Euclidean Geometry

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Summary

In this notebook we develop some linear algebraic tools which can be applied to calculations in any dimension, and to creating graphics in the two- and three-dimensional case. The notebook contains new modules implementing spheres as objects of Euclidean and Riemannian spherical geometry. As an application the construction and plotting of a sphere through four points in the Euclidean 3-space is given. Furthermore, it contains a recursive definition of the generalized geographical parameter representations of n-spheres in the (n+1)-dimensional Euclidean space. Some concepts needed in Möbius geometry, the conformal geometry of the n-sphere, are introduced in Euclidean terms. These concepts are: stereographic projection, its inversion, and reflections at hyperspheres (also called inversions). A version of Erhard Schmidt's orthogonalization: esorthonorm, is introduced in section 2; it has some other features as the function Orthogonalize now built-in *Mathematica*.

Keywords

vector objects, random vectors, rank, orthoframes, unit vectors, norming, cross product, nullvector, standard base, hyperplanes, stereographic projection, inverse stereographic projection, spheres, spheres through four points, hyperspheres, parameter representation for n-spheres, inversion at hyperspheres, torus, geodesics on the flat torus, spiral, spiral group, spiral cylinder, Erhard Schmidt's othogonalization, orthogonal complement.

Version

This notebook is a new version of the notebook eusphere.nb, first published in the item [2] in MathSource. The same is true for the accompanying package euvec.m. In the new version the pacckages vectorcalc.m and euvec.m together replace the old package euvec.m. The package vectorcalc.m contains mainly concepts belonging to affine geometry, while the metric concepts are gathered within the new euvec.m, which needs vectorcalc.m. Since these packages contain the most important procedures used in this and other notebooks, published first in the item Spheres, and the contents as well as some names have been changed, the two versions should not be mixed.   The notebook has been developed with *Mathematica* v. 4.2 and is revised now, as version 5, with *Mathematica* v. 7.0. The actual versions of the notebook and the packages may be downloaded from my homepage.

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■ Introduction

Initialization

The operation of this section are necessary only at the first use of this notebook on your computer. If this is done once, and if the conditions of the subsection "The needed packages" are fulfilled, you may start calling the initialization as described in the next subsection.

■ Preparation

■ Initialization

■ The usages

■ Comparison of some new defined with similar built-in functions

1 The vector space. Basic operations

Summary

In this section we declare vector objects, fix the dimension dim of the vector space, control the linear operations and discuss the Euclidean scalar product which is the Dot product of *Mathematica* applied to vector objects. The cross product is applied to find the equation of the hyperplane through n points in general position.

■ 1.1 Vector objects. The dimension

■ 1.2 The Cross product

■ 1.3 Hyperplanes through n points: Hesse*s normal form

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2 Erhard Schmidt's orthogonalization

Summary

The orthogonalization procedure esorthonorm is contained in the package euvec.m. It follows Erhard Schmidt's orthogonalization, the only difference to which is that the given vector sequence m_List needs not to be linearly independent. If a vector linearly depends on the foregoing vectors, esorthonorm generates the zero vector and eliminates it. Like Erhard Schmidt's orthogonalization it is independant of the dimension; it may be applied to infinite dimensional spaces with a positive semidefinite scalar product, too. The numerical behavior of esorthonorm can be influenced by the option neglect, which sets the parameter of Chop. We describe the Module esorthonorm in the first subsection. After some simple tests we apply it to find the Legendre polynomials. We compare the action of the function esorthonotm with the built-in function Orthogonalize. The last subsection is devoted to find an orthonormal basis of the orthogonal complement of a subspace of a finite dimensional Euclidean vector space.

■ 2.1 The procedure esorthonorm

■ 2.2 The Legendre polynomials generated by esorthonorm and by Orthogonalize

■ 2.3 The Euclidean orthogonal complement

3 Euclidean representations of spheres

Summary

In 3.1 we recursively define a parameter representation of the n-dimensional unit sphere. Subsection 3.2 contains parameter representations for arbitrary hyperspheres, and, for n=3, plot commands for spheres or simple parts of them. Finally, in subsection 3, we construct modules for calculating center, radius and therewith the sphere through four points of the 3-space in general position.

■ 3.1 A parameter representation of the unit n-sphere S^n

■ 3.2 The standard parameter representations of spheres in E^3 and of hyperspheres in E^n

■ **3.3 Cirles in the Euclidean plane**

■ **3.4 Center and radius of the hypersphere in E^n through n+1 points**

■ **3.5 Parameter representations of spheres in the 3-sphere S^3**

4 Stereographic projection

Summary

We want to visualize the geometry of the 3-sphere S^3. By stereographic projection we go to the Euclidean 3-space E^3. Then we apply the 3D-graphics tools of *Mathematica*. For this and other applications we construct a module for the stereographic projection in dimension n, and another for its inversion. We show that these maps are conformal, and preserve k-spheres. The maps are applied to study spheres and tori in S^3 and E^3.

■ **4.1 Definition of the general stereographic projection**

■ **4.2 The inverse stereographic projection**

■ **4.3 Properties of the stereographic projection**

■ **4.3.1 Invariance of k-spheres**

■ **4.3.2 Conformity**

■ **4.4 Example: Tori. Geodesics on the flat torus**

5 Inversions at hyperspheres

Summary

Spherical reflections are fundamental in Möbius geometry: every Möbius transformation is a finite product of "Inversions" (= spherical reflections). They appear also in complex function

theory as the simplest conformal maps not being isometries (= motions). We construct a module for the spherical reflection in n dimension, and show the basic properties of these maps: they map circles in lines or circle, generally k-dimensional subspheres into k-planes or k-subspheres, and are conformal. Since inversions map the infinite outer region of a sphere onto the inner region, and the infinite point to the center of the sphere, these maps may be used to visualize the behavior of figures or functions at infinity. An example is the image of the spriral cylinder constructed in subsection 5.4.

■ 5.1 Spherical reflections

■ 5.2 Planes and spheres are transformed into planes or spheres

■ 5.3 Conformity

■ 5.4 Spirals, spiral cylinder and their inversions

References

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