Elementary Möbius Geometry I

Points and Spheres

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Introduction

In this notebook we apply *Mathematica* to create tools for working in n-dimensional Möbius geometry. The three-dimensional case is emphasized. The calculations and examples illustrate and complete the presentation of the subject in our book [OS3], Section 2.7. There one finds the foundations of elementary Möbius geometry in terms of linear algebra; also hyperbolic and elliptic geometry as used in this notebook are contained there. We intend to apply the functions and modules developed here to Möbius differential geometry in future notebooks.

In the elementary context the Möbius group is defined to be the the group of projective transformations leaving an elliptic hyperquadric S^n of the (n+1)-dimensional real projective space invariant. In homogeneous coordinates the equation of such a hyperquadric has the normal form (for n = 3)

$$-x(1)^2 + x(2)^2 + x(3)^2 + x(4)^2 + x(5)^2 = 0$$

In [OS3) it is proved that the Möbius group is isomorphic to the pseudo-orthogonal group $O(1, n+1))^+$, that is the group leaving the pseudo-orthogonal scalar product of index 1 in the (n+2)-dimensional real vector space

$$\mathbf{pssp}(\mathbf{vec}(x), \mathbf{vec}(y)) := -x(1)\, y(1) + x(2)\, y(2) + x(3)\, y(3) + x(4)\, y(4) + x(5)\, y(5)$$

and a time orientation invariant; these linear transformations generate the projective maps forming the Möbius group. Therefore our consideration are based on the linear algebra of pseudo-Euclidean vector spaces of index 1, sometimes named Minkowski spaces. The tools enhancing *Mathematica* for applications in this field are contained in the package neuvec.m, which needs the packages vectorcalc.m and euvec.m containing useful modules for working in elementary geometry and linear algebra. A notebook "pseuklid.nb" about pseudo-Euclidean geometry is contained in the packed files sphs4.tgz or sphs4.zip on my homepage, see [Spheres]. In Chapter 1 below the usages of the new Modules and Functions defined in the loaded packages and in the Global Context can be seen after evaluating the initialialization cells of the notebook.

The Möbius group acts transitively on the Möbius space: the sphere S^n as a point manifold, and on the manifold of the k-dimensional subspheres. From the latter the case of the hyperspheres, k = n-1, is the simplest and most important. In this notebook we consider mainly the Möbius geometry as the geometry of the point manifold and the geometry of the sphere manifold, the hyperspheres in case n=3 being the spheres. A pseudo-orthonormal frame as the standard basis is mostly appropriate if one treats subspheres; for the geometry of the point manifold the isotropic-orthonormal bases are often useful, because they may be better adapted to point configurations. Both methods are described and applied in the notebook.

Chapter 2 contains the description of the Möbius space and the conformal models of the simply connected space forms. Besides of Euclidean geometry the hyperbolic geometry is presented in greater detail. For the hyperbolic plane geometry we created a special notebook, see [hyp2D]. There one finds the classification of quadrics in the hyperbolic plane.

The Chapters 3, 4, 5 may be considered as an extended update of the notebook mspheres.nb in the collection [Spheres]. Chapter 3 describes the Möbius Geometry of Spheres in pseudo-Euclidean termes. The manifold of all oriented spheres is identified with the set of all spacelike unit vectors. This is the one-sheeted hyper-hyperboloid defined in the 5-dimensional pseudo-Euclidean vector space of index 1 by the equation

```
pssp[vec[x], vec[x]] == -x[1]^2 + x[2]^2 + x[3]^2 + x[4]^2 + x[5]^2 == 1
```

We remark that in Möbius geometry planes are special spheres, more exactly, they are not defined. Considering the model of the Eucldean space as the complement of a distinguished "infinite" point in S^n, the planes are the Möbius geometric spheres containing this point. This is exactly as they appear under stereographic projection with the infinite point as North pole. The stereographic projection is a conformal map of the not directly visible Möbius space S^3 onto the usual Euclidean space E^3 often used in our *Mathematica* code.

In Chapter 4 we study the mutual position of two hyperspheres up to Möbius equivalence.It is characterized by a single invariant, their generalized angle or the Coxeter invariant, which is well known as

$$\text{coxeterinv}[r, R, d] == \frac{-d^2 + r^2 + R^2}{2 r R}$$

```
? coxeterinv
```

coxeterinv[r1, r2, d] is the conformal invariant of two hyperspheres
with radii r1, r2 and distance d of their centers.

