Cubic surfaces with double points in positive characteristic

Marko Roczen

0 Introduction

The classification of singular cubic surfaces, made by Schläfli and Cayley in the last century, and reconsidered by Bruce and Wall [BW] from the viewpoint of modern singularity theory (both over the complex numbers) gives rise to the following question: Let k be an algebraically closed field of arbitrary characteristic p, $f = f(x_o, x_1, x_2, x_3)$ an irreducible homogeneous polynomial of degree 3.
Let $X \subseteq \mathbb{P}_k^3$ be the set of zeros of f in the projective space.
If X has no triple point (in a way, this is the most general case), it has at most double points. They are seen to be rational singularities from the list of Artin [Art], but in general, they do not appear in these normal forms. Hence, it is useful to have a possibility of finding their type. This is given by a "geometric" extension of the "recognition principle" of Bruce and Wall (loc. cit.). An equivalent condition is found via the description of the "local resolution graph" and provides a possibility to avoid some awful coordinate transformations.
Now, configurations of double points and the corresponding normal forms can be calculated.

1 Two characterizations of rational double points

Let R be a complete local Cohen Macaulay k-algebra with residue field k of dimension $d \geq 2$. Spec R is said to be absolutely isolated if there is a resolution of singularities consisting of blowing ups $\varphi_i : X_i \to X_{i-1}$ $(i = 1, \ldots, t)$, $X_o =$ Spec R, X_t smooth. Sing(X_i) finite and φ_i the blowing up of the reduced singular locus Sing(X_i) of X_i. The set (φ_i) of morphisms is essentially unique and said to be the canonical resolution. We associate to R the "local resolution graph" Γ: This is a directed graph having as vertices the components of the formal scheme $\coprod_{i=0}^{t-1} (X_i)^{\wedge}_{\mathrm{Sing}(X_i)}$; its arrows correspond to the morphisms of complete local rings induced by the φ_i. Thus, e.g. the graph

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] & \bullet \\ & \bullet & & \end{tikzcd}$$

comes from an isolated singularity which can be resolved by 4 blowing ups as above, the singular locus of X_2 consists of 2 points, and X_1, X_3 both have one singular point.
Now let R be a double point (i.e. of multiplicity 2), then $R \simeq k[\![x]\!]/(f)$, where $x = (x_o, \ldots, x_d)$ are indeterminates, $f \in k[\![x]\!]$ of order 2. Consider any $w = (w_o, \ldots, w_d) \in \mathbb{R}_+^{d+1}$, such that $w_i \leq \frac{1}{2}$. f is said to be semiquasihomogeneous (sqh) of weight w if $f = \sum\limits_{\nu} a_\nu x^\nu$ such that
(1) $\quad f_1 = \sum\limits_{\nu, w(\nu)=1} a_\nu x^\nu$ defines an isolated singularity,
(2) $\quad f - f_1 = \sum\limits_{\nu, w(\nu)>1} a_\nu x^\nu$.
Spec R is said to be sqh of weight w if there exists such an f as above.

1.1 Characterization:

For a complete local Cohen Macaulay double point Spec R of dimension $d > 1$, the following conditions are equivalent:
(i) Spec R is absolutely isolated.
(ii) Spec R is sqh of some weight w such that $w_o + \ldots + w_d > \frac{d}{2}$.
Further, in (ii) the weight is up to permutation one of the following:

$$\begin{aligned}
A_n &= (\frac{1}{n+1}, \frac{1}{2}, \ldots, \frac{1}{2}), \quad n \geq 1 \\
D_n &= (\frac{1}{n-1}, \frac{n-2}{2(n-1)}, \frac{1}{2}, \ldots, \frac{1}{2}), \quad n \geq 4 \\
E_6 &= (\frac{1}{3}, \frac{1}{4}, \frac{1}{2}, \ldots, \frac{1}{2}) \\
E_7 &= (\frac{1}{3}, \frac{2}{9}, \frac{1}{2}, \ldots, \frac{1}{2}) \\
E_8 &= (\frac{1}{5}, \frac{1}{3}, \frac{1}{2}, \ldots, \frac{1}{2}).
\end{aligned}$$

