= \frac{1}{\#G}\sum_{l\in G} h^*(\lambda_{2,lh^{-1}})^{-1}\circ l^*f\circ h^*\lambda_{1,l} \qquad (l:=gh) \\
&= \frac{1}{\#G}\sum_{l\in G} h^*(\lambda_{2,h^{-1}})^{-1}\circ\lambda_{1,l}^{-1}\circ l^*f\circ\lambda_{1,l} \\
&= \lambda_{2,h}\circ\vartheta(f)
\end{aligned}$$

which shows that $\vartheta(f)$ is indeed G-invariant. We obviously have $\vartheta(f) = f$ if and only if f is a G-invariant morphism. In particular, ϑ is surjective. The assumption that the functor FM_P is full then implies the same property for $\mathsf{FM}_{G \cdot P}$.

(iv) follows from (ii) and Example 3.15: let $\mathsf{FM}_P : \mathrm{D^b}(Y) \xrightarrow{\sim} \mathrm{D^b}(Y')$ be an equivalence. Then, $Q = \mathcal{H}om(P, \mathcal{O}_{Y\times Y'}) \otimes p_Y^* \omega_Y[\dim(Y)]$ is the Fourier-Mukai kernel of a quasi-inverse for FM_P. Using the canonical G-linearisation for ω_Y (see Example 3.3), Q inherits a G_Δ-linearisation such that $(G \cdot Q) \star (G \cdot P) \cong (\mathcal{O}_{\Delta_Y}, \mathsf{can})$ and $(G \cdot P) \star (G \cdot Q) \cong (\mathcal{O}_{\Delta_{Y'}}, \mathsf{can})$. Hence, $G \cdot P$ is an equivalence kernel as was P. $\square$

Let us introduce some groups of autoequivalences; the last one by means of kernels:

$$\begin{aligned}
\mathrm{Aut}(\mathrm{D}(Y)) &= \{\mathsf{FM}_P : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Y)\} \\
\mathrm{Aut}(\mathrm{D}(Y))^{G_\Delta} &:= \{\mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D}(Y)) \ : \ P \text{ is } G_\Delta\text{-invariant}\} \\
\mathrm{Aut}(\mathrm{D}^G(Y)) &= \{\mathsf{FM}_{\tilde{P}} : \mathrm{D}^G(Y) \xrightarrow{\sim} \mathrm{D}^G(Y) \ : \ \tilde{P} \in \mathrm{D}^{G^2}(Y \times Y)\} \\
\mathrm{Aut}^{G_\Delta}(\mathrm{D}(Y)) &:= \{(P, \lambda) \in \mathrm{D}^{G_\Delta}(Y \times Y) \ : \ \mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D}(Y))\}.
\end{aligned}$$

$\mathrm{Aut}(\mathrm{D}(Y))$ and $\mathrm{Aut}(\mathrm{D}^G(Y))$ are the usual groups of exact autoequivalences of the triangulated categories $\mathrm{D}(Y)$ and $\mathrm{D}^G(Y)$. The description of $\mathrm{Aut}(\mathrm{D}(Y))$ given above uses Orlov's Theorem 1.7 that all equivalences have Fourier-Mukai kernels. In [50], Kawamata proves a similar theorem for smooth stacks associated to normal projective varieties with quotient singularities. Since $[Y/G]$ is such a stack, and due to $\mathrm{Coh}([Y/G]) \cong \mathrm{Coh}^G(Y)$, the derived category $\mathrm{D}^G(Y)$ possesses likewise Fourier-Mukai kernels for all equivalences. Using $\mathsf{FM}_{(g,h)^*P} = g^* \circ \mathsf{FM}_P \circ (h^{-1})^*$, we see from

$$\begin{aligned}
\mathrm{Aut}(\mathrm{D}(Y))^{G_\Delta} &= \{\mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D}(Y)) \ : \ P \text{ is } G_\Delta\text{-invariant}\} \\
&= \{\mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D}(Y)) \ : \ (g,g)^*P \cong P \ \forall g \in G\} \\
&= \{\mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D}(Y)) \ : \ g^* \circ \mathsf{FM}_P = \mathsf{FM}_P \circ g^* \ \forall g \in G\}
\end{aligned}$$

that $\mathrm{Aut}(\mathrm{D}(Y))^{G_\Delta}$ is the centraliser of the subset $\{g^* \ : \ g \in G\}$ inside $\mathrm{Aut}(\mathrm{D}(Y))$, hence a subgroup. Finally, we have already endowed $\mathrm{Aut}^{G_\Delta}(\mathrm{D}(Y))$ with a product in Remark 3.16 above. Example 3.15 shows that $(\mathcal{O}_\Delta, \mathsf{can})$ is a neutral element. Inverses exist in $\mathrm{Aut}^{G_\Delta}(\mathrm{D}(Y))$ because of the arguments in the proof of part (iv) above. Some relations between these groups follow:

Proposition 3.18. *Let the finite group G act on the smooth projective variety Y.*

(i) The construction of inflation gives a group homomorphism inf which fits in the following exact sequence, where $Z(G) \subset G$ is the centre of G:

$$\begin{array}{c}
0 \longrightarrow Z(G) \longrightarrow \mathrm{Aut}^{G_\Delta}(\mathrm{D}(Y)) \xrightarrow{\ inf\ } \mathrm{Aut}(\mathrm{D}^G(Y)) \\
\qquad\qquad\qquad\qquad (P, \rho) \longmapsto \mathsf{FM}^G_{(P,\rho)}
\end{array}$$

(ii) *Forgetting the G_Δ-linearisation gives a group homomorphism for which sits in the following exact sequence; here $G_{\mathrm{ab}} := G/[G,G]$ is the abelianised group:*

$$\begin{array}{ccccccccc} 0 \longrightarrow G_{\mathrm{ab}} \longrightarrow & \mathrm{Aut}^{G_\Delta}(\mathrm{D}(Y)) & \xrightarrow{for} & \mathrm{Aut}(\mathrm{D}(Y))^{G_\Delta} & \longrightarrow & \mathrm{H}^2(G,\mathbb{C}^*) \\ & (P,\rho) & \longmapsto & \mathsf{FM}_P & \longmapsto & [P] \end{array}$$

Remark 3.19. Compare two ways to associate an abelian group to the finite group G. Firstly, the centre $Z(G)$ is a commutative subgroup of G. Secondly, there is the quotient $G_{\mathrm{ab}} = G/[G,G]$. If G is abelian, then we have $G = Z(G) = G_{\mathrm{ab}}$, of course. In general, there is no relation between $Z(G)$ and G_{ab}. Note that $\hat{G} = \mathrm{Hom}(G,\mathbb{C}^*) = \mathrm{Hom}(G_{\mathrm{ab}},\mathbb{C}^*) \cong G_{\mathrm{ab}}$ where the last isomorphism is not canonical. Also note that $\mathrm{Hom}(G,\mathbb{C}^*) = \mathrm{H}^1(G,\mathbb{C}^*)$ as G acts trivially on $\mathbb{C}^*$.

Remark 3.20. The image of *inf* is rather big as the map has finite kernel. On the other hand, there is no reason to expect it to be surjective. See the next section on Hilbert schemes or Section 4.3 on Kummer surfaces for specific examples.

Question 3.21. Is the morphism $\mathrm{Aut}(\mathrm{D}(Y))^{G_\Delta} \to \mathrm{H}^2(G,\mathbb{C}^*)$ surjective in general?

Proof. (i) It follows from Lemma 3.17 and Example 3.15 that *inf* is a group homomorphism. The kernel $\ker(\mathit{inf})$ consists of $(P,\lambda) \in \mathrm{D}^{G_\Delta}(Y^2))$ giving equivalences such that $G \cdot P \cong G \cdot \mathcal{O}_\Delta$. Obviously, this forces P to be a sheaf of type $P \cong (g,1)^*\mathcal{O}_\Delta$ for some $g \in G$. Now $(g,1)^*\mathcal{O}_\Delta$ is G_Δ-invariant if and only if $g \in Z(G)$ as $(h,h)^*(g,1)^*\mathcal{O}_\Delta = (gh,h)^*\mathcal{O}_\Delta \cong (gh,h)^*(h^{-1},h^{-1})^*\mathcal{O}_\Delta = (h^{-1}gh,1)^*\mathcal{O}_\Delta$. This in turn implies that the isomorphism $(g,1) : Y^2 \xrightarrow{\sim} Y^2$ is a G_Δ-map. In particular, $P \cong (g,1)^*\mathcal{O}_\Delta$ gets the pulled back G_Δ-linearisation. Giving $\mathcal{O}_\Delta$ a G_Δ-linearisation $\lambda \in \hat{G}$ different from the unit character yields $G \cdot (\mathcal{O}_\Delta,\lambda) \ncong G \cdot (\mathcal{O}_\Delta,\mathrm{id})$; this follows for example from the uniqueness of Fourier-Mukai kernels. Both facts together imply $\ker(\mathit{inf}) \cong Z(G)$.

