are said to be H_k-oriented equivalent if there exist a $g \in G$ and a diffeomorphism $\varphi: Y \to \tilde{Y}$ such that for the corresponding canonical frames we have

$$\tilde{F} = L_g \circ F \circ \varphi^{-1}. \tag{47}$$

Now one easily proves

Theorem 1.3. Two H_k-oriented immersions f, $\tilde{f}$ of the canonical type A_k are H_k-oriented equivalent if and only if there exists a diffeomorphism $\varphi: Y^m \to \tilde{Y}^m$ suchtthat for the corresponding canoniwal structure forms the condition $\tilde{\sigma}_k = \varphi^* \sigma_k$ holds true.

Proof. The necessity of the condition follows from $\sigma = F^* \omega$ and the left invariance of the structure form ω of G. Conversely, if f, $\tilde{f}$ are H_k-oriented, we have $E_k = H_k \times Y^m$, $\tilde{E}_k = H_k \times \tilde{Y}^m$, and the 1-form σ_k, $\tilde{\sigma}_k = \varphi^* \sigma_k$, defines the structure forms ω_k, $\tilde{\omega}_k$ on the frame bundles E_k, $\tilde{E}_k$ uniquely by the transformation (3) with $h \in H_k$. Thus theorem 1.3 is a direct consequence of the immersion theorem 4.2 of [5] if we define the isomorphism $\tilde{\varphi}$ of the G, H_k-structures by

$$z = h \times z(y) \in E_k \longmapsto \hat{z} := h \times \tilde{z}(\varphi(y)) \in \tilde{E}_k, \quad h \in H_k. \quad \square \tag{48}$$

We recall that the type A_k in the case 4 is defined by a certain submanifold $s(A_k/H_{k-1}) \subset G_m(\mathfrak{g}/\mathfrak{h}_{k-1})$. From (34) and the definition of σ_k we obtain

$$\sigma_k(y) = X_I \sigma^I(y) \quad \text{with} \quad \sigma^1 \wedge \dots \wedge \sigma^m(y) \neq 0. \tag{49}$$

Therefore theofem 1.1 implies the following

Corollary 1.4. Let Y^m be simply connected, and $\sigma: TY^m \to \mathfrak{g}$ be a 1-form on Y^m with values in $\mathfrak{g}$. a) The necessary and sufficient conditions that there exist an immersion $f: Y^m \to X^n = G/H$ of the canonical type A_k such that σ becomes the canonical structure form $\sigma = \sigma_k$ under an appropri= ate orientation of f are the following:

1. $d\sigma = -[\sigma, \sigma]/2$,

2. rank $\sigma(y) = n$ for all $y \in Y^m$ and Im $\sigma(y) \in s_k(A_k/H_{k-1})$.

b) Each Y^m which admits an H_k-orientable immersion $f: Y^m \longrightarrow X^n$ of type A_k must be parallelizable. $\square$

The statement b) follows from the fact that by (49) (σ^α), $\alpha = 1, \ldots, m$, is a globally defined cobasis over Y^m.

We remark that in the case $m = 1$ the first condition is trivially fulfilled. For example, in the theory of generic curves $f: \mathbb{R} \longrightarrow E^3$ in the euclidean space E^3 conditions 1. and 2. reduce to the well known shape of the matrix in the Frenet formulas:

$$(50) \qquad \sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sigma^1 & 0 & -k\sigma^1 & 0 \\ 0 & k\sigma^1 & 0 & -\kappa\sigma^1 \\ 0 & 0 & \kappa\sigma^1 & 0 \end{pmatrix} \in m(3), \qquad k > 0.$$

If f is an immersion of a canonical type, but not H_k-orientable, one has to pass to an appropriate covering $q: \tilde{Y} \longrightarrow Y$ and to consider the immersion $\tilde{f} = f \circ q: \tilde{Y} \longrightarrow X^n$. As $\tilde{Y}$ one can take one of the connected components of E_k, or the universal covering of Y. Then the (non oriented) equivalence of immersions $Y \longrightarrow X^n$ is reduced to the equivalence of H_k-oriented immersions $\tilde{Y} \longrightarrow X^n$, in the canonical case 4.

2. An Equivalence Theorem for Immersions of Frenet Type

<u>Definition 2.1.</u> Let a class $\mathfrak{I}(A_q)$ of immersions $f: Y^m \longrightarrow G/H$ of a certain type A_q of order q be given. An immersion $f \in \mathfrak{I}(A_q)$ is called <u>regular of order</u> $\leq q$ (with <u>rank</u> N) if at each point $y \in Y^m$ one has

(51) $\quad \text{rank}\,(d\sigma^\nu)_{\nu \leq \mu_{q+1}} = \text{rank}\,(d\sigma^\nu)_{\nu \leq \mu_q} = N.$

Clearly we have $0 \leq N \leq m$.

