1-Semiquasihomogeneous Singularities of Hypersurfaces – Pathology in Characteristic 2 *†

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Abstract

In [7], the classification of 1-semiquasihomogeneous singularities of hypersurfaces in arbitrary characteristic p was given. They turn out to be defined (up to quadratic suspensions) by the equations given by K. Saito [9] over the base field of complex numbers, as far as $p \neq 2$. For $p = 2$, the even- and odd dimensional case have to be distinguished, and there are nontrivial superdiagonal deformations in the odd- dimensional case. The singularity $\tilde{E}_6$ gives an infinite family of nonisomorphic singularities with fixed principal part, contrary to the classical case of simple elliptic singularities, which have modality 1 (coming from the absolute invariant in the principal part).[1]

0. The problem

k denotes an algebraically closed field. Consider indeterminates $(X_0, \ldots, X_n) =: X$ equipped with positive weights $w_i := w(X_i) \in \mathbb{Q}$ and a formal power series $f \in k[[X]]$ consisting of monomials m_ν of weight $w(m_\nu) := w(\nu) := \sum \nu_i w_i \geq 1$ such that the "prinicipal part" f_1 (:= sum of terms of weight 1) defines an isolated singularity. Associate to f the "Saito-invariant" $s := |X| - 2\sum_{x \in X} w(x)$. We say "$f$ is s-semiquasihomogeneous" (or short: "s-sqh") with respect to the given weights. The case of $s < 1$ characterizes the rational double points. The "boundary case" of $s = 1$ gives in the complex-analytic case (dimension 2) the simple elliptic singularities (cf. [9]); the classification was extended in [7] to arbitrary characteristic of k. The only interesting new feature is the case of $char(k) = 2$, where we obtain nonisomorphic types of equations with fixed

*Corrected version of the paper in An. St. Univ. Ovidius Constantza, Vol 5, f.2 (1997), 99 - 104"; that paper contains a mistake in the comment for the table on page 3: The group G is in any case the same and of order 54. The proof in the original article has mentioned this, already.

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principal part f for two of the possible weights $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. From the known cases, one might be tempted to conjecture that there are only finitely many isomorphism classes with fixed principal part (for $k = \mathbb{C}$ the condition $s \leq 1$ implies that any s-sqh-power series is contact equivalent to its principal part), and the list of sqh-singularities with $s < 1$ seems to indicate this (cf. [4]) in positive characteristic as well. Here we show: The above conjecture is not true in for $\tilde{E}_6$ in odd dimensions ≥ 3.

1. Coordinate transformation for sqh power series

Let $w(g)$ be the weight of the initial term of a power series g with respect to w. Further, write $g = \sum_{q \in \mathbb{Q}} g_q$, where $supp(g_q)$ is contained in the set of monomials of weight q.

Essential tool is the following

Lemma: Let f_1 be quasihomogeneous of some weight w and choose monomials $m_1, \ldots, m_s$ of weights $q_1, \ldots, q_s$ respectively, $q_i > 1$, such that their classes form a base of the k-subspace S of the Tjurina-algebra

$$T(f_1) := k[[X]]/(f_1, \frac{\partial f_1}{\partial X_0}, \ldots, \frac{\partial f_1}{\partial X_n}),$$

where S is generated by the classes of all monomials of weight > 1.
Now take any power series f which is sqh with respect to w and assume $f = f_1 + \sum_{q>1} f_q$. Choose a rational number $q' > 1$ and suppose

$$f_{q'} \equiv \sum_{i, q_i = q'} \alpha_i m_i \ mod\ (f_1, \frac{\partial f_1}{\partial X_0}, \ldots, \frac{\partial f_1}{\partial X_n})$$

for some $\alpha_i \in k$ (the sum over an empty set is defined to be 0). Then f is contact equivalent with

$$f_1 + \sum_{q, 1<q<q'} f_q + \sum_{i, q_i = q'} \alpha_i m_i + g,$$

where $w(g) > q'$.