(ii) It is obvious from the definition of $\mathrm{Aut}^{G_\Delta}(\mathrm{D}(Y))$ that *for* is a group homomorphism. The kernel of *for* corresponds to the G_Δ-linearisations on $\mathcal{O}_\Delta$. From Lemma 3.5, we see that they form a group isomorphic to $\hat{G}$. Given $\mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D}(Y))^{G_\Delta}$, we know from Lemma 1.12 that its Fourier-Mukai kernel P is simple. Furthermore, it is G_Δ-invariant by assumption, so that the map $\mathsf{FM}_P \mapsto [P]$ is defined as in Lemma 3.5. To see that it is a group homomorphism, take two G_Δ-invariant kernels $P, Q \in \mathrm{D}(Y^2)$. Choose isomorphisms $\lambda_g : P \xrightarrow{\sim} (g,g)^*P$ and $\mu_g : Q \xrightarrow{\sim} (g,g)^*Q$ for all $g \in G$. Then we have $\lambda_{gh} = (h,h)^*\lambda_g \circ \lambda_h \cdot [P]_{g,h}$ and likewise for Q, μ. Furthermore, the composition of λ_g and μ_g gives an isomorphism $\mu_g \star \lambda_g : Q \star P \xrightarrow{\sim} ((g,g)^*Q) \star ((g,g)^*P)$ and the latter term is canonically isomorphic to $(g,g)^*(Q \star P)$. Then $\mu_{g,h} \star \lambda_{gh} = (h,h)^*(\mu_g \star \lambda_g) \circ (\mu_h \star \lambda_h) \cdot [Q \star P]_{g,h}$. The formula $(B \circ A) \star (D \circ C) \cong (B \star D) \circ (A \star C)$ for morphisms $P'' \xrightarrow{B} P' \xrightarrow{A} P$ and $Q'' \xrightarrow{D} Q' \xrightarrow{C} Q$ implies $[Q \star P]_{g,h} = [Q]_{g,h} \cdot [P]_{g,h}$. Now it is obvious that $[\cdot] \circ \mathit{for} = 0$ and finally $\mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))^G$ with $[P] = 0$ implies that P is G_Δ-linearisable by Lemma 3.5. $\square$

3.4 An application: Hilbert schemes of points

Let Y be a variety with trivial canonical bundle and consider its n-fold product Y^n. We consider the obvious S_n-action on Y^n via permutation of factors. If $P \in \mathrm{D}(Y \times Y)$ is the kernel of a Fourier-Mukai equivalence, then $P^{\boxtimes n} = p_1^*P \otimes \cdots \otimes p_n^*P \in \mathrm{D}((Y^2)^n)$ gives another equivalence $\mathsf{FM}_{P^{\boxtimes n}} : \mathrm{D}(Y^n) \to \mathrm{D}(Y^n)$. By permuting tensor factors, we obtain a $(S_n)_\Delta$-linearisation on $P^{\boxtimes n}$. We get a group homomorphism

$$\mathrm{Aut}(\mathrm{D}(Y)) \to \mathrm{Aut}^{(S_n)_\Delta}(\mathrm{D}(Y^n)) \to \mathrm{Aut}(\mathrm{D}^{S_n}(Y^n));$$

the first map sends an equivalence $\mathsf{FM}_P : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Y)$ to $P^{\boxtimes n} \in \mathrm{D}^{(S_n)_\Delta}(Y^n \times Y^n)$ with the canonical linearisation from permuting factors; the second map is *inf*.

In some cases, the right hand side can be described by derived McKay correspondence. For example, if Y is a surface, then its Hilbert scheme of n points $H_n := \mathrm{Hilb}_n(Y)$ is a crepant resolution of $S^nY = Y^n/S_n$. Furthermore, there is the following well-known result from Haiman [34, §5.1]:

Theorem 3.22 (Haiman). *For a smooth projective surface Y, there is an isomorphism*

$$\mathrm{Hilb}_n(Y) \cong S_n\text{-Hilb}(Y^n)$$

of the Hilbert scheme of n points on Y with the Nakamura-Hilbert scheme of S_n-clusters in Y^n.

In order to invoke the derived McKay correspondence, we assume that $\omega_Y \cong \mathcal{O}_Y$ is trivial and make use of the symplectic setting: Y^n is a holomorphic symplectic manifold with symplectic form $p_1^*\omega + \cdots + p_n^*\omega \in \mathrm{H}^0(\Omega^2_{Y^n})$ where $0 \neq \omega \in \mathrm{H}^0(\Omega^2_Y)$ and $p_i : Y^n \to Y$ are the projections. Obviously the permutation automorphisms leave this form fixed, so that we have a symplectic S_n-action on Y^n. Furthermore, the quotient $S^nY = Y^n/S_n$ has trivial canonical bundle, and hence $\mathrm{Hilb}_n(Y) \to S^nY$ is a crepant resolution. Summing up, we are now in a position to cite Remark 3.13, in order to obtain an equivalence $\mathrm{D}(\mathrm{Hilb}_n(Y)) \cong \mathrm{D}^{S_n}(Y^n)$. The resulting map

$$\mathrm{Aut}(\mathrm{D}(Y)) \to \mathrm{Aut}(\mathrm{D}(\mathrm{Hilb}_n(Y)))$$

is injective as the centre of S_n is trivial for $n > 2$, and because for $n = 2$ the sheaf $\mathcal{O}_{\Delta\times\Delta}$ with the non-canonical $(S_2)_\Delta$-linearisation is not in the image of $\mathrm{Aut}(\mathrm{D}(Y)) \to \mathrm{Aut}^{(S_2)_\Delta}(\mathrm{D}(Y^2))$.

Now we specialise to the case of K3 surfaces. See Section 2.1 for basic facts about those surfaces and their derived categories.

Proposition 3.23. *Let X_1 and X_2 be two projective K3 surfaces, and let $H_1 := \mathrm{Hilb}_n(X_1)$ and $H_2 := \mathrm{Hilb}_n(X_2)$ be their Hilbert schemes of n points. If H_1 and H_2 are birationally equivalent, then the derived categories are equivalent:* $\mathrm{D}(H_1) \cong \mathrm{D}(H_2)$.

Proof. The birational isomorphism $f : H_1 \dashrightarrow H_2$ induces a map $f^* : \mathrm{H}^2(H_1) \to \mathrm{H}^2(H_2)$, because the locus of indeterminacy has codimension ≥ 2. Actually, f^* is an isomorphism respecting the Hodge structures since H_1 and H_2 are symplectic [33]. From the crepant resolution $H_1 \to X_1^n/S_n$, we find $\mathrm{H}^2(X_1) \subset \mathrm{H}^2(H_1)$, and only the exceptional divisor $E_1 \subset H_1$ is missing: $\mathrm{H}^2(H_1) = \mathrm{H}^2(X_1) \oplus \mathbb{Z} \cdot \delta_1$ with $2\delta_1 = E_1$. In particular, as $[E_1]$ is obviously an algebraic class, the transcendental sublattices coincide: $T(X_1) = T(H_1)$. Hence, the birational isomorphism furnishes an isometry $T(X_1) \cong T(X_2)$. Orlov's derived Torelli Theorem 2.3 in turn implies $\mathrm{D}(X_1) \cong \mathrm{D}(X_2)$. But now we are in a position to apply the above result on lifting equivalences from K3 surfaces to Hilbert schemes. In explicit terms, a Fourier-Mukai equivalence $\mathsf{FM}_P : \mathrm{D}(X_1) \xrightarrow{\sim} \mathrm{D}(X_2)$ gives rise to the canonically $(S_n)_\Delta$-linearised kernel $P^{\boxtimes n} := P \boxtimes \cdots \boxtimes P \in \mathrm{D}((X_1 \times X_2)^n)$. Using Lemma 3.17, one sees $\mathsf{FM}^{S_n}_{P^{\boxtimes n}} : \mathrm{D}^{S_n}(X_1^n) \xrightarrow{\sim} \mathrm{D}^{S_n}(X_2^n)$. Utilising Haiman's result and derived McKay correspondence as above, we find $\mathrm{D}(H_1) \cong \mathrm{D}(H_2)$, as claimed. ☐

Remark 3.24. A K3 surface has only finitely many Fourier-Mukai partners [14, Proposition 5.3], hence there are also only finitely many birationally equivalent Hilbert schemes of n points. Note how this remark and the proposition support the conjectures (C) and (DK) stated in Section 1.4.

Example 3.25. We give a very concrete construction for a birational map of Hilbert schemes. Let X be a smooth hypersurface of degree 4 in $\mathbb{P}^3$, so that X is a K3 surface. For two points $x, y \in X$, we denote by $l(x, y)$ the line in $\mathbb{P}^3$ through x and y. As X is a quartic, the intersection $l(x, y) \cap X$ contains two further points x' and y', at least generically. This procedure yields a rational map

$$f : \mathrm{Hilb}_2(X) \dashrightarrow \mathrm{Hilb}_2(X), \quad z \mapsto z' \text{ such that } l(z) \cap X = z \cup z'$$

where we extend the above description in the obvious way to tangent vectors. The exceptional locus consists of those 2-cycles z such that the total multiplicity of z at X is more than 2. For sufficiently general hypersurfaces X, this locus is actually empty; in this case f is a morphism of varieties. Note that $f \circ f = \mathrm{id}$, as rational maps. In particular, if f is defined everywhere, it is an involution.

It would be very nice though, to have a concrete example of non-isomorphic birational Hilbert schemes.

Example 3.26. However, one should not expect a homomorphism $\mathrm{Bir}(\mathrm{Hilb}_n(X)) \to \mathrm{Aut}(\mathrm{D}(\mathrm{Hilb}_n(X)))$: the passage from an isometry of the transcendental K3 lattice to an autoequivalence of $\mathrm{D}(X)$ is not canonical.