<u>Definition 2.2.</u> An immersion $f \in \mathfrak{I}(A_q)$ is said to be of a Frenet type of order $\leq q$ if it is regular of order $\leq q$, and

if $H_{q+1} = H_q$, i.e. if the actions (38), (41) of H_q are trivial for $z \in E_q$, $h \in H_q$.

Remark. If H_q is not connected it may happen that only the connected cpmponents$(H_q)_o$ of the unity acts trivially, i.e. that the coefficients are constant only on the connected compo= nents of the fibres, but that it is impossible to reduce E_q to $(H_q)_o$ globally. In this case one has to introduce H_q-<u>orientations</u> and to pass to an appropriate covering $\tilde{Y}$ of Y as it was done at the end of section 1. For the following we assume that this is carried out if necessary. Obviously such a reduction is always possible if Y is simply connected, or if the considerations are local.

<u>Theorem 2.1.</u> If $f \in \mathfrak{F}(A_q)$ is of Frenet type of order q, then it is also of Frenet type of order q+1 (with $A_{q+1} = A_q$), and consequently, for all higher orders $q' > q$.

Proof. The differential invariants of order q+1 are 1) the coefficients b'^{κ}_{α}, $\alpha = 1,\ldots, m$, $\kappa = n_{q-1}+1,\ldots, n_q$, and 2) the coefficients b^{ν}_{α}, $\nu = \mu_{q-1}+1,\ldots, \mu_q$, in the decompo= sitions (40). Indeed, since by assumption we have $H_{q+1} = H_q$, all these coefficients are constant along the fibres of E_q. Let c be one of the invariants of order q+1. From the implicite function theorem one easily deduces the following well known

<u>Lemma 2.1.</u> Let c, $c^{\varrho}: Y^m \to \mathbb{R}$, $\varrho = 1,\ldots, N$, be real functions such that for a point $y_o \in Y^m$ we have a neighborhood $U \subseteq Y$, $y_o \in U$, with

(52) $\quad \operatorname{rank}(dc, dc^1,\ldots, dc^N) = \operatorname{rank}(dc^1,\ldots, dc^N) = N.$

Then there existsa neighborhood $V \subseteq Y$, $y_o \in V$, and a smooth function F defined on an open set $W \subseteq \mathbb{R}^N$ such that

(53) $\quad c|V = F(c^1|V,\ldots, c^N|V).$ $\square$

(1) the coefficients b'^{κ}_{α}, $\alpha = 1, \dots, m$, $\kappa = n_{q-1} + 1, \dots, n_q$, and

(2) the coefficients b^{ν}_{α}, $\nu = \mu_{q-1} + 1, \dots, \mu_q$, in the decompositions (40). Indeed, since by assumption we have $H_{q+1} = H_q$, all these coefficients are constant along the fibres of E_q. Let c be one of the invariants of order $q+1$. From the implicit function theorem one easily deduces the following well-known

Lemma 2.1. *Let $c, c^{\rho}: Y^m \to \mathbf{R}$, $\rho = 1, \dots, N$, be real functions such that for a point $y_0 \in Y^m$ we have a neighbourhood $U \subseteq Y$, $y_0 \in U$, with*

(52) $$\operatorname{rank}(dc, dc^1, \dots, dc^N) = \operatorname{rank}(dc^1, \dots, dc^N) = N.$$

Then there exist a neighbourhood $V \subseteq Y$, $y_0 \in V$, and a smooth function F defined on an open set $W \subseteq \mathbf{R}^N$ such that

(53) $$c \mid V = F(c^1 \mid V, \dots, c^N \mid V). \blacksquare$$

We have to show that for every $y \in Y^m$ the coefficients c_{α} in

$$dc(z) = c_{\alpha}(z)\omega^{\alpha}(z)$$

are constant along the fibres. By assumption (51) we find a neighbourhood $U \subseteq Y$ of $y = y_0$ and N differential invariants c^{ρ} of order $\leqslant q$ such that the conditions of Lemma 2.1 are fulfilled. It follows that $c(y) = F(c^1(y), \dots, c^N(y))$ for $y \in V$. The pullbacks of these functions give $c(z) = F(c^1(z), \dots, c^N(z))$. Deriving these equations we obtain

(54) $$dc(z) = c_{\alpha}(z)\omega^{\alpha}(z) = \partial_{\rho}F(c^{\nu}(z))c^{\rho}{}_{\alpha}(z)\omega^{\alpha}(z).$$

Since the $c^{\nu}(z)$ are differential invariants of order $\leqslant q$ the $c^{\nu}(z)$, $c^{\rho}{}_{\alpha}(z)$ are constant along the fibres (by $H_{q+1} = H_q$). Therefore the $c_{\alpha}(z) = \partial_{\rho}F(c^{\nu}(z))c^{\rho}{}_{\alpha}(z)$ have the same property. Thus we can consider them as functions on $V \subseteq Y^m$:

