The Fundamental Theorem for Curves in the n-Dimensional Euclidean Space.

Rolf Sulanke

February 11, 2020

Abstract

In this paper we give a complete detailed proof of the fundamental theorem for curves in the Euclidean n-space $\boldsymbol{E}^n$. As an application we find all curves with constant curvatures in $\boldsymbol{E}^n$.

1 Introduction

The existence- and uniqueness theorem for a curve whose curvatures are given as functions of its arclength is the key tool for the curve theory in elementary Euclidean differential geometry. Unfortunately, I don't know a textbook that contains the necessary conceptual framework and a precise complete proof. For understanding the programs developed under St. Wolfram's Mathematica in my notebook [8] and the corresponding Mathematica packages such a theoretical background is useful, if not necessary. This is the reason for writing this paper, which may be read independently of the mentioned Mathematica tools. The notebook contains programs for the calculation of the Euclidean invariants of curves in arbitrary dimension and for graphical presentations in dimensions 2 and 3. Together with the needed packages it can be downloaded from my homepage.

2 Basic Definitions

We consider curves in the n-dimensional Euclidean space $\boldsymbol{E}^n$, given by *parameter representations*

$$\boldsymbol{x} : t \in I \longmapsto \boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \boldsymbol{E}^n, \tag{1}$$

where the $x_i(t)$ are the components of the radius vector $\boldsymbol{x}(t)$ with respect to a fixed orthonormal frame of E^n with the fixed origin $\boldsymbol{o}$. Here $I \subset \boldsymbol{R}$ denotes an open interval which may be infinite. The vector function $\boldsymbol{x}(t)$ is assumed to be *smooth*, i. e. its components are sufficiently often continuously differentiable: all its derivatives which appear in our formulas exist, are continuous, and continuously differentiable once more. For the curve theory in $\boldsymbol{E}^n$ it suffices that the parameter representations are n times continuously differentiable.

Definition 1. Two parameter representations (1) and $\boldsymbol{y}(s), s \in I_1$, are said to be *equivalent*, if there exists a *parameter transformation*, i.e. a smooth bijective function

$$s \in I_1 \longmapsto t(s) \in I \text{ with } dt/ds(s) \neq 0 \text{ for all } s \in I_1$$

such that

$$\boldsymbol{y}(s) = \boldsymbol{x}(t(s)). \tag{2}$$

One easily sees that (2) defines an equivalence relation in the class of all smooth parameter representations in $\boldsymbol{E}^n$. The corresponding equivalence classes are called *smooth curves*. If in (2) instead of $dt/ds(s) \neq 0$ is required $dt/ds(s) > 0$, the corresponding equivalence classes are named *oriented smooth curves*. A point $\boldsymbol{x}(t)$ of the curve is said to be *regular* if the *tangent vector* $d\boldsymbol{x}/dt(t) \neq \boldsymbol{o}$ is well defined, and *singular* else. □

Clearly, equivalent parameter representations have the same image, whose points are understood as the points of the curve. From the definition is clear by the chain rule:

$$d\boldsymbol{y}/ds = (d\boldsymbol{x}/dt)(dt/ds) \tag{3}$$

that the regularity of the point of the curve does not depend on the chosen parameter representation. We always assume that all points of the curve are regular if speaking about *regular curves*; if there appear singularities, we have to restrict on regular parts of the curve. Singularities must be considered in particular.

Definition 2. Let $\boldsymbol{x} = \boldsymbol{x}(t) \in \boldsymbol{E}^n$ be a point of the curve represented by the smooth parameter representation (1). The *k-th osculating space of the curve at the point* $\boldsymbol{x}$ is defined as the linear hull of the first k derivatives of $\boldsymbol{x} = \boldsymbol{x}(t)$ at the point $\boldsymbol{x}$;

$$T_{\boldsymbol{x},k} := \operatorname{span}(\boldsymbol{x}', \boldsymbol{x}'', \ldots, \boldsymbol{x}^{(k)})_{\boldsymbol{x}=\boldsymbol{x}(t)}. \tag{4}$$

The point is said to be *k-flat* if there exists a k, $k = 1, 2, \ldots, n-1$, with the property

$$\dim T_{\boldsymbol{x},k} = k = \dim T_{\boldsymbol{x},k+1}.$$

The *osculating k-plane* is defined as the Euclidean k-plane spanned by the point $\boldsymbol{x}(t)$ and the vector space $T_{\boldsymbol{x}(t),k}$. □

The first osculating space is the *tangent space* of the curve at $\boldsymbol{x}$ denoted by $T_{\boldsymbol{x}} := T_{\boldsymbol{x},1}$. Clearly, the osculating spaces form a flag of subspaces of $\boldsymbol{E}^n$:

$$T_{\boldsymbol{x}} \subset T_{\boldsymbol{x},2} \subset \ldots \subset T_{\boldsymbol{x},n} \subset \boldsymbol{E}^n.$$

