

On the Approximate Solution of Random Operator Equations

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1. Introduction

In the late fifties the Prague school of probabilists around Špaček and Hanš began to investigate random operator equations (cf. [19]), which resulted in a unified treatment of various stochastic models (see [1]).

Especially the contributions by A. T. Bharucha-Reid, his book [1] and the survey [2] initiated an essential improvement of the theory and the approximation of random operator equations some years ago. The works by Engl [15] and Nowak [28] satisfactorily explained the measurability of solutions of such equations on the basis of the fast development of the theory of measurable multifunctions and measurable selectors (see [21, 26, 33, 37]). But in the meantime there have been published several works on the approximation of random operator equations and their random solutions, too. Let us mention [3, 10, 16 and 27, 25, 29, 30], which contain iteration methods, stochastic projectional schemes and general approximation schemes. On the one hand this paper aims at contributing to the further development of such abstract approximation schemes, where we consistently apply the concept of random operators on random (or 'stochastic') domains introduced by Engl [13], which seems to be especially suitable for application as well as for approximation. Chapter 3 contains a general approximation theorem for "locally" (in a certain random domain) unique random solutions, where we understand by 'approximation' within the scope of this paper a.s. — convergence. In chapter 4 this result is applied to random fixed point problems with contractive random operators on random domains. Thereby [25, Theorem 3.1.] is generalized and it turns out that in [10] and [30] the contraction principle is finally applied. At the end of chapter 4 we present an iteration method with a sequence of uniformly contractive random operators. The results of the chapters 3 and 4 can be applied to stochastic projectional schemes (Engl/Nashed [16]) as well as to "discretization schemes" for nonlinear random operator equations. The latter aspect represents the second aim of this paper. Continuing the ideas of [31] we refer to further advantages of the approximation by "discrete random operators" in chapter 5. They essentially consist in the fact that the approximate problems prove to be 'completely deterministic' and that statistical characteristics of the approximate solutions can easily be calculated. A suitable practical and constructive realization of such a discretization method seems to be the replacement of the random variable z contained in the random equation by certain conditional expectations. Consequently, chapter 6 provides some hints how to approximate Banach space-valued random variables constructively by a sequence of discrete random variables generated by conditional expectations. "Constructive" means here that the conditional expectations may be calculated from finite-dimensional distribution functions of z . Finally we apply the results to the approximate solution of random ordinary differential equations (see also [31]).

We start chapter 2 with a short introduction to the necessary notations and statements from the theory of measurable multifunctions and random operators.

2. Measurability and random operators

Let $(\Omega, \mathfrak{A}, P)$ be a complete probability space and X, Y be real separable Banach spaces. By $\mathfrak{B}(X)$ we denote the σ -algebra of Borel sets of X , i.e. the one generated by the open sets of X , and by $\mathfrak{A} \otimes \mathfrak{B}(X)$ the smallest σ -algebra containing $\{A \times B \mid A \in \mathfrak{A}, B \in \mathfrak{B}(X)\}$. Further we define $\mathfrak{P}(X) := \{D \mid D \subseteq X, D \neq \emptyset\}$ and $\text{Cl}(X) := \{D \mid D \in \mathfrak{P}(X), D \text{ closed}\}$. A map $C: \Omega \rightarrow \mathfrak{P}(X)$ is called a multifunction and we define

$\text{Gr } C := \{(\omega, x) \in \Omega \times X \mid x \in C(\omega)\}$. $C: \Omega \rightarrow \mathfrak{P}(X)$ is called measurable if for all open $B \subseteq X$ $\{\omega \in \Omega \mid C(\omega) \cap B \neq \emptyset\} \in \mathfrak{A}$ (weakly measurable in [21]). Let

$S(C) := \{x \mid x: \Omega \rightarrow X \text{ measurable, } x(\omega) \in C(\omega) \text{ for all } \omega \in \Omega\}$

be the set of all measurable selectors of C and we say that C has a Castaing representation if there exist $x_m \in S(C)$, $m \in \mathbb{N}$, such that $\{x_m(\omega) \mid m \in \mathbb{N}\}$ is dense in $C(\omega)$ for all $\omega \in \Omega$ ([37]). Here \mathbb{N} denotes the set of natural numbers.

Lemma 1

Let $C: \Omega \rightarrow \mathfrak{P}(X)$ and we consider the following properties:

- C is measurable;
- C has a Castaing representation;
- $\text{Gr } C \in \mathfrak{A} \otimes \mathfrak{B}(X)$.

Then:

- (i) $a) \Leftrightarrow c) \Rightarrow b)$
- (ii) If $C: \Omega \rightarrow \text{Cl}(X)$, then all properties are equivalent.

Proof

- (i) [21, Theorem 3.4.] and [26, p. 408].
- (ii) [21, Theorem 3.5. and Theorem 5.6.].

Remark 1

If $(\Omega, \mathfrak{A}, P)$ is not necessarily complete, one can show by using the technique of [15, p. 72] that $\text{Gr } C \in \mathfrak{A} \otimes \mathfrak{B}(X)$ implies the existence of a countable set $x_m: \Omega \rightarrow X$, $m \in \mathbb{N}$, of measurable functions such that for P -almost all $\omega \in \Omega$ $\{x_m(\omega) \mid m \in \mathbb{N}\}$ is a dense subset of $C(\omega)$.

In this case it holds for $C: \Omega \rightarrow \text{Cl}(X)$ that

- $a) \Leftrightarrow b) \Rightarrow c)$ ([21]).

Definition 1 ([13])

a) Let $C: \Omega \rightarrow \mathfrak{P}(X)$. $T: \text{Gr } C \rightarrow Y$ will be called *random operator* if for all $x \in X$ and open $D \subseteq Y$, $\{\omega \in \Omega \mid x \in C(\omega), T(\omega, x) \in D\} \in \mathfrak{A}$. If in addition C is measurable, T is called a *random operator with random domain* C .

(For $T(\omega, x)$ we will also write $T(\omega) x$).

b) Let $C: \Omega \rightarrow \mathfrak{P}(X)$. $T: \text{Gr } C \rightarrow Y$ will be called $\mathfrak{A} \otimes \mathfrak{B}(X)$ -measurable if for all $B \in \mathfrak{B}(Y)$, $T^{-1}(B) = \{(\omega, x) \in \text{Gr } C \mid T(\omega, x) \in B\} \in \mathfrak{A} \otimes \mathfrak{B}(X)$.