(55) $$c_{\alpha}(y) = \partial_{\rho}F(c^{\nu}(y))c^{\rho}{}_{\alpha}(y) = H_{\alpha}(c^{\nu}(y)), \qquad \nu = 1, \dots, N,$$

since the $c^{\rho}{}_{\alpha}(y)$ being differential invariants of order $\leqslant q+1$ are functions of the c^{ν}, may be, on a smaller neighbourhood $V_1 \subseteq V$, $y \in V_1$. Differentiating (55) we obtain $dc_{\alpha}(y)$ as a linear expression in $dc^{\nu}(y)$, from which (51) follows with μ_{q+1} instead of μ_q. Since all the new coefficients of order $q+2$ are constant along the fibres and functions of $c^1, \dots, c^N$, we obviously have $A_{q+1} = A_q$ and $H_{q+2} = H_{q+1} = H_q$. $\blacksquare$

We call *the order* of the immersion f the smallest integer q for which definition 2.2 is fulfilled.

Theorem 2.2. *Let $f, \tilde{f} \in \mathfrak{F}(A_q)$ be immersions of the same Frenet type of order q and rank N, and $\varphi: Y^m \to \tilde{Y}^m$ a diffeomorphism with the property*

(56) $$\varphi^*(\tilde{c}^{\nu}) = c^{\nu}, \qquad \nu = 1, \dots, \mu_{q+1}.$$

To every choice $y_0, z_0, \tilde{z}_0$ with $y_0 \in Y$, $z_0 \in p_q^{-1}(y_0) \subset E_q$, $\tilde{z}_0 \in \tilde{p}_q^{-1}(\varphi(y_0)) \subset \tilde{E}_q$ there exist open neighbourhoods $V, \tilde{V}$ with $y_0 \in V \subseteq Y$, $\tilde{y}_0 = \varphi(y_0) \in \tilde{V} \subseteq \tilde{Y}$ such that $f \mid V$, $\tilde{f} \mid \tilde{V}$ are G-equivalent. There exist exactly one $g \in G$ with

(57) $$\tilde{f}(\tilde{V}) = l_g f(V) \quad \textit{and} \quad \hat{\tilde{f}}(\tilde{z}_0) = g \cdot \hat{f}(z_0).$$

Remark. Of course, if $\tilde{f}: \tilde{Y}^m \to G/H$ is any immersion G-equivalent to f, then $\tilde{f} \in \mathfrak{F}(A_q)$ and (56) is fulfilled. Thus the assumptions of Theorem 2.2 are necessary.

Proof of Theorem 2.2. To prove Theorem 2.2 it suffices to prove the corresponding local isomorphism of the corresponding G, H_q-structures $E_q, \tilde{E}_q$. We shall characterize the graphs of the local isomorphisms F: $E_q \mid V \to \tilde{E}_q \mid \tilde{V}$ by an involutive distribution on a certain submanifold $W \subseteq E_q \times \tilde{E}_q$ which we are going to construct now. First we define a subset $W_0 \subseteq Y \times \tilde{Y}$ by $(y, \tilde{y}) \in W_0$ if and only if there exist neighbourhoods $U, \tilde{U}$, $y \in U$, $\tilde{y} \in \tilde{U}$, and a diffeomorphism $\varphi_U: U \to \tilde{U}$ such that

(58) $$\varphi_U^*(\tilde{c}^{\sigma} \mid \tilde{U}) = c^{\sigma} \mid U, \qquad \sigma = 1, \dots, \mu_{q+1}.$$

Lemma 2.2. *The set $W_0 \subseteq Y \times \tilde{Y}$ is a $(2m - N)$-dimensional sub-*

manifold of $Y \times \widetilde{Y}$. *It contains a connected component which is projected onto* Y *and onto* $\widetilde{Y}$.

Proof. We consider an arbitrary pair $(y_0, \widetilde{y}_0) \in W_0$. Diminishing U if necessary, we can assume that N of the differential invariants $c^\varkappa$, $\nu \leqslant \mu_q$, say $c^1, \ldots, c^N$, are functionally independent on U. We state that $W_0 \cap (U \times \widetilde{U})$ is defined by the system

(59) $$\widetilde{c}^\nu(\widetilde{y}) = c^\nu(y), \quad (y, \widetilde{y}) \in U \times \widetilde{U}, \quad \nu = 1, \ldots, N.$$

Indeed, (58) implies (59) for all $(y, \widetilde{y}) \in W_0$. To prove the converse let us apply Lemma 2.1. We get functions F^σ (if necessary diminishing U again) such that for $y \in U$

(60) $$c^\sigma(y) = F^\sigma(c^1(y), \ldots, c^N(y)), \quad \sigma = 1, \ldots, \mu_{q+1}.$$