Formally, definition (4) depends on the parameter representation of the curve. Using the chain rule and the product rule of the derivation one obtaines recursively starting with (3):

$$\begin{aligned} \tfrac{d^2\boldsymbol{y}}{ds^2} &= \tfrac{d^2\boldsymbol{x}}{dt^2}\left(\tfrac{dt}{ds}\right)^2 + \tfrac{d\boldsymbol{x}}{dt}\tfrac{d^2t}{ds^2}, \\ \tfrac{d^3\boldsymbol{y}}{ds^3} &= \tfrac{d^3\boldsymbol{x}}{dt^3}\left(\tfrac{dt}{ds}\right)^3 + 3\tfrac{d^2\boldsymbol{x}}{dt^2}\tfrac{dt}{ds}\tfrac{d^2t}{ds^2} + \tfrac{d\boldsymbol{x}}{dt}\tfrac{d^3t}{ds^3}, \\ &\ldots\ldots\ldots\ldots\ldots \\ \tfrac{d^k\boldsymbol{y}}{ds^k} &= \tfrac{d^k\boldsymbol{x}}{dt^k}\left(\tfrac{dt}{ds}\right)^k \bmod T_{\boldsymbol{x},k-1}, \end{aligned}$$

where $\bmod T_{\boldsymbol{x},k-1}$ means that the equality holds up to addition of a vector $\boldsymbol{w} \in T_{\boldsymbol{x},k-1}$. These equations imply the first statement of

Proposition 1. *The osculating spaces $T_{\boldsymbol{x},l}$ of a smooth regular curve in $\boldsymbol{E}^n$ don't depend on its parameter representation. They correspond equivariantly to the curve, that means: If $g \in \mathfrak{A}(n)$ is an affine transformation of the space $\boldsymbol{E}^n$ one has*

$$T_{g\boldsymbol{x},l} = dg(T_{\boldsymbol{x},l}), \tag{5}$$

dg denoting the corresponding linear transformation. If all points of the curve are k-flat $(1 \leq k \leq n)$ the curve is contained in a k-dimensional subspace of $\boldsymbol{E}^n$ and not in a subspace of lower dimension.

P r o o f. The equivariance (5) follows easily by deriving the affine transformation g. We define the k-vector

$$\Pi_k(t) := \boldsymbol{x}'(t) \wedge \boldsymbol{x}''(t) \wedge \ldots \wedge \boldsymbol{x}^{(k)}(t).$$

Since $\Pi_k(t_0)$ vanishes if and only if its factors are linearily dependent it follows that a point $\boldsymbol{x}(t_0)$ is k-flat if $\Pi_k(t_0) \neq 0$ and $\Pi_{k+1}(t_0) = 0$ are fulfilled. Furthermore, at points with $\Pi_k(t) \neq 0$ the k-th osculating space is characterised by

$$T_{\boldsymbol{x},k} = \{\boldsymbol{z} | \Pi_k(t) \wedge \boldsymbol{z} = 0\} \text{ with } \boldsymbol{x} = \boldsymbol{x}(t).$$

Now assume that all points of the curve are k-flat. Deriving $\Pi_k(t)$ using the product rule one obtains by the properies of the $\wedge$-product:

$$\frac{d\Pi_k}{dt} = \boldsymbol{x}' \wedge \boldsymbol{x}'' \wedge \ldots \wedge \boldsymbol{x}^{(k-1)} \wedge \boldsymbol{x}^{(k+1)}.$$

Since by our assumption the vector $\boldsymbol{x}^{(k+1)}$ belongs to $T_{\boldsymbol{x},k}$ it follows: there exists a real function $\lambda(t)$ such that the differential equation

$$\frac{d\Pi_k}{dt} = \Pi_k(t)\lambda(t) \tag{6}$$

is satisfied. Now we remember the

Lemma 2. *If all tangent vectors of a parameter representation $\boldsymbol{y}(t)$ satisfy the equation*

$$\boldsymbol{y}' = \boldsymbol{y}\lambda(t), \tag{7}$$

then the curve is a line: there exists a function $\mu(t) > 0$ such that $\boldsymbol{y}(t) = \boldsymbol{y}_0\mu(t)$, $\boldsymbol{y}_0$ a constant vector.