Proof: Follows from section 2. in [7].

The above lemma turns out to be a surprisingly strong tool for "numerical" coordinate transformations. It implies (for $k = \mathbb{C}$) the celebrated theorem (12.6. in [1]) of Arnold, Gusejn-Zade and Varchenko.

2. Non-quasihomogeneous normal forms for $s = 1$

In [7], the classification of quasihomogeneous singularities with $s = 1$ is given. Only possible weights are $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. They are described by Saito's equations (cf. [9]) of simple elliptic singularities for $char(k) \neq 2$. From now on let

$char(k) = 2$. In even dimensions, all sqh-singularities of weight $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, have quasihomogeneous forms in their contact class.

Here is the the complete list in the odd-dimensional case (we prefer to use the "symmetric form" in case of $\tilde{E}_6$, cf. [7], section 1):

$\tilde{E}_6(t)$: $X_0^3 + X_1^3 + X_2^3 + tX_0X_1X_2 + X_3^2$, $t^3 \neq 1$
$\tilde{E}_7(t)$: $X_0X_1(X_1 + X_0)(X_1 + tX_0)$, $t \neq 0,\ 1$
$\tilde{E}_8(t)$: $X_0(X_0 + X_1^2)(X_0 + tX_1^2)$, $t \neq 0,\ 1$

Now, all contact classes of the corresponding sqh series are found by adding linear combinations of the superdiagonal monomials in the Tjurina algebra of the corresponding quasihomogeneous part. But the parameters are not unique, in general. Here is the table of normal forms, where τ denotes the Tjurina number:

sqh-singularities with $s = 1$ for $char(k) = 2$ in odd dimension
(up to quadratic suspension)

name	τ	defining sqh-polynomial
$\tilde{E}_6^0$	16	$X_0^3 + X_1^3 + X_2^3 + tX_0X_1X_2 + X_3^2$, $t^3 \neq 1$
$\tilde{E}_6^1$	14	$X_0^3 + X_1^3 + X_2^3 + tX_0X_1X_2 + X_3^2 + X_0X_1X_2X_3$, $t^3 \neq 1$
$\tilde{E}_6^2$	12	$X_0^3 + X_1^3 + X_2^3 + tX_0X_1X_2 + X_3^2 + \alpha_0X_1X_2X_3 +$
		$+\alpha_1X_0X_2X_3 + \alpha_2X_0X_1X_3$, $t^3 \neq 1$
$\tilde{E}_7$	9	$X_0X_1(X_1 + X_0)(X_1 + tX_0)$, $t \neq \{0, 1\}$
$\tilde{E}_8^0$	12	$X_0(X_0 + X_1^2)(X_0 + tX_1^2)$, $t \neq \{0, 1\}$
$\tilde{E}_8^1$	10	$X_0(X_0 + X_1^2)(X_0 + tX_1^2) + X_0X_1^5$, $t \neq \{0, 1\}$

Contact classes of type $\tilde{E}_6$: For any sqh series with principal part $\tilde{E}_6^0$, fix any $t \in k$. The parameters α_i do not give an isomorphism classification of the corresponding germs, but the classes in the following set are in bijection with the contact classes if we choose them as $G(\alpha_0 : \alpha_1 : \alpha_2) \in I\!P_k^2/G$, where

$G \subseteq PGL(3,k)$ is a finite subgroup of order 54, is generated by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu^2 \end{pmatrix}$ and the Weyl-group of permutation matrices. (Here μ denotes a primitive third unit root.)

Proof: We perform the details for $t \neq 0$.