$T: \text{Gr } C \rightarrow Y$ is called continuous if $T(\omega, \cdot): C(\omega) \rightarrow Y$ is continuous for all $\omega \in \Omega$. A multifunction $C: \Omega \rightarrow \text{Cl}(X)$ is called separable [13] if C is measurable and if there exists a countable

set $Z \subseteq X$ such that for all $\omega \in \Omega$ $\text{cl}(Z \cap C(\omega)) = C(\omega)$. We remark that every measurable $C: \Omega \rightarrow \text{Cl}(X)$ with $C(\omega) = \text{cl}(\text{int } C(\omega))$ for all $\omega \in \Omega$ is separable [15, p. 70].

Lemma 2

a) Let $C: \Omega \rightarrow \mathfrak{B}(X)$ and $T: \text{Gr } C \rightarrow Y$ be $\mathfrak{A} \otimes \mathfrak{B}(X)$ -measurable. Then:

- (i) T is a random operator with random domain C ;
- (ii) for all $x \in S(C)$ $T(\cdot, x(\cdot)): \Omega \rightarrow Y$ is measurable.

b) If $C: \Omega \rightarrow \text{Cl}(X)$ is separable and $T: \text{Gr } C \rightarrow Y$ is a continuous random operator with random domain C , then T is $\mathfrak{A} \otimes \mathfrak{B}(X)$ -measurable.

Proof

a) The first part results from $\text{Gr } C = T^{-1}(Y) \in \mathfrak{A} \otimes \mathfrak{B}(X)$ and Lemma 1 and from

$G := \{(\omega, x) \in \text{Gr } C | T(\omega, x) \in D\} \in \mathfrak{A} \otimes \mathfrak{B}(X)$ for all open $D \subseteq Y$ and the well-known projection theorem ([33, Theorem 4, p. 121]). In fact it holds for all $x \in X$ that

$$\{\omega \in \Omega | x \in C(\omega), T(\omega, x) \in D\} = \text{proj}_\Omega (G \cap (\Omega \times \{x\})).$$

The second part is obvious by definition.

b) This follows from [21, Theorem 6.1.] with a simple modification by taking into account the separability of C .

q. e. d.

Remark 2

If in Lemma 2 b) T is the restriction to $\text{Gr } C$ of a continuous random operator $T: \Omega \times X \rightarrow Y$, it suffices that $C: \Omega \rightarrow \text{Cl}(X)$ is measurable. In this case we can omit the separability condition for C (cf. [6, p. 11/12]).

Lemma 2 shows that the $\mathfrak{A} \otimes \mathfrak{B}(X)$ -measurability of a random operator plays an important role (see also [14, Lemma 10]). In the following we will see that this measurability property is essential for the existence of measurable solutions of random operator equations, too. Let $C: \Omega \rightarrow \mathfrak{B}(X)$ and $T: \text{Gr } C \rightarrow Y$ be a random operator with random domain C and we consider the random operator equation

$$T(\omega) x = 0. \tag{1}$$

We use the usual notions of solution (see [19, 1], i.e. we say $x: \Omega \rightarrow X$ is a 'wide-sense solution' of (1) if for P-almost all $\omega \in \Omega$ $x(\omega) \in C(\omega)$ and $T(\omega) x(\omega) = 0$, and $x: \Omega \rightarrow X$ is a 'random solution' of (1) if x is a wide-sense solution and x is measurable.

In a lot of practical cases the existence of a wide-sense solution of (1) can be proved by means of the theory of deterministic operator equations. The following theorem ([28]) establishes conditions under which the existence of random solutions results from the existence of wide-sense solutions.

Theorem 1

Let $C: \Omega \rightarrow \mathfrak{B}(X)$ and $T: \text{Gr } C \rightarrow Y$ be such that $T^{-1}(\{0\}) \in \mathfrak{A} \otimes \mathfrak{B}(X)$ and $S(\omega) := \{x \in C(\omega) | T(\omega) x = 0\} \neq \emptyset$ a. s. Then:

There exists an at most countable set of random solutions $x_m: \Omega \rightarrow X$, $m \in N$, of (1) such that $\{x_m(\omega) | m \in N\}$ is dense in $S(\omega)$ a. s.

Proof

Analogous to [28, Theorem 1] we choose $\Omega_1 \in \mathfrak{A}$ such that for all $\omega \in \Omega_1$ $S(\omega) \neq \emptyset$ and $S(\omega) = \emptyset$ for all $\omega \in \Omega \setminus \Omega_1$.

Let $\mathfrak{A}_1 := \mathfrak{A} \cap \Omega_1$ and we consider the multifunction $S: \Omega_1 \rightarrow \mathfrak{B}(X)$. We have

$$\begin{aligned} \text{Gr } S &:= \{(\omega, x) \in \Omega_1 \times X | x \in S(\omega)\} \\ &= T^{-1}(\{0\}) \in \mathfrak{A}_1 \otimes \mathfrak{B}(X) \subseteq \mathfrak{A}_1^* \otimes \mathfrak{B}(X), \end{aligned}$$

where $\mathfrak{A}_1^* \subseteq \mathfrak{A}$ is the completion of \mathfrak{A}_1 with respect to P . By Lemma 1 S has a Castaing representation with respect to $(\Omega_1, \mathfrak{A}_1^*, P)$. Extensions of these measurable functions to measurable functions from Ω to X provide the desired random solutions of (1). q. e. d.

Remark 3

a) If $(\Omega, \mathfrak{A}, P)$ is not necessarily complete. Theorem 1 remains valid by using the technique of [15, p. 72].

b) If the condition of Lemma 2 b) is satisfied and $S(\omega) \neq \emptyset$ a. s., then Theorem 1 is valid. For random fixed point problems such a result was obtained in [15, Corollary 7] and [5, Theorem 1]. The condition $S(\omega) \neq \emptyset$ a. s. may be replaced by arbitrary sufficient existence results for operator equations [15, p. 73].

c) If in Theorem 1 a. s. $S(\omega)$ are singletons, then there exists a unique random solution of (1).