P r o o f. If the statement is true, we have

$$\boldsymbol{y}'(t) = \boldsymbol{y}_0\mu'(t) = \boldsymbol{y}\lambda(t) = \boldsymbol{y}_0\mu(t)\lambda(t).$$

Thus, solving the differential equation $\mu' = \mu\lambda(t)$, we obtain

$$\mu(t) = \mathrm{e}^{\int \lambda(t)dt}.$$

Now defining $\boldsymbol{w}(t) := \boldsymbol{w}_0\mu(t)$ for any constant vector $\boldsymbol{w}_0$ one easily verifies that $\boldsymbol{w}(t)$ is a solution of (7) too. Choosing the starting value $\boldsymbol{w}_0 = \boldsymbol{y}(t_0)$

one concludes by the uniqueness theorem for the solutions of differential equations $\boldsymbol{y}(t) = \boldsymbol{w}(t)$, what proves the lemma. □

Remark. The parameter transformation $s = \mu(t)$ gives the usual representation $\boldsymbol{y}(s) = \boldsymbol{y}_0 s$ of a line throug the origin.

Now we return to the proof of Proposition 1. Applying Lemma 2 and (6) to the k-vector $\Pi_k(t)$ we conclude $\Pi_k(t) = \Pi_k(t_0)\mu(t) \neq 0$. Since proportional k-vectors define the same subspace, all derivatives up to order $k \geq 1$ of the parameter representation $\boldsymbol{x}(t)$ for all values $t \in I$ belong to the fixed vector subspace W^k defined by $\Pi_k(t_0)$. We consider a vector of the orthogonal complement $\boldsymbol{v} \in W^{\perp}$. Deriving we obtain

$$d\langle \boldsymbol{x}(t), \boldsymbol{v}\rangle / dt = \langle \boldsymbol{x}'(t), \boldsymbol{v}\rangle = 0.$$

It follows $\langle \boldsymbol{x}(t), \boldsymbol{v}_\kappa\rangle = a_\kappa$ are constants for a basis $(\boldsymbol{v}_\kappa)$ of $W^{\perp}$: The parameter representation of the curve satisfies the linear system

$$\langle \boldsymbol{x}(t), \boldsymbol{v}_\kappa\rangle = a_\kappa, \quad \kappa = k+1, \ldots, n,$$

and the curve belongs to the subspace spanned by one of its points and the space W^k. Since $\Pi_k(t_0) \neq 0$, the first k derivatives are linearly independent, thus the curve may not lie in a subspace of lower dimension. □.

3 The Frenet Frame

The concepts and results of the previous section are valid for real affine spaces; considering $W^{\perp}$ as a subspace of the dual vector space the scalar psoduct used there is well defined. Now we will prove the existence and uniqueness of a Frenet frame, that is a certain family of orthonormal bases $(e_i(t))$, $i = 1, \ldots, n$, attached to the points $\boldsymbol{x}(t)$ of the curve. Here the positive definite scalar product of the Euclidean space becomes essential. Interpreting the parameter t as the time and the parameter representation as a physical motion one sometimes speaks about a *moving frame*. For proving the existence of Frenet frames we need the following concept which also belongs to affine geometry:

Definition 3. A curve with the parameter representation $\boldsymbol{x}(t) \in \boldsymbol{E}^n$ is said to be *generally curved* if at all of its points one has

$$\dim T_{\boldsymbol{x},n-1} = n - 1. \tag{8}$$

□

Clearly, by Proposition 1, this property does not depend on the parameter representation of the curve; geometrically it means that the curve even locally, in small neighbourhoods of its points, may not lie in a k-dimensional subspace with $k < n - 1$; the case $k = n - 1$ is admitted.

Proposition 3. *Let $\boldsymbol{x}(t) \in \boldsymbol{E}^n$ be a parameter representation of a generally curved oriented curve in the n-dimensional oriented Euclidean space. Then there exists a uniquely defined orthonormal, positively oriented, moving frame $(\boldsymbol{e}_i(t))$, $i = 1, \ldots, n$, with the following properties:*

$$span(\boldsymbol{e}_1(t), \ldots, \boldsymbol{e}_k(t)) = (T_{\boldsymbol{x},k})_{\boldsymbol{x}=\boldsymbol{x}(t)}, \quad k = 1, \ldots, n-1, \tag{9}$$

$$\langle \boldsymbol{x}^{(k)}(t), \boldsymbol{e}_k(t) \rangle > 0, \quad k = 1, \ldots, n-1. \tag{10}$$

The moving frame $(\boldsymbol{e}_i(t))$ is independent of the parameter representation and equivariantly associated to the curve: Is $g \in \mathfrak{E}(n)$ an Euclidean orientation preserving motion and $(\hat{\boldsymbol{e}}_i(t))$ the moving frame associated to the curve $\hat{\boldsymbol{x}}(t) = g\boldsymbol{x}(t)$, one has

$$(\hat{\boldsymbol{e}}_i(t)) = (dg(\boldsymbol{e}_i(t))), \tag{11}$$

where dg denotes the differential of g.

