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ON E. CARTAN'S METHOD OF MOVING FRAMES

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INTRODUCTION

In the last years E. Cartan's method of moving frames has been discussed several times, compare Y. Bossard [1], G.R. Jensen [3], R. Sulanke and A. Švec [5] (with a more complete bibliography). The paper [5] contains an immersion theorem (Satz 3.2) which allows to derive global existence and equivalence theorems for immersions in homogeneous spaces. Our aim here is to prove such a theorem for immersions of constant type of compact and simply connected manifolds. In Section 1 we review the results of [5] applied in the following. Section 2 contains the proof of a local equivalence theorem of E. Cartan within the framework of the theory of G, H-structures developed in [5]. In Section 3 we give a global version of this theorem under certain additional assumptions, the necessity of which is indicated by some simple examples. Finally, the last Section 4 contains a simple and direct proof of the homogeneity theorem of E. Cartan.

This paper contains no examples. The results in [2], [3], [4] concerning immersions in concrete homogeneous spaces can easily be interpreted in the terms of G, H-structures. Thus they contribute a lot of examples to the method described here.

1. G, H-STRUCTURES

In the following we tacitly assume that all manifolds, maps, bundles, and actions under consideration are of a sufficiently high class of differentiability, say C_∞ for simplicity.

Let $X^n = G^r / H^{r-n}$ denote the homogeneous space under consideration; the Lie group G acts transitively on X^n, and H is the isotropy group of the point $x_0 \in X^n$. By $\mathfrak{F}_0$ we denote the class of all immersions $f\colon Y^m \to X^n$, m fixed, Y^m variable, but always assumed to be connected. The principal fibre bundle $\pi\colon G \to G/H$ induces the bundle of *frames of order* 0 over Y^m:

$$(1)\qquad f \in \mathfrak{F}_0 \longmapsto E_0 = E_0(f), \qquad \begin{array}{ccc} E_0 & \xrightarrow{\hat f} & G \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y^m & \xrightarrow{f} & X^n = G/H. \end{array}$$

The Maurer – Cartan structure form $\omega\colon TG \to \mathfrak{g}$ on G with values in the Lie algebra $\mathfrak{g}$ of G, defined by

$$(2)\qquad \mathfrak{t} \in T_g G \longmapsto \omega(g, \mathfrak{t}) := (\mathrm{d}L_g^{-1})_g(\mathfrak{t}) \in \mathfrak{g},$$

induces the *structure form of order* 0 $\omega_0\colon TE_0 \to \mathfrak{g}$ on E_0, $\omega_0 := \hat f^*\omega$. The structure $E_0(Y^m, \omega_0)$ is called the *G, H-structure of order* 0 *induced by* $f \in \mathfrak{F}_0$; it has the properties required in the following

Definition 1.1. Let $H \subset G$ be a closed Lie subgroup of the Lie group G. A pair $[E(Y^m, p, H), \omega_0]$ is called a *G, H-structure over* Y^m if the following conditions are fulfilled:

1. $p\colon E \to Y^m$ is a principal fibre bundle with the structure group H over the connected manifold Y^m;

2. $\omega_0\colon TE \to \mathfrak{g}$ is a $\mathfrak{g}$-valued 1-form on E of type $\mathrm{Ad}\,|\,H$, where Ad denotes the adjoint representation of G: if $\gamma_h\colon z \in E \longmapsto h \times z \in E$

denotes the action of H on E, then

$$(3)\qquad \gamma_h^*\omega_0 = \mathrm{Ad}(h) \circ \omega_0 \qquad (h \in H);$$

3. for each $A \in \mathfrak{h}$, $\mathfrak{h}$ the Lie algebra of H, one has

$$(4)\qquad \omega_0(\widetilde{A}) = A,$$

where $\widetilde{A}$ is the fundamental vector field on E corresponding to A;