Obviously, this is an expression in Eucliden termes r,R,d making no sense in Möbius geometry. Introducing in Chapter 4 the normed representing vector of a sphere with center (x,y,z) and radius r:

```
? spherevec
```

spherevec[x, y, z, r] yields a spacelike unit vector of the pseudo-Euclidean 5-space corresponding to the sphere with center {x, y, z} and radius
r of the Euclidean 3-space.

```
spherevec[x, y, z, r]
```

$$\left\{\frac{1 - r^2 + x^2 + y^2 + z^2}{2 r}, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}, \frac{-1 - r^2 + x^2 + y^2 + z^2}{2 r}\right\}$$

```
Simplify[pssp[%, %] == 1]
```

```
True
```

one easily obtains the Coxeter invariant as the scalar product

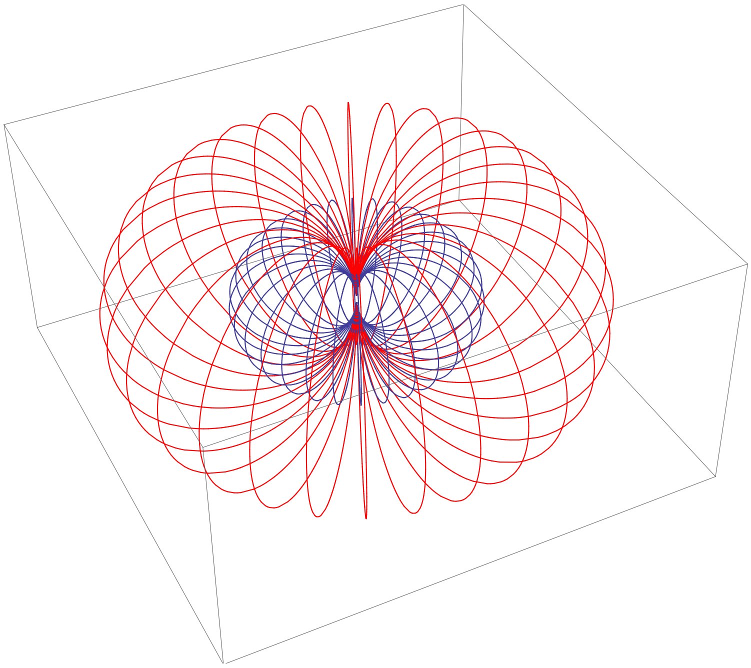
```
Simplify[pssp[spherevec[x, y, z, r], spherevec[X, Y, Z, R]]]
```

$$-\frac{1}{2\,r\,R}\left(-r^2-R^2+x^2-2\,x\,X+X^2+y^2-2\,y\,Y+Y^2+z^2-2\,z\,Z+Z^2\right)$$

This is a new derivation of the Coxeter invariant showing that it is a conformal invariant.

In Chapter 5 the geodesics of the sphere space are classified from an elementary viewpoint. Their are three classes of geodesics corresponding to the types of the central planes whose intersections with the hyper-hyperboloid they are. In each of these classes the Möbius group acts transitively.

In Chapter 6 we describe the Lie algebra of the Möbius group. We apply the *Mathematica* function MatrixExp to pass to the Möbius group and consider some geometric groups as its subgroups. The considerations can be applied to visualize the actions of interesting 1-parametric subgroups on the Möbius space, e.g.



Keywords

Möbius group, space forms (conformal models), isotropic cone, isotropic-orthogonal coordinates, pseudo-orthonormal coordinates, spherical reflections, stereographic projection, pseudo-Euclidean space, pseudo-orthogonal Lie algebra, hyperbolic space, spheres, plaanes, angles, Coxeter invariant, geodesics of the sphere manifold

Initialization

Before starting to work interactively with this notebook first time, read this section carefully and make the necessary preparation. Later it suffices to call the menu item "Evaluation. Evaluate Initialization Cells".

■ The needed packages

For working with the notebook you need the packages
euvec.m, vectorcalc.m, neuvec.m, mspher.m, moeb.m.
You may download these and other packages from my Homepage . From there you find all the packages mentioned above together with the code of the notebook in the file emg.tgz.

Before initializing the notebook ensure that these packages and the current notebook are laying in your working directory, which should be on the $Path of *Mathematica* on your Computerm or

Insert the path of your working directory into the appropriate of the next two cells, e.g.