Proof. The first vector of the Frenet frame must be the normed tangent vector: $\boldsymbol{e}_1 = \boldsymbol{x}'/|\boldsymbol{x}'|$; indeed, a generally curved curve must be regular, the vectors $\pm\boldsymbol{e}_1$ are the only unit vectors in the tangent space, and since the curve is oriented it follows from (10) that $\boldsymbol{e}_1$ is uniquely defined. Assume that the orthonormed vector sequence $(\boldsymbol{e}_i(t))$, $i = 1, \ldots, m$, $m < n - 1$, satisfying (9) and (10) for $k \leq m$ exists and is uniquely defined. Since $m < n - 1$ and the curve is generally curved we have

$$\dim T_{\boldsymbol{x},m+1} = \dim T_{\boldsymbol{x},m} + 1 = m + 1. \tag{12}$$

Therefore the orthogonal complement of $T_{\boldsymbol{x},m}$ in $T_{\boldsymbol{x},m+1}$ is one-dimensional, thus its unit basis vector $\boldsymbol{e}_{m+1}(t)$ is uniquely defined up to sign. By (12), the derivative $\boldsymbol{x}^{(m+1)}(t)$ can not lie in $T_{\boldsymbol{x},m}$, it follows $\langle \boldsymbol{x}^{(m+1)}(t), \boldsymbol{e}_{m+1}(t) \rangle \neq 0$, and the next vector $\boldsymbol{e}_{m+1}(t)$ is uniquely defined by condition (10). We remark that the construction coincides with E. Schmidt's orthogonalization, see e.g. [5], Proposition 6.2.2. Finally, since the moving frame is required to be positively oriented, the last vector $\boldsymbol{e}_n(t)$ must be the cross product of the sequence $(\boldsymbol{e}_i(t))$, $i = 1, \ldots, n-1$. This follows from the properties of the cross product, see e.g. [5], Proposition 6.3.2. Thus the existence and uniqueness of a moving frame with the required properties is proved. The whole construction depends only on the osculating spaces; by Proposition 1 and (5) the last two statements follow. □

Definition 4. The uniquely define moving frame $(\boldsymbol{e}_i(t))$, $i = 1, \ldots, n$, of a generally curved curve described in Proposition 3 is named the *Frenet frame* of the curve. □

Remark. Considering the action of the Euclidean group $\mathfrak{E}(n)$ on the points space $\boldsymbol{E}^n$ speaking about frames one has to add the point $\boldsymbol{x}(t)$ to the vector frame. Let $z_o := (\boldsymbol{o}; \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ be the fixed orthonormal frame consisting of the origin and the orthonormed basis vectors on the axes of the given coordinate system. Then the map

$$g \in \mathfrak{E}(n) \longmapsto z(g) = g z_o := (g\boldsymbol{o}; dg(\boldsymbol{a}_1), \ldots, dg(\boldsymbol{a}_n)) \tag{13}$$

is a bijection which allows to identify the group space $\mathfrak{E}(n)$ with the space of all orthonormed frames $z = (\boldsymbol{x}; \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$. Since $\mathfrak{E}(n)$ acts transitively on $\boldsymbol{E}^n$ with the isotropy group $\boldsymbol{SO}(n)$ of the origin $\boldsymbol{o}$ (remember: we consider orientation preserving motions only) we may identify $\boldsymbol{E}^n$ with the coset space $\mathfrak{E}(n)/\boldsymbol{SO}(n)$. The canonical projection then appears as the map

$$z = (\boldsymbol{x}; \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) \in \mathfrak{E}(n) \longmapsto \boldsymbol{x} \in \boldsymbol{E}^n. \tag{14}$$

The Frenet frame of a curve in $\boldsymbol{E}^n$ is a canonically defined equivariant lift of the curve in the Euclidean space to a curve in the group space allowing to apply the rich structure of the Lie group $\mathfrak{E}(n)$ to Euclidean curve theory. This is an elementary example of E. Cartan's general method in the differential geometry of homogeneous spaces, see E. Cartan [1], see also R. Sulanke [6] with hints to further literature.

4 The Frenet Formulas

The Frenet formulas show the decomposition of the first derivatives of the components of the Frenet frame at the point $\boldsymbol{x} = \boldsymbol{x}(t)$ with respect to the Frenet frame $(\boldsymbol{e}_i(t))$ itself. Of course, these derivatives depend on the parameter. Therefore the first step is to overcome this difficulty by introducing a distinguished natural, that means invariant parameter: the arclength of the curve; a parameter s is called an *arclength* of the regular curve, if for the norm of the tangential vector

$$|\frac{d\boldsymbol{x}}{ds}(s)| = 1 \text{ for all } s \in I_1 \tag{15}$$

is satisfied.