Furthermore we have

$$\widetilde{c}^\sigma \mid \widetilde{U} = c^\sigma \circ \varphi_U^{-1} = F^\sigma(c^1 \circ \varphi_U^{-1}, \ldots, c^N \circ \varphi^{-1}),$$

and we conclude

(61) $$\widetilde{c}^\sigma(\widetilde{y}) = F^\sigma(\widetilde{c}^1(\widetilde{y}), \ldots, \widetilde{c}^N(\widetilde{y})), \quad \sigma = 1, \ldots, \mu_{q+1}, \quad \widetilde{y} \in \widetilde{U},$$

with the same functions F^σ as in (59). In particular, the $\widetilde{c}^1, \ldots, \widetilde{c}^N$ are a maximal system of independent invariants on $\widetilde{U}$. From this we obtain: All $(y, \widetilde{y}) \in U \times \widetilde{U}$, which satisfy (59) also satisfy the system

(62) $$\widetilde{c}^\sigma(\widetilde{y}) = c^\sigma(y), \quad \sigma = 1, \ldots, \mu_{q+1}.$$

Obviously, the system (59) is of rank N on $U \times \widetilde{U}$. It defines a $(2m - N)$-dimensional topological submanifold $W_1 \subseteq U \times \widetilde{U}$ containing $W_0 \cap$ $\cap (U \times \widetilde{U})$. On the other hand, every solution $(y_1, \widetilde{y}_1)$ of (59) is contained in W_0: We can assume that a chart $\widetilde{\psi}(\widetilde{y}) = (\widetilde{y}^\alpha(\widetilde{y})) \in \boldsymbol{R}^m$ is chosen, the first N coordinate functions of which are $\widetilde{y}^\nu = \widetilde{c}^\nu(\widetilde{y})$, $\nu = 1, \ldots, N$. Then (59) reduces to $\widetilde{y}^\nu = c^\nu(y)$. Choosing $m - N$ arbitrary real functions $a^\rho(y)$, $\rho > N$, such that

$$\det(\partial_\alpha c^\nu, \partial_\alpha a^\rho)(y_1) \neq 0, \quad a^\rho(y_1) = \widetilde{y}^\rho(\widetilde{y}_1),$$

we define a diffeomorphism φ_1 of a certain neighbourhood $U_1 \subseteq U$, $y_1 \in U_1$, on a neighbourhood $\widetilde{U}_1 \subset \widetilde{U}$, $\widetilde{y}_1 \in \widetilde{U}_1$, by

$$\varphi_1 : y \in U_1 \mapsto \widetilde{y} = \widetilde{\psi}^{-1}(c^\nu(y), a^\rho(y)) \in \widetilde{U}_1.$$

By definition we have $\widetilde{c}^\nu(\varphi_1(y)) = c^\nu(y)$ and $\varphi_1(y_1) = \widetilde{y}_1$, and, since (59) implies (62), we obtain $(y_1, \widetilde{y}_1) \in W_0$. The last statement of Lemma 2.2 follows from the fact that the graph $\{(y, \varphi(y)) \mid y \in Y\}$ of the given diffeomorphism φ is connected and defines the desired connected component of W_0. ∎

Let us now consider the connected component constructed in Lemma 2.2 and denote it by W_0 again. We summarize the properties of W_0 needed in the following:

1. $W_0 \subseteq Y \times \widetilde{Y}$ is a connected submanifold of $Y \times \widetilde{Y}$ of dimension $2m - N$, the manifold topology of which coincides with its relative topology.

2. W_0 contains the graph of the given diffeomorphism $\varphi: Y \to \widetilde{Y}$.

3. For $(y, \widetilde{y}) \in W_0$ condition (62) is fulfilled.

Now $\beta := p_q \times \widetilde{p}_q : E_q \times \widetilde{E}_q \to Y \times \widetilde{Y}$ defines a principal fibre bundle with the structure group $H_q \times H_q$ over $Y \times \widetilde{Y}$, the action of $H_q \times H_q$ being defined by

(63) $$(h_1, h_2) \times (z_1, z_2) = (h_1 \times z_1, h_2 \times z_2).$$

By W let us denote the restriction of this bundle to W_0. Obviously, we have

(64) $$\dim W = 2(m + r - n_q) - N$$

(remember $\dim H_q = r - n_q$). The projections $\mathrm{pr}_1 : E_q \times \widetilde{E}_q \to E_q$, $\mathrm{pr}_2 : E_q \times \widetilde{E}_q \to \widetilde{E}_q$ and the embedding $\iota: W \to E_q \times \widetilde{E}_q$ induce the 1-forms with values in $\mathfrak{g}$:

