

Lie Algebras

by

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Summary

In this notebook we define basic operations for Lie algebras and calculate the Killing forms of the following Lie algebras:

the general linear Lie algebras $\mathfrak{gl}(n, K)$,

the special linear Lie algebras $\mathfrak{sl}(n, K)$,

the orthogonal Lie algebras $\mathfrak{o}(n)$,

the pseudo-orthogonal Lie algebras $\mathfrak{o}(n, 1)$,

the symplectic Lie algebras $\mathfrak{sp}(2n, K)$,

the unitary Lie algebras $\mathfrak{u}(n)$,

the special unitary Lie algebras $\mathfrak{su}(n)$.

Here K denotes the fields of the real or the complex numbers. Other real forms or complex Lie algebras can be treated similarly; we leave the details to the interested user. Furthermore we construct an orthonormalization procedure appropriate for arbitrary symmetric bilinear forms defined on spaces of matrices, in particular for Killing forms. By the help of this orthonormalization we calculate the index of the considered Killing forms (in the real case), and compare it with the index listed in [3]. It is well known that the Killing forms play an important role in the classification of semisimple Lie algebras and the geometric applications of semisimple Lie groups, see e.g. the book S. HELGASON [1]. Applications for finding geodesics in the Möbius geometry of spaces of m -spheres are contained in [2]. In principle the tools developed in this notebook give the possibility to perform any calculation in the series of Lie algebras listed above.

Keywords

indefinite scalar products, Lie algebra, semisimple Lie algebra, compact Lie algebra, orthogonal Lie algebra, pseudo-orthogonal Lie algebra, linear Lie algebra, special linear Lie algebra, symplectic Lie algebra, unitary Lie algebra, special unitary Lie algebra, orthogonalization, Killing form, commutator, Jacoby identity, structure constants, center.

1 Introduction, Initialization

■ 1.1 The Killing form

The **Killing form** is a symmetric bilinear form on the Lie algebra defined as the trace of the linear endomorphism

$$\text{tr}[a,b] := \text{trace}[\text{ad}[a] \circ \text{ad}[b]],$$

where

$$a \mapsto \text{ad}[a], \quad \text{ad}[a][x] := \text{commute}[a, x],$$

denotes the **adjoint representation** of the Lie algebra. In general, the Killing form is neither definite nor non degenerate. By E. Cartan's criterion the Lie algebra is **semisimple** if and only if its Killing form is non degenerate; a semisimple Lie algebra is **compact** (that means, the corresponding connected Lie groups are compact) if and only if its Killing form is (negative) definite. The semisimple Lie algebras may be decomposed into a direct sum of simple Lie algebras, which are classified, see [1], [3]. In the summary we indicated the main types of infinite series of **simple** Lie algebras: $\mathfrak{sl}(n, \mathbb{K})$, $\mathfrak{o}(n)$ ($n \neq 2, 4$), $\mathfrak{sp}(2n, \mathbb{K})$; besides of these there exist yet a finite number of so called exceptional simple Lie algebras, which we shall not consider in this notebook. The Lie algebras $\mathfrak{o}(n, 1)$ ($n \neq 2, 4$), $\mathfrak{su}(n)$ are real simple Lie algebras; besides of them there exists yet other simple real Lie algebras not considered here. The Lie algebras $\mathfrak{gl}(n, \mathbb{K})$, $\mathfrak{u}(n)$ are not simple; they are direct products of their one-dimensional center and the simple Lie algebras $\mathfrak{sl}(n, \mathbb{K})$ resp. $\mathfrak{su}(n)$. Below we consider the Lie algebras over the complex or the real numbers mentioned in the summary, and calculate their Killing forms.

■ 1.2 Initialization

Activite Kernel/Evaluation/Evaluate/Initialization from the menu!

The concepts considered in this notebook depend on the dimension `dim` of the underlying vector space (= the order of the square matrices) and, in the pseudo-euclidean case, the index `ind` of the (non-degenerate, symmetric) scalar product on this space. These basic constants should be defined after the initialization; then their values with $0 \leq \text{ind} \leq \text{dim}$ may be changed later. If fixed before the initialization, these constants could not be changed later for initialized functions in which they appear. **Thus the initialization cells should not be evaluated if `dim` and `ind` have defined values already.**

```
In[46]:= dim = 3; ind = 1;
```

■ 1.3 The packages `liealg.m`, `liealgun.m`

■ 1.4 Some tools

A general vector, random vectors, random matrices

Smoothing

2 The general and the special linear Lie algebras

■ 2.1 Basic definitions

In *Mathematica* the addition and the multiplication of matrices are implemented linear operations. We combine them in the definition of the commutator, which is the basic operation for Lie algebras. We test the properties of some combinations of these operations in *Mathematica* and apply them to define basic concepts for the general linear Lie algebra $gl(n, K)$, which is defined as the vector space of all square matrices of order n over the field K , provided with the commutator operation.

Symbolic matrices and commutator

We want to distinguish vector Arrays from matrix Arrays, and to have a simple construction of square matrices with given dimension. Therefore we define:

```
In[6]:= matrix[a_, n_, m_] := Array[a, {n, m}];
       matrix[a_, n_:dim] := matrix[a, n, n]
```

```
In[60]:= matrix[a] // MatrixForm
```

```
Out[60]//MatrixForm=
```

$$\begin{pmatrix} a[1, 1] & a[1, 2] & a[1, 3] \\ a[2, 1] & a[2, 2] & a[2, 3] \\ a[3, 1] & a[3, 2] & a[3, 3] \end{pmatrix}$$

```
In[8]:= nullmatrix[k_, n_:dim] := Table[0, {k}, {n}];
       nullmatrix[n_] := nullmatrix[n, n];
       nullmatrix[] := nullmatrix[dim]
```

```
In[59]:= nullmatrix[] // MatrixForm
```

```
In[87]:= nullmatrix[3, 5] // MatrixForm
```

The commutator is the basic operation in Lie algebra; for Lie algebras of matrices the commutator is defined as follows:

```
In[11]:= commute[m_, l_] := Simplify[m.l - l.m]
```

The following true equation is the [Jacoby identity](#):

```
In[61]:= TrueQ[Simplify[commute[commute[matrix[a], matrix[b]], matrix[c]] +
               commute[commute[matrix[b], matrix[c]], matrix[a]] +
               commute[commute[matrix[c], matrix[a]], matrix[b]]] == nullmatrix[]]
```

```
Out[61]= True
```

Basis and cobasis of $\mathfrak{gl}(n,K)$

The vector space of the general linear Lie algebra $\mathfrak{gl}(n,K)$ over the field K has dimension n^2 : it is the space of all square matrices of order n . The standard basis consists of all matrices with only one element 1 and all the others 0:

```
In[12]:= basismatrix[i_, j_, n_: dim] := Module[{k, l},
  If[i < 1 || j < 1 || i > n || j > n, Print["Incorrect indices!"]; Return[]];
  Table[KroneckerDelta[i, k] * KroneckerDelta[j, l], {k, n}, {l, n}]]
```

The standard basis:

```
In[13]:= glbasis[n_: dim] :=
  Flatten[Table[basismatrix[i, j, n], {i, n}, {j, n}], 1]
```

Here is the basis of $\mathfrak{gl}(3,K)$:

```
In[47]:= Map[MatrixForm, glbasis[]]
```

```
Out[47]= {
  {
     $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,
     $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 
  }
}
```

The corresponding dual basis consists of the linear forms which yield the element $m[[i,j]]$ of the matrix m :

```
In[14]:= cobasis[i_, j_, n_: dim][m_] := Module[{k, l, h = n},
  If[i < 1 || j < 1 || i > h || j > h || Dimensions[m] != {h, h},
  Print["Incorrect indices!"]; Return[]];
  Sum[KroneckerDelta[i, k] * KroneckerDelta[j, l] * m[[k, l]],
  {k, 1, n}, {l, 1, n}]]
```

Example

```
In[48]:= Sum[basismatrix[i, j] * cobasis[i, j][matrix[b]], {i, dim}, {j, dim}]
```

```
Out[48]= {{b[1, 1], b[1, 2], b[1, 3]},
  {b[2, 1], b[2, 2], b[2, 3]}, {b[3, 1], b[3, 2], b[3, 3]}}
```

```
In[198]:= matrix[b] ==
```

```
Sum[basismatrix[i, j] * cobasis[i, j][matrix[b]], {i, dim}, {j, dim}]
```

```
Out[198]= True
```

The structure constants

are the coefficients of the commutator of the basis elements:

```
In[15]:= strconst[i_, j_, r_, s_, u_, v_, n_: dim] :=
  cobasis[i, j][commute[basismatrix[r, s, n], basismatrix[u, v, n]]]
```

```
In[47]:= Table[strconst[i, j, dim, 1, 1, 1], {i, dim}, {j, dim}] // MatrixForm
```

```
Out[47]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

```

```
In[49]:= strconst[i, j, r, s, r, s]
```

```
Out[49]= 0
```

```
In[52]:= strconst[i, j, 3, 1, 1, 1]
```

```
Out[52]= KroneckerDelta[1, j] KroneckerDelta[3, i]
```

```
In[54]:= strconst[3, 1, 3, 1, 1, 1]
```

```
Out[54]= 1
```

```
In[55]:= strconst[3, 1, 3, 1, 1, 1] + strconst[3, 1, 1, 1, 3, 1]
```

```
Out[55]= 0
```

The Killing form of the general linear Lie algebra

Any element of the Lie algebra defines a linear endomorphism of the Lie algebra:

```
In[48]:= ad[a_][x_] := commute[a, x]
```

The correspondence $a \mapsto \text{ad}[a]$ is called the [adjoint representation](#) of the Lie algebra.

The [Killing form](#), or trace form, is the trace of the composition: $\text{killing}[a, b] := \text{trace}[\text{ad}[a] \circ \text{ad}[b]]$, where $\text{trace}[C]$ denotes the trace of the linear endomorphism C . In case of the general linear Lie algebra this leads to

```
In[16]:= glkilling[a_, b_, n_:dim] :=
Sum[
  Sum[cobasis[i, j, n][commute[a, commute[b, basismatrix[i, j, n]]],
    {i, 1, n}], {j, 1, n}]
```

The general shape of glkilling as a bilinear or quadratic form can be shown by

```
In[56]:= Simplify[glkilling[matrix[a], matrix[b]]]
```

```
Out[56]= -2 a[3, 3] b[1, 1] + 6 a[2, 1] b[1, 2] + 6 a[3, 1] b[1, 3] + 6 a[1, 2] b[2, 1] -
2 a[3, 3] b[2, 2] + 6 a[3, 2] b[2, 3] + 6 a[1, 3] b[3, 1] + 6 a[2, 3] b[3, 2] +
4 a[3, 3] b[3, 3] - 2 a[2, 2] (b[1, 1] - 2 b[2, 2] + b[3, 3]) +
a[1, 1] (4 b[1, 1] - 2 (b[2, 2] + b[3, 3]))
```

```
In[58]:= Simplify[glkilling[matrix[a], matrix[a]]]
```

```
Out[58]= 4 (a[1, 1]^2 + 3 a[1, 2] a[2, 1] + a[2, 2]^2 + 3 a[1, 3] a[3, 1] +
3 a[2, 3] a[3, 2] - a[2, 2] a[3, 3] + a[3, 3]^2 - a[1, 1] (a[2, 2] + a[3, 3]))
```

Also the standard basis of $\mathfrak{gl}(2, K)$ is not orthogonal with respect to the Killing form:

```
In[65]:= dim = 2
```

```
Out[65]= 2
```

```
In[66]:= Simplify[glkilling[matrix[a], matrix[a]]]
```

```
Out[66]= 2 (a[1, 1]^2 + 4 a[1, 2] a[2, 1] - 2 a[1, 1] a[2, 2] + a[2, 2]^2)
```

In section 2.2 we construct a orthogonalization procedure for symmetric matrix bilinear forms.

The elements of the center

```
In[68]:= dim = 3;
```

By definition, the center of the Lie algebra is the kernel of the adjoint representation; with other words, it consists of all elements a with $\text{commute}[a, x] = \text{nullmatrix}$ for all elements x of the Lie algebra. By

```
In[59]:= Solve[Flatten[Table[commute[basismatrix[i, j], matrix[x]] == 0,
    {i, dim}, {j, dim}]], Flatten[matrix[x]]]
```

Solve::svars: Equations may not give solutions for all "solve" variables. >>

```
Out[59]= {{x[1, 1] -> x[3, 3], x[1, 2] -> 0, x[1, 3] -> 0, x[2, 1] -> 0,
    x[2, 2] -> x[3, 3], x[2, 3] -> 0, x[3, 1] -> 0, x[3, 2] -> 0}}
```

```
In[60]:= MatrixForm[matrix[x] /. Flatten[%]]
```

```
Out[60]//MatrixForm=
```

$$\begin{pmatrix} x[3, 3] & 0 & 0 \\ 0 & x[3, 3] & 0 \\ 0 & 0 & x[3, 3] \end{pmatrix}$$

one has an example that it consists of all scalar multiples of the identity matrix:

```
In[17]:= zelm[z_, n_: dim] := IdentityMatrix[n] * z
```

```
In[61]:= zelm[p] // MatrixForm
```

```
Out[61]//MatrixForm=
```

$$\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

As well known, the defect subspace of the Killing form of $\mathfrak{gl}(n, K)$ coincides with the center of $\mathfrak{gl}(n, K)$. Indeed, we solve

```
In[62]:= Solve[Flatten[Table[glkilling[basismatrix[i, j], matrix[x]] == 0,
    {i, dim}, {j, dim}]], Flatten[matrix[x]]]
```

Solve::svars: Equations may not give solutions for all "solve" variables. >>

```
Out[62]= {{x[1, 1] -> x[3, 3], x[1, 2] -> 0, x[1, 3] -> 0, x[2, 1] -> 0,
    x[2, 2] -> x[3, 3], x[2, 3] -> 0, x[3, 1] -> 0, x[3, 2] -> 0}}
```

```
In[63]:= (matrix[x] /. Flatten[%]) // MatrixForm
```

```
Out[63]//MatrixForm=
```

$$\begin{pmatrix} x[3, 3] & 0 & 0 \\ 0 & x[3, 3] & 0 \\ 0 & 0 & x[3, 3] \end{pmatrix}$$

and obtain the center again. Obviously

```
In[64]:= commute[matrix[a], zelm[p]] // MatrixForm
```

```
Out[64]//MatrixForm=
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
In[71]:= glkilling[matrix[a], zelm[p]]
```

```
Out[71]= 0
```

■ 2.2 Matrix orthonormalization

Applying orthonormalization with respect to the Killing form on any basis of a Lie algebra of matrices we find the rank and the index of the Killing form which are fundamental invariants of the Lie algebra. The orthonormalization procedure defined in this section can be applied to arbitrary symmetric bilinear forms on matrix spaces; it is independent of the Lie algebra structure.