The weight X_n ($X = A, D$ or E, respectively) is uniquely determined by R and called the "type" of the singularity. The local resolution graphs are the following ones and correspond to the type as indicated:
(m = number of vertices)

graph	type (condition)
$\begin{array}{ccccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \dots & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$	$A_{2m-1},\ m \geq 1$ (S) $A_{2m},\quad m \geq 1$ (NS) $E_6,\qquad m = 4$ (NI)
$\begin{array}{ccccccccc} & & & & & & \bullet & & \\ & & & & & & \uparrow & & \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \dots & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & & & & \downarrow & & \\ & & & & & & \bullet & & \end{array}$	$D_m,\ m \geq 4,\ m$ even
$\begin{array}{ccccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \dots & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & & & & \downarrow & & \\ & & & & & & \bullet & & \end{array}$	$D_{m+1},\ m \geq 4,\ m$ even
$\begin{array}{ccccccccc} & & & & & & \bullet & & \\ & & & & & & \uparrow & & \\ & & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & & \downarrow & & \downarrow & & \\ & & & & \bullet & & \bullet & & \end{array}$	$E_7,\ m = 7$
$\begin{array}{ccccccccc} & & & & & & \bullet & & \\ & & & & & & \uparrow & & \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & & & & \downarrow & & \\ & & & & & & \bullet & & \end{array}$	$E_8,\ m = 8$

The conditions (in brackets) are

- S: The exceptional locus of the last blowing up φ_t in the canonical resolution is smooth.
- NS: The exceptional locus of φ_t is not smooth.
- NI: For the quadratic suspension of dimension $d+2$, the exceptional locus of the first blowing up φ_1 has nonisolated singularities (if $R = k[\![x]\!]/(f)$ for any f, then $R' = k[\![x, x_{d+1}, x_{d+2}]\!]/(f + x_{d+1} \cdot x_{d+2})$ is said to be the quadratic suspension of dimension $d+2$).

Proof: For the equivalence of (i), (ii) and the uniqueness of w cf. ([R], 3.3). The remaining conditions follow from the proof of ([R], 3.2.).
Now let $d = 2$. The absolutely isolated double points are known to be rational. Their equations have been computed by Artin ([Art], 3.) and are given in the following list.

1.2 Artin's equations of absolutely isolated double points:

I) $p \neq 2$

$$A_n^o : \qquad x_o^{n+1} - x_1 x_2, \quad n \geq 1$$

$$
\begin{array}{ll}
D_n^o: & x_o^{n-1} + x_o x_1^2 + x_2^2, \quad n \geq 4 \\
E_6^o: & x_o^3 + x_1^4 + x_2^2 \\
E_6^1: & x_o^3 + x_1^4 + x_2^2 + x_o^2 x_1^2 \quad \text{(additionally for } p = 3) \\
E_7^o: & x_o^3 + x_o x_1^3 + x_2^2 \\
E_7^1: & x_o^3 + x_o x_1^3 + x_2^2 + x_o^2 x_1^2 \quad \text{(additionally for } p = 3) \\
E_8^o: & x_o^3 + x_1^5 + x_2^2 \\
E_8^1: & x_o^3 + x_1^5 + x_2^2 + x_o^2 x_1^3 \quad \text{(additionally for } p = 3) \\
E_8^2: & x_o^3 + x_1^5 + x_2^2 + x_o^2 x_1^2 \quad \text{(additionally for } p = 3) \\
E_8^1: & x_o^3 + x_1^5 + x_2^2 + x_o x_1^4 \quad \text{(additionally for } p = 5)
\end{array}
$$