For Windows:

```
SetDirectory["C:\\Dokumente und Einstellungen\\rolf\\Eigene Dateien\\mymath\\wdir"];
```

For Linux:

```
SetDirectory["~/mymath/wdir"]
/home/rolf/mymath/wdir
```

Now give the cell corresponding to your operating system the properties "Cell Evaluatable" and "Initialization cell" (Menu Cell/ Cell properties), and inactivate these properties for the cells corresponding to the other operating system. If this is done, save the notebook. Next time you may start the notebook directly with the evaluation of the initialization, as follows:

- **The Initialization**

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1. List of Symbols and their Usages

In this Chapter one finds tables of the symbols introduced in the imported packages and in the Global Context.
To get the usages click on the name! If this does not work correctly, enable Dynamic Updating in the Evaluation Menu.

- **1.1. Symbols in the Package vectorcalc.m**
- **1.2. Symbols in the Package euvec.m**
- **1.3. Symbols in the Package neuvec.m**
- **1.4. Symbols in the Package mspher.m**
- **1.5. Symbols in the Package moeb.m**
- **1.5. Symbols in the Global Context**

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2. The Möbius Space and the Space Forms

In this Chapter we describe the Möbius space and the Möbius group. Furthermore we consider the Euclidean, spherical and hyperbolic geometries as subgeometries of Möbius geometry.

- **2.1. The Pseudo-Euclidean Model of the Möbius Space. The Möbius Group**
- **2.2. Spherical Riemann Geometry as Subgeometry of Möbius Geometry**
- **2.3. A Conformal Model of the Euclidean Space**
- **2.4. Isotropic-Orthonormal Coordinates**

2.5. The Möbius Model of the Euclidean Space in Isotropic–Orthonormal Coordinates

2.6. Hyperbolic Geometry as Subgeometry of Möbius Geometry

2.6.1. The Hyperbolic Space

2.6.2. Möbius Geometric Embedding and F. Klein's Conformal Disk Model of the Hyperbolic Space

2.6.3. Poincare´ Model of the Hyperbolic Space

2.6.4. Hyperbolic Lines in the Hyperboliic Space

2.6.5. Hyperbolic Lines in the Hyperbolic Plane

2.6.6. Hyperbolic Planes in the Hyperbolic Space

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3. Spheres

In this Chapter we establish the basic bijection between spheres (including planes as spheres of infinite radius) of the Euclidean 3–space E^3 and one–dimensional Euclidean subspaces of the 5–dimensional pseudo–Euclidean vector space V^5 of index 1. Any spacelike vector defines the orthogonal complement of its span, being a 4–dimensional pseudo–Euclidean subspace, the isotropic vectors of which correspond to the points of a sphere S^2 contained in the Möbius space S^3. In the converse direction, the span of the isotropic vectors representing the points of such a sphere is a pseudo–Euclidean 4–space, defining its normal, an Euclidean 1–space.
Remark: Center and radius are metric concepts. From the viewpoint of Möbius geometry they serve as parameters only.

3.1. Spheres through four points and corresponding spacelike vectors: vradius, vcenter, sphereplot3D

3.1.1. The definitions of the spacelike vector function sph4ptsvec

3.1.2. The spacelike vector as a function of center and radius of the sphere: spherevec, hspherevector

3.1.3. Some tests. A graphic application.

3.1.4. Inversion of hspherevector: vradius, vcenter

3.1.5. Oriented spheres

3.1.6. Parameter representation of the sphere corresponding to a spacelike vector

3.2. Planes

We shortly consider some special aspects of the correspondence between planes and their spacelike vectors, see the Corollary in section 3.1.

3.2.1. The Definition planevec

3.2.2. Random Planes

3.2.3. The Plane through Three Points

3.2.4. Planes through Three Random Points

■ 3.3. The Sphere or Plane Corresponding to a Spacelike Vector

We used the Module euklidsphereplot3D already very often. In this subsection we explain its definition.

■ 3.3.1. The Definition of the Parameter Representation euklidsphere

■ Examples

■ The Module euklidsphereplot3D

■ 3.4. Examples, Comparision of Sphere Plot Methods

■ 3.4.1. Example: Equidistant Unit Spheres

■ 3.4.2. Example: The Spacelike Frame Vectors

■ 3.4.3. Example : A Plane Bundle

■ 3.4.4. Evaluation Problems: Floating Point Numbers

■ 3.5. The Nanifold of All Spheres, Random Spheres

■ 3.5.1. The sphere manifold

■ 3.5.2. A random spacelike unit vector

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4. The Generalized Angle

■ 4.1. Definition. Cases. Coxeter Invariant

■ 4.2. Intersecting Spheres

■ 4.3. Non-intersecting Spheres

■ 4.4. Tangential Spheres

■ 4.5. Intersection of a Sphere and a Plane

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5. Geodesics of the Sphere Manifold

