Vector and Tensor Algebra

by

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Started June 26, 2012
Version 3, May 14, 2015
Finished May 26, 2017
Created with *Mathematica* v. 8.0.1, ..., 11.1.1

Introduction

This notebook and the package tensalgv3.m contain besides of elementary vector algebra a complete tensor algebra as a part of affine geometry. In *Mathematica* there doesn't exist a built-in Tensor definition, but some tensor operations, like e.g. TensorProduct are built-in. In the notebook I define a symbolic tensor object which admits to introduce covariant, contravariant and mixed type tensors. The concept can be used for symbolic calculations as well as for calculations with tensors whose coordinates are numerically or functionally specified. Tensor products, the wedge product as the base of exterior algebra, and contraction of tensors are introduced. Their properties are deduced and compared with the corresponding *Mathematica* built-in tensor functions. Since also in the commercial packages of Harald H. Soleng, Tensors in Physics, and Steven M. Christensen, MathTensor, an abstract tensor concept is missing I hope that the notebook and the package presented here deliver useful tools for applications of *Mathematica to problems in algebra, geometry and physics needing tensor calculus*. The analytic part of tensor calculus will be treated in connection with differential geometric applications in future notebooks.

This notebook starts with basic concepts of vector and matrix calculus as far as they are independent of scalar products, developed and used in affine or projective geometry. These concepts are contained and treated more detailed in the *Mathematica* notebook vectoralgebra.nb. The present notebook is a revision and enlargement of the notebook vectoralgebra.nb now completed with basic concepts of tensor algebra. The new defined *Mathematica* functions are collected in the package tensalgv3.m which contains the package vectorcalc.m. The theoretical background can be found in textbooks of linear algebra, e.g. in [Mal], [OSI] and [OSII].

For all comments, in particular for hints about possible failures or deficiencies, the author will be very grateful. Please send all comments to the address mentioned in the References.

Keywords

vector objects, null vector, basis vectors, covector, random vector, linear forms, matrix, random matrix, rank, transposed matrix, tensor, symmetric, antisymmetric, covariant, contravariant, mixed type, symmetrize, alternate, tensor product, p-forms, p-vectors, contraction, epsilon tensor, basis tensors, tensor coordinates, wedge product, strict coordinates, Array.

Hints

Copyright

Initialization

1. List of Symbols and their Usages

Remark

For the notations *Mathematica* commonly uses the convention that the names of built-in functions or objects start with a capital letter. In my notebooks and packages I name all the newly introduced functions starting with small letters.

1.1. Algebraic Operations for Functions

1.2. Symbols in the Package tensalgv3.m

1.3. Symbols in the Global Context

2. Vectors and Linear Forms

2.1. Vectors, Bases, Coordinates

We consider finite-dimensional vector spaces over an arbitrary field, mostly the field of real or complex numbers. For the mathematical concepts used here see any textbook about linear algebra or analytic geometry, e.g. [OSI].

2.1.1. The Definition

2.1.2. Checking the Vector Space Axioms

2.1.3. Bases and Coordinates

2.1.4. Appendix. Other Vector Definitions

The Obsolete Definition “vector”

Changing the Head of Arrows

The domain Vectors

2.2. Covectors and Linear Forms. The Dual Basis

2.2.1. Linear Maps. Linear Forms. Covectors

2.2.2. Transformations of Bases and Coordinates

3. Tensors

In this chapter we introduce a tensor object into *Mathematica* as the basis for all applications of tensors in algebra, geometry and physics. The tensor object allows to distinguish covariant, contravariant, and mixed type tensors. In *Mathematica* an explicit tensor object isn't built-in, in spite of the fact that many useful tensor operations can be found, see below Chapter 5. They refer to Arrays of special type: the coordinate Arrays of the tensor. Tensors of mixed type aren't considered. We assume that the user already knows the basic concepts of tensor algebra. A beginner should first read a textbook containing tensor concepts, e.g. [OSII], §§ 10,11, where one finds more hints to the literature.

3..1. Definition. Examples. Properties

3..2. Tensors of the same Type Form a Vector Space

3.2.1. Addition of Tensors

3.2.2. Multiplication with a Scalar

3.2.3. Tensor Basis

3.2.4. Traditional Notation

3.3. The Tensor Product

The tensor product $\otimes$ is the most general bilinear map between tensor spaces:

$$\otimes: (Ta, Tb) \in T^{(k,h)} V^d \times T^{(m,n)} V^d \mapsto Ta \otimes Tb \in T^{(k+m,h+n)} V^d.$$

It is the heart of tensor algebra. In this Section we explain the definition given in tensalgv2.m and deduce its properties.

3.3.1. The Definition

3.3.2. Examples. Standard and Traditional Notation

3.3.3. Properties of tensorproduct

3.3.4. Functions as coordinates

3.4. Basis Transformations. Tensor Invariants

3.4.1. Basis and Coordinate Transformations

3.4.2. Example. Linear Endomorphisms

3.4.3. Contractions

3.5. Examples of Contractions

3.5.1. Example tensor[a, 3, 2, 3]

3.5.2. Example tensor[a, 5, 8, 3]

3.5.3. Example tensor[a,1,1]

3.5.4. Multilinear Functions and Forms

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4. Symmetry Properties of Tensors

4.1. CovariantTensors and their Coordinate Arrays

4.2. Symmetric and Antisymmetric Tensors

4.3. Symmetrization and Alternation of Tensors

Symmetrization and alternation of tensors are operations for tensors making them symmetric resp. antisymmetric in their covariant or Contravariant parts. In this Subsection we create such operations and show their properties,

4.3.1. Symmetrization of Coordinate Functions

4.3.2. Alternation of Coordinate Functions

4.3.3. The Definitions covsym, contrasym, covalt, contralt

4.3.4. Linearity of the Symmetry Operations

4.3.5. Iteration of the Symmetry Operations

4.4. Exterior Algebra

4.4.1. Introduction

4.4.2. extprod

4.4.3. The General wedge Product

5. Tensors in Mathematica

References