Here is a matrix version of `euvec`outzero` (from `euvec.m`, in [2]), which we need for the orthonormalization procedure; the procedure deletes the null matrices of the list `b` of matrices:

```
In[18]:= matrixoutzero[b_List] :=
  Module[{v = {}, l = Length[b], k = Length[b[[1]]], i},
    For[i = 1, i <= l, i++,
      If[Not[smoothing[b[[i]]] == nullmatrix[k]],
        AppendTo[v, b[[i]], Null[], AppendTo[v, b[[i]]]]]; Return[v]]
```

The following module contained in the package `pseuvec.m` in [2] normalises non isotropic elements:

```
In[19]:= normalize[v_, prod_: glkilling] := Module[{a = Simplify[prod[v, v]], w},
  If[a == 0, w = v, w = v / Sqrt[Abs[a]], w = v / Sqrt[Abs[a]]];
  Return[w]]
```

We created the following procedure `matrixorthonorm` from `orthonorm`, `pseuvec.m` (contained in [2]), replacing `outzero` by `matrixoutzero`:

```
In[20]:= Options[matrixorthonorm] =
  {innerprod -> glkilling, normed -> True, print -> False, neglect -> -10};
```

```

In[21]:= matrixorthonorm[b_List, opts___] :=
Module[{v, w, va, vb, l, ort = {}, i, j, f, h,
  sm, aus, prod, nor, pr, bn, follow, iso, control},
  {prod, nor, pr, bn} = {innerprod, normed, print, neglect} /. {opts} /.
  Options[matrixorthonorm];
  f[h_] := smoothing[h, 10^bn];
  sm[aus_] := MapAll[f, aus];
  w = matrixoutzero[sm[b]];
  Label[follow];
  l = Length[w];
  If[l == 0, Goto[control]];
  For[i = 1, i ≤ l, i++, If[
    Not[Simplify[sm[prod[w[[i]], w[[i]]]]] == 0], AppendTo[ort, w[[i]]];
    If[pr == True, Print[w[[i]] // MatrixForm]];
    v = Simplify[sm[Table[w[[j]] -
      w[[i]] * prod[w[[i]], w[[j]]] / prod[w[[i]], w[[i]], {j, l}]]];
    w = matrixoutzero[v]; Goto[follow], Null, AppendTo[ort, w[[i]]];
    If[pr == True, Print[w[[i]] // MatrixForm]];
    v = Simplify[sm[Table[w[[j]] -
      w[[i]] * prod[w[[i]], w[[j]]] / prod[w[[i]], w[[i]], {j, l}]]];
    w = matrixoutzero[v]; Goto[follow]]];
  Label[iso]; l = Length[w];
  If[l == 0, Goto[control]];
  v = w;
  If[l == 1, AppendTo[ort, v[[1]]];
    If[pr == True, Print[v[[1]] // MatrixForm]];
    Goto[control]];
  For[i = 1, i ≤ l, i++, For[j = i + 1, j ≤ l, j++,
    If[Not[Simplify[sm[prod[v[[i]], v[[j]]]]] == 0], va = v[[j]] - v[[i]];
      vb = v[[j]] + v[[i]]; v[[i]] = va; v[[j]] = vb;
      w = matrixoutzero[Simplify[sm[v]]]; Goto[follow]]];
  ort = Join[ort, v];
  Label[control];
  l = Length[ort];
  If[l == 0, Return[ort]];
  If[nor == True, ort =
    sm[Table[Simplify[normalize[ort[[i]], prod]], {i, Length[ort]}]];
  For[i = 1, i ≤ l - 1, i++, For[j = i + 1, j ≤ l, j++,
    If[Not[Simplify[sm[prod[ort[[i]], ort[[j]]]]] == 0],
      Return[Print["I cannot orthogonalize the given sequence!"]],
      Null, Return[Print["I cannot orthogonalize the
        given sequence!"]]]]]; Return[ort]]

```

Tests

The central elements are isotropic, thus they are returned by normalize:

```
In[69]:= normalize[zelm[a]]
```

```
Out[69]= {{a, 0, 0}, {0, a, 0}, {0, 0, a}}
```



```

In[70]:= gkilling[%, %]
Out[70]= 0

In[71]:= normalize[nullmatrix[] * a]
Out[71]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

In[72]:= normalize[basismatrix[1, 3] * a]
Out[72]= {{0, 0, a}, {0, 0, 0}, {0, 0, 0}}

In[77]:= gkilling[%, %]
Out[77]= 0

In[78]:= normalize[basismatrix[1, 1] * a]
Out[78]= { {  $\frac{a}{2\sqrt{2}\text{Abs}[a]}$ , 0, 0, 0, 0 }, {0, 0, 0, 0, 0 },
           {0, 0, 0, 0, 0 }, {0, 0, 0, 0, 0 } }

In[79]:= gkilling[%, %]
Out[79]=  $\frac{a^2}{\text{Abs}[a]^2}$ 

In[73]:= tlist = Table[basismatrix[i, i], {i, dim}]
Out[73]= { { {1, 0, 0}, {0, 0, 0}, {0, 0, 0} },
           { {0, 0, 0}, {0, 1, 0}, {0, 0, 0} }, { {0, 0, 0}, {0, 0, 0}, {0, 0, 1} } }

In[74]:= matrixorthonorm[tlist, print -> True]


$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$


Out[74]= { { {  $\frac{1}{2}$ , 0, 0 }, {0, 0, 0}, {0, 0, 0} },
           { {  $\frac{1}{2\sqrt{3}}$ , 0, 0 }, {0,  $\frac{1}{\sqrt{3}}$ , 0 }, {0, 0, 0} }, { {1, 0, 0}, {0, 1, 0}, {0, 0, 1} } }

In[75]:= Table[gkilling[%[[i]], %[[j]]], {i, dim}, {j, dim}] // MatrixForm
Out[75]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$


```

2.3 An orthonormal basis of the general linear Lie algebra $gl(n,K)$

To apply the procedure orthonorm to the general case we rename the Killing form in an appropriate way

```
In[22]:= sprod[n_] [a_, b_] := glkilling[a, b, n]
```

```
In[76]:= sprod[2] [{{1/sqrt[2], 0}, {0, 0}}, {{1/sqrt[2], 0}, {0, 0}}]
```

```
Out[76]= 1
```

The standard basis is not orthogonal:

```
In[77]:= dim
```

```
Out[77]= 3
```

```
In[78]:= Table[sprod[3] [glbasis[3] [[i]], glbasis[3] [[j]]], {i, 9}, {j, 9}] //  
MatrixForm
```

```
Out[78]//MatrixForm=
```

$$\begin{pmatrix} 4 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 4 \end{pmatrix}$$

Now we orthogonalize it:

```
In[79]:= matrixorthonorm[glbasis[3], innerprod -> sprod[3]]
```

```
Out[79]= {{{1/2, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {{1/(2*sqrt[3]), 0, 0}, {0, 1/sqrt[3], 0}, {0, 0, 0}},  
{{0, -1/(2*sqrt[3]), 0}, {1/(2*sqrt[3]), 0, 0}, {0, 0, 0}},  
{{0, 1/(2*sqrt[3]), 0}, {1/(2*sqrt[3]), 0, 0}, {0, 0, 0}},  
{{0, 0, -1/(2*sqrt[3])}, {0, 0, 0}, {1/(2*sqrt[3]), 0, 0}},  
{{0, 0, 1/(2*sqrt[3])}, {0, 0, 0}, {1/(2*sqrt[3]), 0, 0}},  
{{0, 0, 0}, {0, 0, -1/(2*sqrt[3])}, {0, 1/(2*sqrt[3]), 0}},  
{{0, 0, 0}, {0, 0, 1/(2*sqrt[3])}, {0, 1/(2*sqrt[3]), 0}},  
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}}
```

```
In[80]:= test3 = %;
```

```
In[81]:= Table[test3[[i]] // MatrixForm, {i, Length[test3]}]
```

$$\text{Out[81]} = \left\{ \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 0 & \frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 \end{pmatrix},$$

$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

```
In[82]:= Table[sprod[3][test3[[i]], test3[[j]]],
           {i, Length[test3]}, {j, Length[test3]}] // MatrixForm
```

```
Out[82]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The last element is isotropic; it is the basis of the center. Generally, the standard basis is far from being orthogonal:

```
In[84]:= dim = 5
```

```
Out[84]= 5
```

```
In[85]:= Table[glkilling[glbasis][[i]], glbasis][[j]],
           {i, Length[glbasis[]]}, {j, Length[glbasis[]]} // MatrixForm
```

```
Out[85]//MatrixForm=
```

$$\begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 10 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here and in the following cell we used the global definition of dim, and for the next cell we remember that glkilling is the default inner product.

```
In[86]:= Timing[matrixorthonorm[glbasis[]]]
```

```
Out[86]= {19.468,
```

$$\left\{ \left\{ \left\{ \frac{1}{2\sqrt{2}}, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\} \right\},$$

$$\{0, 0, 0, 0, 0\}, \left\{ \left\{ \frac{1}{2\sqrt{30}}, 0, 0, 0, 0 \right\}, \left\{ 0, \sqrt{\frac{2}{15}}, 0, 0, 0 \right\},$$

$$\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\} \right\},$$

$$\left\{ \left\{ \frac{1}{2\sqrt{15}}, 0, 0, 0, 0 \right\}, \left\{ 0, \frac{1}{2\sqrt{15}}, 0, 0, 0 \right\},$$

$$\left\{ 0, 0, \sqrt{\frac{3}{5}}, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\} \right\},$$

$$\left\{ \left\{ \frac{1}{2\sqrt{5}}, 0, 0, 0, 0 \right\}, \left\{ 0, \frac{1}{2\sqrt{5}}, 0, 0, 0 \right\}, \left\{ 0, 0, \frac{1}{2\sqrt{5}}, 0, 0 \right\},$$

$$\begin{aligned}
& \left\{0, 0, 0, \frac{1}{\sqrt{5}}, 0\right\}, \{0, 0, 0, 0, 0\}\}, \left\{\left\{0, -\frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \right. \\
& \left. \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\left\{0, 0, -\frac{1}{2\sqrt{5}}, 0, 0\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\left\{0, 0, \frac{1}{2\sqrt{5}}, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\left\{0, 0, 0, -\frac{1}{2\sqrt{5}}, 0\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\left\{0, 0, 0, \frac{1}{2\sqrt{5}}, 0\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\left\{0, 0, 0, 0, -\frac{1}{2\sqrt{5}}\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}\right\}, \\
& \left\{\left\{0, 0, 0, 0, \frac{1}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, -\frac{1}{2\sqrt{5}}, 0, 0\right\}, \right. \\
& \left. \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, \frac{1}{2\sqrt{5}}, 0, 0\right\}, \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\left\{0, 0, 0, 0, 0\right\}, \left\{0, 0, 0, 0, -\frac{1}{2\sqrt{5}}\right\}, 0\right\}, \\
& \left. \{0, 0, 0, 0, 0\}, \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, \frac{1}{2\sqrt{5}}, 0\right\}, \{0, 0, 0, 0, 0\}, \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, -\frac{1}{2\sqrt{5}}\right\}\right\},
\end{aligned}$$

$$\begin{aligned}
& \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{1}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{0, 0, 0, -\frac{1}{2\sqrt{5}}, 0\right\}, \left\{0, 0, \frac{1}{2\sqrt{5}}, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, \frac{1}{2\sqrt{5}}, 0\right\}, \right. \\
& \left. \left\{0, 0, \frac{1}{2\sqrt{5}}, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{0, 0, 0, 0, -\frac{1}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \left\{0, 0, \frac{1}{2\sqrt{5}}, 0, 0\right\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{1}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{0, 0, \frac{1}{2\sqrt{5}}, 0, 0\right\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, -\frac{1}{2\sqrt{5}}\right\}, \left\{0, 0, 0, \frac{1}{2\sqrt{5}}, 0\right\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{1}{2\sqrt{5}}\right\}, \right. \\
& \left. \left\{0, 0, 0, \frac{1}{2\sqrt{5}}, 0\right\}\right\}, \left\{\{1, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0\}, \right. \\
& \left. \{0, 0, 1, 0, 0\}, \{0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 1\}\right\}
\end{aligned}$$

At 1999, *Mathematica* needed 185 s for the orthonormalization. Now, on a Dell Precision 370 from 2004, *Mathematica* v. 7.0 needed 19 s only.

```
In[87]:= %[[1]]
```

```
Out[87]= 19.468
```

```
In[88]:= glortho = %%[[2]];
```

```
In[89]:= Table[glortho[[i]] // MatrixForm, {i, 25}]
```


2.4 The special linear Lie algebra $\mathfrak{sl}(n, K)$

The special linear Lie algebra $\mathfrak{sl}(n, K)$ consists of all square matrices of order n with trace zero. It is the kernel of the trace map

$$\text{trace}: \mathfrak{a} \in \mathfrak{gl}(n, K) \mapsto \text{trace}(\mathfrak{a}) \in K = \mathfrak{gl}(1, K)$$

which is an homomorphism of Lie algebras, and therefore an ideal within the general linear Lie algebra.

From the definitions it follows easily that the Killing form of any ideal is the restriction of the Killing form of the whole Lie algebra. Thus, for finding the index of the Killing form of $\mathfrak{sl}(n, K)$ we first define a basis of $\mathfrak{s}(\mathfrak{ln}, K)$, and then orthonormalize it with respect to $\mathfrak{gkilling}$.

A basis of $\mathfrak{sl}(n, K)$

```
In[23]:= slbasismatrix[i_, j_, n_: dim] :=
      basismatrix[i, j, n] /; i != j;
slbasismatrix[i_, j_, n_: dim] :=
      (basismatrix[i, j, n] - basismatrix[n, n, n]) /; (i == j && j < n)
```

```
In[25]:= slbasis[n_: dim] :=
      Join[Flatten[Table[slbasismatrix[i, j, n], {i, n-1}, {j, n}], 1],
      Table[slbasismatrix[n, h, n], {h, n-1}]]
```

```
In[92]:= slbasis[2]
```

```
Out[92]= {{{1, 0}, {0, -1}}, {{0, 1}, {0, 0}}, {{0, 0}, {1, 0}}}
```

```
In[93]:= slbasis[3]
```

```
Out[93]= {{{1, 0, 0}, {0, 0, 0}, {0, 0, -1}}, {{0, 1, 0}, {0, 0, 0}, {0, 0, 0}},
      {{0, 0, 1}, {0, 0, 0}, {0, 0, 0}}, {{0, 0, 0}, {1, 0, 0}, {0, 0, 0}},
      {{0, 0, 0}, {0, 1, 0}, {0, 0, -1}}, {{0, 0, 0}, {0, 0, 1}, {0, 0, 0}},
      {{0, 0, 0}, {0, 0, 0}, {1, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 1, 0}}}
```