■ 5.1. Introduction

In subsection 3.5 we identified the space of oriented spheres with the hypersurface H: pssp[x,x] = 1 of spacelike unit vectors in the pseudo-Euclidean vector space of index 1: a "pseudo-hypersphere". Since the pseudo-orthogonal group O(1,4) acts transitively on thia pseudosphere, it is a pseudo-Riemannian symmetric space of constant curvature. Since the origin as the center of H is fixed under the action of O(1,4) we define as in the sphere geometry: the geodesics of H are the intersections of the pseudo-hypersphere H with 2-planes through the center o. But now we have three kinds of such planes: pseudo-Euclidean, Euclidean, and isotropic, and the corresponding geodesics. We take the standard unit sphere stb[5] as starting point of the geodesics. To find the types of geodesic we may restrict to the linear hull[stb[1],stb[2]], as a subspace of the tangent space of this hyper-hyperboloid at stb[5]: look at the isotropy action, the action of the subgroup of

the Möbius group preserving stb[5] on the tangential space of H at stb[5]. Obviously, this is the usual standard action of the pseudo-orthogonal group O(3,1) on the 4-dimensional pseudo-Euclidean vector space of index 1.
In the next subsection we interpret the three types of geodesics geometrically.

5.2. Timelike Geodesics

5.3. Spacelike Geodesics

5.4. Isotropic Geodesics

6. The Möbius Group

As remarked in the Introduction already, the Möbius group consists of all time-orientation preserving elements of the pseudo-orthogonal group, denoted by $O(1, 4)^+$, see [OS3]. Proposition 2.7.1. In this Chapter we give two matrix representations of its Lie algebra and consider how some well known geometric groups are realized as subgroups of the Möbius group.

6.1. The Lie Algebra o(1.4) of the Möbius Group

6.2. Lines. The Conformal Representation of the Translation Group and the Dilation Group

6.3. Circles. The Conformal Representation of the Rotation Group.

6.4. The Conformal and isometry Groups of the Space Forms

6.5. A Moving Frame on the Sphere Manifold: pstg5frame

References

[BT] W. Blaschke, G. Thomsen. Volesungen über Differentialgeometrie III. Differentialgeometrie der Kreise und Kugeln. Verlag Springer. Berlin 1929. (This book is available in the WEB: http://www.gbv.de/dms/goettingen/494341041.pdf)

[G06] Alfred Gray, Simon Salamon, Elsa Abbena. Modern Differential Geometry of Curves and Surfaces with *Mathematica*.
Third ed. CRC Press. 2006.

[G94] Alfred Gray. Differentialgeometrie. Klassische Theorie in moderner Darstellung. (Übersetzung aus dem Amerikanischen H. Gollek).
Spektrum Akademischer Verlag, Heidelberg.Berlin.Oxford. 1994.

[H-J] Udo Hertrich -Jeromin. Introduction to Möbius Differential Geometry. London
Mathematical Society Lecture Notes Series, vol. 300, Cambridge University Press,
Cambridge, UK, 2003. (A review of this book (by Thomas E. Cecil) with some hints to the literature is published in BULLETIN (New Series) OF THE AMERICAN MATHEMATICAL SOCIETY. Volume 42, Number 4, Pages 549–554. S 0273-0979(05)01067-0. Article electronically published on July 1, 2005)

[Ma] A. I. Maltzev. Fundaments of Linear Algebra (Russian), Moscou 1956.

[ECG] Rolf Sulanke. The Fundamental Theorem for Curves in the n-Dimensional Euclidean Space. 2009. On my homepage.

[OS3] A. L. Onishchik, R. Sulanke. Projective and Cayley-Klein Geometries. Springer-Verlag. Berlin,Heidelberg. 2006 (German version on my homepage).

[MGII] R. Sulanke. Submanifolds of the Möbius space II. Frenet formulas and curves of constant curvatures. Math. Nachr. 100 (1981), 235–247.

[MGV] R. Sulanke. Möbius geometry V. Homogeneous surfaces in the Möbius space.
Coll. Math. Soc. J. Bolyai 46. Topics in Differential Geometry. Debrecen 1984, 1141–1154.

[Spheres] R. Sulanke. Spheres. A collection of *Mathematica* notebooks and packages. 2004.
Download here. An older version is available from Wolfram's Library Archive. 2000.

[MDG] R. Sulanke. Möbius Differential Geometry. (Only curve theory, on my homepage).

[lox] R. Sulanke. loxodromes-nb. *Mathematica* notebook. (On my homepage.)

[hyp2D] R. Sulanke. Non-Euclidean Geometry. noeuklid2D.nb. *Mathematica* notebook. 2005

Homepage

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