Projective Symplectic Geometry

by

Rolf Sulanke

January 11, 2006, revised March 18,2008
Created with *Mathematica* v. 4.2

Summary

This notebook describes symplectic invariants for elementary configurations in the 3-dimensional real or complex projective space or the 3-sphere. Most of the basic constructs may also be applied in higher dimensions. The theoretical background can be found in [OSIII], section 2.8. The notebook contains examples, mentioned in this book, in greater detail; some calculations omitted in the book are carried out here. In particular, the classification of complex and real symmetric bilinear forms and the description of the corresponding quadrics is given. Some additional material, e. g. the symplectic othogonalization of a vector sequence, is contained too.

Introduction

It is well known that symplectic scalar products, i. e. non degenerated skew symmetric bilinear forms, exist only in even-dimensional vector spaces. A vector space is called symplectic, if a symplectic scalar product for its vectors is distinguished. The symplectic group is defined as the group of linear transformations of the vector space preserving the symplectic scalar product; it acts transitively on the corresponding odd-dimensional projective space, defining the projective symplectic geometry as the theory of geometric properties invariant under this action, in the sense of F. Klein's Erlanger Programm. Since in dimension 2 the symplectic group coincides with the special linear group, the symplectic geometry of a projective line coincides with its projective geometry. The first interesting case is the three-dimensional one, the main subject of this notebook. Closely related to the projective geometry is the spherical geometry. One easily verifies, that the symplectic groups act transitively also on the odd-dimensional spheres, being double coverings of the projective spaces of the same dimension. The tools developed in this notebook can also be applied to explore the spherical symplectic geometries.
In the first section basic concepts of symplectic linear algebra are presented. An algorithm called symplectic orthogonalization seems to be new: it constructs for a given sequence of vectors a symplectic or optional an orthosymplectic sequence of vectors with the same span. This also gives a method to define adapted bases for subspaces. Rank and index of the scalar product restricted to the subspace are calculated, and the symplectic vector sequence consists of a basis of the defect subspace and a symplectic basis for a complementary symplectic subspace within the span.
The second section contains some considerations of symplectic transformations. In particular, we introduce a very simple class of these transformations, namely the symplectic transvections, which generate the symplectic group.
Section 3 is devoted to symplectic line geometry. The absolute of projective symplectic geometry is the complex of the isotropic lines, called the absolute null system. The complement of the nullsystem is the set of symplectic lines, on which the restriction of the scalar product does not vanish. (Remember that the projective lines are the two-dimensional vector subspaces.) For pairs of symplectic lines there exists a symplectic invariant being similar to the distance of two lines in metric geometries. The value of this invariant is the function sym defined and studied in this section.

The aim of the last two sections is the classification of the quadrics in the 3-dimensional complex or real projective symplectic spaces. As in Euclidean or affine geometries one classifies the symmetric bilinear forms whose corresponding quadratic forms define the quadrics. They are equivariantly associated with a special class of endomorphisms, the skew symmetric operators, of the underlying vector space. These operators can be classified using their Jordan decompositions. In section 4 this is done for complex symplectic spaces; for each class normal forms of the operators and the bilinear forms are found. Section 5 describes the refinements necessary for real spaces. Also in this case normal forms are obtained with the help of which one may discuss the shape of the quadrics.
The appendix contains modules useful for any application of *Mathematica* to linear algebra and the corresponding geometries. They are contained in the package vectorcalc.m. The modules specific for symplectic linear algebra are collected in the package symplecticgeo.m. Both packages are not needed for evaluating the present notebook; it is not recommended to import them, since then context or protection conflicts would arise with the equally named functions or modules created in this notebook. User who want to create their own notebooks or packages varying or continuing the considerations presented here may find it useful to import these packages as a starting point of their own new notebooks.
The References only contain a few items often used in this notebook; a more detailed bibliography can be found in [OSIII]. The author will be very thankful for any hints or comments, especially for critical ones. Please send all questions or comments to the address mentioned at the end of the notebook.