II) $p = 2$

$$
\begin{array}{rl}
A_n^o: & x_o^{n+1} + x_1 x_2 \\
D_{2n}^o: & x_o^n x_1 + x_o x_1^2 + x_2^2, \quad n \geq 2 \\
D_{2n}^r: & x_o^n x_1 + x_o x_1^2 + x_2^2 + x_o^{n-r} x_1 x_2, \quad n \geq 2,\ 1 \leq r \leq n-1 \\
D_{2n+1}^o: & x_o^n x_2 + x_o x_1^2 + x_2^2, \quad n \geq 2 \\
D_{2n+1}^r: & x_o^n x_2 + x_o x_1^2 + x_2^2 + x_o^{n-r} x_1 x_2, \quad n \geq 2, 1 \leq r \leq n-1 \\
E_6^o: & x_o^3 + x_1^2 x_2 + x_2^2 \\
E_6^1: & x_o^3 + x_1^2 x_2 + x_2^2 + x_o x_1 x_2 \\
E_7^o: & x_o^3 + x_o x_1^3 + x_2^2 \\
E_7^1: & x_o^3 + x_o x_1^3 + x_2^2 + x_o^2 x_1 x_2 \\
E_7^2: & x_o^3 + x_o x_1^3 + x_2^2 + x_1^3 x_2 \\
E_7^3: & x_o^3 + x_o x_1^3 + x_2^2 + x_o x_1 x_2 \\
E_8^o: & x_o^3 + x_1^5 + x_2^2 \\
E_8^1: & x_o^3 + x_1^5 + x_2^2 + x_o x_1^3 x_2 \\
E_8^2: & x_o^3 + x_1^5 + x_2^2 + x_o x_1^2 x_2 \\
E_8^3: & x_o^3 + x_1^5 + x_2^2 + x_1^3 x_2 \\
E_8^4: & x_o^3 + x_1^5 + x_2^2 + x_o x_1 x_2
\end{array}
$$

Obviously, X_n^r is sqh of weight X_n, i.e. by 1.1. and ([Art], 3.) we obtain

1.3 Remark:

(i) The map $X_n^r \mapsto X_n$ gives the type of the singularity.
(ii) The Tjurina number $\tau : \{X_n^r |\ \text{all}\ r\} \to I\!N$ is injective for a fixed type X_n.
The symbol X_n^r will be used for the corresponding complete local ring and (by abuse of language) its spectrum, too.

2 Singularities and normal forms

The singularities of the cubic surface X give rise to the possible normal forms (depending on parameters, in some cases). Though differences from the classical case can appear only in some characteristics $p \neq 0$, the application of 1.1. simplifies coordinate transformations sometimes.
Let $S = S(X) := X^{\wedge}_{\mathrm{Sing}(X)}$ be the formal scheme obtained from X by completion along the singular locus. S will be called the type of the cubic surface X. The classification can be done via S: If X has only isolated singularities and contains a triple point, this is the only singularity, and X is the projective closure of the cone over a smooth plane cubic. In any other case, X contains at most double points. This is the situation considered here. The following description extends the list in the paper of Bruce and Wall [BW], and some of the cases (which remain unchanged) are only listed for completeness. Let $P \in X$ be singular, $P = (0:0:0:1) \in \mathbb{P}^3$ and $(x_o : x_1 : x_2 : x_3)$ the homogeneous coordinates. We write

$$f = x_3 f_2 + f_3, \quad f_i = f_i(x_o, x_1, x_2) \quad \text{homogeneous of degree } i.$$

The classification of quadratic forms (in arbitrary characteristic) gives us the following possibilities:
A) $f_2 = x_1^2 - x_o x_2$
B) $f_2 = x_o x_1$
C) $f_2 = x_o^2$
Let $L := V^+(f_2, f_3) \subseteq \mathbb{P}^2$ be the space of lines in X passing P, $\mathbb{P}^2$ with the coordinates $(x_o : x_1 : x_2)$.

Case A: Obviously, P is an A_1 singularity of X. Further, $\mathrm{Sing}(X - \{P\})$ is in bijective correspondence with $\mathrm{Sing}(L)$, where a point $Q \in L$ of mutliplicity k is mapped to an A_{k-1} singularity of $X - \{P\}$ (cf. [BW], Lemma 2). Thus all possibilities for S are

$$\begin{aligned} S \;=\; & A_1,\ 2A_1,\ A_1 \amalg A_2,\ 3A_1,\ A_1 \amalg A_3,\ 2A_1 \amalg A_2, \\ & 4A_1,\ A_1 \amalg A_4,\ 2A_1 \amalg A_3,\ A_1 \amalg 2A_2,\ A_1 \amalg A_5. \end{aligned}$$

Here, the symbol nX always denotes $X \amalg \ldots \amalg X$ (n disjoint copies).