```
In[94]:= dim
```

```
Out[94]= 5
```

In[95]:= **slbasis[]**

Out[95]= {{{1, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, -1}}, {{0, 1, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
 {{0, 0, 1, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}}, {{0, 0, 0, 1, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 1}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0}, {1, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0}, {0, 1, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, -1}}, {{0, 0, 0, 0, 0}, {0, 0, 1, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0}, {0, 0, 0, 1, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 1},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {1, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 1, 0, 0, 0}, {0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, -1}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 1, 0}, {0, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 1}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {1, 0, 0, 0, 0}},
 {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 1, 0, 0, 0},
 {0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 1, 0, 0}},
 {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 1, 0},
 {0, 0, 0, 0, -1}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 0, 1}},
 {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {1, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 1, 0, 0, 0}},
 {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0}, {0, 0, 0, 1, 0}}}

Orthonormalization of the basis

In[96]:= **matrixorthonorm[slbasis[3], innerprod → sprod[3]]**

Out[96]= $\left\{ \left\{ \left\{ \frac{1}{2\sqrt{3}}, 0, 0 \right\}, \{0, 0, 0\}, \left\{ 0, 0, -\frac{1}{2\sqrt{3}} \right\} \right\}, \right.$
 $\left. \left\{ \left\{ -\frac{1}{6}, 0, 0 \right\}, \left\{ 0, \frac{1}{3}, 0 \right\}, \left\{ 0, 0, -\frac{1}{6} \right\} \right\}, \right.$
 $\left. \left\{ \left\{ 0, -\frac{1}{2\sqrt{3}}, 0 \right\}, \left\{ \frac{1}{2\sqrt{3}}, 0, 0 \right\}, \{0, 0, 0\} \right\}, \right.$
 $\left. \left\{ \left\{ 0, \frac{1}{2\sqrt{3}}, 0 \right\}, \left\{ \frac{1}{2\sqrt{3}}, 0, 0 \right\}, \{0, 0, 0\} \right\}, \right.$
 $\left. \left\{ \left\{ 0, 0, -\frac{1}{2\sqrt{3}} \right\}, \{0, 0, 0\}, \left\{ \frac{1}{2\sqrt{3}}, 0, 0 \right\} \right\}, \right.$
 $\left. \left\{ \left\{ 0, 0, \frac{1}{2\sqrt{3}} \right\}, \{0, 0, 0\}, \left\{ \frac{1}{2\sqrt{3}}, 0, 0 \right\} \right\}, \right.$
 $\left. \left\{ \left\{ 0, 0, 0 \right\}, \left\{ 0, 0, -\frac{1}{2\sqrt{3}} \right\}, \left\{ 0, \frac{1}{2\sqrt{3}}, 0 \right\} \right\}, \right.$
 $\left. \left\{ \left\{ 0, 0, 0 \right\}, \left\{ 0, 0, \frac{1}{2\sqrt{3}} \right\}, \left\{ 0, \frac{1}{2\sqrt{3}}, 0 \right\} \right\} \right\}$

In[97]:= **stest3 = %;**

In[98]:= **Length[stest3]**

Out[98]= 8

In[99]:= **Map[MatrixForm, stest3]**

Out[99]= $\left\{ \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2\sqrt{3}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{6} \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right.$
 $\left. \begin{pmatrix} 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & 0 \end{pmatrix} \right\}$

```
In[100]:= Table[glkilling[stest3[[a]], stest3[[b]], 3], {a, 8}, {b, 8}] //
MatrixForm
```

```
Out[100]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
In[101]:= Timing[matrixorthonorm[slbasis[5], innerprod → sprod[5]]]
```

```
Out[101]= {18.031,
```

$$\begin{aligned} & \left\{ \left\{ \left\{ \frac{1}{2\sqrt{5}}, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \right. \\ & \left. \left\{0, 0, 0, 0, -\frac{1}{2\sqrt{5}}\right\} \right\}, \left\{ \left\{ -\frac{1}{2\sqrt{15}}, 0, 0, 0, 0 \right\}, \left\{ 0, \frac{1}{\sqrt{15}}, 0, 0, 0 \right\}, \right. \\ & \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{ 0, 0, 0, 0, -\frac{1}{2\sqrt{15}} \right\} \right\}, \\ & \left\{ \left\{ -\frac{1}{2\sqrt{30}}, 0, 0, 0, 0 \right\}, \left\{ 0, -\frac{1}{2\sqrt{30}}, 0, 0, 0 \right\}, \right. \\ & \left. \left\{ 0, 0, \frac{\sqrt{\frac{3}{10}}}{2}, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \left\{ 0, 0, 0, 0, -\frac{1}{2\sqrt{30}} \right\} \right\}, \\ & \left\{ \left\{ -\frac{1}{10\sqrt{2}}, 0, 0, 0, 0 \right\}, \left\{ 0, -\frac{1}{10\sqrt{2}}, 0, 0, 0 \right\}, \left\{ 0, 0, -\frac{1}{10\sqrt{2}}, 0, 0 \right\}, \right. \\ & \left. \left\{ 0, 0, 0, \frac{\sqrt{2}}{5}, 0 \right\}, \left\{ 0, 0, 0, 0, -\frac{1}{10\sqrt{2}} \right\} \right\}, \left\{ \left\{ 0, -\frac{1}{2\sqrt{5}}, 0, 0, 0 \right\}, \right. \\ & \left. \left\{ \frac{1}{2\sqrt{5}}, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\} \right\}, \\ & \left\{ \left\{ 0, \frac{1}{2\sqrt{5}}, 0, 0, 0 \right\}, \left\{ \frac{1}{2\sqrt{5}}, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \right. \\ & \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\} \right\}, \left\{ \left\{ 0, 0, -\frac{1}{2\sqrt{5}}, 0, 0 \right\}, \right. \\ & \left. \{0, 0, 0, 0, 0\}, \left\{ \frac{1}{2\sqrt{5}}, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\} \right\}, \\ & \left\{ \left\{ 0, 0, \frac{1}{2\sqrt{5}}, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \left\{ \frac{1}{2\sqrt{5}}, 0, 0, 0, 0 \right\}, \right. \\ & \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\} \right\}, \left\{ \left\{ 0, 0, 0, -\frac{1}{2\sqrt{5}}, 0 \right\}, \right. \end{aligned}$$

$$\begin{aligned}
& \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \\
& \left\{\left\{0, 0, 0, \frac{1}{2\sqrt{5}}, 0\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\left\{0, 0, 0, 0, -\frac{1}{2\sqrt{5}}\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}\right\}, \\
& \left\{\left\{0, 0, 0, 0, \frac{1}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, -\frac{1}{2\sqrt{5}}, 0, 0\right\}, \right. \\
& \left. \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, \frac{1}{2\sqrt{5}}, 0, 0\right\}, \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, -\frac{1}{2\sqrt{5}}, 0\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, \frac{1}{2\sqrt{5}}, 0\right\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, -\frac{1}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{1}{2\sqrt{5}}\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, \frac{1}{2\sqrt{5}}, 0, 0, 0\right\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, -\frac{1}{2\sqrt{5}}, 0\right\}, \right. \\
& \left. \left\{0, 0, \frac{1}{2\sqrt{5}}, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{0, 0, 0, \frac{1}{2\sqrt{5}}, 0\right\}, \left\{0, 0, \frac{1}{2\sqrt{5}}, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, -\frac{1}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \right.
\end{aligned}$$

$$\left\{ \left\{ 0, 0, \frac{1}{2\sqrt{5}}, 0, 0 \right\} \right\}, \left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\} \right\},$$

$$\left\{ \left\{ 0, 0, 0, 0, \frac{1}{2\sqrt{5}} \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, \frac{1}{2\sqrt{5}}, 0, 0 \right\} \right\},$$

$$\left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, -\frac{1}{2\sqrt{5}} \right\} \right\},$$

$$\left\{ \left\{ 0, 0, 0, \frac{1}{2\sqrt{5}}, 0 \right\} \right\}, \left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\} \right\},$$

$$\left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, \frac{1}{2\sqrt{5}} \right\}, \left\{ 0, 0, 0, \frac{1}{2\sqrt{5}}, 0 \right\} \right\} \right\}$$

```
In[102]:= %[[1]]
```

```
Out[102]= 18.031
```

This calculation of test5 on the Dell Inspiron 3200 under Linux, *Mathematica* 4.0, needed about 175 s. Now, on the Dell Precision 370 under *Mathematica* 7.0, SuSE Linux 11.1, we needed 16 s only.

```
In[103]:= stest5 = %[[2]];
```

```
In[60]:= Length[stest5]
```

```
Out[60]= 24
```

```
In[104]:= Table[glkilling[stest5[[a]], stest5[[b]]], {a, 24}, {b, 24}] // MatrixForm
```

```
Out[104]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The index of the Killing form of $\mathfrak{sl}(5, \mathbb{R})$ is 10. As for all semisimple Lie algebras the form is non degener-

ate. The [Cartan index](#) of a semi simple real Lie algebra is defined as

$$\text{CartanIndex} := \text{pos} - \text{neg},$$

where pos denotes the number of the spacelike and neg the number of the timelike elements of an orthogonalized basis. Obviously one has

$$\text{CartanIndex} = \text{dimension} - 2 \text{ neg}$$

The result $\text{CartanIndex}[\text{sl}(5, \mathbb{R})] = 4$ coincides with the value given in [3], p. 27.

3 The orthogonal Lie algebras $\mathfrak{o}(n)$

■ 3.1 Basis of $\mathfrak{o}(n)$, general element

The orthogonal Lie algebra $\mathfrak{o}(n)$ is the space of all skew symmetric matrices of order n . The standard basis of $\mathfrak{o}(n)$ is:

```
In[26]:= obm[i_, j_, n_:dim] :=
          basismatrix[i, j, n] - basismatrix[j, i, n] /; i < j
```

```
In[47]:= dim
```

```
Out[47]= 3
```

```
In[48]:= obm[1, 2, 3]
```

```
Out[48]= {{0, 1, 0}, {-1, 0, 0}, {0, 0, 0}}
```

```
In[49]:= obm[3, 4, 4] // MatrixForm
```

```
Out[49]//MatrixForm=
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$\text{obm}[i, j, n]$ will not be evaluated for invalid arguments:

```
In[51]:= obm[2, 2, 3]
```

```
Out[51]= obm[2, 2, 3]
```

The standard basis is

```
In[27]:= obasis[n_:dim] :=
```

```
Flatten[Table[Table[obm[i, j, n], {j, i+1, n}], {i, n}], 1]
```

```
In[52]:= Map[MatrixForm, obasis[3]]
```

```
Out[52]= { { ( 0 1 0 ), ( 0 0 1 ), ( 0 0 0 ) },
            { (-1 0 0 ), ( 0 0 0 ), ( 0 0 1 ) },
            { ( 0 0 0 ), (-1 0 0 ), ( 0 -1 0 ) } }
```

The arbitrary element of the orthogonal Lie algebra $\mathfrak{o}(n)$:

```
In[28]:= oelm[a_, n_: dim] := Sum[Sum[obm[i, j, n] a[i, j], {i, 1, j - 1}],
      {j, 2, n}]
```

```
In[53]:= oelm[a] // MatrixForm
```

```
Out[53]//MatrixForm=
```

$$\begin{pmatrix} 0 & a[1, 2] & a[1, 3] \\ -a[1, 2] & 0 & a[2, 3] \\ -a[1, 3] & -a[2, 3] & 0 \end{pmatrix}$$

■ 3.2 The Killing form of $\mathfrak{o}(n)$

The Killing form:

```
In[29]:= okilling[a_, b_, n_: dim] :=
  Sum[Sum[
    cobasis[i, j, n][Evaluate[commute[a, commute[b, obm[i, j, n]]]],
    {i, 1, j - 1}], {j, 2, n}]
```

The standard basis is orthogonal; therefore an orthogonalization is not necessary.

```
In[58]:= okilling[oelm[a, 6], oelm[a, 6], 6]
```

```
Out[58]= -8 a[1, 2]^2 - 8 a[1, 3]^2 - 8 a[1, 4]^2 - 8 a[1, 5]^2 - 8 a[1, 6]^2 -
  8 a[2, 3]^2 - 8 a[2, 4]^2 - 8 a[2, 5]^2 - 8 a[2, 6]^2 - 8 a[3, 4]^2 -
  8 a[3, 5]^2 - 8 a[3, 6]^2 - 8 a[4, 5]^2 - 8 a[4, 6]^2 - 8 a[5, 6]^2
```

The restriction of gkilling to $\mathfrak{o}(n)$, $n = 2, 3, 4, 6$, is conformal to the Killingform okilling .

```
In[59]:= Simplify[gkilling[oelm[a, 6], oelm[a, 6], 6]]
```

```
Out[59]= -24 (a[1, 2]^2 + a[1, 3]^2 + a[1, 4]^2 +
  a[1, 5]^2 + a[1, 6]^2 + a[2, 3]^2 + a[2, 4]^2 + a[2, 5]^2 + a[2, 6]^2 +
  a[3, 4]^2 + a[3, 5]^2 + a[3, 6]^2 + a[4, 5]^2 + a[4, 6]^2 + a[5, 6]^2)
```

```
In[60]:= Simplify[% / %%]
```

```
Out[60]= 3
```

```
In[83]:= Simplify[% / okilling[oelm[a, 6], oelm[b, 6], 6]]
```

```
Out[83]= 3
```

The Lie algebra $\mathfrak{o}(2)$ is abelian, therefore one has

```
In[84]:= okilling[oelm[a, 2], oelm[b, 2], 2]
```

```
Out[84]= 0
```

Since $\mathfrak{o}(n)$ for $n > 2$ is semisimple and compact, the Killing forms are negative definite, e.g.