(65) $$\sigma = \mathrm{pr}_1^* \omega_q, \quad \widetilde{\sigma} = \mathrm{pr}_2^* \widetilde{\omega}_q \quad \text{on} \quad E_q \times \widetilde{E}_q,$$

(66) $$\tau = \iota^* \sigma, \quad \widetilde{\tau} = \iota^* \widetilde{\sigma} \quad \text{on} \quad W.$$

Lemma 2.3. *For* $w \in W$ *we define*

$$(67)\qquad t \in D_w : \leftrightarrow t \in T_w W \quad \text{and} \quad \tilde{\tau}(t) = \tau(t).$$

Then

(a) *the correspondence* $w \in W \mapsto D_w \subseteq T_w W$ *defines a distribution on* W *for which the projections*

$$(68)\qquad \mathrm{pr}_1 \mid D_w : D_w \to T_z E_q, \quad \mathrm{pr}_2 \mid D_w : D_w \to T_{\tilde{z}} \tilde{E}_q, \quad w = (z, \tilde{z}),$$

are linear isomorphisms. This implies

$$(69)\qquad \dim D_w = m + r - n_q.$$

(b) *The distribution* D *is involutive.*

Proof. Let $w_0 = (z_0, \tilde{z}_0) \in W$ and $(y_0, \tilde{y}_0) = \beta(w_0)$. We take a neighbourhood $U \times \tilde{U}$ of $(y_0, \tilde{y}_0)$ as in Lemma 2.2. Considering the differential invariants $c^\nu, \tilde{c}^\nu$ as functions on E_q and $\tilde{E}_q$, respectively, we obtain that $W \cap \beta^{-1}(U \times \tilde{U})$ is characterized by the system $\tilde{c}^\nu(\tilde{z}) = c^\nu(z)$, $\nu = 1, \ldots, N$. Therefore on $W \mid U \times \tilde{U}$ the equations

$$(70)\qquad dc^\nu(t) = d\tilde{c}^\nu(t), \qquad \nu = 1, \ldots, N, \qquad t \in T_w W,$$

are valid. Assume that in the decompositions

$$(71)\qquad dc^\nu(z) = c^\nu{}_\alpha(z)\omega^\alpha(z), \qquad \nu = 1, \ldots, N,$$

we have (enumerating the ω^α in an appropriate manner):

$$(72)\qquad \det(c^\nu{}_\mu(z)) \neq 0, \qquad \nu, \mu = 1, \ldots, N.$$

Since the $c^\nu{}_\alpha$ are invariants of order $\leqslant q + 1$, the analogous relation holds true for $\tilde{c}^\nu{}_\mu(\tilde{z})$. By an elementary consideration one proves that the forms

$$(73)\qquad \begin{array}{l} dc^1, \ldots, dc^N, \quad \omega^{N+1}, \ldots, \omega^m, \quad \omega^{n_q+1}, \ldots, \omega^r, \\ \qquad \tilde{\omega}^{N+1}, \ldots, \tilde{\omega}^m, \quad \tilde{\omega}^{n_q+1}, \ldots, \tilde{\omega}^r \end{array}$$

make up a basis of $T^*_w W$ for each $w \in W \mid U \times \tilde{U}$. Now we will show that $D_w \subseteq T_w W$ is defined by the $m - N + r - n_q$ independent 1-forms

$$(74)\qquad \begin{array}{l} \tilde{\omega}^\rho - \omega^\rho = 0, \quad \tilde{\omega}^k - \omega^k = 0, \\ \qquad \rho = N + 1, \ldots, m, \quad k = n_q + 1, \ldots, r. \end{array}$$

Indeed, if we decompose $\tilde{\tau} - \tau = 0$ with respect to the adapted basis (31) of $\mathfrak{g}$, we obtain (74) as a consequence of (67). Conversely, let (74) be fulfilled. Then (71) and the fact that $c^\nu{}_\alpha(z) = \tilde{c}^\nu{}_\alpha(\tilde{z})$ for all $(z, \tilde{z}) \in W$ yield together with the first series of (74):

$$(75)\qquad c^\nu{}_\mu(z)(\tilde{\omega}^\mu(t) - \omega^\mu(t)) = 0$$

for all $t \in T_w W$ satisfying (74), compare (70). From (72) we get

$$(76)\qquad \tilde{\omega}^\mu(t) - \omega^\mu(t) = 0, \quad \mu = 1, \ldots, N.$$