Case B (cf. [BW], Lemma 3): The singularities of $X - \{P\}$ correspond to the points of $\mathrm{Sing}(L - \{Q\})$, $Q := (0:0:1)$, and under this bijection, a point of multiplicity k is mapped to an A_{k-1} singularity. Further, P is an $A_{k_o+k_1+1}$ singularity if k_i denotes the multiplicity of $L_i = V(x_i, f_3)$ at Q. The only possible k_i are $\{k_o, k_1\} = \{1\}, \{1,2\}, \{1,3\}$.
Thus, all possible cases are:

$$\begin{aligned} S \;=\; & A_2,\ A_2 \amalg A_1,\ 2A_2,\ A_2 \amalg 2A_1,\ 2A_2 \amalg A_1, 3A_2, \\ & A_3,\ A_3 \amalg A_1,\ A_3 \amalg 2A_1,\ A_4,\ A_4 \amalg A_1,\ A_5, A_5 \amalg A_1. \end{aligned}$$

To see why the singularity at P is of the type claimed above, we may use the local resolutions as follows:
Without loss of generality

$$\begin{aligned} f &= x_3 x_o x_1 + f_3(x_o, x_1, x_2), \\ f_3 &= x_o(a_o x_o^2 + a_1 x_o x_2 + a_2 x_2^2) + x_1(a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2) + a_6 x_2^3. \end{aligned}$$

Putting $x_3 = 1$, we obtain an equation

$$x_o x_1 + x_o \cdot a(x_o, x_2) + x_1 \cdot b(x_o, x_2) = 0$$

for X near P corresponding to the origin in $\mathbf{A}^3$.
(1) $f_3(Q) \neq 0$, then $a_6 \neq 0$ and P is A_2 (weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$).
(2) $f_3(Q) = 0$, then $a_6 = 0$, $k_1 = 1$ implies $a_2 \neq 2$.
(2.1) $k_o = 1$ (equivalently $a_5 \neq 0$), P is A_3 (weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$).
(2.2) $k_o = 2$ (equivalently $a_5 = 0$ and $a_4 \neq 0$):
Blow up P; after an obvious coordinate transformation you obtain a point that is sqh of weight $(\frac{1}{2}, \frac{1}{3}, \frac{1}{2})$, and therefore P is A_4.
(2.3) $k_o = 3$ (equivalently $a_4 = a_5 = 0$ and $a_3 \neq 0$):
One blowing up leads to a point of sqh weight $(\frac{1}{2}, \frac{1}{4}, \frac{1}{2})$, i.e. P is A_5.

Case C: This case will be performed in full detail, including the relevant normal forms.
Let $i := 4 - \#L$, $\#L$ the number of (closed points) of $V^+(x_o, f_3)$. Then $i \in \{1, 2, 3\}$, P is the only singularity of X and has type D_4, D_5, E_6 for $i = 1, 2, 3$, respectively: Let (Ci) be the corresponding case, $f = x_3 x_o^2 + x_o \cdot g_2(x_1, x_2) + g_3(x_1, x_2)$, g_ℓ homogeneous of degree ℓ. Depending on $p = \operatorname{char} k$, we obtain after a linear homogeneous transformation
(C1) $g_3 = x_1^3 + x_2^3$ for $p \neq 3$, and $g_3 = x_1^2 x_2 + x_3^2$ for $p = 3$.
(C2) $g_3 = x_1^2 x_2$ and $g_2(0, 1) \neq 0$
(C3) $g_3 = x_1^3$ and $g_2(0, 1) \neq 0$.
Using 1.1.(ii), in each case we obtain for P a sqh-singularity of the following type and initial term (i.e. term of weight 1):
(C1) $D_4 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$,
$x_o^2 + x_1^3 + x_2^3$ for $p \neq 3$, and $x_o^2 + x_1^2 x_2 + x_2^3$ for $p = 3$
(C2) $D_5 = (\frac{1}{2}, \frac{3}{8}, \frac{1}{4})$,
$x_o^2 + c x_o x_2^2 + x_1^2 x_o$, $c \neq 0$
(C3) $E_6 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$,
$x_o^2 + c x_o x_2^2 + x_1^3$, $c \neq 0$