```
In[61]:= okilling[oelm[a], oelm[a]]
```

```
Out[61]= -2 a[1, 2]^2 - 2 a[1, 3]^2 - 2 a[2, 3]^2
```

We obtain an orthonormed basis using `normalize`:


```
In[62]:= dim = 5
```

```
Out[62]= 5
```

```
In[63]:= obasis[]
```

```
Out[63]= {{{0, 1, 0, 0, 0}, {-1, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}, {{0, 0, 1, 0, 0}, {0, 0, 0, 0, 0},
           {-1, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
           {{0, 0, 0, 1, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0},
           {0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 1}, {0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0}},
           {{0, 0, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, -1, 0, 0, 0}, {0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 1, 0},
           {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, -1, 0, 0, 0}},
           {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 1}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
           {0, -1, 0, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
           {0, 0, 0, 1, 0}, {0, 0, -1, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 1},
           {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
           {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 1}, {0, 0, 0, 0, 0},
           {0, 0, -1, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0}, {0, 0, 0, 0, 1}, {0, 0, 0, -1, 0}}}
```

In[64]:= **Table[normalize[obasis][[i]], okilling], {i, 10}]**

Out[64]= $\left\{ \left\{ \left\{ 0, \frac{1}{\sqrt{6}}, 0, 0, 0 \right\}, \left\{ -\frac{1}{\sqrt{6}}, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \right. \right.$
 $\left. \left\{ 0, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\} \right\}, \left\{ \left\{ 0, 0, \frac{1}{\sqrt{6}}, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \right.$
 $\left. \left\{ -\frac{1}{\sqrt{6}}, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\} \right\},$
 $\left\{ \left\{ 0, 0, 0, \frac{1}{\sqrt{6}}, 0 \right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right.$
 $\left. \left\{ -\frac{1}{\sqrt{6}}, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\} \right\}, \left\{ \left\{ 0, 0, 0, 0, \frac{1}{\sqrt{6}} \right\}, \right.$
 $\left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{ -\frac{1}{\sqrt{6}}, 0, 0, 0, 0 \right\} \right\},$
 $\left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, \frac{1}{\sqrt{6}}, 0, 0 \right\}, \left\{ 0, -\frac{1}{\sqrt{6}}, 0, 0, 0 \right\}, \right.$
 $\left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\} \right\}, \left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, \frac{1}{\sqrt{6}}, 0 \right\}, \right.$
 $\left. \{0, 0, 0, 0, 0\}, \left\{ 0, -\frac{1}{\sqrt{6}}, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\} \right\},$
 $\left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, \frac{1}{\sqrt{6}} \right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right.$
 $\left. \left\{ 0, -\frac{1}{\sqrt{6}}, 0, 0, 0 \right\} \right\}, \left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \right.$
 $\left. \left\{ 0, 0, 0, \frac{1}{\sqrt{6}}, 0 \right\}, \left\{ 0, 0, -\frac{1}{\sqrt{6}}, 0, 0 \right\}, \{0, 0, 0, 0, 0\} \right\},$
 $\left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \left\{ 0, 0, 0, 0, \frac{1}{\sqrt{6}} \right\}, \{0, 0, 0, 0, 0\}, \right.$
 $\left. \left\{ 0, 0, -\frac{1}{\sqrt{6}}, 0, 0 \right\} \right\}, \left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0\}, \right.$
 $\left. \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, \frac{1}{\sqrt{6}} \right\}, \left\{ 0, 0, 0, -\frac{1}{\sqrt{6}}, 0 \right\} \right\}$

In[65]:= **otest5 = %;**

4 The pseudo-orthogonal Lie algebras $\mathfrak{o}(n,k)$

■ 4.1 Basis, arbitrary element

The pseudo-orthogonal Lie algebras $\mathfrak{o}(n,k)$ is the Lie algebra of the pseudo-orthogonal group $O(n,k)$, which is the group of all linear automorphisms of the n -dimensional real vector space leaving a non degenerate symmetric bilinear form of index k invariant. The group $O(4,1)$ is the Lorentz group applied in relativity theory. The Lie algebra $\mathfrak{o}(5,1)$ is the Lie algebra of the conformal group of the 3-sphere (Möbius group of the 3-sphere). The standard basis of $\mathfrak{o}(n, k)$ is:

```
In[30]:= psobm[i_, h_, k_: ind, n_: dim] :=
  If[i <= n - k && h > n - k, basismatrix[i, h, n] + basismatrix[h, i, n],
    obm[i, h, n]] /; (i < h && k <= n)
```

```
In[31]:= psobasis[k_: ind, n_: dim] :=
  Flatten[Table[Table[psobm[i, j, k, n], {j, i + 1, n}], {i, n}], 1]
```

The arbitrary element of the pseudo-orthogonal Lie algebra $\mathfrak{o}(k, n)$:

```
In[32]:= psoelm[a_, k_: ind, n_: dim] := Sum[Sum[psobm[i, j, k, n] a[i, j],
  {i, 1, j - 1}], {j, 2, n}] /; k <= n
```

```
In[69]:= psoelm[a, 2, 5] // MatrixForm
```

Out[69]/MatrixForm=

$$\begin{pmatrix} 0 & a[1, 2] & a[1, 3] & a[1, 4] & a[1, 5] \\ -a[1, 2] & 0 & a[2, 3] & a[2, 4] & a[2, 5] \\ -a[1, 3] & -a[2, 3] & 0 & a[3, 4] & a[3, 5] \\ a[1, 4] & a[2, 4] & a[3, 4] & 0 & a[4, 5] \\ a[1, 5] & a[2, 5] & a[3, 5] & -a[4, 5] & 0 \end{pmatrix}$$

Examples, tests

```
In[91]:= dim
```

Out[91]= 5

```
In[93]:= ind
```

Out[93]= 1

```
In[85]:= psoelm[a] // MatrixForm
```

Out[85]/MatrixForm=

$$\begin{pmatrix} 0 & a[1, 2] & a[1, 3] & a[1, 4] & a[1, 5] \\ -a[1, 2] & 0 & a[2, 3] & a[2, 4] & a[2, 5] \\ -a[1, 3] & -a[2, 3] & 0 & a[3, 4] & a[3, 5] \\ -a[1, 4] & -a[2, 4] & -a[3, 4] & 0 & a[4, 5] \\ a[1, 5] & a[2, 5] & a[3, 5] & a[4, 5] & 0 \end{pmatrix}$$

```
In[86]:= psokilling[%, %]
```

```
Out[86]= -6 a[1, 2]^2 - 6 a[1, 3]^2 - 6 a[1, 4]^2 + 6 a[1, 5]^2 - 6 a[2, 3]^2 -
        6 a[2, 4]^2 + 6 a[2, 5]^2 - 6 a[3, 4]^2 + 6 a[3, 5]^2 + 6 a[4, 5]^2
```

```
In[90]:= psobasis[1, 3]
```

```
Out[90]= {{{0, 1, 0}, {-1, 0, 0}, {0, 0, 0}},
          {{0, 0, 1}, {0, 0, 0}, {1, 0, 0}}, {{0, 0, 0}, {0, 0, 1}, {0, 1, 0}}}
```

```
In[68]:= Map[MatrixForm, psobasis[]]
```

```
Out[68]= {
  ( 0 1 0 0 0 ) ( 0 0 1 0 0 ) ( 0 0 0 1 0 )
  (-1 0 0 0 0 ) ( 0 0 0 0 0 ) ( 0 0 0 0 0 )
  ( 0 0 0 0 0 ) (-1 0 0 0 0 ) ( 0 0 0 0 0 )
  ( 0 0 0 0 0 ) ( 0 0 0 0 0 ) (-1 0 0 0 0 )
  ( 0 0 0 0 0 ) ( 0 0 0 0 0 ) ( 0 0 0 0 0 )

  ( 0 0 0 0 1 ) ( 0 0 0 0 0 ) ( 0 0 0 0 0 )
  ( 0 0 0 0 0 ) ( 0 0 1 0 0 ) ( 0 0 0 1 0 )
  ( 0 0 0 0 0 ) ( 0 -1 0 0 0 ) ( 0 0 0 0 0 )
  ( 0 0 0 0 0 ) ( 0 0 0 0 0 ) ( 0 -1 0 0 0 )
  ( 1 0 0 0 0 ) ( 0 0 0 0 0 ) ( 0 0 0 0 0 )

  ( 0 0 0 0 0 ) ( 0 0 0 0 0 ) ( 0 0 0 0 0 ) ( 0 0 0 0 0 )
  ( 0 0 0 0 1 ) ( 0 0 0 0 0 ) ( 0 0 0 0 0 ) ( 0 0 0 0 0 )
  ( 0 0 0 0 0 ) ( 0 0 0 1 0 ) ( 0 0 0 0 1 ) ( 0 0 0 0 0 )
  ( 0 0 0 0 0 ) ( 0 0 -1 0 0 ) ( 0 0 0 0 0 ) ( 0 0 0 0 1 )
  ( 0 1 0 0 0 ) ( 0 0 0 0 0 ) ( 0 0 1 0 0 ) ( 0 0 0 1 0 )
}
```

```
In[100]:= psoelm[a, 2, 6] //MatrixForm
```

```
Out[100]//MatrixForm=
```

```
( 0      a[1, 2]  a[1, 3]  a[1, 4]  a[1, 5]  a[1, 6] )
(-a[1, 2]  0      a[2, 3]  a[2, 4]  a[2, 5]  a[2, 6] )
(-a[1, 3] -a[2, 3]  0      a[3, 4]  a[3, 5]  a[3, 6] )
(-a[1, 4] -a[2, 4] -a[3, 4]  0      a[4, 5]  a[4, 6] )
(a[1, 5]  a[2, 5]  a[3, 5]  a[4, 5]  0      a[5, 6] )
(a[1, 6]  a[2, 6]  a[3, 6]  a[4, 6] -a[5, 6]  0      )
```

■ 4.2 The Killing form of $\mathfrak{o}(n, k)$

```
In[33]:= psokilling[a_, b_, k_:ind, n_:dim] :=
```

```
Sum[Sum[cobasis[i, j, n][
```

```
  Evaluate[commute[a, commute[b, psobm[i, j, k, n]]]],
```

```
{i, 1, j - 1}], {j, 2, n}]
```

Using the tables [3] one concludes that the index of inertia of this Killing form is

```
In[34]:= psoindex[k_, n_] := k^2 - k*n + n*(n-1)/2
```

Comparing this value with the index of the Killing forms calculated below controls our constructions.

```
In[91]:= psoindex[1, 5]
```

```
Out[91]= 6
```

```
In[102]:= psokilling[psaelm[a], psaelm[a]]
```

```
Out[102]:= -6 a[1, 2]^2 - 6 a[1, 3]^2 - 6 a[1, 4]^2 + 6 a[1, 5]^2 - 6 a[2, 3]^2 -
6 a[2, 4]^2 + 6 a[2, 5]^2 - 6 a[3, 4]^2 + 6 a[3, 5]^2 + 6 a[4, 5]^2
```

Again the Killing form psokilling is conformal to the restriction of glkilling:

```
In[103]:= Simplify[% / glkilling[psaelm[a], psaelm[a]]]
```

```
Out[103]:=  $\frac{3}{10}$ 
```

```
In[104]:= psindex[2, 5]
```

```
Out[104]:= 4
```

```
In[105]:= psokilling[psaelm[a, 2, 5], psaelm[a, 2, 5], 2, 5]
```

```
Out[105]:= -6 a[1, 2]^2 - 6 a[1, 3]^2 + 6 a[1, 4]^2 + 6 a[1, 5]^2 - 6 a[2, 3]^2 +
6 a[2, 4]^2 + 6 a[2, 5]^2 + 6 a[3, 4]^2 + 6 a[3, 5]^2 - 6 a[4, 5]^2
```

For $n > 2$ all the pseudo-orthogonal Lie algebras have a non-degenerate, and, for $n > k > 0$, indefinite Killing form; the standard Basis is orthogonal.

Timing: The Killing form of $\mathfrak{o}(19, 1)$:

```
In[106]:= Timing[psokilling[psaelm[a], psaelm[a]]]
```

```
Out[106]:= {0.028002, -6 a[1, 2]^2 - 6 a[1, 3]^2 - 6 a[1, 4]^2 + 6 a[1, 5]^2 -
6 a[2, 3]^2 - 6 a[2, 4]^2 + 6 a[2, 5]^2 - 6 a[3, 4]^2 + 6 a[3, 5]^2 + 6 a[4, 5]^2}
```

```
In[96]:= Timing[psokilling[psaelm[a, 1, 10], psaelm[a, 1, 10], 1, 10]]
```

```
Out[96]:= {2.157, -16 a[1, 2]^2 - 16 a[1, 3]^2 - 16 a[1, 4]^2 - 16 a[1, 5]^2 - 16 a[1, 6]^2 -
16 a[1, 7]^2 - 16 a[1, 8]^2 - 16 a[1, 9]^2 + 16 a[1, 10]^2 - 16 a[2, 3]^2 -
16 a[2, 4]^2 - 16 a[2, 5]^2 - 16 a[2, 6]^2 - 16 a[2, 7]^2 - 16 a[2, 8]^2 -
16 a[2, 9]^2 + 16 a[2, 10]^2 - 16 a[3, 4]^2 - 16 a[3, 5]^2 - 16 a[3, 6]^2 -
16 a[3, 7]^2 - 16 a[3, 8]^2 - 16 a[3, 9]^2 + 16 a[3, 10]^2 - 16 a[4, 5]^2 -
16 a[4, 6]^2 - 16 a[4, 7]^2 - 16 a[4, 8]^2 - 16 a[4, 9]^2 + 16 a[4, 10]^2 -
16 a[5, 6]^2 - 16 a[5, 7]^2 - 16 a[5, 8]^2 - 16 a[5, 9]^2 + 16 a[5, 10]^2 -
16 a[6, 7]^2 - 16 a[6, 8]^2 - 16 a[6, 9]^2 + 16 a[6, 10]^2 - 16 a[7, 8]^2 -
16 a[7, 9]^2 + 16 a[7, 10]^2 - 16 a[8, 9]^2 + 16 a[8, 10]^2 + 16 a[9, 10]^2}
```

```
In[97]:= %[[1]]
```

```
Out[97]:= 2.157
```


To get an orthonormal basis it suffices to normalize the elements of `psobasis[1,10]`: By the result above we may define:

```
In[105]:= psnormbasis = psobasis[1, 10] / 4;
```

```
In[107]:= psnormbasis[[10]] // MatrixForm
```

```
Out[107]//MatrixForm=
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[108]:= psokilling[psnormbasis[[10]], psnormbasis[[10]], 1, 10]
```

```
Out[108]= -1
```

5 The symplectic Lie algebras $\mathfrak{sp}(2m, K)$

The Lie algebra $\mathfrak{sp}(2m, K)$ is the Lie algebra of the symplectic group $\mathrm{Sp}(2m, K)$, which consists of all linear automorphisms of a $2m$ -dimensional vector space leaving a non degenerate alternating bilinear form invariant.

■ 5.1 The symplectic bilinear form

The symplectic bilinear form on a finite dimensional vector space is non degenerate and skew symmetric (alternating); it exists only for even-dimensional vector spaces. All such forms are equivalent under the action of the general linear group. Its matrix with respect to a standard basis is

```
In[35]:= alt[n_?EvenQ] := Module[{m = nullmatrix[n], i},
  For[i = 1, i <= n/2, i++,
    (m[[i, i + n/2]] = 1; m[[i + n/2, i]] = -1);];
```

```
m]
```

```
In[123]:= alt[10] // MatrixForm
```

```
Out[123]//MatrixForm=
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[124]:= Det[%]
```

```
Out[124]= 1
```

The symplectic bilinear form is

```
In[36]:= altprod[x_List, y_List] :=
      x.(alt[Length[x]].y) /; (Length[x] == Length[y] && EvenQ[Length[x]])
```

On a 2-dimensional vector space altprod coincides with the determinant:

```
In[125]:= altprod[vec[x, 2], vec[y, 2]]
```

```
Out[125]= -x[2] y[1] + x[1] y[2]
```

```
In[126]:= altprod[vec[x, 6], vec[y, 6]]
```

```
Out[126]= -x[4] y[1] - x[5] y[2] - x[6] y[3] + x[1] y[4] + x[2] y[5] + x[3] y[6]
```

```
In[127]:= altprod[vec[x, 4], vec[y, 2]]
```

```
Out[127]= altprod[{x[1], x[2], x[3], x[4]}, {y[1], y[2]}]
```

```
In[128]:= altprod[vec[x, 4], vec[y, 4]] + altprod[vec[y, 4], vec[x, 4]]
```

```
Out[128]= 0
```

```
In[129]:= altprod[vec[x, 4], vec[x, 4]]
```

```
Out[129]= 0
```

■ 5.2 Basis and general element

The set of the elements defined as follows contains a basis of the Lie algebra $\mathfrak{sp}(2n, K)$; for simplicity we admit that some of the elements $\text{spbasis}[i, j, n]$ are nullmatrices.