3. A general approximation scheme for random solutions

Of central interest of this chapter is a general approximation theorem for random solutions of random operator equations. With the notations from chapter 2 let

$C, C_m: \Omega \rightarrow \mathfrak{B}(X)$, $m \in N$, and $T: \text{Gr } C \rightarrow Y$, $T_m: \text{Gr } C_m \rightarrow Y$ be given, where T and T_m , $m \in N$, be $\mathfrak{A} \otimes \mathfrak{B}(X)$ -measurable.

Now we consider the random operator equations

$$\begin{aligned} T(\omega) x &= 0 & (1) \\ T_m(\omega) x_m &= 0, \quad m \in N, & (1m). \end{aligned}$$

Now our attention is focussed on the conditions to the operators T, T_m , $m \in N$, under which random solutions of (1m) converge a. s. to a random solution of (1). Such a problem was also considered in [10, 30], in [16] for the case of stochastic projectional schemes in Hilbert spaces, in [4, 25] for contractive random fixed point problems.

Theorem 2

a) For all $x \in S(C)$ let a sequence $x'_m \in S(C_m)$, $m \in N$, exist such that $\lim_{m \rightarrow \infty} \|x(\omega) - x'_m(\omega)\|_X = 0$ a. s. and

$$\lim_{m \rightarrow \infty} \|T(\omega) x(\omega) - T_m(\omega) x'_m(\omega)\|_Y = 0 \text{ a. s.}$$

b) Let $\alpha: \Omega \times [0, \infty) \rightarrow [0, \infty)$ exist such that a. s. $\alpha(\omega, \cdot)$ is a function continuous in $t = 0$ with $\alpha(\omega, 0) = 0$, and that a. s. and for all $m \in N$, all $\bar{x}_m, \tilde{x}_m \in C_m(\omega)$ one has the condition:

$$\|\bar{x}_m - \tilde{x}_m\|_X \leq \alpha(\omega, \|T_m(\omega) \bar{x}_m - T_m(\omega) \tilde{x}_m\|_Y).$$

c) Let there exist random solutions $x, x_m: \Omega \rightarrow X$, $m \in N$ of (1) and (1m), respectively.

$$\text{Then we have } \lim_{m \rightarrow \infty} \|x(\omega) - x_m(\omega)\|_X = 0 \text{ a. s.}$$

Proof

We choose a random solution $x^* \in S(C)$ of (1) such that $x^*(\omega) = x(\omega)$ a. s. Then there exists a sequence $x'_m \in S(C_m)$, $m \in N$, with the property:

$$\lim_{m \rightarrow \infty} \|x(\omega) - x'_m(\omega)\|_X = 0 \text{ a. s.} \quad \text{and}$$

$$\lim_{m \rightarrow \infty} \|T_m(\omega) x'_m(\omega)\|_Y = 0 \text{ a. s.}$$

From b) we obtain:

$$\|x_m(\omega) - x'_m(\omega)\|_X \leq \alpha(\omega, \|T_m(\omega) x'_m(\omega)\|_Y) \text{ a. s., } m \in N.$$

The continuity property of $\alpha(\omega, \cdot)$ a. s. proves the theorem.

q. e. d.

Remark 4

a) Conditions a) and b) of Theorem 2 are in some sense stochastic versions of a "consistency condition for $T, T_m, m \in N$," and an "inverse stability condition for $T_m, m \in N$," respectively (see [36]). The motivation of the above concept originates from [36].

b) Under the conditions a) and b) of Theorem 2 the random solutions of (1) and (1m) are unique. In this sense, the approximation of locally unique random solutions of (1) is possible by means of Theorem 2. Condition c) can be replaced by assuming the existence of wide-sense solutions (Theorem 1).

c) The consistency condition a) can be replaced by the sufficient condition of Lemma 3 for the convergence of measurable selectors, which is a certain generalization of [34, Theorem 4.3.] to the case of Banach spaces, and by the condition: For P-almost all $\omega \in \Omega$ and all $x \in C(\omega), x_m \in C_m(\omega), m \in N$, with the property $\lim_{m \rightarrow \infty} \|x_m - x\|_X = 0$ one has $\lim_{m \rightarrow \infty} \|T_m(\omega)x_m - T(\omega)x\|_Y = 0$. Condition a) simplifies itself essentially in the case of $C_m(\omega) \cong C(\omega)$ a.s., $m \in N$. Then we can choose $x'_m := x$ and a) reduces itself to:

$$\lim_{m \rightarrow \infty} \|T(\omega)x - T_m(\omega)x\|_Y = 0, \text{ for all } x \in C(\omega), \text{ a.s.}$$

Of course it is sufficient for the validity of Theorem 2 if a) is only fulfilled for the random solution $x \in S(C)$ of (1).

Lemma 3

Let $C_m: \Omega \rightarrow Cl(X), m \in N$, be separable and such that:

$$\lim_{m \rightarrow \infty} D(C(\omega), C_m(\omega)) = 0 \text{ a.s. } (D \text{ denotes the Hausdorff-distance})$$

Then, for all $x \in S(C)$, there exist $x'_m \in S(C_m), m \in N$, such that $\lim_{m \rightarrow \infty} \|x(\omega) - x'_m(\omega)\|_X = 0$ a.s.

Proof

We choose $x \in S(C)$ and define the following multifunctions ($\varepsilon_m > 0, m \in N$):

$$S_m: \Omega \rightarrow Cl(X) \quad S_m(\omega) := \{x' \in C_m(\omega) \mid \|x' - x(\omega)\|_X \leq d(x(\omega), C_m(\omega)) + \varepsilon_m\}.$$

From [20, Lemma 2.1.] we know that $d(x(\cdot), C_m(\cdot)): \Omega \rightarrow R^1$ is measurable and from [6, Theorem 1] we conclude that the multifunctions S_m are measurable.

Now, for every sequence $x'_m \in S(S_m), m \in N$ (Lemma 1), it holds that

$$\|x'_m(\omega) - x(\omega)\|_X \leq d(x(\omega), C_m(\omega)) + \varepsilon_m, m \in N, \omega \in \Omega.$$

If we choose $\varepsilon_m \rightarrow 0$, the proof is finished. q. e. d.