Remark 3.27. Markman [54] gives an example of non-birational Hilbert schemes H_1 and H_2 such that $\mathrm{H}^2(H_1) \cong \mathrm{H}^2(H_2)$. This shows that the first step of the above proof is not reversible. As we still find $\mathrm{D}(H_1) \cong \mathrm{D}(H_2)$ using the further reasoning from the above proof, this produces another example for non-birational D-equivalent varieties.

4 Autoequivalences for abelian varieties and Kummer surfaces

4.1 Derived categories of abelian varieties: results of Mukai and Orlov

Let A be an abelian variety of dimension g. We introduce the ubiquitous morphisms:

$$\begin{array}{rlr} m : A \times A \to A, & (a_1, a_2) \mapsto a_1 + a_2 & \text{group law} \\ \mu : A \times A \to A \times A, & (a_1, a_2) \mapsto (a_1 + a_2, a_2) & \\ -1 : A \to A, & a \mapsto -a & \text{group inversion} \\ t_a : A \to A, & x \mapsto a + x & \text{translation for fixed } a \in A \end{array}$$

Topologically A is a $2g$-torus, hence we have $\dim \mathrm{H}^i(A, \mathbb{Z}) = \binom{2g}{i}$. Furthermore the cotangent bundle is free, and in particular $\mathrm{td}_A = 1$. We denote the dual abelian variety by $\hat{A} := \mathrm{Pic}^0(A)$. It consists of all line bundles of trivial first Chern class. These fulfill furthermore $(-1)^*\alpha \cong \alpha^\vee$ and $t_a^*\alpha \cong \alpha$ for all $a \in A$ and $\alpha \in \hat{A}$. The latter condition can also be taken as an algebraic definition of $\hat{A} \subset \mathrm{Pic}(A)$. Any line bundle $L \in \mathrm{Pic}(A)$ gives rise to a morphism

$$\varphi_L : A \to \hat{A}, \quad a \mapsto t_a^* L \otimes L^\vee.$$

The pair (A, L) is called principally polarised if $\varphi_L : A \xrightarrow{\sim} \hat{A}$ is an isomorphism. This is the case if and only if L is ample and $\dim \mathrm{H}^0(L) = 1$ is minimal. The other extreme is $\varphi_L = 0$ which is the case precisely if $L \in \hat{A}$. See [63] for details.

The Poincaré bundle is a distinguished line bundle $\mathcal{P} \in \mathrm{Pic}(A \times \hat{A})$. It is uniquely characterised by the two properties $\mathcal{P}|_{A \times \{\alpha\}} \cong \alpha$ for all $\alpha \in \hat{A}$ and $\mathcal{P}|_{\{0\} \times \hat{A}} \cong \mathcal{O}_{\hat{A}}$. Mukai used in [61] $\mathcal{P}$ as a kernel in both directions, i.e. $\mathsf{FM}_\mathcal{P} : \mathrm{D}(\hat{A}) \to \mathrm{D}(A)$ and $\mathsf{FM}^t_\mathcal{P} : \mathrm{D}(A) \to \mathrm{D}(\hat{A})$. Note that the universal property of $\mathcal{P}$ can be restated by $\mathsf{FM}_\mathcal{P}(k(\alpha)) \cong \alpha$ for all $\alpha \in \hat{A}$ and $\mathsf{FM}^t_\mathcal{P}(k(0)) = \mathcal{O}_{\hat{A}}$.

Theorem 4.1 (Mukai). *For all $a \in A$ and $\alpha \in \hat{A}$*

(i) $\mathsf{FM}^t_\mathcal{P} \circ \mathsf{FM}_\mathcal{P} \cong (-1_{\hat{A}})^*[-g]$, $\qquad \mathsf{FM}_\mathcal{P} \circ \mathsf{FM}^t_\mathcal{P} \cong (-1_A)^*[-g]$,

(ii) $\mathsf{M}_{\alpha^\vee} \circ \mathsf{FM}_\mathcal{P} \cong \mathsf{FM}_\mathcal{P} \circ t_\alpha^*$, $\qquad \mathsf{FM}_\mathcal{P} \circ \mathsf{M}_{\mathcal{P}_a} \cong t_a^* \circ \mathsf{FM}_\mathcal{P}$.

Proof. For the crucial formula (i) see [61, Theorem 2.2]. Formula (ii) appears together with many more relations in [61, §3]. In (ii), we denote $\mathcal{P}_a := \mathcal{P}_{\{a\} \times \hat{A}} \in \mathrm{Pic}^0(\hat{A})$. ☐

In particular, $\mathsf{FM}_\mathcal{P} : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(\hat{A})$ is an equivalence. And in case (A, L) is principally polarised, we get an autoequivalence $\mathsf{FM}_\mathcal{P} \in \mathrm{Aut}(\mathrm{D}(A))$. Under this condition, Mukai established the relation $(\mathsf{M}_L \circ \mathsf{FM}_\mathcal{P})^3 = [-g]$. Together with $\mathsf{FM}^4_\mathcal{P} = [-2g]$ from the theorem, this induces an action $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathrm{D}(A))/\mathbb{Z} \cdot [1]$ via $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \mapsto \mathsf{M}_L$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \mapsto \mathsf{FM}_\mathcal{P}$.

A big achievement towards the understanding of $\mathrm{Aut}(\mathrm{D}(A))$ has been made by Orlov in [72]. For example, he shows the following very remarkable property of Fourier-Mukai kernels on abelian varieties:

Theorem 4.2 (Orlov). *Let A and B be two abelian varieties. If $\mathsf{FM}_P : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(B)$ is a Fourier-Mukai equivalence with kernel P, then $P \in \mathrm{D}(A \times B)$ is isomorphic to a shifted sheaf, i.e. $P \cong \mathrm{h}^i(P)[-i]$ for an $i \in \mathbb{Z}$.*

Proof. Confer [72, Proposition 2.9]. Among manifolds with $K = 0$, this feature seems to be peculiar for abelian varieties. However, it also holds for varieties of Fano type or with trivial canonical class. Iterating the spherical reflection functors of Section 1.3 produces Fourier-Mukai kernels which necessarily must have homology at several places (in dimension at least 2 and if orthogonal objects exist): $\mathsf{T}_E^5(E) = E[5-5d]$ and $\mathsf{T}_E^5(C) = C$ if $\mathrm{Hom}^*(E, C) = 0$; but no shifted sheaf on $Y \times Y$ can have homology in degrees $5d - 5 > 2d$ apart. Note how the absence of spherical objects for abelian varieties of Lemma 1.32 ties in with the general statement of the theorem. □

To describe Orlov's picture of $\mathrm{Aut}(\mathrm{D}(A))$, we have to introduce the following group:

$$\mathrm{Sp}(A \times \hat{A}) := \left\{ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in \mathrm{Aut}(A \times \hat{A}) \ : \ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \begin{pmatrix} \hat{f}_4 & -\hat{f}_2 \\ -\hat{f}_3 & \hat{f}_1 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_{\hat{A}} \end{pmatrix} \right\}$$

which is the group of symplectic automorphisms of $A \times \hat{A}$ with respect to the natural symplectic form on $A \times \hat{A}$. It was also studied by Mukai [62].

Example 4.3. Consider the case of a principally polarised abelian variety (A, L). Then, there is a faithful representation $\mathrm{SL}(2, \mathbb{Z}) \hookrightarrow \mathrm{Sp}(A \times \hat{A})$ using the obvious morphisms $\mathrm{id}_A : A \xrightarrow{\sim} A$, $\varphi_L : A \xrightarrow{\sim} \hat{A}$, $\hat{\varphi}_L : \hat{A} \xrightarrow{\sim} A$ and $\mathrm{id}_{\hat{A}} : \hat{A} \xrightarrow{\sim} \hat{A}$. Additionally, let us assume that A is an abelian variety with $\mathrm{End}(A) = \mathbb{Z}$. Then, the just listed four isomorphisms are unique up to integer scalars, so $f_1 = n_1 \mathrm{id}_A$, $f_4 = n_4 \mathrm{id}_{\hat{A}}$, $f_3 = n_3 \varphi_L$ and $f_2 = n_2 \hat{\varphi}_L$ for some $n_1, n_2, n_3, n_4 \in \mathbb{Z}$. The relation of $\mathrm{Sp}(A \times \hat{A})$ translates to $n_1 n_4 - n_2 n_3 = 1$. Hence $\mathrm{Sp}(A \times \hat{A}) \cong \mathrm{SL}(2, \mathbb{Z})$ in this situation.