```
In[37]:= spbasis[i_, j_, n_?EvenQ] := Module[{m = nullmatrix[n], k = n/2},
```

```
  If[i <= k && j <= k, m[[i, j]] = 1; m[[k+j, k+i]] = -1; Return[m]];
  If[i <= k && i <= j-k <= k, m[[i, j]] = 1;
  m[[j-k, i+k]] = 1; Return[m]];
  If[i > k && i-k <= j <= k, m[[i, j]] = 1;
  m[[k+j, i-k]] = 1; Return[m]];
  Return[nullmatrix[n]]]
```

Applying `matrixoutzero` we may construct a basis of $\mathfrak{sp}(2n, K)$:

```
In[38]:= basislist[n_?EvenQ] :=
  matrixoutzero[Flatten[Table[spbasis[i, j, n], {i, n}, {j, n}], 1]]
```

Since the nullmatrices do not contribute to the sum, the general element of $\mathfrak{sp}(2n, K)$ can be defined as

```
In[39]:= spelm[a_, n_?EvenQ] := Sum[spbasis[i, j, n] * a[i, j], {i, n}, {j, n}]
```

Examples, tests

```
In[130]:= spbasis[1, 1, 2] // MatrixForm
```

```
Out[130]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

```
In[131]:= Table[spbasis[i, j, 2], {i, 2}, {j, 2}]
```

```
Out[131]= {{{{1, 0}, {0, -1}}, {{0, 1}, {0, 0}}}, {{{0, 0}, {1, 0}}, {{0, 0}, {0, 0}}}}
```

```
In[132]:= Map[MatrixForm, basislist[2]]
```

```
Out[132]= { { (1 0), (0 1) }, { (0 -1), (0 0) }, { (0 0), (1 0) } }
```

```
In[133]:= spbasis[2, 1, 2]
```

```
Out[133]= {{0, 0}, {1, 0}}
```

```
In[134]:= Table[spbasis[i, j, 4] // MatrixForm, {i, 4}, {j, 4}]
```

```
Out[134]= { { { (1 0 0 0), (0 1 0 0), (0 0 1 0), (0 0 0 1) },
  { (0 0 0 0), (0 0 0 0), (0 0 0 0), (0 0 1 0) },
  { (0 0 -1 0), (0 0 0 0), (0 0 0 0), (0 0 0 0) },
  { (0 0 0 0), (0 0 -1 0), (0 0 0 0), (0 0 0 0) } },
  { { (0 0 0 0), (0 0 0 0), (0 0 0 0), (0 0 0 0) },
  { (1 0 0 0), (0 1 0 0), (0 0 0 0), (0 0 0 0) },
  { (0 0 0 -1), (0 0 0 0), (0 0 0 0), (0 0 0 0) },
  { (0 0 0 0), (0 0 0 -1), (0 0 0 0), (0 0 0 0) } },
  { { (0 0 0 0), (0 0 0 0), (0 0 0 0), (0 0 0 0) },
  { (0 0 0 0), (0 0 0 0), (0 0 0 0), (0 0 0 0) },
  { (1 0 0 0), (0 1 0 0), (0 0 0 0), (0 0 0 0) },
  { (0 0 0 0), (1 0 0 0), (0 0 0 0), (0 0 0 0) } },
  { { (0 0 0 0), (0 0 0 0), (0 0 0 0), (0 0 0 0) },
  { (0 0 0 0), (0 0 0 0), (0 0 0 0), (0 0 0 0) },
  { (0 0 0 0), (0 0 0 0), (0 0 0 0), (0 0 0 0) },
  { (0 0 0 0), (0 1 0 0), (0 0 0 0), (0 0 0 0) } } }
```

```
In[135]:= Map[MatrixForm, basislist[4]]
```

$$\text{Out[135]} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}$$

Evaluating the next cell we learn that $\text{sp}(2, \mathbb{K}) = \text{sl}(2, \mathbb{K})$:

```
In[136]:= spelm[a, 2] // MatrixForm
```

```
Out[136]//MatrixForm=
```

$$\begin{pmatrix} a[1, 1] & a[1, 2] \\ a[2, 1] & -a[1, 1] \end{pmatrix}$$

```
In[138]:= spelm[a, 4] // MatrixForm
```

```
Out[138]//MatrixForm=
```

$$\begin{pmatrix} a[1, 1] & a[1, 2] & a[1, 3] & a[1, 4] \\ a[2, 1] & a[2, 2] & a[1, 4] & a[2, 4] \\ a[3, 1] & a[3, 2] & -a[1, 1] & -a[2, 1] \\ a[3, 2] & a[4, 2] & -a[1, 2] & -a[2, 2] \end{pmatrix}$$

Here is a test of the invariance condition:

```
In[139]:= Transpose[%].alt[4] + alt[4].%
```

```
Out[139]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

```
In[141]:= basislist[10];
```

```
In[142]:= Length[%]
```

```
Out[142]= 55
```

```
In[143]:= sptt4 = basislist[4]
```

```
Out[143]= {{{1, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, 0}},
  {{0, 1, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, -1, 0}},
  {{0, 0, 1, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  {{0, 0, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, -1}, {0, 0, 0, 0}},
  {{0, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, -1}},
  {{0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  {{0, 0, 0, 0}, {0, 0, 0, 0}, {1, 0, 0, 0}, {0, 0, 0, 0}},
  {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}},
  {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 1, 0, 0}}}
```

```
In[144]:= Length[sptt4]
```

```
Out[144]= 10
```

Test of the invariance condition:


```
In[145]:= spelm[a, 6] // MatrixForm
```

```
Out[145]//MatrixForm=
```

$$\begin{pmatrix} a[1, 1] & a[1, 2] & a[1, 3] & a[1, 4] & a[1, 5] & a[1, 6] \\ a[2, 1] & a[2, 2] & a[2, 3] & a[1, 5] & a[2, 5] & a[2, 6] \\ a[3, 1] & a[3, 2] & a[3, 3] & a[1, 6] & a[2, 6] & a[3, 6] \\ a[4, 1] & a[4, 2] & a[4, 3] & -a[1, 1] & -a[2, 1] & -a[3, 1] \\ a[4, 2] & a[5, 2] & a[5, 3] & -a[1, 2] & -a[2, 2] & -a[3, 2] \\ a[4, 3] & a[5, 3] & a[6, 3] & -a[1, 3] & -a[2, 3] & -a[3, 3] \end{pmatrix}$$

```
In[146]:= %.alt[6] + alt[6].Transpose[%]
```

```
Out[146]= {{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}
```

The dimension of the Lie algebra $\mathfrak{sp}(2m, K)$ is (with $n = 2m$), see [3], p.32:

```
In[147]:= spdim[n_?EvenQ] := n * (n + 1) / 2
```

```
In[150]:= spdim[10]
```

```
Out[150]= 55
```

This coincides with the lengths of our basislists.

■ 5.2 The Killingform spkilling

Since nullmatrices do not contribute terms to the Killing form we may define

```
In[40]:= spkilling[a_, b_, n_?EvenQ] :=
          Sum[Sum[cobasis[i, j, n][commute[a, commute[b, spbasis[i, j, n]]],
                {i, 1, n}], {j, 1, n}]
```

```
In[153]:= Simplify[spkilling[spelm[a, 2], spelm[b, 2], 2]]
```

```
Out[153]= 4 (2 a[1, 1] b[1, 1] + a[2, 1] b[1, 2] + a[1, 2] b[2, 1])
```

Of course, this coincides with the Killing form of $\mathfrak{sl}(2, K)$:

```
In[152]:= glkilling[spelm[a, 2], spelm[b, 2], 2]
```

```
Out[152]= 8 a[1, 1] b[1, 1] + 4 a[2, 1] b[1, 2] + 4 a[1, 2] b[2, 1]
```

```
In[160]:= Simplify[spkilling[spelm[a, 6], spelm[a, 6], 6]]
```

```
Out[160]= 16 (a[1, 1]^2 + 2 a[1, 2] a[2, 1] + a[2, 2]^2 + 2 a[1, 3] a[3, 1] +
            2 a[2, 3] a[3, 2] + a[3, 3]^2 + a[1, 4] a[4, 1] + 2 a[1, 5] a[4, 2] +
            2 a[1, 6] a[4, 3] + a[2, 5] a[5, 2] + 2 a[2, 6] a[5, 3] + a[3, 6] a[6, 3])
```

```
In[161]:= Simplify[glkilling[spelm[a, 6], spelm[a, 6], 6]]
```

```
Out[161]= 24 (a[1, 1]^2 + 2 a[1, 2] a[2, 1] + a[2, 2]^2 + 2 a[1, 3] a[3, 1] +
            2 a[2, 3] a[3, 2] + a[3, 3]^2 + a[1, 4] a[4, 1] + 2 a[1, 5] a[4, 2] +
            2 a[1, 6] a[4, 3] + a[2, 5] a[5, 2] + 2 a[2, 6] a[5, 3] + a[3, 6] a[6, 3])
```

```
In[162]:= Simplify[% / %]
```

```
Out[162]=  $\frac{3}{2}$ 
```

Again we obtained the conformality to the restriction of the Killing form of the general linear Lie algebra.

```
In[163]:= spdim[6]
```

```
Out[163]= 21
```

```
In[164]:= spdim[12]
```

```
Out[164]= 78
```

```
In[165]:= Timing[spkilling[spelm[a, 12], spelm[a, 12], 12]]
```

```
Out[165]= {33.6341, 20 a[1, 1]^2 + 44 a[1, 2] a[2, 1] +
  20 a[2, 2]^2 + 44 a[1, 3] a[3, 1] + 44 a[2, 3] a[3, 2] + 20 a[3, 3]^2 +
  44 a[1, 4] a[4, 1] + 44 a[2, 4] a[4, 2] + 44 a[3, 4] a[4, 3] +
  20 a[4, 4]^2 + 44 a[1, 5] a[5, 1] + 44 a[2, 5] a[5, 2] + 44 a[3, 5] a[5, 3] +
  44 a[4, 5] a[5, 4] + 20 a[5, 5]^2 + 44 a[1, 6] a[6, 1] + 44 a[2, 6] a[6, 2] +
  44 a[3, 6] a[6, 3] + 44 a[4, 6] a[6, 4] + 44 a[5, 6] a[6, 5] +
  20 a[6, 6]^2 + 20 a[1, 7] a[7, 1] + 44 a[1, 8] a[7, 2] + 44 a[1, 9] a[7, 3] +
  44 a[1, 10] a[7, 4] + 44 a[1, 11] a[7, 5] + 44 a[1, 12] a[7, 6] +
  4 (2 a[1, 1]^2 + a[1, 2] a[2, 1] + a[1, 3] a[3, 1] + a[1, 4] a[4, 1] +
    a[1, 5] a[5, 1] + a[1, 6] a[6, 1] + a[1, 7] a[7, 1] + a[1, 8] a[7, 2] +
    a[1, 9] a[7, 3] + a[1, 10] a[7, 4] + a[1, 11] a[7, 5] + a[1, 12] a[7, 6]) +
  2 (a[1, 2] a[2, 1] + a[1, 3] a[3, 1] + a[1, 4] a[4, 1] + a[1, 5] a[5, 1] +
    a[1, 6] a[6, 1] + 2 a[1, 7] a[7, 1] + a[1, 8] a[7, 2] + a[1, 9] a[7, 3] +
    a[1, 10] a[7, 4] + a[1, 11] a[7, 5] + a[1, 12] a[7, 6]) +
  20 a[2, 8] a[8, 2] + 44 a[2, 9] a[8, 3] + 44 a[2, 10] a[8, 4] +
  44 a[2, 11] a[8, 5] + 44 a[2, 12] a[8, 6] +
  4 (a[1, 2] a[2, 1] + 2 a[2, 2]^2 + a[2, 3] a[3, 2] + a[2, 4] a[4, 2] +
    a[2, 5] a[5, 2] + a[2, 6] a[6, 2] + a[1, 8] a[7, 2] + a[2, 8] a[8, 2] +
    a[2, 9] a[8, 3] + a[2, 10] a[8, 4] + a[2, 11] a[8, 5] + a[2, 12] a[8, 6]) +
  2 (a[1, 2] a[2, 1] + a[2, 3] a[3, 2] + a[2, 4] a[4, 2] + a[2, 5] a[5, 2] +
    a[2, 6] a[6, 2] + a[1, 8] a[7, 2] + 2 a[2, 8] a[8, 2] + a[2, 9] a[8, 3] +
    a[2, 10] a[8, 4] + a[2, 11] a[8, 5] + a[2, 12] a[8, 6]) +
  20 a[3, 9] a[9, 3] + 44 a[3, 10] a[9, 4] + 44 a[3, 11] a[9, 5] +
  44 a[3, 12] a[9, 6] +
  4 (a[1, 3] a[3, 1] + a[2, 3] a[3, 2] + 2 a[3, 3]^2 + a[3, 4] a[4, 3] +
    a[3, 5] a[5, 3] + a[3, 6] a[6, 3] + a[1, 9] a[7, 3] + a[2, 9] a[8, 3] +
    a[3, 9] a[9, 3] + a[3, 10] a[9, 4] + a[3, 11] a[9, 5] + a[3, 12] a[9, 6]) +
  2 (a[1, 3] a[3, 1] + a[2, 3] a[3, 2] + a[3, 4] a[4, 3] + a[3, 5] a[5, 3] +
    a[3, 6] a[6, 3] + a[1, 9] a[7, 3] + a[2, 9] a[8, 3] + 2 a[3, 9] a[9, 3] +
    a[3, 10] a[9, 4] + a[3, 11] a[9, 5] + a[3, 12] a[9, 6]) +
  20 a[4, 10] a[10, 4] + 44 a[4, 11] a[10, 5] + 44 a[4, 12] a[10, 6] +
  4 (a[1, 4] a[4, 1] + a[2, 4] a[4, 2] + a[3, 4] a[4, 3] +
    2 a[4, 4]^2 + a[4, 5] a[5, 4] + a[4, 6] a[6, 4] +
    a[1, 10] a[7, 4] + a[2, 10] a[8, 4] + a[3, 10] a[9, 4] +
    a[4, 10] a[10, 4] + a[4, 11] a[10, 5] + a[4, 12] a[10, 6]) +
  2 (a[1, 4] a[4, 1] + a[2, 4] a[4, 2] + a[3, 4] a[4, 3] + a[4, 5] a[5, 4] +
    a[4, 6] a[6, 4] + a[1, 10] a[7, 4] + a[2, 10] a[8, 4] + a[3, 10] a[9, 4] +
    2 a[4, 10] a[10, 4] + a[4, 11] a[10, 5] + a[4, 12] a[10, 6]) +
  20 a[5, 11] a[11, 5] + 44 a[5, 12] a[11, 6] +
  4 (a[1, 5] a[5, 1] + a[2, 5] a[5, 2] + a[3, 5] a[5, 3] + a[4, 5] a[5, 4] +
```

$$\begin{aligned}
& 2a[5, 5]^2 + a[5, 6]a[6, 5] + a[1, 11]a[7, 5] + a[2, 11]a[8, 5] + a[3, 11] \\
& \quad a[9, 5] + a[4, 11]a[10, 5] + a[5, 11]a[11, 5] + a[5, 12]a[11, 6]) + \\
2 & (a[1, 5]a[5, 1] + a[2, 5]a[5, 2] + a[3, 5]a[5, 3] + a[4, 5]a[5, 4] + \\
& \quad a[5, 6]a[6, 5] + a[1, 11]a[7, 5] + a[2, 11]a[8, 5] + a[3, 11]a[9, 5] + \\
& \quad a[4, 11]a[10, 5] + 2a[5, 11]a[11, 5] + a[5, 12]a[11, 6]) + \\
20 & a[6, 12]a[12, 6] + 4(a[1, 6]a[6, 1] + a[2, 6]a[6, 2] + \\
& \quad a[3, 6]a[6, 3] + a[4, 6]a[6, 4] + a[5, 6]a[6, 5] + 2a[6, 6]^2 + \\
& \quad a[1, 12]a[7, 6] + a[2, 12]a[8, 6] + a[3, 12]a[9, 6] + \\
& \quad a[4, 12]a[10, 6] + a[5, 12]a[11, 6] + a[6, 12]a[12, 6]) + \\
2 & (a[1, 6]a[6, 1] + a[2, 6]a[6, 2] + a[3, 6]a[6, 3] + a[4, 6]a[6, 4] + \\
& \quad a[5, 6]a[6, 5] + a[1, 12]a[7, 6] + a[2, 12]a[8, 6] + a[3, 12]a[9, 6] + \\
& \quad a[4, 12]a[10, 6] + a[5, 12]a[11, 6] + 2a[6, 12]a[12, 6]) \}
\end{aligned}$$

In[166]= %[[1]]

Out[166]= 33.6341

Mathematica needed 30.15 s.

In[167]= Simplify[%[[2]]]

$$\begin{aligned}
\text{Out[167]= } & 28(a[1, 1]^2 + 2a[1, 2]a[2, 1] + a[2, 2]^2 + 2a[1, 3]a[3, 1] + 2a[2, 3]a[3, 2] + \\
& a[3, 3]^2 + 2a[1, 4]a[4, 1] + 2a[2, 4]a[4, 2] + 2a[3, 4]a[4, 3] + \\
& a[4, 4]^2 + 2a[1, 5]a[5, 1] + 2a[2, 5]a[5, 2] + 2a[3, 5]a[5, 3] + \\
& 2a[4, 5]a[5, 4] + a[5, 5]^2 + 2a[1, 6]a[6, 1] + 2a[2, 6]a[6, 2] + \\
& 2a[3, 6]a[6, 3] + 2a[4, 6]a[6, 4] + 2a[5, 6]a[6, 5] + a[6, 6]^2 + \\
& a[1, 7]a[7, 1] + 2a[1, 8]a[7, 2] + 2a[1, 9]a[7, 3] + 2a[1, 10]a[7, 4] + \\
& 2a[1, 11]a[7, 5] + 2a[1, 12]a[7, 6] + a[2, 8]a[8, 2] + 2a[2, 9]a[8, 3] + \\
& 2a[2, 10]a[8, 4] + 2a[2, 11]a[8, 5] + 2a[2, 12]a[8, 6] + a[3, 9]a[9, 3] + \\
& 2a[3, 10]a[9, 4] + 2a[3, 11]a[9, 5] + 2a[3, 12]a[9, 6] + \\
& a[4, 10]a[10, 4] + 2a[4, 11]a[10, 5] + 2a[4, 12]a[10, 6] + \\
& a[5, 11]a[11, 5] + 2a[5, 12]a[11, 6] + a[6, 12]a[12, 6])
\end{aligned}$$

In[41]= sprod[n_?EvenQ][a_, b_] := spkilling[a, b, n]

In[169]= matrixorthonorm[basislist[2], innerprod → sprod[2]]

$$\begin{aligned}
\text{Out[169]= } & \left\{ \left\{ \left\{ \frac{1}{2\sqrt{2}}, 0 \right\}, \left\{ 0, -\frac{1}{2\sqrt{2}} \right\} \right\}, \right. \\
& \left. \left\{ \left\{ 0, -\frac{1}{2\sqrt{2}} \right\}, \left\{ \frac{1}{2\sqrt{2}}, 0 \right\} \right\}, \left\{ \left\{ 0, \frac{1}{2\sqrt{2}} \right\}, \left\{ \frac{1}{2\sqrt{2}}, 0 \right\} \right\} \right\}
\end{aligned}$$

In[170]= Table[sprod[2][%[[i]], %[[j]], {i, 3}, {j, 3}]

Out[170]= {{1, 0, 0}, {0, -1, 0}, {0, 0, 1}}