Remark 5

a) It seems to be possible to apply Theorem 3 to stochastic projectional schemes in Hilbert spaces ([16]) for constructing measurable approximations to random solutions of nonlinear random operator equations and a result like [16, Propos. 3.3.] seems to be available for the nonlinear case.

b) A fundamental purpose in the approximation of (1) is to solve the problems (1m) in a certain sense "simpler". Stochastic projectional schemes reduce (1) to a finite-dimensional but still stochastic problem (1m). That is the reason why so-called "discretization schemes", in which (1m) represents a deterministic problem, are suggested in chapter 5.

c) So far we have restricted ourselves to the a.s.-convergence, but the formulation of Theorem 3 and its proof give some hints how the convergence in probability, the L_p -convergence a. s. o. can be obtained under other convergence assumptions.

4. Approximation of random fixed points of contractive random operators

As a special case of the results of chapter 3 we now turn to the approximation of random fixed point problems. We consider $C: \Omega \rightarrow \mathfrak{P}(X)$ and a random operator $T: Gr C \rightarrow X$ with random domain C and the random fixed point problem

$$x = T(\omega)x. \tag{2}$$

For the following investigations we generally assume the following:

- (i) $C: \Omega \rightarrow Cl(X)$ is separable;
- (ii) for all $(\omega, x) \in Gr C: T(\omega)x \in C(\omega)$;
- (iii) there exists an $\alpha: \Omega \rightarrow [0, 1)$ such that for all $(\omega, x), (\omega, y) \in Gr C$:

$$\|T(\omega)x - T(\omega)y\|_X \leq \alpha(\omega)\|x - y\|_X$$

("contractive random operator", [14]).

The notions "wide-sense fixed point" and "random fixed point" of (2) are defined analogous to the related notion of solution for (1) ([1, 19]).

Under the above assumptions there exists, according to [14, Theorem 11] resp. Theorem 1 (Lemma 2b), Remark 3c) a unique random fixed point of (2).

Moreover, let $C_m: \Omega \rightarrow \mathfrak{P}(X)$ and random operators $T_m: Gr C_m \rightarrow X$ with random domains C_m be given for all $m \in N$ and we consider

$$x_m = T_m(\omega)x_m. \tag{2m}$$

Theorem 3

- a) Let condition a) of Theorem 2 be fulfilled;
- b) for all $m \in N$ let $C_m: \Omega \rightarrow Cl(X)$ be separable, for all $(\omega, x) \in Gr C_m$ let $T_m(\omega)x \in C_m(\omega)$ and for all $(\omega, x), (\omega, y) \in Gr C_m$:

$$\|T_m(\omega)x - T_m(\omega)y\|_X \leq \alpha(\omega)\|x - y\|_X.$$

Then we have $\lim_{m \rightarrow \infty} \|x(\omega) - x_m(\omega)\|_X = 0$ a.s., where x and x_m are the unique random fixed points of (2) and (2m), respectively, $m \in N$.

Proof

We use Theorem 2 and have to prove only condition b) of this Theorem. We write (2m) as $(I - T_m(\omega))x_m = 0$. Then it holds for all $\omega \in \Omega, m \in N$, and $\bar{x}_m, \tilde{x}_m \in C_m(\omega)$ because of condition b) that:

$$\begin{aligned} & \|(I - T_m(\omega))\bar{x}_m - (I - T_m(\omega))\tilde{x}_m\|_X \\ & \geq \|\bar{x}_m - \tilde{x}_m\|_X - \|T_m(\omega)\bar{x}_m - T_m(\omega)\tilde{x}_m\|_X \\ & \geq (1 - \alpha(\omega))\|\bar{x}_m - \tilde{x}_m\|_X. \end{aligned}$$

This proves the Theorem. q. e. d.

Remark 6

a) In [2, p. 653] A. T. Bharucha-Reid asked for Theorems like the one above. Since that time a number of results has been obtained, e.g. [25, Theorem 3.1.; 4, Theorem 4]. Theorem 3 is a generalization of [25] to the case of operators on random domains. We remark that we restrict ourselves to the case of a.s.-convergence, but refer to Remark 5c).

b) A number of random fixed point problems, where the random operators are not necessarily contractive, can be traced back to problems with locally contractive random operators under corresponding assumptions (cf. [10]).

Iteration methods for the approximate solution of (1) or (2) were often investigated, too (e.g. [19, 25, 27, 29]). In connection with Theorem 3 it seems to be interesting to combine the well-known fixed point iteration with the approximation process (2m).

Such a result is already contained in [19, Theorem 2]. We give a more general version of such a result for operators on random domains. Let us consider:

$$\bar{x}_m(\omega) := T_m(\omega) P_m(\omega, \bar{x}_{m-1}(\omega)), \quad m = 1, 2, \dots, \omega \in \Omega, \quad (3)$$

where

- (i) $\bar{x}_0: \Omega \rightarrow X$ is measurable;
- (ii) $P_m: \Omega \times X \rightarrow X, m \in N$, are random operators with the properties:

$$\left. \begin{aligned} P_m(\omega, x) &\in C_m(\omega) \text{ for all } x \in X \\ P_m(\omega, x) &= x \text{ for all } x \in C_m(\omega) \\ \|P_m(\omega, x) - P_m(\omega, y)\|_X &\leq \|x - y\|_X \text{ for all } x, y \in X \end{aligned} \right\} \begin{array}{l} m \in N, \\ \omega \in \Omega. \end{array}$$

(3) is a so-called approximation-iteration process and represents a stochastic version of the iteration method in [24, p. 61].

Theorem 4

If the assumptions of Theorem 3 are fulfilled and $\bar{x}_m: \Omega \rightarrow X, m \in N$, are the random variables defined by (3) (with the assumptions (i), (ii)), then

$$\lim_{m \rightarrow \infty} \|x(\omega) - \bar{x}_m(\omega)\|_X = 0 \quad \text{a.s.},$$

where x is the unique random fixed point of (2).