Proposition 4. *Any regular curve in the n-dimensional Euclidean space possesses parameter representations $\boldsymbol{x} = \boldsymbol{x}(s)$ with an arclength s as parameter. An arclength of an oriented curve is uniquely defined up to an additive constant. It is invariant under Euclidean transformations.*

Proof. Taking the norm on both sides of (3) we obtain the condition

$$1 = |\frac{d\boldsymbol{y}}{ds}| = |\frac{d\boldsymbol{x}}{dt}|\frac{dt}{ds}. \tag{16}$$

Here we used $dt/ds > 0$ considering orientation preserving parameter transformations only. Starting with an arbitrary parameter t of the curve we define the parameter s by

$$\frac{ds}{dt} = |\frac{d\boldsymbol{x}}{dt}|, \tag{17}$$

$$s(t) = \int_{t_0}^{t} |\frac{d\boldsymbol{x}}{dt}| dt.$$

Obviously, $s(t)$ is a strictly monotonically increasing function uniquely defined up to an additive constant as the indefinite integral of (17). Inserting the inverse function $t = t(s)$ into the original parameter representation and calculating the norm shows that (16) is fulfilled. The invariance statement follows immediately by the invariance of the Euclidean norm. Onviously, setting $\hat{s} = -s$, the parameter $\hat{s}$ also is an arclength corresponding to the inverse orientation of the curve. □

Proposition 5. *Assume that $\boldsymbol{x} = \boldsymbol{x}(s) \in \boldsymbol{E}^n$ is a parameter representation of a generally curved curve with arclength s. Then the following* derivation equations are valid:

$$\begin{array}{l}
\frac{d\boldsymbol{x}}{ds} = \boldsymbol{e}_1, \\
\frac{d\boldsymbol{e}_1}{ds} = \qquad \boldsymbol{e}_2 k_1, \\
\frac{d\boldsymbol{e}_2}{ds} = -\boldsymbol{e}_1 k_1 + \boldsymbol{e}_3 k_2, \\
\dots\dots\dots\dots\dots\dots\dots, \\
\frac{d\boldsymbol{e}_{n-1}}{ds} = \qquad -\boldsymbol{e}_{n-2} k_{n-2} + \boldsymbol{e}_n k_{n-1}, \\
\frac{d\boldsymbol{e}_n}{ds} = \qquad\qquad -\boldsymbol{e}_{n-1} k_{n-1}.
\end{array} \tag{18}$$

P r o o f. The first equation follows from the definition of the arclength. The orthonormality of the Frenet frame implies

$$0 = \frac{d\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle}{ds} = \langle \frac{d\boldsymbol{e}_i}{ds}, \boldsymbol{e}_j \rangle + \langle \boldsymbol{e}_i, \frac{d\boldsymbol{e}_j}{ds} \rangle. \tag{19}$$

Therefore the coefficient matrix of the decomposition of the derivatives $d\boldsymbol{e}_i/ds$ is skew symmetric. On the other hand, since $\boldsymbol{e}_i \in T_{\boldsymbol{x},i}$, its derivative must lie in $T_{\boldsymbol{x},i+1}$, and the shape (18) of the coefficient matrix follows by (9). □

Definition 5. Equations (18) are named the *Frenet formulas* of the generally curved curve. The coefficient

$$k_i(s) = \langle \frac{d\boldsymbol{e}_i}{ds}, \boldsymbol{e}_{i+1} \rangle, \quad i = 1, \dots, n-1, \tag{20}$$

is named the *i-th curvature* of the curve at the point $\boldsymbol{x}(s)$. In case of the plane we have only one curvature; we write $k = k(s)$. In case of the three-dimensional space we have besides of the *curvature* $k = k_1$ still the second curvature $\tau = k_2$, in this case named the *torsion* of the curve. □

Proposition 6. *The curvatures $k_i(s)$ of a generally curved curve in $\boldsymbol{E}^n$ are invariant under Euclidean orientation preserving motions. They are functions of the points of the curve satisfying*

$$k_j(s) > 0 \text{ for } j = 1, \dots, n-2. \tag{21}$$

The curve lies in a hyperplane of $\boldsymbol{E}^n$ if and only if $k_{n-1}(s) = 0$ for all $s \in I$.

P r o o f. . The invariance statement follows directly from the equvariance of the Frenet frame, proved in Proposition 4, since derivation permutes with

Euclidean motions being linear transformations. For the arclength the only possible orientation preserving transformations of the arclength parameter are $\hat{s} = s + a$ with a constant. It follows

$$\hat{\boldsymbol{e}}_1(\hat{s}) = \frac{d\boldsymbol{y}(\hat{s})}{d\hat{s}} = \frac{d\boldsymbol{x}(s+a)}{ds}\frac{ds}{d\hat{s}} = \frac{d\boldsymbol{x}(s+a)}{ds} = \boldsymbol{e}_1(s+a).$$