```
In[171]:= matrixorthonorm[basislist[4], innerprod → sprod[4]]
```

```
Out[171]= {{ { 1/(2*sqrt(3)), 0, 0, 0 }, { 0, 0, 0, 0 }, { 0, 0, -1/(2*sqrt(3)), 0 }, { 0, 0, 0, 0 } },
  { { 0, 0, 0, 0 }, { 0, 1/(2*sqrt(3)), 0, 0 }, { 0, 0, 0, 0 }, { 0, 0, 0, -1/(2*sqrt(3)) } },
  { { 0, -1/(2*sqrt(6)), 0, 0 }, { 1/(2*sqrt(6)), 0, 0, 0 }, { 0, 0, 0, -1/(2*sqrt(6)) }, { 0, 0, 1/(2*sqrt(6)), 0 } },
  { { 0, 1/(2*sqrt(6)), 0, 0 }, { 1/(2*sqrt(6)), 0, 0, 0 }, { 0, 0, 0, -1/(2*sqrt(6)) }, { 0, 0, -1/(2*sqrt(6)), 0 } },
  { { 0, 0, -1/(2*sqrt(3)), 0 }, { 0, 0, 0, 0 }, { 1/(2*sqrt(3)), 0, 0, 0 }, { 0, 0, 0, 0 } },
  { { 0, 0, 1/(2*sqrt(3)), 0 }, { 0, 0, 0, 0 }, { 1/(2*sqrt(3)), 0, 0, 0 }, { 0, 0, 0, 0 } },
  { { 0, 0, 0, -1/(2*sqrt(6)) }, { 0, 0, -1/(2*sqrt(6)), 0 }, { 0, 1/(2*sqrt(6)), 0, 0 }, { 1/(2*sqrt(6)), 0, 0, 0 } },
  { { 0, 0, 0, 1/(2*sqrt(6)) }, { 0, 0, 1/(2*sqrt(6)), 0 }, { 0, 1/(2*sqrt(6)), 0, 0 }, { 1/(2*sqrt(6)), 0, 0, 0 } },
  { { 0, 0, 0, 0 }, { 0, 0, 0, -1/(2*sqrt(3)) }, { 0, 0, 0, 0 }, { 0, 1/(2*sqrt(3)), 0, 0 } },
  { { 0, 0, 0, 0 }, { 0, 0, 0, 1/(2*sqrt(3)) }, { 0, 0, 0, 0 }, { 0, 1/(2*sqrt(3)), 0, 0 } } }
```

```
In[172]:= sptest[4] = %;
```

```
In[173]:= Length[sptest[4]]
```

```
Out[173]= 10
```

```
In[174]:= Table[sprod[4][sptest[4][[i]], sptest[4][[j]]], {i, 10}, {j, 10}] //
  MatrixForm
```

```
Out[174]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Mathematica needed 222 s for the next calculation, on the Dell Inspiron 3200 under Linux, now, on the Dell Precision 370, 20 s. only:

```
In[175]:= Timing[matrixorthonorm[basislist[6], innerprod → sprod[6]]]
```

$$\begin{aligned}
\text{Out[175]} = & \left\{ 20.4493, \left\{ \left\{ \frac{1}{4}, 0, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\}, \right. \right. \\
& \left. \left\{ 0, 0, 0, -\frac{1}{4}, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\} \right\}, \\
& \left\{ \{0, 0, 0, 0, 0, 0\}, \left\{ 0, \frac{1}{4}, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \right. \\
& \left. \{0, 0, 0, 0, 0, 0\}, \left\{ 0, 0, 0, 0, -\frac{1}{4}, 0 \right\}, \{0, 0, 0, 0, 0, 0\} \right\}, \\
& \left\{ \{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\}, \left\{ 0, 0, \frac{1}{4}, 0, 0, 0 \right\}, \right. \\
& \left. \{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\}, \left\{ 0, 0, 0, 0, 0, -\frac{1}{4} \right\} \right\}, \\
& \left\{ \left\{ 0, -\frac{1}{4\sqrt{2}}, 0, 0, 0, 0 \right\}, \left\{ \frac{1}{4\sqrt{2}}, 0, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{ 0, 0, 0, 0, -\frac{1}{4\sqrt{2}}, 0 \right\}, \left\{ 0, 0, 0, \frac{1}{4\sqrt{2}}, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\} \right\}, \\
& \left\{ \left\{ 0, \frac{1}{4\sqrt{2}}, 0, 0, 0, 0 \right\}, \left\{ \frac{1}{4\sqrt{2}}, 0, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{ 0, 0, 0, 0, -\frac{1}{4\sqrt{2}}, 0 \right\}, \left\{ 0, 0, 0, -\frac{1}{4\sqrt{2}}, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\} \right\}, \\
& \left\{ \left\{ 0, 0, -\frac{1}{4\sqrt{2}}, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \left\{ \frac{1}{4\sqrt{2}}, 0, 0, 0, 0, 0 \right\}, \right. \\
& \left. \left\{ 0, 0, 0, 0, 0, -\frac{1}{4\sqrt{2}} \right\}, \{0, 0, 0, 0, 0, 0\}, \left\{ 0, 0, 0, \frac{1}{4\sqrt{2}}, 0, 0 \right\} \right\}, \\
& \left\{ \left\{ 0, 0, \frac{1}{4\sqrt{2}}, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \left\{ \frac{1}{4\sqrt{2}}, 0, 0, 0, 0, 0 \right\}, \right. \\
& \left. \left\{ 0, 0, 0, 0, 0, -\frac{1}{4\sqrt{2}} \right\}, \{0, 0, 0, 0, 0, 0\}, \left\{ 0, 0, 0, -\frac{1}{4\sqrt{2}}, 0, 0 \right\} \right\}, \\
& \left\{ \left\{ 0, 0, 0, -\frac{1}{4}, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{ \frac{1}{4}, 0, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\} \right\}, \\
& \left\{ \left\{ 0, 0, 0, \frac{1}{4}, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{ \frac{1}{4}, 0, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\} \right\}, \\
& \left\{ \left\{ 0, 0, 0, 0, -\frac{1}{4\sqrt{2}}, 0 \right\}, \left\{ 0, 0, 0, -\frac{1}{4\sqrt{2}}, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{ 0, \frac{1}{4\sqrt{2}}, 0, 0, 0, 0 \right\}, \left\{ \frac{1}{4\sqrt{2}}, 0, 0, 0, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\} \right\}, \\
& \left\{ \left\{ 0, 0, 0, 0, \frac{1}{4\sqrt{2}}, 0 \right\}, \left\{ 0, 0, 0, \frac{1}{4\sqrt{2}}, 0, 0 \right\}, \{0, 0, 0, 0, 0, 0\}, \right.
\end{aligned}$$

$$\begin{aligned}
& \left\{0, \frac{1}{4\sqrt{2}}, 0, 0, 0, 0\right\}, \left\{\frac{1}{4\sqrt{2}}, 0, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0, 0\}, \\
& \left\{\left\{0, 0, 0, 0, 0, -\frac{1}{4\sqrt{2}}\right\}, \{0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, -\frac{1}{4\sqrt{2}}, 0, 0\right\},\right. \\
& \left.\left\{0, 0, \frac{1}{4\sqrt{2}}, 0, 0, 0\right\}, \{0, 0, 0, 0, 0, 0\}, \left\{\frac{1}{4\sqrt{2}}, 0, 0, 0, 0, 0\right\}\right\}, \\
& \left\{\left\{0, 0, 0, 0, 0, \frac{1}{4\sqrt{2}}\right\}, \{0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, \frac{1}{4\sqrt{2}}, 0, 0\right\},\right. \\
& \left.\left\{0, 0, \frac{1}{4\sqrt{2}}, 0, 0, 0\right\}, \{0, 0, 0, 0, 0, 0\}, \left\{\frac{1}{4\sqrt{2}}, 0, 0, 0, 0, 0\right\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0, 0\}, \left\{0, 0, -\frac{1}{4\sqrt{2}}, 0, 0, 0\right\}, \left\{0, \frac{1}{4\sqrt{2}}, 0, 0, 0, 0\right\},\right. \\
& \left.\{0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, 0, -\frac{1}{4\sqrt{2}}\right\}, \left\{0, 0, 0, 0, \frac{1}{4\sqrt{2}}, 0\right\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0, 0\}, \left\{0, 0, \frac{1}{4\sqrt{2}}, 0, 0, 0\right\}, \left\{0, \frac{1}{4\sqrt{2}}, 0, 0, 0, 0\right\},\right. \\
& \left.\{0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, 0, -\frac{1}{4\sqrt{2}}\right\}, \left\{0, 0, 0, 0, -\frac{1}{4\sqrt{2}}, 0\right\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, -\frac{1}{4}, 0\right\}, \{0, 0, 0, 0, 0, 0\},\right. \\
& \left.\{0, 0, 0, 0, 0, 0\}, \left\{0, \frac{1}{4}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0, 0\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{1}{4}, 0\right\}, \{0, 0, 0, 0, 0, 0\},\right. \\
& \left.\{0, 0, 0, 0, 0, 0\}, \left\{0, \frac{1}{4}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0, 0\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, 0, -\frac{1}{4\sqrt{2}}\right\}, \left\{0, 0, 0, 0, -\frac{1}{4\sqrt{2}}, 0\right\},\right. \\
& \left.\{0, 0, 0, 0, 0, 0\}, \left\{0, 0, \frac{1}{4\sqrt{2}}, 0, 0, 0\right\}, \left\{0, \frac{1}{4\sqrt{2}}, 0, 0, 0, 0\right\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, 0, \frac{1}{4\sqrt{2}}\right\}, \left\{0, 0, 0, 0, \frac{1}{4\sqrt{2}}, 0\right\},\right. \\
& \left.\{0, 0, 0, 0, 0, 0\}, \left\{0, 0, \frac{1}{4\sqrt{2}}, 0, 0, 0\right\}, \left\{0, \frac{1}{4\sqrt{2}}, 0, 0, 0, 0\right\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, 0, -\frac{1}{4}\right\},\right. \\
& \left.\{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\}, \left\{0, 0, \frac{1}{4}, 0, 0, 0\right\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, 0, \frac{1}{4}\right\},\right. \\
& \left.\{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\}, \left\{0, 0, \frac{1}{4}, 0, 0, 0\right\}\right\}
\end{aligned}$$

In[176]:= %[[1]]

Out[176]:= 20.4493

In[177]:= **sptest[6] = %[[2]];**

In[178]:= **Length[sptest[6]]**

Out[178]:= 21

In[179]:= **Table[sptest[6][[i]] // MatrixForm, {i, 21}]**

$$\text{Out[179]= } \left\{ \begin{array}{l} \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} \end{pmatrix}, \\ \\ \begin{pmatrix} 0 & -\frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{4\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{4\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \\ \begin{pmatrix} 0 & 0 & -\frac{1}{4\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4\sqrt{2}} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{4\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4\sqrt{2}} & 0 & 0 \end{pmatrix}, \\ \\ \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{4\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{4\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \right\}$$


```
In[180]:= Table[spprod[6][sptest[6][[i]], sptest[6][[j]]], {i, 21}, {j, 21}] //
MatrixForm
```

```
Out[180]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From the tables [3], p. 33, it follows that the index of the Killingform $\text{skilling}[a,b,n]$, n even, is

```
In[181]:= spindex[n_?EvenQ] := n^2 / 4
```

and this coincides with our results:

```
In[183]:= Table[{2 k, spindex[2 k]}, {k, 3}] // TableForm
```

```
Out[183]//TableForm=
```

```
2 1
4 4
6 9
```

6 The unitary and the special unitary Lie algebras $u(n)$, $su(n)$

Attention: Not all the needed cells of this section are initialized, since first the package [4] must be downloaded from *MathSource* (see below). Only if this is done, the section may be evaluated completely!

■ 6.1 A basis of $u(n)$

```
In[42]:= ubasisa[i_, j_, n_:dim] := obm[i, j, n] /; (i < j && j <= n)
```

```

In[43]:= ubasisb[i_, j_, n_: dim] :=
  (basismatrix[i, j, n] + basismatrix[j, i, n]) * I /; (i < j && j <= n)

In[44]:= ubasisc[i_, n_: dim] := basismatrix[i, i, n] * I /; (i <= n)

In[45]:= ubasis[n_: dim] :=
  Join[Flatten[Join[Table[ubasisa[i, j, n], {j, 2, n}, {i, 1, j-1}],
    Table[ubasisb[i, j, n], {j, 2, n}, {i, 1, j-1}]],
  1], Table[ubasisc[i, n], {i, n}]]

```

Tests

The dimension of the real vector space $u(n)$ is n^2 .