Proof

First let us remark that (3) is well-defined and that all $\bar{x}_m: \Omega \rightarrow X$ are measurable by Lemma 2. If $x_m, m \in N$, are the random solutions of (2m), it holds for all $m \in N$ and a.s. that:

$$\begin{aligned} \|x_m(\omega) - \bar{x}_m(\omega)\|_X &= \|T_m(\omega) x_m(\omega) - T_m(\omega) P_m(\omega, \bar{x}_{m-1}(\omega))\|_X \\ &\leq \alpha(\omega) \|x_m(\omega) - P_m(\omega, \bar{x}_{m-1}(\omega))\|_X \\ &\leq \alpha(\omega) \|x_m(\omega) - \bar{x}_{m-1}(\omega)\|_X \end{aligned}$$

and the proof can be continued as in [38, p. 78]. We obtain

$$\lim_{m \rightarrow \infty} \|x_m(\omega) - \bar{x}_m(\omega)\|_X = 0 \quad \text{a.s.} \quad \text{q. e. d.}$$

Remark 7

a) If $C_{m-1}(\omega) \subseteq C_m(\omega), \omega \in \Omega, m = 2, 3, \dots$, we can choose $P_m = I$ in (3). If X is a Hilbert space and $C_m: \Omega \rightarrow \text{Cl}(X)$ is such that for all $\omega \in \Omega$ $C(\omega)$ is convex, we can choose $P_m(\omega, x) = P_{C_m}(\omega, x)$, where P_{C_m} is the random metric projector ([17, chapter 3]).

b) In chapter 5 we give an application of the approximation iteration method (3), which is completely deterministic (comp. Remark 9).

5. Discretization schemes for random operators and equations

In this chapter we investigate methods for the discrete approximation of random operator equations and their random solutions. The basic notion is a "discretization scheme" for random operators, which is based on a "discretization" of the underlying probability space $(\Omega, \mathfrak{A}, P)$, i.e. on partitions of Ω into a finite number of measurable sets, and goes back to ideas from [31]. As in chapter 3 let $C: \Omega \rightarrow \mathfrak{B}(X)$ and $T: \text{Gr } C \rightarrow Y$ be $\mathfrak{A} \otimes \mathfrak{B}(X)$ -measurable.

Definition 2

$\{\mathfrak{A}_m, C_m, T_m\}_{m \in N}$ will be called a "discretization scheme" for T if for all $m \in N$ there exist

- (i) a finite partition $A_{ml} \in \mathfrak{A}, l = 1, \dots, m$, of Ω ,

$$\text{i.e. } \bigcup_{l=1}^m A_{ml} = \Omega \text{ and } A_{ml} \cap A_{mk} = \emptyset, l \neq k,$$

such that $\mathfrak{A}_m = \sigma(\{A_{ml}\}_{l=1, \dots, m})$.

($\sigma(\mathfrak{E})$ denotes the smallest σ -algebra containing $\mathfrak{E} \subseteq \mathfrak{A}$).

- (ii) $C_{ml} \in \mathfrak{B}(X), T_{ml}: C_{ml} \rightarrow Y, l = 1, \dots, m$, such that:

$$\left. \begin{aligned} C_m: \Omega \rightarrow \mathfrak{B}(X) \quad C_m(\omega) &= C_{ml} \\ T_m: \text{Gr } C_m \rightarrow Y \quad T_m(\omega) &= T_{ml} \end{aligned} \right\} \omega \in A_{ml}, l = 1, \dots, m.$$

The operators T_m are called "discrete random operators".

Now let a discretization scheme $\{\mathfrak{A}_m, C_m, T_m\}_{m \in N}$ for T be given and we consider the random operator equations:

$$T(\omega) x = 0 \quad (1)$$

$$T_m(\omega) x_m = 0, \quad m \in N. \quad (1m)$$

Remark 8

a) For all $m \in N$ T_m is a random operator on random domain C_m .

b) Every wide-sense solution of (1m) is also a random solution and is of the form $x_m: \Omega \rightarrow X, x_m(\omega) = x_{ml}, \omega \in A_{ml}, l = 1, \dots, m$, where $x_{ml} \in C_{ml}$ and $T_{ml}x_{ml} = 0, l = 1, \dots, m$.

Consequently (1m) reduces itself to m deterministic operator equations.

c) If $g: X \rightarrow R^1$ and if E denotes the mean value with respect to $(\Omega, \mathfrak{A}, P)$, it holds that

$$E(g(x_m(\cdot))) = \sum_{l=1}^m g(x_{ml}) P(A_{ml}),$$

i.e. the computation of statistical characteristics of random solutions of (1m) is easily possible if the probabilities $P(A_{ml}), l = 1, \dots, m$, are known. Under suitable convergence properties of $\{x_m(\cdot)\}_{m \in N}$ to a random solution x for (1) (Theorem 3; Remark 5 c)) there also result convergence statements for the statistical characteristics:

$$\lim_{m \rightarrow \infty} E(g(x_m(\cdot))) = E(g(x(\cdot))).$$

Remark 9

We consider the approximation iteration method (3) with discrete random operators $T_m, m \in N$, under the simplifying assumption that $C_{m-1}(\omega) \subseteq C_m(\omega), \omega \in \Omega$, i.e. (3) is of the form:

$$\bar{x}_m(\omega) := T_m(\omega) \bar{x}_{m-1}(\omega), \quad m = 2, 3, \dots, \bar{x}_1 \in C_1.$$

If we assume $\mathfrak{A}_m \subseteq \mathfrak{A}_{m+1}, m \in N$, then this iteration process can be described completely deterministic. If we additionally assume that for all $m \in N$ there exists a $k_m \in \{1, \dots, m\}$ such that $A_{m, k_m} = A_{m+1, k_m} \cup A_{m+1, m+1}, A_{ml} = A_{m+1, l}, l \neq k_m$, then the following iteration scheme results:

$$\bar{x}_m(\omega) = \bar{x}_{ml} := \begin{cases} T_{ml} \bar{x}_{m-1, l} & \omega \in A_{ml}, l = 1, \dots, m-1, \\ T_{mm} \bar{x}_{m-1, k_m-1} & \omega \in A_{mm}, l = m. \end{cases}$$

In the following we want to refer to a simple possibility to construct discretization schemes.

For this purpose we consider a real separable Banach space Z , a random variable $z: \Omega \rightarrow Z$ with range

$$R(z) := \{z(\omega) | \omega \in \Omega\} \subseteq Z',$$

a multifunction $C: Z' \rightarrow \mathfrak{B}(X)$, and an operator $T: Z' \times X \rightarrow Y$. Then, under suitable assumptions, the equation

$$T(z(\omega), x) = 0 \quad (4)$$

represents a random operator equation.