4. the following structure equation holds true:

$$(5)\qquad \mathrm{d}\omega_0 = -\frac{1}{2}[\omega_0, \omega_0];$$

5. for $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ and ω_0 one has

$$(6)\qquad \operatorname{rank} \pi \circ \omega_0(z) = m, \qquad z \in E.$$

For shortness we simply write $E(Y^m, \omega_0)$ for a G, H-structure, and $E_0 = E_0(f)$ for the G, H-structure induced by f. That $E_0(f)$ satisfies the conditions 1-4 is a direct consequence of its definition. We recall that the action of H on the bundle space $\pi\colon G \to G/H$ is defined by

$$(7)\qquad (h, g) \in H \times G \longmapsto h \times g := gh^{-1} \in G.$$

To prove Condition 5 let us first introduce the *canonical form*

$$(8)\qquad \theta_0(z) := \pi \circ \omega_0(z)\colon T_z(E) \to \mathfrak{g}/\mathfrak{h}.$$

From the commutative diagram (1) and $\pi \circ L_g = l_g \circ \pi$ we deduce

$$(9)\qquad \theta_0(z) = \mathrm{d}l_g^{-1} \circ \mathrm{d}f_y \circ (\mathrm{d}p_0)_z, \qquad \text{with} \quad g = \hat f(z), \quad y = p_0(z).$$

Since $(\mathrm{d}p_0)_z$ is surjective, we obtain for the image

$$(10)\qquad \operatorname{Im} \theta_0(z) = \mathrm{d}l_g^{-1}(\operatorname{Im} \mathrm{d}f_y),$$

and (6) follows from $\operatorname{rank} \mathrm{d}f_y = m$. Here and in the following we identify the vector spaces

$$(11)\qquad \mathfrak{g}/\mathfrak{h} = T_eG / T_eH \equiv T_{x_0}X^n$$

via the canonical isomorphism corresponding to $\pi\colon \mathfrak{g} = T_eG \to T_{x_0}$. Consequently, the linear isotropy representation is identified with the induced action of H over $\mathfrak{g} / \mathfrak{h}$

(12) $\qquad h \in H \mapsto (dl_h)_{x_0} = \dot{\mathrm{Ad}}(h) \in GL(\mathfrak{g} / \mathfrak{h}),$

(13) $\qquad \dot{\mathrm{Ad}}(B \bmod \mathfrak{h}) := (\mathrm{Ad}(h)(B)) \bmod \mathfrak{h}, \quad B \in \mathfrak{g}, \quad h \in H.$

Obviously, the map from E into the Grassmannian manifold $G_m(\mathfrak{g} / \mathfrak{h})$

(14) $\qquad z \in E \mapsto \mathrm{Im}\, \theta_0(z) \in G_m(\mathfrak{g} / \mathfrak{h})$

is an H-invariant, which means that

(15) $\qquad \mathrm{Im}\, \theta_0(h \times z) = \dot{\mathrm{Ad}}(h) \cdot \mathrm{Im}\, \theta_0(z),$

where $\dot{\mathrm{Ad}}(h)$ now denotes the action of H induced by (12) on $G_m(\mathfrak{g} / \mathfrak{h})$.

An isomorphism $\hat{\varphi}: E \to \tilde{E}$ of principal H-bundles is said to be an *isomorphism of the* G, H*-structures* $E(Y, \omega_0)$ on $\tilde{E}(\tilde{Y}, \tilde{\omega}_0)$ if it fulfils $\hat{\varphi}^* \tilde{\omega}_0 = \omega_0$. Now we can formulate the following theorem which is a special case of the 'Immersionstheorem' 3.2 proved in [5]:

Theorem 1.1. *Let* Y^m *be simply connected and* $E(Y^m, \omega_0)$ *a* G, H*-structure over* Y^m. *For fixed elements* $(z_0, g_0) \in E \times G$ *there exists a unique immersion* $\hat{f}: E \to G$ *such that* $\hat{f}(z_0) = g_0$, $\hat{f}$ *is the inducing map of the immersion* $f: Y^m \to G / H$ *defined by*