Theorem 4.4 (Orlov). *For any abelian variety A, there is an exact sequence of groups*

$$0 \longrightarrow \mathbb{Z} \times A \times \hat{A} \xrightarrow{\ \eta\ } \mathrm{Aut}(\mathrm{D}(A)) \xrightarrow{\ \gamma\ } \mathrm{Sp}(A \times \hat{A}) \longrightarrow 0.$$

Proof. See [72, Theorem 3.7]. The first morphism sends $\eta(n, a, \alpha) := t_a^* \circ \mathsf{M}_\alpha[n]$, using that shifts, translations and twists by line bundles of degree 0 commute. The morphism γ is constructed below. □

We proceed by explaining how one can associate a symplectic automorphism $\gamma(F)$ to an autoequivalence $F : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(A)$. The description below differs slightly from the one in [72] in that we suppress mentioning kernels as far as possible. Orlov's procedure actually works for a pair of abelian varieties. It pays anyway to distinguish between source and target in the technical description below. Accordingly, we will temporarily work with two D-equivalent abelian varieties A and B. An equivalence $F : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(B)$ induces $F^{-1} : \mathrm{D}(B) \xrightarrow{\sim} \mathrm{D}(A)$ and $(F^{-1})^t : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(B)$. Recall that the latter has the same Fourier-Mukai kernel as F^{-1} and is just considered as a transform in the other direction. Next, we get an equivalence $F \times (F^{-1})^t : \mathrm{D}(A \times A) \xrightarrow{\sim} \mathrm{D}(B \times B)$ with the crucial property

$$\mathsf{FM}_{(F\times(F^{-1})^t)(P)} = (F^{-1})^t \circ \mathsf{FM}_P \circ F^t : \mathrm{D}(B) \to \mathrm{D}(B)$$

for any kernel $P \in \mathrm{D}(A \times A)$, as explained in Example 1.6(4). Orlov associates to F the new transform $F^{\mathrm{sp}} : \mathrm{D}(A \times \hat{A}) \times \mathrm{D}(B \times \hat{B})$ via the following diagram

$$\begin{array}{ccc}
\mathrm{D}(A \times \hat{A}) & \xrightarrow{\quad F^{\mathrm{sp}} \quad} & \mathrm{D}(B \times \hat{B}) \\
{\scriptstyle \mathrm{id}_A \times \mathsf{FM}_{\mathcal{P}_A}} \downarrow & & \uparrow {\scriptstyle \mathrm{id}_B \times \mathsf{FM}^{-1}_{\mathcal{P}_B}} \\
\mathrm{D}(A \times A) & & \mathrm{D}(B \times B) \\
{\scriptstyle \mu_{A*}} \downarrow & & \uparrow {\scriptstyle \mu^{-1}_{B*}} \\
\mathrm{D}(A \times A) & \xrightarrow[F\times(F^{-1})^t]{} & \mathrm{D}(B \times B)
\end{array}$$

Note that in the case $A = B$ the functor F^{sp} is obtained from $F \times (F^{-1})^t$ by twofold conjugation. The deep fact [72, Proposition 2.3] is that F^{sp} takes skyscraper sheaves to skyscrapers and is by Remark 1.9 of the form $F^{\mathrm{sp}} = \mathsf{M}_{N(F)} \circ \gamma(F)^*$, up to shift; here $\gamma(F) : A \times \hat{A} \xrightarrow{\sim} B \times \hat{B}$ is an isomorphism and $N(F) \in \mathrm{Pic}(B \times \hat{B})$ a line bundle. The isomorphism $\gamma(F)$ is symplectic by [72, Proposition 2.6].

An important feature of this construction is that γ is compatible with composition [72, Proposition 2.4]. So for equivalences $F_1 : \mathrm{D}(A_1) \xrightarrow{\sim} \mathrm{D}(A_2)$ and $F_2 : \mathrm{D}(A_2) \xrightarrow{\sim} \mathrm{D}(A_3)$ we have $\gamma(F_2) \circ \gamma(F_1) = \gamma(F_2 \circ F_1)$ as elements of $\mathrm{Sp}(A_1 \times \hat{A}_1, A_3 \times \hat{A}_3)$. In particular for $A = B$, this gives the homomorphism $\gamma : \mathrm{Aut}(\mathrm{D}(A)) \to \mathrm{Sp}(A \times \hat{A})$ using that $\gamma(\mathrm{id}_{\mathrm{D}(A)}) = \mathrm{id}_{A\times\hat{A}}$, which is a special case of the first example below.

Examples 4.5. We will compute the functor $F^{\mathrm{sp}} : \mathrm{D}(A \times \hat{A}) \xrightarrow{\sim} \mathrm{D}(B \times \hat{B})$ for a couple of Fourier-Mukai equivalences $F : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(B)$. In the case of an autoequivalence $F \in \mathrm{Aut}(\mathrm{D}(A))$, this also yields the map $\gamma(F) : A \times \hat{A} \xrightarrow{\sim} A \times \hat{A}$. In every example we will explicitly calculate $F^{\mathrm{sp}}(k(a, \alpha))$ for a pair $(a, \alpha) \in A \times \hat{A}$. As can be seen from the diagram defining F^{sp}, the first two steps are always the same:

$$(\mathrm{id} \times \mathsf{FM}_{\mathcal{P}})(k(a, \alpha)) = k(a) \boxtimes \mathcal{P}_\alpha = k(a) \boxtimes \alpha$$
$$\mu_*(k(a) \boxtimes \alpha) = (\mu^{-1})^*(p_1^* k(a) \otimes p_2^* \alpha) = p_2^* \alpha|_{(p_1\mu^{-1})^{-1}(a)} = j_* t_{-a}^* \alpha = j_* \alpha$$

where $j : A \hookrightarrow A^2$, $x \mapsto (x, x-a)$; this uses $p_2\mu^{-1} = p_2$ and that the preimage $(p_1\mu^{-1})^{-1}(a) = \{(x, x-a) : x \in A\}$ is both the graph of t_{-a} and the image of j. Note that $\mathsf{FM}_{j_*\alpha} = t_a^* \circ \mathsf{M}_\alpha = \mathsf{M}_\alpha \circ t_a^*$. The next step in the construction of F^{sp} is given by $(F \times (F^{-1})^t)(j_*\alpha)$. As recalled above, calculating this kernel amounts to understanding the map $(F^{-1})^t \circ \mathsf{M}_\alpha t_a^* \circ F^t$.

(1) Let us examine the line bundle twist $F := \mathsf{M}_L \in \mathrm{Aut}(D(A))$ for $L \in \mathrm{Pic}(A)$. Its Fourier-Mukai kernel $\Delta_* L$ is symmetric, so that $\mathsf{M}_L^t = \mathsf{M}_L$. Accordingly, we find $\mathsf{FM}_{(\mathsf{M}_L \times \mathsf{M}_L^{-1})(j_*\alpha)} = \mathsf{M}_L^{-1} \circ \mathsf{M}_\alpha \circ t_a^* \circ \mathsf{M}_L = \mathsf{M}_{\varphi_L(a)\otimes\alpha} \circ t_a^*$. Going the first two steps backwards, this in turn implies $(\mathsf{M}_L)^{\mathrm{sp}}(k(a,\alpha)) = k(a, \varphi_L(a) + \alpha)$, so that

$$\gamma(\mathsf{M}_L) = \begin{pmatrix} \mathrm{id}_A & 0 \\ \varphi_L & \mathrm{id}_{\hat{A}} \end{pmatrix}.$$

Note that $\gamma(\mathsf{M}_L) = 1$ in $\mathrm{Sp}(A \times \hat{A})$ if and only if $L \in \hat{A}$, in complete accordance with Orlov's sequence from Theorem 4.4.

(2) A translation $t_b : A \xrightarrow{\sim} A$ gives the autoequivalence $F := t_{b*} = t_{-b}^* \in \mathrm{Aut}(\mathrm{D}(A))$. Then we have $F^t = t_b^*$ and thus $\mathsf{FM}_{(t_{-b}^* \times t_{-b}^*)(j_*\alpha)} = t_{-b}^* \circ \mathsf{M}_\alpha \circ t_a^* \circ t_b^* = \mathsf{M}_\alpha \circ t_a^*$. So we get $(t_{b*})^{\mathrm{sp}}(k(a,\alpha)) = k(a,\alpha)$ and $\gamma(t_{b*}) = 1$ in $\mathrm{Sp}(A \times \hat{A})$.

(3) Let $f : A \to A$ be an automorphism of abelian varieties, i.e. f is a group homomorphism as well as an isomorphism of schemes, and consider the equivalence $F := f^*$. In analogy with the preceding example, we have $(f^*)^t = f_*$ and this gives here $\mathsf{FM}_{(f^* \times f^*)(j_*\alpha)} = f^* \circ \mathsf{M}_\alpha \circ t_a^* \circ (f^{-1})^* = \mathsf{M}_{f^*(\alpha)} \circ t_{f^{-1}(a)}^*$. Accordingly, we derive

$$\gamma(f_*) = \begin{pmatrix} f^{-1} & 0 \\ 0 & \hat{f} \end{pmatrix}.$$

(4) Lastly, we turn to the Poincaré bundle $\mathcal{P} \in \mathrm{Pic}(A \times \hat{A})$. For notational sanity's sake, let us consider the transpose, i.e. $F := \mathsf{FM}_{\mathcal{P}}^t : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(\hat{A})$. Then we find $\mathsf{FM}_{(F \times (F^{-1})^t)(j_*\alpha)} = \mathsf{FM}_{\mathcal{P}}^{-1} \circ \mathsf{M}_\alpha \circ t_a^* \circ \mathsf{FM}_{\mathcal{P}} = t_{\alpha^\vee}^* \circ \mathsf{FM}_{\mathcal{P}}^{-1} \circ \mathsf{FM}_{\mathcal{P}} \circ \mathsf{M}_{\mathcal{P}_a} = t_{-\alpha}^* \circ \mathsf{M}_{\mathcal{P}_a}$, using Theorem 4.1(ii). This shows

$$\gamma(\mathsf{FM}_{\mathcal{P}}^t) = \begin{pmatrix} 0 & -\mathrm{id}_{\hat{A}} \\ \mathrm{id}_A & 0 \end{pmatrix}, \qquad \gamma(\mathsf{FM}_{\mathcal{P}}) = \begin{pmatrix} 0 & \mathrm{id}_{\hat{A}} \\ -\mathrm{id}_A & 0 \end{pmatrix}.$$