Remark 10

a) If in (4) z is replaced by simple random variables $z_m: \Omega \rightarrow Z$ with m values and $R(z_m) \subseteq Z'$, then "discretization schemes" for (4) result in a natural way.

b) The discrete random variables $z_m(\omega) = z_{ml}, \omega \in A_{ml}, l = 1, \dots, m$, have to be chosen according to various criteria, e.g. the convergence, the numerical realization a.s.o. To replace z by conditional expectations w.r.t. certain proper σ -algebras (see chapter 6) seems to be a suitable possibility.

6. Some remarks on the approximation of Banach space-valued random variables via conditional expectations

As before let $(\Omega, \mathfrak{A}, P)$ be a probability space, Z a real separable Banach space with the norm $\|\cdot\|$. The spaces $L_p(\Omega, \mathfrak{A}, P; Z)$ ($1 \leq p < \infty$) let be defined as usually. For $z \in L_1(\Omega, \mathfrak{A}, P; Z)$, $A \in \mathfrak{A}$ with $P(A) > 0$, $E(z|A) := \frac{1}{P(A)} \int_A z(\omega) dP$ is called the conditional expected value of z with respect to A . If $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is a further σ -algebra, then $E(z|\mathfrak{A}_0)$ denotes the conditional expectation of z w.r.t. \mathfrak{A}_0 (see e.g. [12]). We note that if $\{A_i\}_i \subset \mathfrak{A}$ is a finite partition of Ω and $\mathfrak{A}_0 := \sigma(\{A_i\}_i)$, then $E(z|\mathfrak{A}_0)(\omega) = E(z|A_i)$, $\omega \in A_i$.

Lemma 4 ([11], Theorems 1 and 4)

Let $z \in L_p(\Omega, \mathfrak{A}, P; Z)$ and σ -algebras $\mathfrak{A}_m \subseteq \mathfrak{A}$, $\mathfrak{A}_m \subseteq \mathfrak{A}_{m+1}$, $m \in N$, be given and define $\mathfrak{A}' := \sigma(\bigcup_{m \in N} \mathfrak{A}_m)$.

Then we have:

$\lim_{m \rightarrow \infty} E(z|\mathfrak{A}_m) = E(z|\mathfrak{A}')$ a.s. and in the sense of L_p ($1 \leq p < \infty$).

Remark 11

a) If $\mathfrak{A}(z) := \sigma\{z^{-1}(B) | B \in \mathfrak{B}(Z)\}$ denotes the smallest σ -algebra in \mathfrak{A} with respect to which $z: \Omega \rightarrow Z$ is measurable and if $\mathfrak{A}(z) \subseteq \mathfrak{A}' := \sigma(\bigcup_{m \in N} \mathfrak{A}_m)$ (for the definition of \subseteq see [31, p. 526]), then it results in Lemma 4:

(*) $\lim_{m \rightarrow \infty} E(z|\mathfrak{A}_m) = z$ a.s. and in the L_p -sense. In case the σ -algebras \mathfrak{A}_m are moreover generated by finite partitions $\{A_{ml}\}_{l=1, \dots, m} \subset \mathfrak{A}$ of Ω , then (*) represents a statement on the approximation of z by discrete random variables (with the values $E(z|A_{ml})$, $l = 1, \dots, m$) (cf. Remark 10).

If $E(z|A_{ml})$, $l = 1, \dots, m$, $m \in N$, can be computed from the distribution of z , then this approximation is constructive.

b) In [35] possibilities are given how such partitions $\{A_{ml}\}_{l=1, \dots, m}$, $m \in N$, can be generated from the corresponding partition sequences of Z . We remark that in this case the in general infinite-dimensional distribution of z is needed to determine $P(A_{ml})$ and $E(z|A_{ml})$, $l = 1, \dots, m$, $m \in N$, and that this method is not constructive in the case of infinite-dimensional spaces. The following notion allows more practicable possibilities to construct discrete approximations of random variables.

Definition 3 [31, p. 527])

A random variable $z: \Omega \rightarrow Z$ will be called 'separably producible' if there exists a sequence $\{z_i\}_{i \in N}$ of real random variables on $(\Omega, \mathfrak{A}, P)$ such that $\mathfrak{A}(z) \subseteq \sigma(\bigcup_{i \in N} \mathfrak{A}(z_i))$.

Examples

a) $Z = R^n$, i.e. $z: \Omega \rightarrow R^n$ is a vector-valued random variable with components z_i , $i = 1, \dots, n$. Then $\mathfrak{A}(z) \subseteq \sigma(\bigcup_{i=1}^n \mathfrak{A}(z_i))$.

b) $Z = C(I, R^n)$, where $I \subset R^1$ is compact. Let S be a denumerable dense subset of I . Then we have ([31, p. 526]):

$$\mathfrak{A}(z) \subseteq \sigma\left(\bigcup_{i \in S} \bigcup_{t=1}^n \mathfrak{A}(z_i(t))\right).$$

c) Let Z be a real separable Banach space, $z: \Omega \rightarrow Z$ a Gaussian random variable. Then there exists a sequence $\{z_i\}_{i \in N}$ of independent, $N(0, 1)$ -distributed real random variables such that:

$$\mathfrak{A}(z) \subseteq \sigma\left(\bigcup_{i \in N} \mathfrak{A}(z_i)\right).$$

(In [23] a.s.-convergent expansions for Gaussian random variables of the form $z(\omega) = \sum_{i=1}^{\infty} z_i(\omega) e_i + E(z)$ a.s. are investigated,

where $\{e_i\}_{i \in N}$ is a proper set of linearly independent elements).

d) Every random variable $z: \Omega \rightarrow Z$ is separably producible with a certain sequence $\{z_i\}_{i \in N}$.