Since on W the differential invariants fulfil $\tilde{c}^\nu(\tilde{z}) = c^\nu(z)$, $(z, \tilde{z}) \in W$, we finally obtain using (74) and (76):

$$\tilde{\omega}^\kappa(t) = \tilde{c}^\kappa{}_\alpha(\tilde{z})\tilde{\omega}^\alpha(t) = c^\kappa{}_\alpha(z)\omega^\alpha(t) = \omega^\kappa(t)$$

for $\kappa = m + 1, \ldots, n_q$. These equations together with (74) and (76) imply $\tilde{\tau}(t) = \tau(t)$, i.e. $t \in D_w$. Therefore (64) implies (69). To prove (68) it suffices to show that e.g. $\mathrm{pr}_1 \mid D_w$ is surjective. Since the ω^α, $\alpha = 1, \ldots, m$, ω^k, $k = n_q + 1, \ldots, r$, make up a coframe field on E_q, we can express the values $dc^\nu(t)$ for each $t \in T_z E_q$ by (71). Then the vector $s \in T_w W$ with the components $dc^\nu(s) = dc^\nu(t)$, $\tilde{\omega}^\rho(s) = \omega^\rho(s) = \omega^\rho(t)$, $\tilde{\omega}^k(s) = \omega^k(s) = \omega^k(t)$ (with abuse of notations) is contained in D_w and projected on t.

The involutivity of the system is a direct consequence of the structure equations which are preserved under inducing:

$$d(\tilde{\tau} - \tau) = -\frac{1}{2}([\tilde{\tau}, \tilde{\tau}] - [\tau, \tau]) = -\frac{1}{2}([\tilde{\tau}, \tilde{\tau} - \tau] - [\tau - \tilde{\tau}, \tau]). \blacksquare$$

Now we can finish the proof of Theorem 2.2. First we remark that the distribution D is H_q-invariant under the diagonal action

$$(77)\qquad \begin{array}{l} \alpha_h : w = (z, \tilde{z}) \in E_q \times \tilde{E}_q \mapsto h \times w = \\ \qquad = (h \times z, h \times \tilde{z}) \in E_q \times \tilde{E}_q, \qquad h \in H_q. \end{array}$$

This easily follows from Property 2, Definition 1.1, of the G, H_q-structure: $\alpha_h^*(\widetilde{\tau} - \tau) = \mathrm{Ad}\,(h) \circ (\widetilde{\tau} - \tau)$. Furthermore, from Condition 3 of the same definition we obtain for each $A \in \mathfrak{h}_q$ and the corresponding fundamental vector field $\widetilde{A}$:

$$(78) \qquad \begin{aligned} \widetilde{\omega}^\rho(\widetilde{A}) &= \omega^\rho(\widetilde{A}) = 0 \qquad \text{for} \quad \rho = 1, \ldots, N, \\ \widetilde{\omega}^k(\widetilde{A}) &= \omega^k(\widetilde{A}) \qquad \text{for} \quad k = n_q + 1, \ldots, r. \end{aligned}$$

Therefore $\widetilde{A}(w) \in D_w$, and the integral curves of $\widetilde{A}$ are integral curves of the distribution D. It follows that the integral manifolds of D are invariant under the action (77) restricted to the connected component of unity $(H_q)_0$. Now we take a point $y_0 \in Y$ and put $\widetilde{y}_0 = \varphi(y_0)$. By Property 2 of W_0 we have $(y_0, \widetilde{y}_0) \in W_0$. For any $w_0 = (z_0, \widetilde{z}_0) \in \beta^{-1}(y_0, \widetilde{y}_0)$ we obtain a maximal connected integral manifold $M = M_{w_0}$ of D with $w_0 \in M$. From (68) we see that $\mathrm{pr}_1 \mid M$ is a submersion, and, since $\dim D_w = \dim D_q$, a local diffeomorphism. Let us put

$$(79) \qquad \bar{M} = \alpha(H_q) M.$$

Then $\bar{M}$ is an eventually non-connected integral manifold of D, since the action of H_q is tangential to D. $\bar{M}$ is projected by pr_1 on an open submanifold of E_q invariant under the action of H_q, since $\mathrm{pr}_1 \circ \alpha_h = \gamma_h \circ \mathrm{pr}_1$. As $w_0 \in \bar{M}$, we find an open neighbourhood V of $y_0 = p_q \circ \beta(w_0)$ such that $E_q \mid V = p_q^{-1}(V) \subseteq \mathrm{pr}_1(\bar{M})$. From Lemma 2.3 we obtain that, for sufficiently small V, the manifold $\bar{M} \cap \mathrm{pr}_1^{-1}(E_q \mid V)$ is the graph of a diffeomorphism $F\colon E_q \mid V \to \widetilde{E}_q \mid \widetilde{V}$, where $\widetilde{V} = p_q \circ \beta(\bar{M} \cap \mathrm{pr}_1^{-1}(E_q \mid V))$; indeed, from the invariance property of $\bar{M}$ we get: if $w = (z, F(z)) \in \bar{M}$, then $\alpha_h(w) = (h \times z, h \times F(z)) \in \bar{M}$, and this yields $F(h \times z) = h \times F(z)$. Therefore F is an isomorphism of $E_q \mid V$ onto $\widetilde{E}_q \mid \widetilde{V}$. Theorem 1.1 implies now Theorem 2.2. Obviously, g is uniquely determined by the second condition (57).∎