Remark 4.6. Assume that (A, L) is a principally polarised abelian variety with minimal endomorphisms: $\mathrm{End}(A) = \mathbb{Z}$. Then, by Example 4.3, we have $\mathrm{Sp}(A \times \hat{A}) \cong \mathrm{SL}(2, \mathbb{Z})$. We compare the two group homomorphisms from Mukai and Orlov, respectively:

$$\begin{array}{lll} \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Aut}(\mathrm{D}(A))/\mathbb{Z}\cdot[1] & \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \mapsto \mathsf{M}_L, & \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \mapsto \mathsf{FM}_{\mathcal{P}} \\ \mathrm{Aut}(\mathrm{D}(A)) \to \mathrm{SL}(2,\mathbb{Z}) & \mathsf{M}_L \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), & \mathsf{FM}_{\mathcal{P}} \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right). \end{array}$$

The second line uses Examples 4.5(1), (4). Note that this gives, up to shifts, a splitting of the exact sequence from Theorem 4.4 in this situation. Actually more is true: said sequence really splits whenever A is principally polarised [75].

4.2 Equivariant equivalences for the involution action

An abelian variety has a natural involution action from the group law, so that the group is $G = \{\mathrm{id}_A, -\mathrm{id}_A\}$. Ultimately, we want to apply Proposition 3.18 from the last chapter in this situation. The statement itself gives two group homomorphisms

$$\begin{aligned} \mathit{inf} &: \mathrm{Aut}^{G_\Delta}(\mathrm{D}(A)) \to \mathrm{Aut}(\mathrm{D}^G(A)) \\ \mathit{for} &: \mathrm{Aut}^{G_\Delta}(\mathrm{D}(A)) \twoheadrightarrow \mathrm{Aut}(\mathrm{D}(A))^{G_\Delta} \end{aligned}$$

both of which have kernel isomorphic to G.

The first question is which autoequivalences of $\mathrm{D}(A)$ are G-linearisable. According to Lemmas 1.12 and 3.5 and aware of $\mathrm{H}^2(G, \mathbb{C}^*) = 0$, we have to look for G_Δ-invariant Fourier-Mukai kernels. These are just equivalence kernels $\mathcal{E} \in \mathrm{D}(A \times A)$ such that $(-1,-1)^*\mathcal{E} \cong \mathcal{E}$, or in other terms, such that $(-1)^* \circ \mathsf{FM}_\mathcal{E} \circ (-1)^* = \mathsf{FM}_\mathcal{E}$. Recall that $\mathrm{Aut}(\mathrm{D}(A))^{G_\Delta} \subset \mathrm{Aut}(\mathrm{D}(A))$ is the subgroup of G-invariant autoequivalences.

Examples 4.7.

(1) Consider the Poincaré bundle $\mathcal{P} \in \mathrm{D}(A \times \hat{A})$. We have by its universal property $(-1,-1)^*\mathcal{P}|_{A\times\{\alpha\}} = (-1)^*\alpha^\vee = \alpha = \mathcal{P}|_{A\times\{\alpha\}}$ and furthermore $\mathcal{P}$ is normalised by $(-1,-1)^*\mathcal{P}|_{\{0\}\times\hat{A}} = (-1,-1)^*\mathcal{O}_{\hat{A}} = \mathcal{O}_{\hat{A}} = \mathcal{P}|_{\{0\}\times\hat{A}}$. The seesaw theorem yields $(-1,-1)^*\mathcal{P} \cong \mathcal{P}$ and if A is principally polarised, then $\mathsf{FM}_\mathcal{P} \in \mathrm{Aut}(\mathrm{D}(A))^{G_\Delta}$.

(2) If we have a product decomposition $A = A_1 \times A_2$ with A_1 principally polarised, then $\mathsf{FM}_{p_1^*\mathcal{P}_1} = \mathsf{FM}_{\mathcal{P}_1} \times \mathrm{id}_{A_2} \in \mathrm{Aut}(\mathrm{D}(A))^{G_\Delta}$.

(3) The twist M_L for a line bundle $L \in \mathrm{Pic}(A)$ is G-invariant if and only if $(-1)^*L \cong L$. This is the case for the $(2g)^2$ different 2-torsion line bundles of degree 0 which correspond to $\hat{A}[2] \subset \mathrm{Pic}(A)$. However, there are many more G-invariant line bundles on A, for example $L \otimes (-1)^*L$ for any $L \in \mathrm{Pic}(A)$. Thus, we find $\hat{A}[2] \subset \mathrm{Aut}(\mathrm{D}(A))^{G_\Delta}$ and $\mathrm{Pic}(A) \to \mathrm{Aut}(\mathrm{D}(A))^{G_\Delta}$, $L \mapsto L \otimes (-1)^*L$. The latter map has kernel $\hat{A} \subset \mathrm{Pic}(A)$, so that we get a faithful representation of the Neron-Severi group $\mathrm{NS}(A) = \mathrm{Pic}(A)/\hat{A} \hookrightarrow \mathrm{Aut}(\mathrm{D}(A))^{G_\Delta}$.

(4) We can also look for G-invariant morphisms of A. Note that a translation t_a is G-invariant if and only if a is 2-torsion, i.e. $2a = 0$. Furthermore, every isomorphism of A as an abelian variety, i.e. an element of $\mathrm{Aut}_{\mathrm{AV}}(A) := \{f : A \xrightarrow{\sim} A : f(0) = 0\}$, is G-invariant. Using the obvious action of $\mathrm{Aut}_{\mathrm{AV}}(A)$ on $A[2]$, we find a semidirect product subgroup $A[2] \rtimes \mathrm{Aut}_{\mathrm{AV}}(A) \hookrightarrow \mathrm{Aut}(\mathrm{D}(A))^{G_\Delta}$.

Returning to general equivalences of $\mathrm{D}(A)$, we use Orlov's exact sequence of Theorem 4.4 to produce a G-invariant equivalent.

Proposition 4.8. *If A is an abelian variety and $G = \{\pm\,\mathrm{id}_A\}$, then there is an exact sequence*

$$0 \longrightarrow \mathbb{Z} \times A[2] \times \hat{A}[2] \xrightarrow{\eta} \mathrm{Aut}(\mathrm{D}(A))^{G_\Delta} \xrightarrow{\gamma} \mathrm{Sp}(A \times \hat{A}) \longrightarrow 0$$

where $A[2] \subset A$ and $\hat{A}[2] \subset \hat{A}$ are the subgroups consisting of all 2-torsion points.

Proof. We start with $0 \to \mathbb{Z} \times A \times \hat{A} \to \operatorname{Aut}(\mathrm{D}(A)) \to \operatorname{Sp}(A \times \hat{A}) \to 0$ which is the exact sequence from Theorem 4.4. Note that G acts in a compatible way on each term: via conjugation by $(-1)^*$ on $\operatorname{Aut}(\mathrm{D}(A))$, hence via conjugation by $\gamma((-1)^*) = -\operatorname{id}_{A \times \hat{A}}$ on $\operatorname{Sp}(A \times \hat{A})$ which is the trivial action, and via $\operatorname{id}_{\mathbb{Z}} \times (-\operatorname{id}_A) \times (-\operatorname{id}_{\hat{A}})$ on $\mathbb{Z} \times A \times \hat{A}$. The last action follows from $(-1)^* \circ t_{a*} \circ \mathsf{M}_\alpha \circ (-1)^* = t_{(-a)*} \circ \mathsf{M}_{\alpha^\vee}$. Taking invariants with respect to this G-action we get the exact sequence from the proposition, possibly without the righthand 0. To check surjectivity at the righthand side, we have to calculate $\mathrm{H}^1(G, \mathbb{Z} \times A \times \hat{A})$. First we get $\mathrm{H}^1(G, \mathbb{Z}) = \operatorname{Hom}(G, \mathbb{Z}) = 0$ because $\mathbb{Z}$ has the trivial G-action. Using the description of group cohomology given in Section 3.1 we find next that $Z^1(G, A) \xrightarrow{\sim} A$, $f \mapsto f(-1)$. On the other hand, under this isomorphism the 1-boundaries correspond to the subgroup $-2A$. Now A is divisible and hence $\mathrm{H}^1(G, A) \cong A/(-2)A = 0$. Performing a similar argument with $\hat{A}$ in place of A, we deduce $\mathrm{H}^1(G, \mathbb{Z} \times A \times \hat{A}) = 0$, so that the sequence of G-invariants above is indeed exact. ☐

Example 4.9. In the case where A is principally polarised we can interpret this new sequence in the following manner: after Polishchuk [75] the exact sequence of Theorem 4.4 splits. Hence every autoequivalence $F \in \operatorname{Aut}(\mathrm{D}(A))$ determines and is determined by an isomorphism $\gamma(F) : A \times \hat{A} \xrightarrow{\sim} A \times \hat{A}$ and a line bundle $N(F) \in \operatorname{Pic}(A \times \hat{A})$ and, strictly speaking, an integer shift. This data $(\gamma(F), N(F))$ gives rise to an element of $\operatorname{Aut}(\mathrm{D}(A))^{G_\Delta}$ if and only if the line bundle $N(F)$ is G-invariant.