```
In[184]:= Length[ubasis[10]]
```

```
Out[184]= 100
```

```
In[185]:= ubasis[3]
```

```
Out[185]= {{{0, 1, 0}, {-1, 0, 0}, {0, 0, 0}},
  {{0, 0, 1}, {0, 0, 0}, {-1, 0, 0}}, {{0, 0, 0}, {0, 0, 1}, {0, -1, 0}},
  {{0, i, 0}, {i, 0, 0}, {0, 0, 0}}, {{0, 0, i}, {0, 0, 0}, {i, 0, 0}},
  {{0, 0, 0}, {0, 0, i}, {0, i, 0}}, {{i, 0, 0}, {0, 0, 0}, {0, 0, 0}},
  {{0, 0, 0}, {0, i, 0}, {0, 0, 0}}, {{0, 0, 0}, {0, 0, 0}, {0, 0, i}}}
```

```
In[186]:= Map[MatrixForm, %]
```

```
Out[186]= {
  {
     $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,
     $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ ,
     $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ ,
     $\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,
     $\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$ ,
     $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$ ,
     $\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,
     $\begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,
     $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}$ 
  }
}
```

■ 6.2 Declare real coefficients

In spite of the fact that $u(n)$ consists of complex matrices it is a real Lie algebra: a complex square matrix of order n belongs to $u(n)$ if and only if it fulfils

```
In[187]:= Transpose[matrix[a]] == -Conjugate[matrix[a]];
```

This condition is preserved under linear combination with real coefficients only.

```
In[188]:= a = ubasis[3][[9]]; Transpose[a] == -Conjugate[a]
```

```
Out[188]= True
```

```
In[189]:= a = ubasis[3][[9]] * I; Transpose[a] == -Conjugate[a]
```

```
Out[189]= False
```

```
In[46]:= Clear[a]
```

Thus we have to declare that the coordinates of the elements of $u(n)$ shall be real. This can be done by the function `Declare` contained in the package [4] by Pekka Janhunen. First download the package, and copy it in the working directory

```
In[47]:= SetDirectory[
  "C:\\Dokumente und Einstellungen\\rolf\\Eigene Dateien\\mymath\\lie"]
Out[47]= C:\Dokumente und Einstellungen\rolf\Eigene Dateien\mymath\lie
```

Then import

```
In[48]:= << Declare.m
  {Declare, NewDeclare, NonPositive, RealQ}
```

We add some rules for the function Conjugate:

```
In[49]:= Unprotect [Conjugate]
Out[49]= {Conjugate}

In[50]:= Conjugate[x_?RealQ + I * y_?RealQ] := x - I * y

In[51]:= Conjugate[x_?RealQ] := x

In[52]:= Conjugate[I * x_?RealQ] := -I * x

In[53]:= Protect [Conjugate]
Out[53]= {Conjugate}
```

Now evaluate the definitions contained in the next subsection.

```
In[54]:= ? Declare
```

Declare[a,Odd] or Declare[{x,y},Real] etc. Possible declarations are Odd,Even,Integer,Real,Positive,Negative,NonPositive,NonNegative. Declare[x,Real,{n,m},Integer] etc. are also possible. Declare does not remove any previous properties, for example after (Declare[x,Integer]; Declare[x,Real]) IntegerQ[x] is still true. See also NewDeclare.

```
In[55]:= ? NewDeclare
```

NewDeclare is the same as Declare except that it removes any previous properties of the symbols using Clear. Only single symbols may be used with NewDeclare.

■ 6.3 General element, cobasis, and Killing form of $u(n)$

Definitions

Now evaluate the subsubsection "Definitions"!

The general element of the Lie algebra $u(n)$ is

```
In[56]:= Clear [dim];
```

```

In[57]:= uelm[a_, n_: dim] :=
  Module[{k, c = Table[a[k], {k, n^2}]}, Declare[c, Real];
  Sum[ubasis[n][[k]] * a[k], {k, n^2}]

In[58]:= ucobasisa[i_, j_, n_: dim][m_?MatrixQ] :=
  Re[cobasis[i, j, n][m]] /; (i < j && j <= n)

In[59]:= ucobasisb[i_, j_, n_: dim][m_?MatrixQ] :=
  Im[cobasis[i, j, n][m]] /; (i < j && j <= n)

In[60]:= ucobasisc[i_, n_: dim][m_?MatrixQ] := Im[cobasis[i, i, n][m]] /; (i <= n)

In[61]:= ucobasis[n_: dim][m_?MatrixQ] :=
  Join[Flatten[Join[Table[ucobasisa[i, j, n][m], {j, 2, n}, {i, 1, j-1}],
  Table[ucobasisb[i, j, n][m], {j, 2, n}, {i, 1, j-1}]], 1],
  Table[ucobasisc[i, n][m], {i, n}]]

In[62]:= ukilling[x_?MatrixQ, y_?MatrixQ, n_: dim] :=
  Sum[ucobasis[n][commute[x, commute[y, ubasis[n][[i]]]]][[i]], {i, n^2}]

```

Examples, Tests

```
In[63]:= uelm[a, 2] // MatrixForm
```

```
Out[63]//MatrixForm=
```

$$\begin{pmatrix} i a[3] & a[1] + i a[2] \\ -a[1] + i a[2] & i a[4] \end{pmatrix}$$

```
In[64]:= TrueQ[Conjugate[uelm[ab, 2]] == -Transpose[uelm[ab, 2]]]
```

```
Out[64]= True
```

```
In[65]:= RealQ[a[1]]
```

```
Out[65]= True
```

```
In[66]:= uelm[a, 4] // MatrixForm
```

```
Out[66]//MatrixForm=
```

$$\begin{pmatrix} i a[13] & a[1] + i a[7] & a[2] + i a[8] & a[4] + i a[10] \\ -a[1] + i a[7] & i a[14] & a[3] + i a[9] & a[5] + i a[11] \\ -a[2] + i a[8] & -a[3] + i a[9] & i a[15] & a[6] + i a[12] \\ -a[4] + i a[10] & -a[5] + i a[11] & -a[6] + i a[12] & i a[16] \end{pmatrix}$$

```
In[67]:= TrueQ[Conjugate[uelm[a, 4]] == -Transpose[uelm[a, 4]]]
```

```
Out[67]= True
```

```
In[68]:= ucobasis[4][uelm[a, 4]]
```

```
Out[68]= {a[1], a[2], a[3], a[4], a[5], a[6], a[7], a[8],
  a[9], a[10], a[11], a[12], a[13], a[14], a[15], a[16]}
```

```
In[69]:= Simplify[ukilling[uelm[a, 2], uelm[a, 2], 2]]
```

```
Out[69]= -2 (4 a[1]^2 + 4 a[2]^2 + (a[3] - a[4])^2)
```

```
In[70]:= Simplify[ukilling[uelm[a, 4], uelm[b, 4], 4]]
```

```
Out[70]= -2 (8 a[1] b[1] + 8 a[2] b[2] + 8 a[3] b[3] + 8 a[4] b[4] +
      8 a[5] b[5] + 8 a[6] b[6] + 8 a[7] b[7] + 8 a[8] b[8] + 8 a[9] b[9] +
      8 a[10] b[10] + 8 a[11] b[11] + 8 a[12] b[12] + 3 a[13] b[13] -
      a[14] b[13] - a[15] b[13] - a[16] b[13] - a[13] b[14] + 3 a[14] b[14] -
      a[15] b[14] - a[16] b[14] - a[13] b[15] - a[14] b[15] + 3 a[15] b[15] -
      a[16] b[15] - a[13] b[16] - a[14] b[16] - a[15] b[16] + 3 a[16] b[16])
```

```
In[71]:= Simplify[glkilling[uelm[a, 4], uelm[b, 4], 4]]
```

```
Out[71]= -2 (8 a[1] b[1] + 8 a[2] b[2] + 8 a[3] b[3] + 8 a[4] b[4] +
      8 a[5] b[5] + 8 a[6] b[6] + 8 a[7] b[7] + 8 a[8] b[8] + 8 a[9] b[9] +
      8 a[10] b[10] + 8 a[11] b[11] + 8 a[12] b[12] + 3 a[13] b[13] -
      a[14] b[13] - a[15] b[13] - a[16] b[13] - a[13] b[14] + 3 a[14] b[14] -
      a[15] b[14] - a[16] b[14] - a[13] b[15] - a[14] b[15] + 3 a[15] b[15] -
      a[16] b[15] - a[13] b[16] - a[14] b[16] - a[15] b[16] + 3 a[16] b[16])
```

```
In[72]:= Simplify[% == %]
```

```
Out[72]= True
```

In this case the unitary Killing form is just the restriction of the (complex) Killingform of the general linear Lie algebra!

```
In[73]:= uprod[x_, y_] := ukilling[x, y, 3]
```

```
In[74]:= Table[uprod[ubasis[3][[i]], ubasis[3][[j]]], {i, 9}, {j, 9}]
```

```
Out[74]= {{-12, 0, 0, 0, 0, 0, 0, 0, 0},
      {0, -12, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -12, 0, 0, 0, 0, 0, 0},
      {0, 0, 0, -12, 0, 0, 0, 0, 0}, {0, 0, 0, 0, -12, 0, 0, 0, 0},
      {0, 0, 0, 0, 0, -12, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -4, 2, 2},
      {0, 0, 0, 0, 0, 0, 2, -4, 2}, {0, 0, 0, 0, 0, 0, 2, 2, -4}}
```

```
In[75]:= % // MatrixForm
```

```
Out[75]//MatrixForm=
```

$$\begin{pmatrix} -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -4 \end{pmatrix}$$

In[76]:= **Timing**[**matrixorthonorm**[**ubasis**[3], **innerprod** -> **uprod**]]

Out[76]= {3.313, {{{{0, $\frac{1}{2\sqrt{3}}$, 0}, {- $\frac{1}{2\sqrt{3}}$, 0, 0}, {0, 0, 0}},
 {{{0, 0, $\frac{1}{2\sqrt{3}}$ }, {0, 0, 0}, {- $\frac{1}{2\sqrt{3}}$, 0, 0}}},
 {{{0, 0, 0}, {0, 0, $\frac{1}{2\sqrt{3}}$ }, {0, - $\frac{1}{2\sqrt{3}}$, 0}}},
 {{{0, $\frac{i}{2\sqrt{3}}$, 0}, { $\frac{i}{2\sqrt{3}}$, 0, 0}, {0, 0, 0}},
 {{{0, 0, $\frac{i}{2\sqrt{3}}$ }, {0, 0, 0}, { $\frac{i}{2\sqrt{3}}$, 0, 0}}},
 {{{0, 0, 0}, {0, 0, $\frac{i}{2\sqrt{3}}$ }, {0, $\frac{i}{2\sqrt{3}}$, 0}}},
 {{{ $\frac{i}{2}$, 0, 0}, {0, 0, 0}, {0, 0, 0}},
 {{{ $\frac{i}{2\sqrt{3}}$, 0, 0}, {0, $\frac{i}{\sqrt{3}}$, 0}, {0, 0, 0}},
 {{{i, 0, 0}, {0, i, 0}, {0, 0, i}}}}

Mathematica needed 3.1 s for this calculation.

In[77]:= **olist** = %[[2]];

In[78]:= **Length**[**olist**]

Out[78]= 9

In[79]:= **Timing**[**Table**[**uprod**[**olist**[[i]], **olist**[[j]], {i, 9}, {j, 9}]]

Out[79]= {1.921, {{{-1, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, -1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, -1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, -1, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, -1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, -1}}}}

For this calculation *Mathematica* needed 17.11 s, now, with Dell Precision 370, 1.9 s only.

In[80]:= **Simplify**[%[[2]]] // **MatrixForm**

Out[80]//**MatrixForm**=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[81]:= **dim = 5**

Out[81]:= 5

In[82]:= **Timing[Simplify[Table[ukilling[ubasis][[i]], ubasis][[j]],
{i, 25}, {j, 25}]] // MatrixForm]**

In[83]:= **%[[1]]**

Out[83]:= 173.063

For the last calculation *Mathematica* in 1999 needed 33 min, with Dell Precision 370 now only 173 s..

In[84]:= **uprod4[a_, b_] := ukilling[a, b, 4]**

In[85]:= **Timing[matrixorthonorm[ubasis[4], innerprod → uprod4]]**

Out[85]= {36.891, {{{{0, $\frac{1}{4}$, 0, 0}, {- $\frac{1}{4}$, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
{{0, 0, $\frac{1}{4}$, 0}, {0, 0, 0, 0}, {- $\frac{1}{4}$, 0, 0, 0}, {0, 0, 0, 0}},
{{0, 0, 0, 0}, {0, 0, $\frac{1}{4}$, 0}, {0, - $\frac{1}{4}$, 0, 0}, {0, 0, 0, 0}},
{{0, 0, 0, $\frac{1}{4}$ }, {0, 0, 0, 0}, {0, 0, 0, 0}, {- $\frac{1}{4}$, 0, 0, 0}},
{{0, 0, 0, 0}, {0, 0, 0, $\frac{1}{4}$ }, {0, 0, 0, 0}, {0, - $\frac{1}{4}$, 0, 0}},
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, $\frac{1}{4}$ }, {0, 0, - $\frac{1}{4}$, 0}},
{{0, $\frac{i}{4}$, 0, 0}, { $\frac{i}{4}$, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
{{0, 0, $\frac{i}{4}$, 0}, {0, 0, 0, 0}, { $\frac{i}{4}$, 0, 0, 0}, {0, 0, 0, 0}},
{{0, 0, 0, 0}, {0, 0, $\frac{i}{4}$, 0}, {0, $\frac{i}{4}$, 0, 0}, {0, 0, 0, 0}},
{{0, 0, 0, $\frac{i}{4}$ }, {0, 0, 0, 0}, {0, 0, 0, 0}, { $\frac{i}{4}$, 0, 0, 0}},
{{0, 0, 0, 0}, {0, 0, 0, $\frac{i}{4}$ }, {0, 0, 0, 0}, {0, $\frac{i}{4}$, 0, 0}},
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, $\frac{i}{4}$ }, {0, 0, $\frac{i}{4}$, 0}},
{{ $\frac{i}{\sqrt{6}}$, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
{{ $\frac{i}{4\sqrt{3}}$, 0, 0, 0}, {0, $\frac{i\sqrt{3}}{4}$, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
{{ $\frac{i}{4}$, 0, 0, 0}, {0, $\frac{i}{4}$, 0, 0}, {0, 0, $\frac{i}{2}$, 0}, {0, 0, 0, 0}},
{{i, 0, 0, 0}, {0, i, 0, 0}, {0, 0, i, 0}, {0, 0, 0, i}}}}

In 1999 *Mathematica* needed 6.3 min for this orthonormalization. New time with Dell Precision 370 now 37 s