The same considerations for the derivatives of $\hat{\boldsymbol{e}}_j(\hat{s}) = \boldsymbol{e}_j(s+a)$ (see Proposition 3) show $\hat{k}_j(\hat{s}) = k_j(s+a)$: the curvatures depend only on the points of the curve, and not on the parameter representation. Now, by Proposition 3 the decomposition of $\boldsymbol{e}_j$ with respect to the derivatives of $\boldsymbol{x}$ satisfies by (9) and (10):

$$\boldsymbol{e}_j = \sum_{i=1}^{j} \boldsymbol{x}^{(i)}\lambda_{i,j},\, \lambda_{j,j} > 0 \text{ for } j = 1, \ldots, n-1, \tag{22}$$

$$\langle \boldsymbol{e}_j, \boldsymbol{x}^{(i)} \rangle = 0 \text{ for } i = 1, \ldots, j-1 \tag{23}$$

Deriving (22) it follows

$$\frac{d\boldsymbol{e}_j}{ds} = \boldsymbol{x}^{(j+1)}\lambda_{j,j} + o_j,$$

where o_j denotes a linear combination of derivatives of $\boldsymbol{x}$ up to order j. Now, applying (22) for $j+1$ and (23), we obtain

$$k_j = \langle \frac{d\boldsymbol{e}_j}{ds}, \boldsymbol{e}_{j+1} \rangle = \langle \boldsymbol{x}^{(j+1)}, \boldsymbol{e}_{j+1} \rangle \lambda_{j,j} = \lambda_{j,j}\lambda_{j+1,j+1} > 0 \text{ for } j = 1, \ldots, n-2. \tag{24}$$

The last statement follows immediately by Proposition 1 and the following

Lemma 7. *A point $\boldsymbol{x}(s) \in \boldsymbol{E}^n$ of a generally curved curve is $(n-1)$-flat if and only if $k_{n-1}(s) = 0$.*

Proof. Deriving (22) for $j = n-1$ we conclude using (9) and the Frenet formulas:

$$k_{n-1}(s) = 0 \Longleftrightarrow \frac{d\boldsymbol{e}_{n-1}}{ds} \in T_{\boldsymbol{x}(s),n-1} \Longleftrightarrow \boldsymbol{x}^{(n)}(s) \in T_{\boldsymbol{x}(s),n-1}.$$

Because for a generally curved curve we have for all of its points

$$\dim T_{\boldsymbol{x},j} = j \text{ for } j = 1, \ldots, n-1, \tag{25}$$

the statement follows.□□

For a geometric interpretation of the curvatures we interpret s as a time parameter. Then the point $\boldsymbol{x}$ moves with constant velocity 1 along the curve. The approximation of second order

$$\boldsymbol{x}(s+\Delta s) = \boldsymbol{x}(s) + \boldsymbol{e}_1(s)\Delta s + \boldsymbol{e}_2(s)k_1(s)(\Delta s)^2/2 + O(\Delta s)^3 \tag{26}$$

shows that k_1 determines how fast the curve deviates from its tangent in the near of the point $\boldsymbol{x}(s)$. The j-th osculating plane let be oriented by the vector sequence $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_j)$. Thus $\boldsymbol{e}_j$ is the normal vector of the $(j-1)$-th osculating space in $T_{\boldsymbol{x},j}$, $j > 1$. Therefore the expression

$$|\frac{d\boldsymbol{e}_j}{ds}| = \sqrt{k_{j-1}^2 + k_j^2}, \quad (k_o = k_n := 0),$$

measures the change velocity of the $j-1$-th osculating space, which is always positive for generally curved curves and $j \leq n-1$. The vanishing of k_{n-1} for all s means the osculating hyperplane is constant; $k_{n-1} > 0$ means that the osculating hyperplane moves up in the sense of $\boldsymbol{e}_n$; it moves down in case $k_{n-1} < 0$. Remember that $\boldsymbol{e}_n$ is uniquely defined by the oriented curve and the orientation of $\boldsymbol{E}^n$. One says that the curve *turns right* in the first case, *and left* in case $k_{n-1} < 0$.

5 The Fundamental Theorem

Now we are going to prove the following proposition, usually named the *Fundamental Theorem of Curve Theory*, since it characterizes the curve up to Euclidean motions, i.e. up to its position in the space.