Example 4.10. Fix an arbitrary line bundle $L \in \operatorname{Pic}(A)$. In general we will have to face $(-1)^*L \ncong L$ so that M_L is not G-invariant. However, the line bundle $L^{-1} \otimes (-1)^*L$ always has degree 0. This is obvious over $\mathbb{C}$ and follows from the theorem of the square in general [63, §8]. As $\hat{A}$ is divisible, we can choose a square root α of $L^{-1} \otimes (-1)^*L$, so that $(-1)^*(L \otimes \alpha) \cong L \otimes \alpha$. This implies $(-1)^* \circ \mathsf{M}_{L\otimes\alpha} = \mathsf{M}_{L\otimes\alpha} \circ (-1)^*$ and hence $\mathsf{M}_{L\otimes\alpha} \in \operatorname{Aut}(\mathrm{D}(A))^{G_\Delta}$ is a G-invariant preimage of $\gamma(\mathsf{M}_L)$.

Remark 4.11. Generalising this example, for any autoequivalence $F : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(A)$ the image $\gamma(F) \in \operatorname{Sp}(A \times \hat{A})$ has a G-invariant preimage $F' \in \operatorname{Aut}(\mathrm{D}(A))^{G_\Delta}$. Hence F and F' differ by an equivalence of the form $\eta(n, a, \alpha)$ for some $(n, a, \alpha) \in \mathbb{Z} \times A \times \hat{A}$ so that $F = F' \circ \eta(n, a, \alpha)$. In other words, $F \circ \eta(0, -a, \alpha^\vee)$ is G-invariant.

Remark 4.12. Consider now two D-equivalent abelian varieties A and B from the outset. As explained after Theorem 4.4, one can to an equivalence $F : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(B)$ still associate a symplectic isomorphism $\gamma(F) : A \times \hat{A} \xrightarrow{\sim} B \times \hat{B}$. However, by leaving the realm of automorphism groups, the methods used above break down. In particular, it is not clear, whether there is a G-invariant equivalence $\mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(B)$. See the last Section 4.4 for a concrete instance of this question.

4.3 The Kummer case

Here we deal with an abelian surface A and its corresponding Kummer surface X. By the derived McKay correspondence as stated in Theorem 3.12, the universal subscheme $\mathcal{Z} \subset X \times A$ gives an equivalence

$$\Xi := \mathsf{FM}_{\mathcal{O}_{\mathcal{Z}}} : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}^G(A).$$

It induces an isomorphism of groups of autoequivalences via

$$\xi : \mathrm{Aut}(\mathrm{D}^G(A)) \xrightarrow{\sim} \mathrm{Aut}(\mathrm{D}(X)), \quad F \mapsto \xi(F) := \Xi^{-1} \circ F \circ \Xi.$$

Examples 4.13. We continue by giving a couple of instances of this.

(1) It is a trivial observation that ξ maps a shift $[n]$ to itself. Note that the left inflations of $\mathcal{O}_\Delta$ and $\mathcal{O}_{(1,-1)\Delta}$ coincide, where $(1,-1)\Delta := \{(a,-a) : a \in A\}$ is the graph of the involution. Hence, they give rise to the same autoequivalence of $\mathrm{D}^G(A)$. This is due to $\mathit{inf} : \mathrm{Aut}^{G_\Delta}(\mathrm{D}(A)) \to \mathrm{Aut}(\mathrm{D}^G(A))$ being $2:1$ by Proposition 3.18.

(2) Consider the translation $t_a : A \xrightarrow{\sim} A$ with a 2-torsion point $a \in A[2]$. Then t_a is G_Δ-invariant as $(-1) \circ t_a = t_a \circ (-1)$. Hence, as in the last example, we obtain an autoequivalence $\mathit{inf}(t_{a*}) \in \mathrm{Aut}(\mathrm{D}^G(A))$. The image $\xi(\mathit{inf}(t_{a*}))$ on the Kummer side is actually given by a surface automorphism, as well. To see this, recall the resolution $\pi : X \to A/G$ whose exceptional divisor $E \subset X$ consists of 16 disjoint (-2)-curves, corresponding to the 16 points in A of order 2. Now for $b \notin A[2]$, the free G-orbit of b is the cycle $k(b) \oplus k(-b)$. It has a unique G-linearisation by Example 3.10 and $\Xi^{-1}(k(b) \oplus k(-b)) = k(x)$ where $x \in X$ is unique with $\pi(x) = b$. This implies that $\xi(t_{a*})$ maps $k(x) \mapsto k(x')$ with $\pi(x') = a + b$. Altogether, we find $\xi(t_{a*}) \in \mathrm{Aut}(X)$ is an automorphism of the Kummer surface, which permutes the 16 rational curves pointwise.

(3) Analogously, an isomorphism $f : A \xrightarrow{\sim} A$ as abelian variety is G-invariant and permutes the 2-torsion points. Hence, it also descends to an automorphism of the singular Kummer surface A/G. Like in the previous item, $\xi(f_*) \in \mathrm{Aut}(X)$ is a surface automorphism, another time fixing the exceptional (-2)-curves.

(4) A 2-torsion line bundle $L \in \hat{A}[2]$ is G-invariant and simple. Hence, it can be linearised in two ways by Lemma 3.5. Choosing one, we find that $(L, \lambda) \in \mathrm{D}^G(A)$ gives actions $\lambda_a : L(a) \xrightarrow{\sim} (-1)^* L(a) = L(a)$ by $+1$ or -1. Defining $\rho(a) \in \mathbb{F}_2 = \{0,1\}$ via $\lambda(a) = (-1)^{\rho(a)}$, we see that each of these 32 G-linearised line bundles corresponds to a map $\rho : A[2] \to \mathbb{F}_2$. Actually, exactly the 32 affine functions $A[2] \to \mathbb{F}_2$ appear. We have $\Xi^{-1}(L,\lambda) = \mathcal{O}_X(\frac{1}{2}\sum_{a \in A[2]} \rho(a) C_a)$ by [13, §10.2]. In particular, Ξ maps $(\mathcal{O}_A, 1) \mapsto \mathcal{O}_X$ and $(\mathcal{O}_A, -1) \mapsto \mathcal{O}_X(\frac{1}{2}\sum_i C_i)$. Note that a linear combination $\sum_a \rho(a) C_a$ is divisible by 2 in $\mathrm{H}^2(X, \mathbb{Z})$ if and only if $\rho : A[2] \to \mathbb{F}_2$ is affine linear [3, §VIII.5]. Because the Fourier-Mukai kernel of M_L is supported on the diagonal, we deduce $\xi(\mathsf{M}_L) = \mathsf{M}_{\Xi(L)}$. This autoequivalence again permutes the exceptional curves on X.

Up to now we have seen some examples in the following automorphism groups:

$$\begin{array}{ccc} & \mathrm{Aut}^{G_\Delta}(\mathrm{D}(A)) & \\ {\scriptstyle for}\swarrow & & \searrow{\scriptstyle inf} \\ \mathrm{Aut}(\mathrm{D}(A))^{G_\Delta} & & \mathrm{Aut}(\mathrm{D}^G(A)) \end{array}$$

Via $2:1$ homomorphisms, the roof $\mathrm{Aut}^{G_\Delta}(\mathrm{D}(A))$ connects the two bottom groups in which lies are our primary interest: the exact sequence of Proposition 4.8 should give a grip on $\mathrm{Aut}(\mathrm{D}(A))^{G_\Delta}$ and we have $\mathrm{Aut}(\mathrm{D}^G(A)) \cong \mathrm{Aut}(\mathrm{D}(X))$, as described above. There does not seem to be an obvious homomorphism between the two bottom groups. Nevertheless, we can at least see part of the picture using cohomology. In order to do this, we have to introduce the orbifold cohomology $\mathrm{H}^*([A/G])$ of $[A/G]$ with rational coefficients. For the general definition refer to [27] or [22]. In our setting, everything boils down to

$$\mathrm{H}^*([A/G]) = \mathrm{H}^*(A)^G \oplus \mathrm{H}^*(A^G) = \mathrm{H}^{2*}(A) \oplus \mathbb{Q}^{A[2]}.$$

Here, G acts on $\mathrm{H}^1(A)$ simply by -1, hence on $\mathrm{H}^i(A)$ by $(-1)^i$. Thus, precisely the even cohomology is G-invariant. The summand $\mathrm{H}^*(A^G)$ refers to the cohomology of the G-invariant subvariety of A. Since $A^G = A[2]$ are just the 16 points of order 2, we see $\mathrm{H}^*(A^G) = \mathrm{H}^0(A[2]) = \mathbb{Q}^{A[2]}$. The grading scheme of orbifold cohomology puts, in this situation, $\mathrm{H}^*(A^G)$ in degree 2. Finally, we get $\mathrm{H}^2([A/G]) = \mathrm{H}^2(A) \oplus \mathbb{Q}^{A[2]}$ and $\mathrm{H}^0([A/G]) = \mathrm{H}^0(A) \cong \mathbb{Q}$, $\mathrm{H}^4([A/G]) = \mathrm{H}^4(A) \cong \mathbb{Q}$. Note that $\mathrm{H}^*(X) \cong \mathrm{H}^*([A/G])$ as $\mathbb{Q}$-vector spaces. The orbifold cohomology $\mathrm{H}^*([A/G])$ inherits a pure Hodge structure of weight 2 from $\mathrm{H}^{2*}(A)$: more explicitly, $\mathrm{H}^{2,0}([A/G]) := \mathrm{H}^{2,0}(A)$. There is a product on $\mathrm{H}^*([A/G])$ which agrees with the usual multiplication in $\mathrm{H}^*(X)$ up to sign; it is even possible to get this with integer coefficients [27, §2]. However, we will skip these issues and just work with rational coefficients. For our purposes it is enough to define $\mathrm{H}^2(A)$ and $\mathbb{Q}^{A[2]}$ to be orthogonal and to declare the product on $\mathbb{Q}^{A[2]}$ as follows: $A[2]$ is an orthogonal basis with $a \cdot a := -[pt]$ for $a \in A[2]$. Thus, we have $\mathrm{H}^*(X) \cong \mathrm{H}^*([A/G])$ including Hodge structure and pairing (but excluding lattice).