(For the proof we start as in [18, p. 72] from a sequence of finite partitions $\{\{B_{mj}\}_{j=1, \dots, m}\}_{m \in N} \subset \mathfrak{B}(Z)$ of Z , which was constructed by means of a sequence $\{u_i\}_{i \in N}$ being dense in Z , so that it holds for $z'_m: \Omega \rightarrow Z$

$$z'_m(\omega) = u_j, \omega \in A_{mj} := z^{-1}(B_{mj}), j = 1, \dots, m, m \in N,$$

that: $z(\omega) = \lim_{m \rightarrow \infty} z'_m(\omega)$, for all $\omega \in \Omega$.

$$[31, Lemma 1] \text{ yields: } \mathfrak{A}(z) \subseteq \sigma\left(\bigcup_P \bigcup_{m \in N} \bigcup_{j=1}^m \mathfrak{A}((1_{A_{mj}}))\right).$$

Remark 12

a) If $\{z_i\}_{i \in N}$ is a sequence of real random variables on $(\Omega, \mathfrak{A}, P)$ and if \mathfrak{A}' is defined by $\mathfrak{A}' := \sigma(\bigcup_{i \in N} \mathfrak{A}(z_i))$, then we can proceed

according to the following principle in order to generate \mathfrak{A}' by a sequence of finite partitions of Ω (see Remark 11 a)):

Let $\{\{I_{i,l}^n\}_{j=1, \dots, r(i,n)}\}_{n \in N} \subset \mathfrak{B}(R^1)$, $i \in N$, be sequences of finite partitions of R^1 . Then we define:

$$A_l^{(n,k)} := \bigcap_{i=1}^k z_i^{-1}(I_{i,l}^n), l = 1, \dots, \prod_{i=1}^k r(i,n),$$

$$l_i \in \{1, \dots, r(i,n)\},$$

$$i = 1, \dots, k,$$

$$l \leftrightarrow (l_1, \dots, l_k), k, n \in N.$$

(see also [31, p. 528; 35]).

If the distribution of $(z_1, \dots, z_k): \Omega \rightarrow R^k$ is known (e.g. from the distribution of z , comp. Examples a), b)), then $P(A_l^{(n,k)})$ can be computed from multiple integrals.

b) In the case of Example c) $E(z|A_l^{(n,k)})$ can also be easily computed:

$$E(z|A_l^{(n,k)}) = \sum_{i=1}^k E(z_i | z_i^{-1}(I_{i,l}^n)) e_i.$$

By specialization of the Banach space Z we thus get a possibility to approximate Gaussian processes with continuous sample paths and Gaussian random vectors, respectively.

Another method for the same purpose is to be found in [31, p. 531; 32].

c) Concerning possibilities to approximate random vectors we want to refer to [35].

More details, further statements and detailed proofs on the subject of this chapter are to be found in [32].

7. Application to the approximate solution of random ordinary differential equations

Let us consider the following initial value problem for random ordinary differential equations

$$\dot{x}(t) = f(t, u(\omega, t), x(t)), t \in [t_0, t_1] \tag{5}$$

$$x(t_0) = x_0(\omega),$$

where u is a stochastic process and x_0 is a random variable on a probability space $(\Omega, \mathfrak{A}, P)$.

Such problems were investigated in detail in [1, chapter 6; 9] and in the framework of random operator equations in [7, 13, 15, 22].

In this chapter we want to show the applicability of the concept from chapter 5 to the discrete approximation of random solutions of (5) under relatively strong assumptions. Then, continu-

ing the ideas from [31], we get a general method for the approximate calculation of statistical characteristics of random solutions, such as the distribution, moments a.s.o. The method is based on the approximation of u and x_0 by certain, suitable conditional expectations (comp. chapter 6).

For the following we assume:

(i) $(\Omega, \mathfrak{A}, P)$ is a complete probability space;

$x_0 \in L_1(\Omega, \mathfrak{A}, P; R^n)$ is a vector-valued random variable with $x_0(\omega) \in R(x_0) \subseteq R^n$ for all $\omega \in \Omega$, and $R(x_0)$ is convex, closed; $u: \Omega \times [t_0, t_1] \rightarrow R^r$ is a stochastic process with continuous sample paths and such that $u \in L_1(\Omega, \mathfrak{A}, P; C([t_0, t_1], R^r))$, and for $(\omega, t) \in \Omega \times [t_0, t_1]$ $u(\omega, t) \in R(u) \subset R^r$, where $R(u)$ is convex, closed;

(ii) $f: [t_0, t_1] \times R(u) \times D \rightarrow R^n$ is uniformly continuous and Lipschitz continuous in the last variable (with constant L), where

$$D := \bigcup_{\tilde{x} \in R(x_0)} \{x \in R^n \mid \|x - \tilde{x}\|_{R^n} \leq r_0\}, \quad L(t_1 - t_0) < 1$$

and

$$0 < (t_1 - t_0)Lr_0 + \sup_{x \in R(x_0)} \int_{t_0}^{t_1} \|f(t, u(\omega, t), x)\|_{R^n} dt \leq r_0.$$

Then we can formulate problem (5) as a random fixed point problem and we define:

$X := C([t_0, t_1]; R^n)$ with the supremum norm $\|\cdot\|$,

$C: \Omega \rightarrow \text{Cl}(X)$ $C(\omega) := \{x \in X \mid \|x - x_0(\omega)\| \leq r_0\}$,

$T: \text{Gr } C \rightarrow X$ $[T(\omega)x](t) := x_0(\omega) + \int_{t_0}^t f(s, u(\omega, s), x(s)) ds,$
 $t \in [t_0, t_1], (\omega, x) \in \text{Gr } C.$

Analogous to [15, p. 75/76] we show that $C: \Omega \rightarrow \text{Cl}(X)$ is separable and T is a random operator (with random domain C). By standard arguments it follows that for all $(\omega, x), (\omega, y) \in \text{Gr } C$:

$$\|T(\omega)x - x_0(\omega)\| \leq \int_{t_0}^{t_1} \|f(t, u(\omega, t), x_0(\omega))\|_{R^n} dt + L(t_1 - t_0)\|x - x_0(\omega)\| \leq r_0$$

$$\|T(\omega)x - T(\omega)y\| \leq L(t_1 - t_0)\|x - y\|.$$

The unique random fixed point $x: \Omega \rightarrow X$ of T can by [15, Lemma 10; resp. 22, Proposition 4.2.] be interpreted as a stochastic process the sample paths of which satisfy (5) a.s.

For the approximation of (5) and its random solution we further assume:

(iii) There exists a sequence $\{\{A_{ml}\}_{l=1, \dots, m}\}_{m \in N} \subset \mathfrak{A}$ of partitions of Ω such that for all $t \in [t_0, t_1]$ $u(\cdot, t)$ and x_0 are measurable with respect to $\mathfrak{A}' := \sigma(\bigcup_{m \in N} \{A_{ml}\}_{l=1, \dots, m})$ (see chapter 6).