Theorem 8. *Let $k_i(s), s = 1, \ldots, n-1$, $s \in I$, be smooth functions satisfying (21). Then there exists a generally curved curve with parameter representation $\boldsymbol{x}(s) \in \boldsymbol{E}^n$ such that s is an arclength parameter and the given functions $k_i(s)$ are its curvatures. Two oriented curves in the Euclidean space $\boldsymbol{E}^n$ having the same curvature functions are congruent under an orientation preserving motion.*

P r o o f. We consider the Frenet formulas (18) as a system of linear differental equations of first order for the moving frame $(\boldsymbol{x}; \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$. It splits in the last n equations for the vectors $(\boldsymbol{e}_i)$ of the frame and the firat equation for the point $\boldsymbol{x}$ which can be solved by an integration, if $\boldsymbol{e}_1(s)$ is known. The general existence and uniqueness theorem for systems of linear differential equations, see e.g. E. Kamke [3], §V.19, admits to formulate:

Lemma 9. *For any family of starting conditions*

$$\boldsymbol{x}(s_0) = \boldsymbol{x}_0,\ \boldsymbol{e}_i(s_0) = \boldsymbol{a}_i \text{ for } i = 1, \ldots, n \tag{27}$$

there exists a uniquely defined solution of (18) defined for all $s \in I$. □

Since the matrix of the linear system for the $\boldsymbol{e}_i$ is skew symmetric, equation (19) holds true, thus the scalar products of the solution vectors are constant. Starting with an orthonormal frame $(\boldsymbol{a}_i)$ the solution $(\boldsymbol{e}_i(s))$ is an orthonormal moving frame of the curve with the parameter representation $\boldsymbol{x}(s)$ obtained by integrating the first equation of the system (18) with the first starting condition (27). By the first equation (18) it follows that the curve is regular and s an arclength parameter. We show that equation (9) is satisfied (with t replaced by s). Indeed, equation (22) is fulfilled for $j = 1$ with $\lambda_{1,1} = 1$. Therefore (9) and (22) are satisfied for $j = 1$. By induction, assuming that both equations are true for the positive integer $j < n - 1$ we show that they are valid for $j + 1$ too. Using (9) and (22) for j we have

$$\boldsymbol{e}_j = \boldsymbol{x}^{(j)} \lambda_{jj} \bmod T_{\boldsymbol{x},j-1}, \lambda_{jj} > 0.$$

Deriving this equation and applying the Frenet formula and the induction assumption we conclude

$$\frac{d\boldsymbol{e}_j}{ds} + \boldsymbol{e}_{j-1} k_{j-1} = \boldsymbol{e}_{j+1} k_j = \boldsymbol{x}^{(j+1)} \lambda_{j,j} \bmod T_{\boldsymbol{x},j}.$$

Taking the scalar product with $\boldsymbol{e}_{j+1}$ and noticing that by the induction assumption equation (9) for j the vector $\boldsymbol{e}_{j+1}$ is orthogonal to $T_{\boldsymbol{x},j}$ we obtain

$$k_j = \langle \boldsymbol{x}^{(j+1)}, \boldsymbol{e}_{j+1} \rangle \lambda_{jj} > 0 \text{ for } j < n - 1.$$

It follows $\langle \boldsymbol{x}^{(j+1)}, \boldsymbol{e}_{j+1} \rangle = \lambda_{jj}/k_j > 0$ for $j + 1 < n$: $\boldsymbol{e}_{j+1}$ is the $j+1$-th vector of the Frenet frame of the curve $\boldsymbol{x} = \boldsymbol{x}(s)$; the n-th vector of the Frenet frame is uniquely defined as the cross product of the others. Therefore the solution gives the Frenet frame of the curve $\boldsymbol{x}(s)$; it is generally curved and the given functions $k_i(s)$ are its curvatures by Propositions 5 and 6: the existence statement is proved.

To prove the uniqueness assume that $\boldsymbol{y}(s)$ is another generally curved curve with the same curvature functions $k_i(s)$,$s \in I$ its arclength. Let $(\boldsymbol{c}_i(s))$ be its Frenet frame and set

$$\boldsymbol{y}_0 = \boldsymbol{y}(s_0),\ \boldsymbol{b}_i = \boldsymbol{c}_i(s_0) \text{ for } i = 1, \ldots, n.$$

From linear algebra it is well known that there exists a uniquely defined orientation preserving transformation $g \in \mathfrak{E}(n)$ such that for $\boldsymbol{a}_i = \boldsymbol{e}_i(s_0)$

$$g(\boldsymbol{x}_0) = \boldsymbol{y}_0,\ dg(\boldsymbol{a}_i) = \boldsymbol{b}_i \text{ for } i = 1, \ldots, n$$

are satisfied. We consider the curve $\boldsymbol{z}(s) = g(\boldsymbol{x}(s))$. Since arclength and curvatures are invariant and the Frene frame is equivariantly attached to the curve, the Frenet franme of $\boldsymbol{z}(s)$ is the image of the Frenet frame of $\boldsymbol{x}(s)$; it satisfies the Frenet formulas with the same starting conditions like $\boldsymbol{y}(s)$. Thus we conclude by the uniqueness of the solutions, see Lemma 9, the equality $g\boldsymbol{x}(s) = \boldsymbol{y}(s)$ what remained to be shown. □

6 Curves of Constant Curvatures

The next proposition describes the relation between curves of constant curvatures and orbits of 1-parameter subgroups of the motion group. More generally, the statement is valid for any class of curves in a homogeneous space, for which a complete system of invariants exists.