There is a natural equivariant Chern character $\mathrm{ch}^G : \mathrm{D}^G(A) \to \mathrm{H}^*([A/G])$ defined as follows; see [2] for the general case. Consider a G-linearised vector bundle (F, λ) and note that such bundles generate the K-group $\mathrm{K}^G(A)$. Then for all 2-torsion points $a \in A[2]$, we have actions $\lambda_a : F(a) \xrightarrow{\sim} (-1)^* F(a) = F(a)$ on the fibre $F(a)$. The cocycle condition for linearisations implies $\lambda_a^2 = \mathrm{id}$, so that only eigenvalues 1 and -1 appear. Let $\mathrm{tr}(F,\lambda)_a := \mathrm{tr}(\lambda_a) \in \mathbb{Z}$ be the trace of λ_a; and let $\mathrm{tr}(F,\lambda) \in \mathbb{Z}^{A[2]}$ be the vector consisting of the traces for all 2-torsion points. Eventually, the equivariant Chern character is given by

$$\mathrm{ch}^G : \mathrm{D}^G(A) \to \mathrm{H}^*([A/G]) = \mathrm{H}^{2*}(A) \oplus \mathbb{Q}^{A[2]}, \quad (F,\lambda) \mapsto (\mathrm{ch}(F), \mathrm{tr}(F,\lambda)),$$

where the first component $\mathrm{ch}(F) \in \mathrm{H}^{2*}(A)$ is the usual Chern character. A kernel $(P, \mu) \in \mathrm{D}^{G^2}(\mathrm{D}(A \times A)$ yields again a map $\mathsf{FM}^{\mathrm{H}}_{(P,\mu)} : \mathrm{H}^*([A/G]) \to \mathrm{H}^*([A/G])$. Just note that $\mathsf{FM}^{\mathrm{H}}_{(P,\mu)}$ acts on transcendental cycles in the old manner. Algebraic cycles (i.e. elements of $\mathrm{H}^0(A) \oplus \mathrm{NS}(A)_{\mathbb{Q}} \oplus \mathbb{Q}^{A[2]} \oplus \mathrm{H}^4(A))$ can be represented as a linear combination of (Chern characters of) G-linearised sheaves, to which we apply $\mathrm{ch}^G \circ \mathsf{FM}_{(P,\mu)}$. Altogether, this produces a homomorphism $(\cdot)^{\mathrm{H}} : \mathrm{Aut}(\mathrm{D}^G(A)) \to \mathrm{Aut}(\mathrm{H}^*([A/G])$. We use the decomposition $\mathrm{H}^2([A/G]) = \mathrm{H}^2(A) \oplus \mathbb{Q}^{A[2]}$ to define the subgroup of $\mathrm{Aut}(\mathrm{H}^*([A/G]))$ respecting summands:

$$\mathrm{Aut}(\mathrm{H}^*_{\mathrm{ex}}([A/G])) := \{\varphi \in \mathrm{Aut}(\mathrm{H}^*([A/G])) \,:\, \varphi(\mathrm{H}^*(A)) = \mathrm{H}^*(A),\ \varphi(\mathbb{Q}^{A[2]}) = \mathbb{Q}^{A[2]}\}.$$

Elements of $\mathrm{Aut}(\mathrm{H}^*_{\mathrm{ex}}([A/G]))$ correspond to Hodge isometries of the Kummer surface X permuting the 16 exceptional curves.

Proposition 4.14. *Let A be a principally polarised abelian variety. Then, the image of the composition $(\cdot)^{\mathrm{H}} \circ \mathit{inf} : \mathrm{Aut}^{G_\Delta}(\mathrm{D}(A)) \to \mathrm{Aut}(\mathrm{D}^G(A)) \to \mathrm{Aut}\,\mathrm{H}^*([A/G])$ is contained in the group $\mathrm{Aut}(\mathrm{H}^*_{\mathrm{ex}}([A/G]))$ and the following diagram commutes:*

$$\begin{array}{ccc} & & \mathrm{Aut}(\mathrm{H}^{2*}(A)) \\ & \overset{(\cdot)^{\mathrm{H}}\circ \mathit{for}}{\nearrow} & \uparrow \scriptstyle{\mathrm{res}} \\ \mathrm{Aut}^{G_\Delta}(\mathrm{D}(A)) & & \\ & \underset{(\cdot)^{\mathrm{H}}\circ \mathit{inf}}{\searrow} & \\ & & \mathrm{Aut}(\mathrm{H}^{2*}_{\mathrm{ex}}([A/G])) \end{array}$$

where $\mathrm{res} : \mathrm{Aut}(\mathrm{H}^{2*}_{\mathrm{ex}}([A/G]) \to \mathrm{Aut}(\mathrm{H}^*(A))$ *is given by restriction.*

Proof. Note how the Examples 4.7 satisfy the claim. Now take a G_Δ-linearised kernel $P \in \mathrm{Aut}^{G_\Delta}(\mathrm{D}(A))$ and consider its left inflation $G \cdot P = P \oplus (1,-1)^*P$. Since $\mathsf{FM}^{\mathrm{H}}_{(1,-1)^*P} = ((-1)^* \circ \mathsf{FM}_P)^{\mathrm{H}} = \mathsf{FM}^{\mathrm{H}}_P$, we find $\mathsf{FM}^{\mathrm{H}}_{G\cdot P} = \mathsf{FM}^{\mathrm{H}}_P$ on $\mathrm{H}^{2*}(A)$. (Note that the final taking of G-invariants reduces the two identical summands in $\mathsf{FM}^{\mathrm{H}}_{G\cdot P}(\cdot)$ to just one.) Next consider an element $(0, 1_a) \in \mathrm{H}^{2*}(A) \oplus \mathbb{Q}^{A[2]}$ where $1_a \in \mathbb{Q}^{A[2]}$ is the vector having just one entry 1 corresponding to the 2-torsion point $a \in A[2]$. Fixing an arbitrary point $p \notin A[2]$, we can write $\frac{1}{2}\,\mathrm{ch}^G(k(p)[1] \oplus k(-p)[1] \oplus k(a)^{\oplus 2}) = (0, 1_a)$. Note that $k(a)$ has the standard G-linearisation $+1$ and $k(p) \oplus k(-p) \in \mathrm{D}^G(A)$ is uniquely G-linearised by Example 3.10. Similar to the reasoning above, we see $\mathrm{ch}^G(\mathsf{FM}_{G\cdot P}(k(p)[1] \oplus k(-p)[1] \oplus k(a)^{\oplus 2})) = (0, *)$ because we also know that $\mathsf{FM}^{\mathrm{H}}_P(k(p)) \in \mathrm{H}^{2*}(A)$ does not depend on the particular point $p \in A$. Hence, $\mathsf{FM}^{\mathrm{H}}_{G\cdot P}$ has indeed the block form claimed.

The commutativity of the triangle follows from Lemma 3.17(i): for $P \in \mathrm{Aut}^{G_\Delta}(\mathrm{D}(A))$ it states $\mathsf{FM}_{G\cdot P}(F, \lambda) \cong \mathsf{FM}_P(F)$ as sheaves on A. The same property also holds for transcendental cycles in $\mathrm{H}^2(A)$ by definition of $\mathsf{FM}^{\mathrm{H}}_{G\cdot P}$. □

Remark 4.15. A casual interpretation of the proposition is given by the diagram

$$\begin{array}{ccc} & \mathrm{Aut}(\mathrm{D}(A)) \xrightarrow{(\cdot)^{\mathrm{H}}} & \mathrm{Aut}(\mathrm{H}^{2*}(A)) \\ {}^{for}\nearrow & & \uparrow \mathrm{res} \\ \mathrm{Aut}^{G_\Delta}(\mathrm{D}(A)) & & \\ {}_{inf}\searrow & & \\ & \mathrm{Aut}(\mathrm{D}^G(A)) \xrightarrow[(\cdot)^{\mathrm{H}}]{} & \mathrm{Aut}(\mathrm{H}^*([A/G])) \end{array}$$

It is inaccurate, however, in that the right-hand map is not well defined.