(16) $\qquad f(y) := \pi(\hat{f}(z)), \quad z \in p^{-1}(y),$

independent of the choice of z, *and one has* $E(Y^m, \omega_0) = E_0(f)$. *The isomorphism classes of* G, H*-structures over simply connected manifolds are in a natural* 1-1 *correspondence with the* G*-equivalence classes of the immersions of these manifolds into* G / H *by* $f \mapsto E_0(f)$. ∎

Remark 1. In the paper [5], we unfortunately called $E(Y^m, \omega_0)$ a $\mathfrak{g}, H$-structure; but this may be misleading, since by (3) it is clear that the structure depends on the embedding of H in G.

Remark 2. In [5] we treated the existence problem more generally, omitting the assumption of simple connectedness. In this case we get a

smallest covering $q: \tilde{Y} \to Y$ such that the lifted G, H-structure $\tilde{E}(\tilde{Y}, \tilde{\omega}_0) = q^*(E(Y, \omega_0))$ can be realized by an immersion $\tilde{f}: \tilde{Y} \to G / H$; $\tilde{f}$ is defined up to G-equivalence.

Now we are going to describe E. Cartan's reduction procedure. From (15) we see that $\mathrm{Im}\, \theta_0(z)$ runs over an H-orbit in the Grassmann manifold $G_{m,n} = G_m(\mathfrak{g} / \mathfrak{h})$ if z runs through a fiber of E_0. Thus we obtain the intrinsically defined map

(17) $\qquad \gamma_1: y \in Y^m \mapsto \gamma_1(y) := H \cdot \mathrm{Im}\, \theta_0(z) \in G_{m,n} / H, \quad p_0(z) = y.$

If the action of H over $G_{m,n}$ is regular, i.e., if there exists a (unique) differentiable structure on $G_{m,n} / H$ such that the canonical map $q: G_{m,n} \to G_{m,n} / H$ is a submersion, then it can easily be shown that the correspondence $f \in \mathfrak{F}_0 \mapsto \gamma_1 = \gamma_1(f) \in C_\infty(Y^m, G_{m,n} / H)$ is a G-invariant of the immersion f, compare definition 1.2 in [5]; indeed, as a section of the trivial bundle $Y^m \times (G_{m,n} / H)$ it is a geometric object over Y^m, and by its definition the conditions

$$\gamma_1(l_g \circ f) = \gamma_1(f), \qquad \gamma_1(\varphi_* f) = \varphi_* \gamma_1(f)$$

are fulfilled for each $g \in G$ and each diffeomorphism $\varphi: Y \to \tilde{Y}$; here denotes $\varphi_* F := F \circ \varphi^{-1}$ for every $F: Y^m \to Z^k$.

Unfortunately, the action of H on $G_{m,n}$ fails to be regular in the general case. So we are forced to impose restrictions to the class of immersions under consideration. Let us consider a general H-manifold B^N of class C_∞, and let $A \subseteq B^N$ denote an H-invariant submanifold. The action of H on A is said to be *normal* if it is regular, and if there exists a section $s: A / H \to A$ such that every point $a = s(\dot{a})$, $\dot{a} \in A / H$, has the same isotropy subgroup $H_1 \subseteq H$. If such a section is chosen, the elements $s(\dot{a})$ are called the *normal forms* for the action of H on A, and $s(A / H)$ is the manifold of normal forms.

It can easily be proved (compare Lemma 2.1 of [5]) that a normal submanifold A is diffeomorphic to $(A / H) \times H / H_1$. In [5] we discussed the problem of the existence of sufficiently large normal submanifolds for the action of H on $G_{m,n}$. If H is compact, they always exist, but practically they exist for all important classical geometries. An important example with noncompact H is the Möbius geometry (conformal geometry of S^n), which is treated by E. Cartan's method in the paper [4]. A description of E. Cartan's reduction procedure in the general case was given by G. R. Jensen [3], but it is only local, not formulated within the framework of fibre bundles, and the choice of normal forms remains quite arbitrary.