Let f be sqh of weight $\tilde{E}_6$. Since the vector subspace of $T(f_1)$ generated by the superdiagonal monomials has a base consisting of the classes of $X_1X_2X_3$, $X_0X_2X_3$, $X_0X_1X_3$ and $X_0X_1X_2X_3$, we obtain for f up to contact equivalence the polynomial

$$X_0^3+X_1^3+X_2^3+tX_0X_1X_2+X_3^2+\alpha_0X_1X_2X_3+\alpha_1X_0X_2X_3+\alpha_2X_0X_1X_3+\alpha_3X_0X_1X_2X_3.$$

case (0): $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$:

This is the quasihomogeneous equation $\tilde{E}_6^0$

case (1): $\alpha_0 = \alpha_1 = \alpha_2 = 0, \alpha_3 \neq 0$:

Using the transformation

$$X_i \mapsto \frac{1}{\alpha_3^{2/3}} \quad (i = 0,1,2), \qquad X_3 \mapsto \frac{1}{\alpha_3},$$

we obtain a multiple of the above equation $\tilde{E}_6^1$.

case (2): $(\alpha_0, \alpha_1, \alpha_2) \neq (0,0,0)$:

Let $f = f^{(\alpha_0,\alpha_1,\alpha_2,\alpha_3)}$ be the above equation. Then we may choose $\alpha_3 = 0$ without changing the contact class of f:

Assume $\alpha_0 \neq 0$. Then

$$X_i \mapsto X_i \ \ (i = 0,1,2), \qquad X_3 \mapsto X_3 + \frac{\alpha_3}{\alpha_0}X_0X_3$$

sends f into a power series $f_1 + f_{7/6} + g$ with $w(g) \geq 3/2$. In the Tjurina-algebra of f_1, the part $g_{3/2}$ of g is 0. Thus, by the lemma on coordinate transformation $f = f^{(\alpha_0,\alpha_1,\alpha_2,\alpha_3)}$ is contact equivalent $f^{(\alpha_0,\alpha_1,\alpha_2,0)}$.

Further, $f = f^{(\alpha_0,\alpha_1,\alpha_2,0)}$ is contact equivalent $f^{(\kappa\alpha_0,\kappa\alpha_1,\kappa\alpha_2,0)}$ for any $\kappa \in k^*$.

To decide whether $f = f^{(\alpha_0,\alpha_1,\alpha_2,0)}$ is contact equivalent $f^{(\alpha_0',\alpha_1',\alpha_2',0)}$ we have to consider the effect of an automorphism of $k[[X]]$ to f, followed by the multiplication with a unit e. Since multiplication by e does not affect terms of weight $7/6$ (corresponding to the α_i with $i < 3$), we may take $e = 1$. Now apply an automorphism φ to f. Obviously, only the linear part of φ can affect the α_i. Further, we may assume $X_3 \mapsto X_3$. For the remaining X_i, we have

$X_i \mapsto \psi(X_i) + \beta_i X_3$, where $\psi \in GL(3,k)$ is a linear automorphism in the variables X_0, X_1, X_2 which is an invariant of $h = X_0^3 + X_1^3 + X_2^3 + tX_0X_1X_2$. The coefficients of the matrix for ψ satisfy a system of polynomial equations. The coordinate ring of its zeroes turns out to be a finite dimensional k-algebra of dimension 54. On the other hand, the above group G with its 3 relevant multiples by 3rd roots of unity gives a group of precisely 54 matrices in $GL(3,k)$. Thus we guessed all solutions, already.

Now we check all 54 cases and obtain (again using lemma 1.): $f^{(\alpha_0,\alpha_1,\alpha_2,0)}$ and $f^{(\alpha_0',\alpha_1',\alpha_2',0)}$ are contact equivalent iff (up to multiplication with a nonzero constant) $(\alpha_0,\alpha_1,\alpha_2,0)$ is transformed to $(\alpha_0',\alpha_1',\alpha_2',0)$ by an element of G.

Note that the case of $t = 0$ is more complicated: The coordinate ring has dimension 648 (i.e. here we have multiple solutions), and the set of all solutions is found computing Groebner bases of the relevant ideal and performing a primary decompositon (the computation - as in the other case for h - is due to G. Pfister and Singular [2]).

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