Remark 4.16. In the case of a principally polarised abelian variety A, one could also make use of the Examples 4.7. In a certain sense, they generate $\mathrm{Sp}(A \times \hat{A})$. Proceeding along these lines becomes worth considering only if the splitting of the exact sequence from Theorem 4.4 holds in general. A first case is provided by a generic abelian variety, i.e. with $\mathrm{NS}(A) = \langle L \rangle$ and $L^2 = k > 1$ and $\mathrm{End}(A) = \mathbb{Z}$. Then, the isogeny $\varphi_L : A \to \hat{A}$ has degree k and $\mathrm{Sp}(A \times \hat{A})$ is isomorphic to a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$:

$$\mathrm{Sp}(A \times \hat{A}) = \left\{ \begin{pmatrix} n_1 \cdot \mathrm{id}_A & n_2 \cdot \hat{\varphi}_L \\ n_3 \cdot \varphi_L & n_4 \cdot \mathrm{id}_{\hat{A}} \end{pmatrix} \ : \ n_1 n_4 - n_2 n_3 = 1, \ n_2 \equiv 0 \ (k) \right\}.$$

4.4 Generalised Kummer varieties

By the following construction of Beauville [6], one can to an abelian surface A associate a new manifold $K_n(A)$, where $n \geq 1$. Let $N_n A \subset A^n$ be the fibre over 0 of the summation morphism $s : A^n \to A$. Then $N_n A$ is obviously S_n-invariant and we define $K_{n-1}(A) := S_n\text{-Hilb}(N_n A)$ as S_n-Hilbert scheme. The variety $K_n(A)$ is called *generalised Kummer variety* corresponding to A. It is a basic fact that $K_n(A)$ is a smooth projective variety of dimension $2n$, see [6]. Generalised Kummer varieties comprise one of the two known series of irreducible holomorphic symplectic manifolds up to deformation equivalence.

Example 4.17. Note that $N_2 A = \{(a, -a) \ : \ a \in A\} \cong A$ and $K_1(A) = S_2\text{-Hilb}(N_2 A)$ is the crepant resolution of $A/S_2 = A/\{\pm 1\}$ as in Section 3.4. Hence, $K_1(A)$ is the usual Kummer surface made from the abelian surface A.

Namikawa [66] raised the question whether D-equivalent abelian surfaces A, B will have D-equivalent generalised Kummer varieties $K_n(A)$ and $K_n(B)$. Most prominent is the special problem where $B = \hat{A}$. We know $\mathrm{D}(A) \cong \mathrm{D}(\hat{A})$ by Theorem 4.1 of Mukai.

Remark 4.18. Concerning the case $n = 1$, we saw already in the first of examples 4.7 that the Kummer surfaces $K_1(A)$ and $K_1(\hat{A})$ are D-equivalent. However, much more is true: Hosono et al. show in [40] for any two abelian surfaces A, B that

$$\mathrm{D}(A) \cong \mathrm{D}(B) \iff K_1(A) \cong K_1(B),$$

i.e. two abelian surfaces are D-equivalent if and only if the corresponding Kummer surfaces are isomorphic. The proof proceeds along the characterisation of derived categories of abelian and K3 surfaces by means of cohomology lattices. In particular, it does not extend to generalised Kummer varieties. Indeed, there is an example by Namikawa [66] (see also [33, Example 25.16]) of an abelian surface A such that $K_2(A)$ and $K_2(\hat{A})$ are not even birational.

Namikawa's question seems to give a good opportunity to invoke the results of Chapter 3. Alas, the techniques are not sufficient: one would like to use the Poincaré bundle $\mathcal{P} \in \mathrm{D}(A \times \hat{A})$ to produce a G_Δ-linearised kernel $Q \in \mathrm{D}^{G_\Delta}(N_n A \times N_n \hat{A})$ with $\mathsf{FM}_Q : \mathrm{D}(\hat{N}) \xrightarrow{\sim} \mathrm{D}(N)$.

The inclusion $i : N_n A \hookrightarrow A^n$ is S_n-equivariant. In particular, as in Section 3.1 we obtain the functor $i^* : \mathrm{D}^{S_n}(A^n) \to \mathrm{D}^{S_n}(N_n A)$. Fixing both A and n, let us introduce the shorthands $N := N_n A$ and $\hat{N} := N_n \hat{A}$. Let $Q := i^* \mathcal{P}^{\boxtimes n} = \mathcal{P}^{\boxtimes n}|_{\hat{N} \times N} \in \mathrm{D}^{S_n}(N \times \hat{N})$, and $(\alpha_1, \ldots, \alpha_n) \in \hat{N}$, i.e. $\alpha_i \in \hat{A}$ such that $\alpha_1 \otimes \cdots \otimes \alpha_n \cong \mathcal{O}_A$. Then we have with the projections $p_i : A^n \to A$ and the inclusion $i : N \hookrightarrow A^n$

$$\begin{aligned}
\mathsf{FM}_Q(\alpha_1, \ldots, \alpha_n) &= p_{N*}(Q \otimes p_{\hat{N}}^*(\alpha_1, \ldots, \alpha_n)) = \mathcal{P}^{\boxtimes n}|_{\{(\alpha_1, \ldots, \alpha_n)\} \times N} \\
&= \alpha_1 \boxtimes \cdots \boxtimes \alpha_n|_N = (p_1 i)^* \alpha_1 \otimes \ldots (p_n i)^* \alpha_n \\
&\cong p_1^* \alpha_1 \otimes \cdots \otimes p_{n-1}^* \alpha_{n-1} \otimes s^*(-1)^*(\alpha_1^\vee \otimes \cdots \otimes \alpha_{n-1}^\vee) \\
&= p_1^* \alpha_1 \otimes \cdots \otimes p_{n-1}^* \alpha_{n-1} \otimes s^*(\alpha_1 \otimes \cdots \otimes \alpha_{n-1}) \\
&= (\alpha_1^2 \otimes \alpha_2 \otimes \cdots \otimes \alpha_{n-1}) \boxtimes \cdots \boxtimes (\alpha_1 \otimes \cdots \otimes \alpha_{n-2} \otimes \alpha_{n-1}^2)
\end{aligned}$$

where the second to last line uses the see-saw principle in the guise of $s^* \alpha = \alpha^{\boxtimes n}$ for the sum map $s : A^n \to A$ and $\alpha \in \hat{A}$. Now the last line immediately shows that FM_Q fails to be faithful if some α_i are of order 2. It seems as if the symmetry condition required by G_Δ-linearised equivalence kernels is too strong. Note that the other obvious candidate $i_* \mathcal{P}^{\boxtimes (n-1)}$ also drops out: it yields an equivalence but it is not G_Δ-invariant.

About the ground field

Although we work exclusively over $\mathbb{C}$, many statements remain true using an algebraically closed field k of characteristic 0 instead. The content of the first chapter is even valid for arbitrary algebraically closed fields, perhaps with the exception of some examples. Just note that one has to replace the cohomology rings $\mathrm{H}^*(Y, \mathbb{Z})$ and $\mathrm{H}^*(Y, \mathbb{Q})$ by the intersection rings $\mathrm{CH}^*(Y)$ and $\mathrm{CH}^*(Y) \otimes \mathbb{Q}$ in Section 1.2. Because of $\mathrm{CH}^*(Y) \otimes \mathbb{Q} \cong \mathrm{K}(Y) \otimes \mathbb{Q}$, one might work with the K-groups instead as well.

The second chapter on K3 surfaces is more problematic when it comes to general fields: the proofs of the Torelli theorem in [5], [3] and [73] all use analytic methods. Even if the local questions about liftings of isomorphisms in families may be tackled purely algebraically, this cannot possibly hold for Orlov's derived Torelli theorem 2.3(ii): it can be expressed by saying that two K3 surfaces are D-equivalent precisely if there exists a Hodge isometry between their transcendental lattices. On the other hand, the proof of the main Theorem 2.2 also works on the level of intersection rings. Accordingly, the following question is a natural generalisation of problem 2.7: can the ring automorphism ι_{CH} of $\mathrm{CH}^*(X) = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathrm{CH}^2(X)$ with $\iota_{\mathrm{CH}}(r, \alpha, p) := (r, -\alpha, p)$ be lifted to an autoequivalence?

The part on G-linearisations and equivariant derived categories is valid in characteristic 0 only. The main problem is the taking of G-invariants which is badly behaved if the characteristic divides the group order. Also, the Schur multipliers of remark 3.6 behave differently in prime characteristic. On the other hand, they do not change if $\mathbb{C}$ is replaced a field which contains all roots of unity; see [41, §23.5]. The example concerning Hilbert schemes given in Section 3.4 uses the transcendental lattice of a K3 surface.

In Section 4.1 of the last chapter, Mukai's crucial relation (4.1) for the Poincaré bundle holds over arbitrary fields. Likewise, Orlov's exact sequence from Theorem 4.4 is exact over an arbitrary field k without the righthand 0. Of course, the product $A \times \hat{A}$ has to be replaced by the group $(A \times \hat{A})_k$ of k-rational points. However, the proof of the surjectivity of the homomorphism $\mathrm{Aut}(\mathrm{D}(A)) \to \mathrm{Sp}(A \times \hat{A})$ uses that the field is algebraically closed and of characteristic 0. As all further computations build on this result, the assumption has to be kept for the remainder.

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