3. A GLOBAL VERSION OF THE EQUIVALENCE THEOREM

The following simple example shows that Theorem 2.2 fails to hold true in the global case, even if Y^m is supposed to be simply connected: Take two open geodesic balls on the sphere S^2 of radius 1; they are equivalent if and only if their radii coincide. But each diffeomorphism φ: $Y^2 \to \widetilde{Y}^2$ of the geodesic balls fulfills (56). The compactness of $Y, \widetilde{Y}$ alone does not suffice either to ensure the global equivalence of two immersions $f, \widetilde{f}$ satisfying the assumptions of Theorem 2.2: Take the circle $f\colon S^1 \subset E^2$ of radius 1 in the euclidean plane and consider the twofold covering $\widetilde{f}\colon \widetilde{Y}^1 \to S^1$. Clearly, f and $\widetilde{f}$ are not equivalent, but all the assumptions of Theorem 2.2 are fulfilled. Now we prove:

Theorem 3.1. *Let the assumptions and notations be the same as in Theorem 2.2. Assume further Y^m to be simply connected and compact, and the structure group H_q be compact too. Then f and $\widetilde{f}$ are equivalent.*

Proof. First we remark that the manifolds W_0, W constructed in the proof of Theorem 2.2 are compact: W_0 as a connected component of the solution of (62) is a closed subset of the compact manifold $Y \times \widetilde{Y}$, hence compact, and the compactness of the structure group H_q implies the compactness of W. Now we take the (not necessarily connected) integral manifold $\bar{M}$ through $w_0 = (z_0, \widetilde{z}_0)$ as defined in the proof of Theorem 2.2. We shall prove that it is projected by $\mathrm{pr}_1 \mid W$ onto E_q. Since it is invariant under the diagonal action of H_q, it suffices to prove the following: Let $\gamma(t)$, $d\gamma / dt \neq 0$, be a regular smooth path joining z_0 with an arbitrary point z_1 of the connected component of E_q containing z_0. Then there exists an integral curve $w(t)$ of D over $\gamma(t)$. To verify this we first remark that for each $z \in E_q$ the set $W_z := (\mathrm{pr}_1 \mid W)^{-1}(z)$ is a closed and therefore compact submanifold of W with dimension $m - N + r - n_q$ (see (64) and take into account Lemma 2.3, which implies that $\mathrm{pr}_1 \mid W$ is a submersion). Again from Lemma 2.3 we conclude, that D induces a field of directions on $(\mathrm{pr}_1 \mid W)^{-1}(\gamma)$. This set carries a natural manifold structure of a manifold with a boundary consisting of $(\mathrm{pr}_1^{-1}(z_0) \cup \mathrm{pr}_1^{-1}(z_1)) \cap W$. The compactness of $W_{\gamma(t)}$ allows to apply the usual continuation arguments showing that the integral curve starting in w_0 ends in W_{z_1}; we remark that the directions of the field are transversal to the manifolds $W_{\gamma(t)}$.

Now $\bar{M}$ is a principal fibre bundle with the structure group H_q. The projection $\beta \mid \bar{M}: \bar{M} \to Y \times \tilde{Y}$ maps $\bar{M}$ on an m-dimensional submanifold $Z^m \subset Y \times \tilde{Y}$. Since $\mathrm{pr}_1 \mid \bar{M}$ is a submersion, the projection $p_1: Z^m \to Y^m$ is a submersion too, and therefore Z^m is a covering manifold. From the definition of $\bar{M}$ follows $\beta(M) = \beta(\bar{M}) = Z^m$, and since M is connected as a maximal integral manifold, Z^m is connected, too. Since Y^m is simply connected, Z^m is the graph of a globally defined diffeomorphism $F_0: Y \to \tilde{Y}$, and $\bar{M}$ is the graph of an isomorphism $F: E_q \to \tilde{E}_q$. Now Theorem 1.1 implies our statement.∎

Corollary 3.1. *Let the same assumptions as in Theorem 3.1 be fulfilled with the exception of the simple connectedness of Y. Then there exists a* $g \in G$ *such that* $\tilde{f}(\tilde{Y}^m) = l_g \circ f(Y^m)$.