Definition 1.2. Let $[A, s]$ be a normal H-invariant submanifold of $G_{m,n}$ with fixed normal forms $s(A / H)$. An immersion $f \in \mathfrak{F}_0$, or a G, H-structure $E_0(Y^m, \omega_0)$ respectively, is said to be *of type* A if $\operatorname{Im} \theta_0(z) \in A$ for all $z \in E_0$. *The first order reduction of* E_0 is defined to be

(18) $\qquad E_1 := \{z \in E_0 \mid \operatorname{Im} \theta_0(z) \in s(A / H)\},$

(19) $\qquad \omega_1 := \omega_0 \mid TE_1, \quad p_1 := p \mid E_1.$

In [5] we proved (Satz 4.2):

Theorem 1.2.

(a) *Let* $E_0(Y^m, \omega_0)$ *be a* G, H*-structure of type* A. *Then* $E_1(Y^m, \omega_0)$ *as defined by* (18), (19) *is a* G, H_1*-structure.*

(b) *If* $E_0(Y, \omega_0) \approx \widetilde{E}_0(\widetilde{Y}, \widetilde{\omega}_0)$ *are isomorphic as* G, H*-structures, then their reductions are isomorphic as* G, H_1*-structures, and conversely:*

(20) $\qquad E_0(Y, \omega_0) \approx \widetilde{E}_0(\widetilde{Y}, \widetilde{\omega}_0) \longleftrightarrow E_1(Y, \omega_1) \approx \widetilde{E}_1(\widetilde{Y}, \widetilde{\omega}_1).$

(c) *Assume* $E_0 = E_0(f)$, f *of type* A, *and put* $\hat{f}_1 := \hat{f} \mid E_1$ *(compare* (1)*). Then*

(21) $\qquad y \in Y^m \longmapsto f_1(y) := \hat{f}_1(z) H_1 \in G / H_1, \quad z \in p_1^{-1}(y),$

is independent of the choice of $z \in p_1^{-1}(y) \subset E_1$, *and defines an immersion* $f_1 : Y^m \to G / H_1$ *of* Y^m *into the homogeneous space* $X_1^{n_1} := G / H_1$; *the immersions* $f, \widetilde{f}$ *of type* A *are* G*-equivalent if and only if the corresponding immersions* $f_1, \widetilde{f}_1$ *are* G*-equivalent.* ∎

Theorem 1.2 is the first step of the reduction procedure introduced

by E. Cartan [2]. To describe the general step of this procedure explicitly let us assume that we already performed k reductions. Thus we obtained:

(A) A sequence of normal types $[A_\lambda, s_\lambda]$ of order λ, $\lambda = 1, \ldots, k$, with $A = A_1$ and

(22) $\qquad \mathfrak{F}_0 \supseteq \mathfrak{F}_1(A_1) \supseteq \ldots \supseteq \mathfrak{F}_k(A_k);$

(B) a sequence of closed Lie subgroups $H_\lambda \subseteq G$, $\lambda = 0, \ldots, k$, with $H_\lambda \supseteq H_{\lambda+1}$, $\dim H_\lambda > \dim H_{\lambda+1}$, $H_0 = H$, and the corresponding sequence of homogeneous spaces

(23) $\qquad X_\lambda^{n_\lambda} := G / H_\lambda, \quad \dim H_\lambda = r - n_\lambda,$

with canonical G-projections

(24) $\qquad \beta_\lambda : gH_\lambda \in X_\lambda^{n_\lambda} \longmapsto gH_{\lambda-1} \in X_{\lambda-1}^{n_{\lambda-1}};$

(C) a sequence of canonical reductions for the G, H-structures $E_0(Y, \omega_0)$ of type A_k:

(25) $\qquad E_k \subset E_{k-1} \subset \ldots \subset E_1 \subset E_0, \quad \omega_\lambda = \omega_0 \mid TE_\lambda, \quad p_\lambda = p_0 \mid E_\lambda,$

and, if $E_0 = E_0(f)$, the corresponding immersions

(26) $\qquad f_\lambda : Y^m \to X_\lambda^{n_\lambda}, \quad f_0 = f,$

such that

(27) $\qquad f_{\lambda-1} = \beta_\lambda \circ f_\lambda, \quad \lambda = 1, \ldots, k.$