■ Keywords

annihilator, associated operator, Gram matrix, index of a subspace, isotropic subspaces, lines, null system, orthosymplectic basis, polar, pole, projective symplectic space, quadrics, rank of a subspace, skew symmetric operator, symmetric bilinear form, spherical symplectic geometry, sym, symplectic basis, symplectic orthogonalization, symplectic scalar product, symplectic transvection, symplectic vector space,

■ Hints

copyrigt

■ Copyright

Initialization

1. Symplectic vector space. Corresponding projective space

In this section we introduce the basic concepts of symplectic linear algebra: the scalar product, the Gram matrix, symplectic and orthosymplectic bases, polar spaces. Especially, in subsection 1.3, we construct a procedure called symortho, being analogous to the orthogonalization procedure in Euclidean geometry. The constructs of subsection 1.1 −1.4 are valid for any even dimension, also if a default dimension is not defined. But for convenience, and for the tests and examples, it is recommended to set the default value dim. For the Graphics3D objects considered in subsection 1.5 the default dimension dim of the underlying vector space must be set: dim = 4

■ 1.1. The symplectic scalar product ssp, grammatrix, gram

■ 1.2. Symplectic and orthosymplectic bases, subspaces

■ 1.3. Symplectic orthogonalization: symortho

■ 1.4. Polar spaces

- ### 1.5. Projective points, spherical points, and the corresponding Graphics3D objects

2. The symplectic group, transvections

The transvections defined in this section should better be called "symplectic transvections": it will be proved that the symplectic scalar product remains invariant und their action.

- **Definitions**
- **Test: transvection**
- **Example of a symplectic transformation not being a transvection:**
- **Example: Affine translations are not symplectic, in general**

3. Lines. The line invariant

In this section we introduce the basic invariant sym of pairs of symplectic lines, and show some of its properties.

- ### 3.1. Symplectic and isotropic lines
- ### 3.2. The symplectic invariant of a pair of symplectic lines
- ### 3.3. Geometric Interpretation
- ### 3.4. The invariant sym and the scalar product for 2-vectors

4. Associated operators and symmetric bilinear forms

The aim of this and the following section is the classification of the quadrics in the 3-dimensional projective symplectic space. As in other linear geometries one classifies the symmetric bilinear forms and applies the results to describe the symplectic geometry of the quadrics. To any symmetric bilinear form corresponds a so-called skew symmetric associated operator of the underlying 4-dimensional symplectic vector space, and this correspondence is equivariant and bijective: The associated operators are symplectically equivalent iff the bilinear forms have this property. Now it can be proved that two complex skew symmetric operators are symplectically equivalent iff they have the same Jordan normal form. Therefore the symplectic classification is not as simple as the Euclidean. For the details of the theory see I. M. Jaglom [Jag1], A. I. Maltzev [Mal], and [OSIII].

- ### 4.1. Definitions. First Examples
- ### 4.2. Bilinear forms with diagonizable skew symmetric operators and the corresponding quadrics
- ### 4.3. Complex classification

5. Real classification of symmetric bilinear forms and the corresponding quadrics

- **5.1. Real classification of bilinear forms with diagonizable skew symmetric operators**
- **5.2. J2-quadrics**
- **5.3. Degenerated bilinear forms**

Appendix. Some tools of linear algebra

- **A.2.1 List of the usages for all newly implemented general constructions used or developed in this notebook**
- **A.2.2 Vector calculus**
- **A.2.3 Matrix calculus**

References

Jag1
[Jag1] I. M. Jaglom. Quadratic and skew-symmetric bilinear forms in a real symplectic space. (Russian). Trudi Sem. Vect.-Tens. Analisu 8:364-381. (1950).

[Jag2] I. M. Jaglom. On the linear subspaces of a symplectic space.(Russian).Trudi Sem.Vect.-Tens. Analisu 9: 309-318. (1952).

[Jan] Pekka Janhunen. Declare.m. http://library.wolfram.com/infocenter/MathSource/4574/

Mal
[Mal] A. I. Maltzev. Grundlagen der linearen Algebra (Russ.) , Moskau 1956.

OSIII
[OSIII] A.L. Onishchik, Sulanke, R. Projektive und Cayley-Kleinsche Geometrien, http://www-irm.mathematik.hu-berlin.de/~sulanke

Address