Proof. All the consideration up to the last one of the foregoing proof remain valid, but now Z^m is a covering of Y and of $\tilde{Y}$. We lift the G, H_q-structure E_q over Y to Z^m. From a local consideration it is clear that this lift can be identified with $\bar{M}$, and its structure form with $\tau \mid T\bar{M}$. Analogously we get $\tilde{\tau} \mid \bar{M}$ lifting $\tilde{E}_q(\tilde{Y}, \tilde{\omega}_q)$ to $\bar{M}$. Now by definition of $\bar{M}$ we have $\tilde{\tau} \mid \bar{M} = \tau \mid \bar{M}$. Thus both lifts give the same G, H_q-structure over Z^m. On the other hand, the immersions $f \circ p_1 \mid Z^m$ and $\tilde{f} \circ p_2 \mid Z^m$ induce this G, H_q-structure. Thus both immersions are equivalent. This yields $l_g(f(p_1(Z^m))) = l_g(f(Y^m)) = \tilde{f}(\tilde{Y}^m)$.∎

4. THE HOMOGENEITY THEOREM

In the last section we will give a simple and direct proof of the well-known homogeneity theorem, (compare [2], section 132, [3], I.12, [5], Theorem 4.4).

Theorem 4.1. *Let* $f \in \mathfrak{F}(A_q)$ *be an immersion with the following property: For the canonical form* θ_q *all the coefficients* b'^{κ}_{α}, $\alpha = 1, \ldots, m$, $\kappa = m+1, \ldots, n_q$, *(compare (36)), are constant. Then it is a Frenet type immersion of order* $\leqslant q$ *with rank* 0. *There exists a*

unique connected Lie subgroup $K \subseteq G$, *the Lie algebra* $\mathfrak{k}$ *of which is defined as a subalgebra of* $\mathfrak{g}$ *by*

$$\omega^{\kappa}(g) - b'^{\kappa}_{\alpha}\omega^{\alpha}(g) = 0, \qquad \kappa = m+1, \ldots, n_q, \tag{80}$$

$$\dim \mathfrak{k} = r - n_q + m, \tag{81}$$

such that $f(Y^m)$ *is G-equivalent to an open submanifold of the orbit* $Kx_0 \subseteq X^n$,

$$Kx_0 \approx K / K \cap H. \tag{82}$$

Proof. The first statement is obvious, since we have $dc^{\nu} = 0$ for all G-invariants, and $b'^{\kappa}_{\alpha} = \text{const.}$ by the assumption, which implies $H_{q+1} = H_q$. Since $f_q: Y^m \to G / H_q$ is an immersion, $\hat{f}_q: E_q \to G$ must be an immersion, too. Therefore we have $\dim d\hat{f}_q(T_z E_q) = r - n_q + m$. On the other hand we consider the distribution defined by (80) on G. It has the same dimension (81), is left invariant, since the b'^{κ}_{α} are constant, and it is involutive: Indeed, there exists an integral manifold of dimension $r - n_q + m$ in the neighbourhood of a point $g = \hat{f}_q(z_0)$ of G, and the left invariance implies the global involutivity. Thus the integral manifold through e is a connected Lie subgroup $K \subseteq G$ (not necessarily closed). Taking a connected component E'_q of E_q we see that $L_g^{-1}\hat{f}(E'_q) \subseteq K$ as an open submanifold. Since Y^m is connected we have $p_q(E'_q) = Y^m$, and we obtain

$$l_g^{-1} f_q(Y) = l_g^{-1} f_q \circ p_q(E'_q) = \pi_q \circ L_g^{-1}\hat{f}_q(E'_q) \subseteq K\pi_q(e) \tag{83}$$

as an open submanifold. The canonical projection $\eta: G / H_q \to G / H$ now yields

$$l_g^{-1} \circ \eta \circ f_q(Y) = l_g^{-1} f(Y) \subseteq \eta(K \cdot H_q) = Kx_0$$

as an open submanifold.∎

Corollary 4.1. *If under the assumptions of Theorem 4.1* Y^m *is compact, the image of* f *is the whole orbit* $f(Y^m) = Kx_1$.

Proof. Clearly, $f(Y^m)$ is open and compact, hence closed. Kx_1 connected implies the statement.∎

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ISOMETRIC ACTIONS WITH ISOTROPY SUBGROUPS OF MAXIMAL RANK ON RIEMANNIAN MANIFOLDS

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In studying differentiable actions of compact Lie groups important basic results are obtained by introducing an invariant Riemannian metric and thus rendering the action isometric. In fact, this way to any given orbit an open invariant neighbourhood can be produced which is the union of orbits having isotropy type equal to or greater than the given one. Thus are the fundamental facts derived concerning the local behaviour of differentiable actions of compact Lie groups. It seems therefore justified to abide by the surroundings of Riemannian geometry and to try to find some results concerning the global behaviour of actions too this way. For the global behaviour of actions the maximal ones among the above mentioned open invariant neighbourhoods of a given orbit seem to be decisive. Since such a maximal neighbourhood is bounded by the cut locus of the given orbit, first cut loci of orbits are studied in entire generality. Then the simple case is examined when the isotropy subgroups are of maximal rank and the Riemannian manifold has non-positive sectional curvature everywhere. A more detailed presentation containing proofs will appear elsewhere.