(D) Let $\theta_\lambda := \pi_\lambda \circ \omega_\lambda : TE_\lambda \to \mathfrak{g} / \mathfrak{h}_\lambda$ be the *canonical form of order* λ. It depends on the $(\lambda + 1)$-jets of f, compare (9). The θ_λ fulfil

(28) $\qquad \theta_{\lambda-1} \mid TE_\lambda = \dot{\beta}_\lambda \circ \theta_\lambda,$

where $\dot{\beta}_\lambda : \mathfrak{g} / \mathfrak{h}_\lambda \to \mathfrak{g} / \mathfrak{h}_{\lambda-1}$ is the canonical map. By the definition of E_λ we have

(29) $\qquad \operatorname{Im} \theta_{\lambda-1}(h \times z) = \operatorname{Im} \theta_{\lambda-1}(z), \quad z \in E_\lambda, \quad h \in H_\lambda,$

such that

$$(30)\qquad \gamma_\lambda : y \in Y^m \mapsto \operatorname{Im} \theta_{\lambda-1}(z) \in s_\lambda(A_\lambda / H_{\lambda-1}), \quad z \in E_\lambda, \quad p_\lambda(z) = y,$$

is a well defined smooth G-invariant of the immersion $f \in \mathfrak{F}_\lambda(A_\lambda)$. From (28) and (30) one can deduce that all invariants γ_μ, $\mu < \lambda$, are included in γ_λ.

Our task is now to describe the next reduction step $E_k \mapsto E_{k+1}$. Let us first introduce a base (X_I) of $\mathfrak{g}$ adapted to the flag $(\mathfrak{h}_\lambda)_{\lambda=1,\dots,k}$ of subalgebras, i.e. such that

$$(31)\qquad X_{n_\lambda+1}, \dots, X_r \in \mathfrak{h}_\lambda.$$

Applying the summation convention we write

$$(32)\qquad \omega_\lambda = X_I \omega^I, \qquad I = 1, \dots, r,$$

where the ω^I are 1-forms on E_λ, and

$$(33)\qquad \theta_\lambda = \dot{X}_i \omega^i, \qquad i = 1, \dots, n_\lambda,$$

is the canonical form of E_λ, $\dot{X}_i = \pi_\lambda(X_i) \in \mathfrak{g}/\mathfrak{h}_\lambda$. By (6) we may assume that the vectors $X_i \in \mathfrak{g}$ and the normal form s_1 of A_1 are chosen in such a manner that

$$(34)\qquad \omega^1(z) \wedge \dots \wedge \omega^m(z) \neq 0 \quad \text{for all} \quad z \in E_1.$$

Then the same is true for each reduction E_λ, $\lambda \geqslant 1$. Since the vertical tangential spaces of E_λ are defined by

$$(35)\qquad t \in Tv_z \leftrightarrow \omega^i(z, t) = 0, \qquad i = 1, \dots, n_\lambda,$$

the forms $(\omega^\alpha(z),\ \alpha = 1, \dots, m;\ \omega^\nu(z),\ \nu = n_k + 1, \dots, r)$ form a basis of $(T_z E_k)^*$, and we obtain for all $z \in E_k$:

$$(36)\qquad \omega^\kappa(z) = b'^\kappa{}_\alpha(z)\omega^\alpha(z), \qquad \alpha = 1, \dots, m, \quad \kappa = m+1, \dots, n_k.$$