```
In[86]:= %[[1]]
Out[86]= 36.891

In[87]:= olist4 = %[[2]];

In[88]:= Length[olist4]
Out[88]= 16

In[89]:= Timing[Table[uprod4[olist4[[i]], olist4[[j]]], {i, 16}, {j, 16}]]
Out[89]= {22.203, {{-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}}}
```

Time: 228.38 s New time: 22 s

```
In[90]:= Simplify[%[[2]]] // MatrixForm
```

```
Out[90]//MatrixForm=
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[91]:= dim = 5
```

```
Out[91]= 5
```


In[92]:= **Timing[matrixorthonorm[ubasis[], innerprod → ukilling]]**

Out[92]= {270.125, {{{{0, $\frac{1}{2\sqrt{5}}$, 0, 0, 0}, {- $\frac{1}{2\sqrt{5}}$, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}, {{0, 0, $\frac{1}{2\sqrt{5}}$, 0, 0}, {0, 0, 0, 0, 0},
{- $\frac{1}{2\sqrt{5}}$, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0}, {0, 0, $\frac{1}{2\sqrt{5}}$, 0, 0}, {0, - $\frac{1}{2\sqrt{5}}$, 0, 0, 0},
{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}, {{0, 0, 0, $\frac{1}{2\sqrt{5}}$, 0}, {0, 0, 0, 0, 0},
{0, 0, 0, 0, 0}, {- $\frac{1}{2\sqrt{5}}$, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0}, {0, 0, 0, $\frac{1}{2\sqrt{5}}$, 0}, {0, 0, 0, 0, 0},
{0, - $\frac{1}{2\sqrt{5}}$, 0, 0, 0}, {0, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
{0, 0, 0, $\frac{1}{2\sqrt{5}}$, 0}, {0, 0, - $\frac{1}{2\sqrt{5}}$, 0, 0}, {0, 0, 0, 0, 0}},
{{0, 0, 0, 0, $\frac{1}{2\sqrt{5}}$ }, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
{- $\frac{1}{2\sqrt{5}}$, 0, 0, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, $\frac{1}{2\sqrt{5}}$ },
{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, - $\frac{1}{2\sqrt{5}}$, 0, 0, 0}},
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, $\frac{1}{2\sqrt{5}}$ }, {0, 0, 0, 0, 0},
{0, 0, - $\frac{1}{2\sqrt{5}}$, 0, 0}}, {{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
{0, 0, 0, 0, 0}, {0, 0, 0, 0, $\frac{1}{2\sqrt{5}}$ }, {0, 0, 0, - $\frac{1}{2\sqrt{5}}$, 0}},
{{0, $\frac{i}{2\sqrt{5}}$, 0, 0, 0}, {- $\frac{i}{2\sqrt{5}}$, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}, {{0, 0, $\frac{i}{2\sqrt{5}}$, 0, 0},
{0, 0, 0, 0, 0}, {- $\frac{i}{2\sqrt{5}}$, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},

$$\begin{aligned}
& \left\{ \{0, 0, 0, 0, 0\}, \left\{0, 0, \frac{i}{2\sqrt{5}}, 0, 0\right\}, \left\{0, \frac{i}{2\sqrt{5}}, 0, 0, 0\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\left\{0, 0, 0, \frac{i}{2\sqrt{5}}, 0\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{\frac{i}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, \frac{i}{2\sqrt{5}}, 0\right\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{0, \frac{i}{2\sqrt{5}}, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \{0, 0, 0, \frac{i}{2\sqrt{5}}, 0\}, \left\{0, 0, \frac{i}{2\sqrt{5}}, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\left\{0, 0, 0, 0, \frac{i}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{\frac{i}{2\sqrt{5}}, 0, 0, 0, 0\right\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{i}{2\sqrt{5}}\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, \frac{i}{2\sqrt{5}}, 0, 0, 0\right\}\right\}, \\
& \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{i}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{0, 0, \frac{i}{2\sqrt{5}}, 0, 0\right\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{i}{2\sqrt{5}}\right\}, \left\{0, 0, 0, \frac{i}{2\sqrt{5}}, 0\right\}\right\}, \\
& \left\{\left\{\frac{i}{2\sqrt{2}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}\right\}, \left\{\left\{\frac{i}{2\sqrt{30}}, 0, 0, 0, 0\right\}, \left\{0, i\sqrt{\frac{2}{15}}, 0, 0, 0\right\}, \right. \\
& \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\left\{\frac{i}{2\sqrt{15}}, 0, 0, 0, 0\right\}, \left\{0, \frac{i}{2\sqrt{15}}, 0, 0, 0\right\}, \right. \\
& \left. \left\{0, 0, \frac{1}{2}i\sqrt{\frac{3}{5}}, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \\
& \left\{\left\{\frac{i}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \left\{0, \frac{i}{2\sqrt{5}}, 0, 0, 0\right\}, \left\{0, 0, \frac{i}{2\sqrt{5}}, 0, 0\right\}, \right. \\
& \left. \left\{0, 0, 0, \frac{i}{\sqrt{5}}, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\{i, 0, 0, 0, 0\}, \right.
\end{aligned}$$


```
In[96]:= Table[ubasis[[i]] // MatrixForm, {i, 25}]
```

$$\text{Out[96]} = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \end{pmatrix} \right\}$$

■ 6.4 The special unitary Lie algebras su(n)

The special unitary Lie algebra su(n) is the subalgebra of the elements of u(n) with trace zero; as kernel of the trace map it is an ideal. Therefore its Killing form is the restriction of ukilling to su(n). We define a basis of su(n):

```
In[97]:= Clear[dim]
```

```
In[98]:= subbasis[n_:dim] := Join[Table[ubasis[n][[i]], {i, n*(n-1)}],
  Table[ubasis[n][[j]] - ubasis[n][[n^2]], {j, n*(n-1)+1, n^2-1}]]
```

```
In[99]:= Table[subbasis[5][[j]] // MatrixForm, {j, 5*(5-1)+1, 5^2-1}]
```

$$\text{Out[99]} = \left\{ \begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & -i \end{pmatrix} \right\}$$

The general element of su(n) is

```
In[100]:= suelm[a_, n_:dim] :=
Module[{k, c = Table[a[k], {k, n^2}]}, Declare[c, Real];
Sum[subbasis[n][[k]] * a[k], {k, n^2-1}]
```

```
In[103]:= dim = 5;
```

```
In[104]:= abc = suelm[a]
```

```
Out[104]= {{i a[21], a[1] + i a[11], a[2] + i a[12], a[4] + i a[14], a[7] + i a[17]},
{-a[1] + i a[11], i a[22], a[3] + i a[13], a[5] + i a[15], a[8] + i a[18]},
{-a[2] + i a[12], -a[3] + i a[13], i a[23], a[6] + i a[16], a[9] + i a[19]},
{-a[4] + i a[14], -a[5] + i a[15], -a[6] + i a[16], i a[24], a[10] + i a[20]},
{-a[7] + i a[17], -a[8] + i a[18], -a[9] + i a[19],
-a[10] + i a[20], -i a[21] - i a[22] - i a[23] - i a[24]}}
```

```
In[105]:= Conjugate[abc] == -Transpose[abc]
```

```
Out[105]= True
```

```
In[106]:= suelm[a] // MatrixForm
```

```
Out[106]//MatrixForm=
```

$$\begin{pmatrix} i a[21] & a[1] + i a[11] & a[2] + i a[12] & a[4] + i a[14] & & & a[7] \\ -a[1] + i a[11] & i a[22] & a[3] + i a[13] & a[5] + i a[15] & & & a[8] \\ -a[2] + i a[12] & -a[3] + i a[13] & i a[23] & a[6] + i a[16] & & & a[9] \\ -a[4] + i a[14] & -a[5] + i a[15] & -a[6] + i a[16] & i a[24] & & & a[10] \\ -a[7] + i a[17] & -a[8] + i a[18] & -a[9] + i a[19] & -a[10] + i a[20] & -i a[21] - i a[22] - i a[23] - i a[24] & & \end{pmatrix}$$

```
In[86]:= Timing[matrixorthonorm[subbasis[], innerprod -> ukilling]]
```

```
Out[86]= {228.99, {{{{0, 1/(2*sqrt(5)), 0, 0, 0}, {-1/(2*sqrt(5)), 0, 0, 0, 0}, {0, 0, 0, 0, 0},
{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}, {{0, 0, 1/(2*sqrt(5)), 0, 0}, {0, 0, 0, 0, 0},
{-1/(2*sqrt(5)), 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0}, {0, 0, 1/(2*sqrt(5)}, 0, 0}, {0, -1/(2*sqrt(5)}, 0, 0, 0},
{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}, {{0, 0, 0, 1/(2*sqrt(5)}, 0}, {0, 0, 0, 0, 0},
{0, 0, 0, 0, 0}, {-1/(2*sqrt(5)}, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}}}
```

$$\begin{aligned}
& \left\{ \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, \frac{1}{2\sqrt{5}}, 0\right\}, \{0, 0, 0, 0, 0\}, \right. \\
& \left. \left\{0, -\frac{1}{2\sqrt{5}}, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \right. \\
& \left. \left. \left\{0, 0, 0, \frac{1}{2\sqrt{5}}, 0\right\}, \left\{0, 0, -\frac{1}{2\sqrt{5}}, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \right\}, \right. \\
& \left. \left\{\left\{0, 0, 0, 0, \frac{1}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \right. \\
& \left. \left. \left\{-\frac{1}{2\sqrt{5}}, 0, 0, 0, 0\right\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{1}{2\sqrt{5}}\right\}, \right. \right. \\
& \left. \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, -\frac{1}{2\sqrt{5}}, 0, 0, 0\right\}\right\}, \right. \\
& \left. \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{1}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \right. \right. \\
& \left. \left. \left\{0, 0, -\frac{1}{2\sqrt{5}}, 0, 0\right\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \right. \\
& \left. \left. \{0, 0, 0, 0, 0\}, \left\{0, 0, 0, 0, \frac{1}{2\sqrt{5}}\right\}, \left\{0, 0, 0, -\frac{1}{2\sqrt{5}}, 0\right\}\right\}, \right. \\
& \left. \left\{\left\{0, \frac{i}{2\sqrt{5}}, 0, 0, 0\right\}, \left\{\frac{i}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \right. \right. \\
& \left. \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\left\{0, 0, \frac{i}{2\sqrt{5}}, 0, 0\right\}, \right. \right. \\
& \left. \left. \{0, 0, 0, 0, 0\}, \left\{\frac{i}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \right. \\
& \left. \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, \frac{i}{2\sqrt{5}}, 0, 0\right\}, \left\{0, \frac{i}{2\sqrt{5}}, 0, 0, 0\right\}, \right. \right. \\
& \left. \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\left\{0, 0, 0, \frac{i}{2\sqrt{5}}, 0\right\}, \right. \right. \\
& \left. \left. \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \left\{\frac{i}{2\sqrt{5}}, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \right. \\
& \left. \left\{\{0, 0, 0, 0, 0\}, \left\{0, 0, 0, \frac{i}{2\sqrt{5}}, 0\right\}, \{0, 0, 0, 0, 0\}, \right. \right. \\
& \left. \left. \left\{0, \frac{i}{2\sqrt{5}}, 0, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \left\{\{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \right. \\
& \left. \left. \left\{0, 0, 0, \frac{i}{2\sqrt{5}}, 0\right\}, \left\{0, 0, \frac{i}{2\sqrt{5}}, 0, 0\right\}, \{0, 0, 0, 0, 0\}\right\}, \right. \\
& \left. \left\{\left\{0, 0, 0, 0, \frac{i}{2\sqrt{5}}\right\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{i}{2\sqrt{5}}, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, \frac{i}{2\sqrt{5}} \right\}, \\
& \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, \frac{i}{2\sqrt{5}}, 0, 0, 0 \right\}, \\
& \left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, \frac{i}{2\sqrt{5}} \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \right. \\
& \left. \left\{ 0, 0, \frac{i}{2\sqrt{5}}, 0, 0 \right\} \right\}, \left\{ \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \right. \\
& \left. \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, \frac{i}{2\sqrt{5}} \right\}, \left\{ 0, 0, 0, \frac{i}{2\sqrt{5}}, 0 \right\} \right\}, \\
& \left\{ \left\{ \frac{i}{2\sqrt{5}}, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \right. \\
& \left. \left\{ 0, 0, 0, 0, -\frac{i}{2\sqrt{5}} \right\} \right\}, \left\{ \left\{ -\frac{i}{2\sqrt{15}}, 0, 0, 0, 0 \right\}, \left\{ 0, \frac{i}{\sqrt{15}}, 0, 0, 0 \right\}, \right. \\
& \left. \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, -\frac{i}{2\sqrt{15}} \right\} \right\}, \\
& \left\{ \left\{ -\frac{i}{2\sqrt{30}}, 0, 0, 0, 0 \right\}, \left\{ 0, -\frac{i}{2\sqrt{30}}, 0, 0, 0 \right\}, \right. \\
& \left. \left\{ 0, 0, \frac{1}{2} i \sqrt{\frac{3}{10}}, 0, 0 \right\}, \left\{ 0, 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0, -\frac{i}{2\sqrt{30}} \right\} \right\}, \\
& \left\{ \left\{ -\frac{i}{10\sqrt{2}}, 0, 0, 0, 0 \right\}, \left\{ 0, -\frac{i}{10\sqrt{2}}, 0, 0, 0 \right\}, \left\{ 0, 0, -\frac{i}{10\sqrt{2}}, 0, 0 \right\}, \right. \\
& \left. \left\{ 0, 0, 0, \frac{i\sqrt{2}}{5}, 0 \right\}, \left\{ 0, 0, 0, 0, -\frac{i}{10\sqrt{2}} \right\} \right\} \} \}
\end{aligned}$$

In[87]:= %[[1]]

Out[87]= 228.99

Time needed for this orthonormalization: 48 min. New time: 229 s.

In[88]:= **su5list = %[[2]];**

In[89]:= **Length[su5list]**

Out[89]= 24

```

In[90]:= Table[ukilling[su5list[[i]], su5list[[j]]], {i, 24}, {j, 24}]
Out[90]= {{-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1}}

```

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