Case (C1): $p \neq 3$: For some linear form ℓ, put

$$x_1 := x_1 - \frac{a}{3} x_o, \; x_2 := x_2 - \frac{c}{3} x_o, \; x_3 := x_3 + \ell(x_o, x_1, x_2)$$

to obtain the
NORMAL FORMS: $f = x_o^2x_3 + rx_ox_1x_2 + x_1^3 + x_2^3$, $r \in \{0,1\}$
SINGULARITY AT P: D_4 if $p \neq 2,3$ and
D_4^r if $p = 2$
(If $p \neq 2,3$, the Hessian of f is $x_o^2x_1x_2$ and $x_o^2(36x_1x_2 - x_o^2)$, respectively. Both cases $r = 0,1$ thus do not provide projectively equivalent surfaces. If $p = 2$, the Tjurina numbers at P are $\tau = 8$ for $r = 0$, $\tau = 6$ for $r = 1$.)

$p = 3$: Use $x_1 := x_1 - \frac{b}{2}x_o$, $x_2 := x_o - \alpha x_o$, $x_3 := x_3 + \ell(x_o, x_1, x_2)$.
NORMAL FORMS: $f = x_o^2x_3 + rx_ox_2^2 + x_1^2x_2 + x_2^3$, $r \in \{0,1\}$
SINGULARITY AT P: D_4
(The Hessians are $x_o^2x_1^2$ and $x_o^2(x_ox_2 - x_1^2)$, respectively, thus $r = 0,1$ provide nonequivalent surfaces).

Case (C2): A substitution $x_2 := x_2 - ax_o$, $x_3 := x_3 + \ell(x_o, x_1, x_2)$ gives for
$p = 2$:
NORMAL FORMS: $f = x_o^2x_3 + rx_ox_1x_2 + x_ox_2^2 + x_1^2x_2$, $r \in \{0,1\}$
SINGULARITY AT P: D_5^r
Further, for $p \neq 2$ we choose $x_1 := x_1 + \alpha x_o$, $x_3 := x_3 + \bar{\ell}(x_o, x_1, x_2)$ and obtain a single
NORMAL FORM: $f = x_o^2x_3 + x_ox_2^2 + x_1^2x_2$ with
SINGULARITY AT P: D_5

Case (C3): f can be transformed into

$$f = x_o^2x_3 + ax_ox_1^2 + bx_ox_1x_2 + x_ox_2^2 + x_1^3.$$

Now we choose a coordinate transformation as before to obtain for
$p = 2$:
NORMAL FORMS: $f = x_o^2x_3 + rx_ox_1x_2 + x_ox_2^2 + x_1^3$, $r \in \{0,1\}$
SINGULARITY AT P: E_6^r
In the remaining cases, we choose

$$x_1 := x_1 + \alpha x_o,\ x_2 := x_2 + \beta x_1,\ x_3 = x_3 + \ell(x_o, x_1, x_2).$$

The condition $a = b = 0$ is expressed by

$$b + 2\beta = 0,\ a + b\beta + \beta^2 + 3\alpha = 0,$$

which is solvable for $p \neq 3$. Thus we obtain for
$p \neq 2,3$:
NORMAL FORM: $f = x_o^2x_3 + x_ox_2^2 + x_1^3$
SINGULARITY AT P: E_6
$p = 3$:
NORMAL FORMS: $f = x_o^2x_3 + rx_ox_1x_2 + x_ox_2^2 + x_1^3$, $r \in \{0,1\}$

SINGULARITY AT P: E_6^r
(since for $r = 0$, the Tjurina number is $\tau = 9$, and for $r = 1$ we have $\tau = 7$).

Note, that Schläfli and Cayley mistakenly give only one normal form for surfaces with a singularity of type D_4 ([S], p. 229). This was already remarked in [BW]. The given form of Schläfli (loc. cit.) is easily seen to be equivalent to the one above for $r = 1$. In characteristic 2, both cases $r = 0, 1$ give even nonisomorphic singularities of type D_4.

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