It can easily be seen that the matrix $(b'^\kappa{}_\alpha(z))$ is the matrix of the coordinates of $\operatorname{Im} \theta_k(z)$ in the natural chart of $G_{m.n_k}$ defined on the

open and dense set of all those m-dimensional subspaces of $\mathfrak{g}/\mathfrak{h}_k$ which are projected bijectively on the linear hull $\mathscr{L}(\dot{X}_1, \dots, \dot{X}_m)$. Taking into account (28), (29) we conclude: The G-invariant γ_λ of order $\lambda \leqslant k$ has the components

$$(37)\qquad b'^\kappa{}_\alpha(z) = b'^\kappa{}_\alpha(y), \qquad z \in p_k^{-1}(y), \qquad \kappa = m+1, \dots, n_{\lambda-1}.$$

Therefore, in order to define the types A_{k+1} of order $k+1$, we have only to consider the coefficients $b'^\kappa{}_\alpha$ with $\kappa > n_{k-1}$, and to investigate the action

$$(38)\qquad b'^\kappa{}_\alpha(z) \mapsto b'^\kappa{}_\alpha(h \times z), \qquad z \in E_k, \qquad h \in H_k, \qquad \kappa = n_{k-1} + 1, \dots, n_k.$$

The other coefficients being constant along the fibres can be divided into two disjoint subsets: those which are the same constants for all immersions of the class $\mathfrak{F}(A_k)$ defining the type A_k, and the others characterizing the normal form of $\operatorname{Im} \theta_{k-1}(z)$ on A_k / H_{k-1} depending on $y \in Y^m$ in general and defining G-invariant functions of the immersion. Let us assume that after we had performed the λ-th reduction we obtained μ_λ G-invariant functions $c^\nu(z)$, $\mu_0 = 0 \leqslant \mu_1 \leqslant \dots \leqslant \mu_k$, with

$$(39)\qquad c^\nu(h \times z) = c^\nu(z) \quad \text{for} \quad h \in H_\lambda, \quad z \in E_\lambda, \quad \nu \leqslant \mu_\lambda;$$

they are called *differential invariants of order* $\leqslant \lambda$ for the immersions $f \in \mathfrak{F}_k(A_k)$. We consider their differentials, which as a consequence of (39), decompose in the following way:

$$(40)\qquad dc^\nu = b^\nu{}_\alpha(z)\omega^\alpha(z), \qquad \nu = 1, \dots, \mu_k.$$

In general, the $b^\nu{}_\alpha(h \times z)$ will depend on $h \in H_k$:

$$(41)\qquad (b^\nu{}_\alpha(z)) \mapsto (b^\nu{}_\alpha(h \times z)), \qquad z \in E_k, \qquad h \in H_k;$$

but since the dc^ν for $\nu \leqslant \mu_{k-1}$ have already been considered performing the foregoing reductions we obtain

$$(42)\qquad b^\nu{}_\alpha(h \times z) = b^\nu{}_\alpha(z) \quad \text{for} \quad \nu \leqslant \mu_{k-1}, \quad z \in E_k, \quad h \in H_k,$$

and we only have to consider the coefficients (41) of the decompositions of the new invariants, i.e. we may assume $\mu_{k-1} < \nu \leqslant \mu_k$ in (41). Now let us distinguish the following four cases:

1. The simultaneous action of H_k given by (38) and (41) is not trivial, and we can find some normal submanifolds $[A_{k+1}, s_{k+1}]$ of this action – then we repeat the procedure described in Theorem 1.2, and get E_{k+1}, H_{k+1} and new invariants c^ν, $\nu = \mu_k + 1, \ldots, \mu_{k+1}$.

2. The action of H_k in (38) or (41) is not trivial, but it is impossible to find appropriate normal invariant subsets for this action. This case can be excluded if H_k is compact. Otherwise we can try to find regular invariant submanifolds of the action and to proceed as described in G.R. Jensen [3].