We define

$$\left. \begin{array}{l} x_{0m}: \Omega \rightarrow R^n \quad x_{0m}(\omega) = E(x_0 | A_{ml}) \\ u_m: \Omega \times [t_0, t_1] \rightarrow R^r \quad u_m(\omega, t) = E(u(\cdot, t) | A_{ml}) \end{array} \right\} \begin{array}{l} \omega \in A_{ml}, \\ l = 1, \dots, m, \\ m \in N, \end{array}$$

and note that $x_{0m}(\omega) \in R(x_0)$, $u_m(\omega, t) \in R(u)$, $\omega \in \Omega$, $t \in (t_0, t_1]$, $m \in N$.

Therefore the following problem makes sense:

$$\begin{aligned} \dot{x}_m(t) &= f(t, u_m(\omega, t), x_m(t)), \quad t \in [t_0, t_1], \\ x_m(t_0) &= x_{0m}(\omega). \end{aligned} \quad (5m)$$

Analogous to Remark 8 a) (5m) reduces itself to m deterministic initial value problems for ordinary differential equations.

Theorem 5

Let the assumptions (i), (ii), (iii) be fulfilled. Then, for all $m \in N$, there exists a unique random solution $x_m: \Omega \rightarrow X$ of (5m) and it holds that:

$$\lim_{m \rightarrow \infty} \|x(\omega) - x_m(\omega)\| = 0 \quad \text{a.s.}$$

Proof

For all $m \in N$ we define $C_m: \Omega \rightarrow \text{Cl}(X)$, $T_m: \text{Gr } C_m \rightarrow X$ analogous to C, T , where we only have to substitute x_{0m}, u_m for x_0, u , and analogous to above we conclude that condition b) of theorem 3 is fulfilled.

To prove condition a) of Theorem 3 we observe that $\lim_{m \rightarrow \infty} \|x_0(\omega) - x_{0m}(\omega)\|_{R^n} = 0$ a.s. (Lemma 5) and that therefore the assumption of Lemma 3 is satisfied. Then, for all $\bar{x} \in S(C)$, there exist $x'_m \in S(C_m)$, $m \in N$, such that $\lim_{m \rightarrow \infty} \|\bar{x}(\omega) - x'_m(\omega)\| = 0$ a.s. Because of $\lim_{m \rightarrow \infty} \|u(\omega) - u_m(\omega)\| = 0$ a.s. (Lemma 5) and the uniform continuity of f it results that $\lim_{m \rightarrow \infty} \|T(\omega)\bar{x}(\omega) - T_m(\omega)x'_m(\omega)\| = 0$ a.s. Theorem 3 provides the assertion. q. e. d.

Remark 13

a) If we choose the partitions $A_{ml} \in \mathfrak{A}$, $l = 1, \dots, m$, $m \in N$, of Ω such that $P(A_{ml}), E(x_0 | A_{ml}), E(u(\cdot, t) | A_{ml}), l = 1, \dots, m, t \in [t_0, t_1]$ can be calculated from finite-dimensional distribution functions of (x_0, u) , then the method explained above is a constructive possibility of the approximate calculation of random solutions of (5) and their statistical properties. The algorithm is analogous to that in [31, Remark 5, p. 529/530]. This proposed method is of the type mentioned in [8, p. 174].

b) In [31] the random differential equations (and random integral equations, respectively) were considered as equations in Banach spaces of random variables. Hence, the above approximation methods could be interpreted as Galerkin methods there. By means of the concept of random operator equations we succeed in giving a more natural foundation of the methods and more general convergence results could be obtained.

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Summary

Werner Römisch

On the approximate solution of random operator equations

In this paper a general approximation scheme for nonlinear random operator equations and their random solutions is developed. As a special case we obtain approximation results for random fixed points of contractive random operators on random domains. As a suitable realization of the general concept we propose so-called "discretization schemes", which lead to constructive, numerical methods for the calculation of statistical characteristics of the random solutions. This procedure is based on the approximation of Banach space-valued random variables by suitable conditional expectations. As an application we consider the approximate solution of random ordinary differential equations.

Werner Römisch

Zur approximativen Lösung stochastischer Operatorgleichungen

Es wird ein allgemeines Approximationsschema für nichtlineare stochastische Operatorgleichungen und deren stochastische Lösungen vorgestellt. Als Spezialfall ergeben sich Approximationsaussagen für stochastische Fixpunkte kontraktiver stochastischer Operatoren. Als geeignete Realisierung des allgemeinen Konzeptes werden „Diskretisierungsschemata“ vorgeschlagen, die zu numerischen Verfahren für die Berechnung statistischer Charakteristiken von stochastischen Lösungen führen. Dieses Vorgehen basiert auf einer Approximation von Banachraum-wertigen Zufallsvariablen durch geeignete bedingte Erwartungen. Als Anwendung wird die approximative Lösung von Differentialgleichungen mit zufälligen Parametern untersucht.

Вернер Рёмиш

Об аппроксимативном решении случайных операторных уравнений

Представляется общая схема аппроксимации нелинейных случайных операторных уравнений и их случайных решений. Частным случаем являются результаты об аппроксимации случайных неподвижных точках случайных операторов сжатия. Как выгодная реализация общей схемы предлагается «методы дискретизации», которые resultent в вычислительных методах для статистических характеристик случайных решений. Названная схема основана на аппроксимации случайных величин со значениями в банаховых пространствах некоторыми условными математическими ожиданиями. В качестве применения исследуется аппроксимативное решение дифференциальных уравнений со случайными параметрами.

Werner Römisch

Sur la solution approximative d'équations à opérateurs stochastiques

L'auteur décrit un schéma d'approximation général pour des équations à opérateurs stochastiques non linéaires et leurs solutions stochastiques. Comme cas spécial, il s'ensuit des énoncés d'approximation pour les points fixes stochastiques d'opérateurs stochastiques contractifs. Pour la réalisation appropriée du concept général, il propose des «schémas de discrétisation» qui conduisent à des procédés numériques pour le calcul de caractéristiques statistiques des solutions stochastiques. Ce processus est fondé sur une approximation des variables aléatoires, à valeur d'espace de Banach, par des espérances conditionnées appropriées. A titre d'exemple appliqué, il étudie la solution approximative d'équations différentielles avec paramètres aléatoires.