Elementary Möbius Geometry III

Pairs of Subspheres in S^3

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Introduction

This notebook continues the notebooks [emg1] and [emg2] about elementary Möbius geometry. We consider here those pairs of subspheres of the Möbius space being not treated in these notebooks under the action of the Möbius group; the titles of sections 2 - 5 show the pairings considered here. Pairs of spheres are treated in the notebook [emg1], and pairs of circles in [emg2]. The last section treates some examplees of geodesics in the space of 0-spheres. In this section some concepts of Lie algebras are needed. They are contained in the package liealgsh.m, added to moebpack.tar.gz. Again the three-dimensional case is emphasized. The calculations and examples illustrate and complete the presentation of the subject in our book [O-S], Section 2.7, or the paper [S4]. There one finds the foundations of elementary Möbius geometry in terms of linear algebra. See also the Introductions to the notebooks mentioned above.

Our consideration are based on the linear algebra of n-dimensionalpseudo-Euclidean vector spaces of index 1, sometimes named Minkowski spaces. The tools enhancing *Mathematica* for applications in this field are contained in the package neuvec.m, which needs the packages vectorcalc.m and euvec.m containinguseful modules for working in elementary geometry and linear algebra. A notebook “pseuklid.nb” about pseudo-Euclidean geometry is contained in the packed files sphs4.tgz or sphs4.zip on my homepage, see [S1]. In Chapter 1 below the usages of the new Modules and Functions defined in the loaded packages and in the Global Context can be seen after evaluating the initializializationcells of the notebook. Loading the package mpairs.m containingfunctions applied to the Möbius geometry of pairs the other packages already used in Elementary Möbius Geometry I, II are loaded automatically. All the packages needed to work with this notebook are contained in the packed file moebpack.tar.gz; download it and unzip the files to your working directory.

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Keywords

Möbius group, isotropic cone, pseudo-orthonormal coordinates, spheres, circles, point pairs, pairs of subspheres, pseudo-Euclidean space, pseudo-orthogonal Lie algebra, Killingform, Möbius invariantsof pairs of subspheres, Möbius invariantsexpressed by Euclidean invariants, geodesics of the 0-spheres manifold.

Initialization

1. List of Symbols and their Usages

In this Chapter one finds tables of the symbols introduced in the imported packages and in the Global Context.
To get the usages click on the name! If this does not work correctly, enable Dynamic Updating in the Evaluation Menu.

- **1.1. Symbols in the Package vectorcalc.m**
- **1.2. Symbols in the Package euvec.m**
- **1.3. Symbols in the Package neuvec.m**
- **1.4. Symbols in the Package mspher.m**
- **1.5. Symbols in the Package mcirc.m**
- **1.6. Symbols in the Package moeb.m**
- **1.7. Symbols in the Package mpairs.m**
- **1.8. Symbols in the Global Context**

2. Spheresand circles

A pair (S, C), consistingof a sphere S and a circle C of the 3-dimensionalEuclidean space or the 3-dimensional sphere, is defined up to a Möbius transform by exactly a single invariant, if C is not contained in S ("general position"). It is easy to show that all pairs (S, C) with C⊂S are Möbius equivalent; indeed, this follows from the transitivityof the action of the Möbius group of the sphere S^2.on the manifold of all circles contained in S^2. In the first subsection we calculate this invariant moebsc, then we give some examples, and finally we derive an expression of moebsc in terms of Euclidean invariantsof the pair (S,C).

- **2.1 Definition of the Invariant moebsc**
- **2.2 Examples**
- **2.3 Euclidean Interpretation of moebsc**

3. Spheresand Point Pairs

- **3.1. Point Pairs as 0-Spheres**
- **3.2. The Invariant moebspp and the Mutual Position**
- **3.3. An Expression of the Invariant moebspp in Euclidean Terms**
- **3.4. Orthogonality**

4. Circlesand Point Pairs

In this section we consider pairs (C, pp) consistingof a Circle C = S_1^1 and a point pair pp = (a1,a2) = S_2^0 contained in the Euclidean 3-space or the 3-sphere. Such a pair is said to be in general position, if the union C⋃pp is not contained in a 2-sphere. Then its mutual position is defined up to Möbius equivalence by two invariants, which we are going to calculate. We study the relation between the values of these invariants and the geometry of the pair. In the last subsection we find formulas expressing the invariants in terms of the Euclidean invariants of the pairs.

- **4.1. The invariants**
- **4.2. Examples**
- **4.3. The Separation Property**
- **4.4. Two Associated Spheres Belonging to a Pair (S_1^1, S_2^0)**
- **4.5. Möbius Invariants Expressed by Euclidean Invariants**

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5. Point Quadrupels

We consider pairs of 0-spheres, i.e. point quadrupels {{a1,a2},{b1,b2}}. Since each point quadrupel belongs to a sphere, we could consider spherical Möbius geometry: dim = 4, but for later applications we will continue with dim =5 here. Since the Möbius group of the sphere coincides with the broken linear transformations in one complex variable of the Riemann sphere, and this is the projective complex line, an invariant of the quadrupel coincides with the complex cross ratio, the basic invariant of complex projective geometry. Historically, the Möbius group has been found by Möbius in this important context. We considered the original complex Möbius group, and its relation to the 2-dimensional real case, in the notebook [RS]. Here we are interested more in the real situation, for later application in differential geometry. Therefore we will apply the general method of Möbius invariants for pairs of subspheres also in this lowest dimensional case. Since the cross ratio is a complex number we may expect to obtain two real invariants describing the mutual position of two point pairs in the Möbius space.

- **5.1.Acompleteinvariantsystem**
- **5.2. Throws**
- **5.3. Four points on the standard unit circle**

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6. Geodesics in the Space of 0-Spheres

- **6.1. Introduction**
- **6.2. Spacelike geodesics**
- **6.3. Timelike geodesics**
- **6.4. Isotropic geodesics**

References

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