3. Suppose $\dim H_k > 0$, and the action (38), (41) of H_k to be trivial. Then the coefficients b, b' entering in (38), (41) define new differential invariants of order $k+1$, say c^ν, $\mu_k < \nu \leqslant \mu_{k+1}$. Put $E_{k+1} = E_k$, $H_{k+1} = H_k$, $\omega_{k+1} = \omega_k$, and consider the action (41) on the new coefficients $b^\nu{}_\alpha$, $\mu_k < \nu \leqslant \mu_{k+1}$. If this is not trivial, we come back to the cases 1 or 2. If it is trivial, we again take their differentials and repeat the consideration as long as we get differential invariants of higher orders which are functionally independent of the invariants obtained before, i.e. as long as

$$\text{rank}\,(dc^\nu)_{\nu \leqslant \mu_L} < \text{rank}\,(dc^\nu)_{\nu \leqslant \mu_{L+1}}; \tag{43}$$

here the c^ν are considered as functions on Y^m. Since for all L the rank cannot exceed m, we finally get after a finite number of steps that

$$\text{rank}\,(dc^\nu)_{\nu \leqslant \mu_L} = \text{rank}\,(dc^\nu)_{\nu \leqslant \mu_{L+1}}, \tag{44}$$

and

$$H_{L+1} = H_L. \tag{45}$$

Then the G, H_L-structure $E_L(Y^m, \omega_L)$ is called the bundle of *Frenet frames* of $f \in \mathfrak{F}_L(A_L)$. It must be remarked that condition (44) is a local one, and that it can fail in certain points or subsets of Y^m. Excluding

this we obtain in the next sections equivalence theorems for immersions of Frenet type.

4. We have $\dim H_k = 0$. Then the bundle E_k has discrete fibres. It is called the bundle of *canonical frames* of the immersion f, and A_k is called a *canonical type*. We define:

Definition 1.3. Let $f \in \mathfrak{F}_k(A_k)$ be an immersion of a canonical type A_k. f is called H_k*-orientable* if the connected components $E_k^{(\epsilon)}$ of E_k cover Y^m simply, i.e. if $p_k \mid E_k^{(\epsilon)}$ is a diffeomorphism of $E_k^{(\epsilon)}$ on Y^m. It is called H_k*-oriented* if it is H_k-orientable and if one of the $E_k^{(\epsilon)}$ is distinguished.

For an H_k-oriented immersion $f \in \mathfrak{F}_k(A_k)$ we have a unique global section $y \in Y^m \mapsto z(y) \in E_k$, $z = (p_k \mid E_k^{(\epsilon)})^{-1}$, which defines a 1-form on Y^m with values in $\mathfrak{g}$: the *canonical structure form* $\sigma_k := z^*\omega_k$, and a uniquely defined immersion of Y^m into G, namely

$$F\colon y \in Y^m \mapsto \hat{f}_k(z(y)) \in G. \tag{46}$$

The mapping F is called the *canonical moving frame* of the H_k-oriented immersion f. Two H_k-oriented immersions $f, \tilde{f} \in \mathfrak{F}_k(A_k)$ are said to be H_k*-oriented equivalent* if there exist a $g \in G$ and a diffeomorphism $\varphi\colon Y \to \tilde{Y}$ such that for the corresponding canonical frames we have

$$\tilde{F} = L_g \circ F \circ \varphi^{-1}. \tag{47}$$

Now one easily proves

Theorem 1.3. *Two H_k-oriented immersions $f, \tilde{f}$ of the canonical type A_k are H_k-oriented equivalent if and only if there exists a diffeomorphism $\varphi\colon Y^m \to \tilde{Y}^m$ such that for the corresponding canonical structure forms the condition $\tilde{\sigma}_k = \varphi^*\sigma_k$ holds true.*

Proof. The necessity of the condition follows from $\sigma = F^*\omega$ and the left invariance of the structure form ω of G. Conversely, if $f, \tilde{f}$ are H_k-oriented, we have $E_k = H_k \times Y^m$, $\tilde{E}_k = E_k \times \tilde{Y}^m$, and the 1-form σ_k, $\tilde{\sigma}_k = \varphi^*\sigma_k$, defines the structure forms $\omega_k, \tilde{\omega}_k$ on the frame bundles $E_k, \tilde{E}_k$ uniquely by the transformation (3) with $h \in H_k$. Thus