Proposition 10. *Any generally curved orbit of a* 1*-parameter subgroup* $g(t) \in \mathfrak{E}(n), t \in \boldsymbol{R}$*, is a curve of constant curvatures* $k_i(t) =$ *const.,* $i = 1, \ldots, n-1$*. Conversely, any generally curved curve of constant curvatures is a part of such an orbit.*

Proof. Let $\boldsymbol{x}(t) = g(t)\boldsymbol{x}_0$ be such an orbit. Since

$$g(s)\boldsymbol{x}(t) = g(s)g(t)\boldsymbol{x}_0 = g(s+t)\boldsymbol{x}_0 = \boldsymbol{x}(s+t), \quad s, t \in \boldsymbol{R},$$

the subgroup acts transitively on the orbit. Since it belongs to the motion group, the curvatures must be constant. Conversely, consider a generally curved curve $\boldsymbol{x}(t) \in \boldsymbol{E}^n$ of constant curvatures, and let

$$z(t) = (\boldsymbol{x}(t); \boldsymbol{e}_1(t), \ldots, \boldsymbol{e}_n(t))$$

be its Frenet frame. By the Fundamental Theorem we may assume that the curve and its Frenet frame are defined for all $t \in \boldsymbol{R}$. Since the motion group acts simply transitively on the manifold of all positively oriented orthonormal frames, there exists a uniquely defined orientation preserving element $g(t) \in \mathfrak{E}(n)$ with $z(t) = g(t)z(0)$. Fixing for the moment $t = s$ and consider the curve $\boldsymbol{y}(t) := g(s)\boldsymbol{x}(t)$. Since the curvatures are invariant, it has the same constant curvatures as $\boldsymbol{x}(t)$. The Frenet frames are equivariantly associated to the curves; therefore the Frenet frame of $\boldsymbol{y}(t)$ is $g(s)z(t)$. This Frenet frame is the solution of the Frenet formulas with start condition $g(s)z(0)$. On the other hand, the frame $h(t) := z(s+t)$ has the same property, and from the uniqueness statement of the Fundamental Theorem it follows

$$z(s+t) = g(s+t)z(0) = g(s)z(t) = g(s)g(t)z(0).$$

Since the action on the frames is simply transitively, we conclude:

$$g(s+t) = g(s)g(t), \tag{28}$$

$g(t)$ is a 1-parameter subgroup of the motion group, and $\boldsymbol{x}(t) = g(t)\boldsymbol{x}(0)$ is its orbit. □

In the second part of the proof of Proposition 10 speaking about Frenet frames we implicitely assumed that t is the arclength of the curve. From (28) it follows that the function

$$s \in \boldsymbol{R} \longmapsto g(s) \in \mathfrak{E}(n)$$

is a homomorphism of the additive group of $\boldsymbol{R}$ onto a 1-parameter subgroup of $\mathfrak{E}(n)$; a parameter representation of the subgroup satisfying (28) we name a *homomorphism parameter*. For any constant $a \in \boldsymbol{R}$, $a \neq 0$, also the parameter transformation $\hat{s} = as$, $\hat{g}(\hat{s}) = g(as)$, yields an homomorphism onto the same 1-parameter subgroup. Thus, considering an orbit $\boldsymbol{x}(t) = g(t)\boldsymbol{x}_0$ of a 1-parameter subgroup we always may assume that t is the arclength on the orbit: indeed, differentiating $g(s)\boldsymbol{x}(t) = \boldsymbol{x}(s+t)$ with respect to t we obtain for the tangential vector

$$dg(s)(\frac{d\boldsymbol{x}(t)}{dt}) = \frac{d\boldsymbol{x}(s+t)}{dt}.$$

Since the differential of a motion is an orthonormal transformation of the Euclidean vector space it follows

$$|\frac{d\boldsymbol{x}(t)}{dt}| = a = \text{ const.}, \ a > 0,$$

and the transformation (17) gives that $s(t) = at$ at the same time is the arclength of the orbit and a homomorphism parameter. In the following we always assume that the parameter on the orbit is its arclength.

Clearly, since the motion group of the Euclidean spaces $\boldsymbol{E}^k \subset \boldsymbol{E}^n$, $k < n$, are subgroups $\mathfrak{E}(k) \subset \mathfrak{E}(n)$ in a natural way, the orbits of 1-parameter subgroups of $\mathfrak{E}(k)$ are also such orbits in dimension n. By Proposition 1, equation (5), the k-th osculating spaces of the curves are equivariantly associated to the curve, the orbit property of a curve implies that all points of an orbit are k-flat if one point of the orbit has this property. By Proposition 1, the k-flat orbits are those which belong to a k-dimensional subspace of $\boldsymbol{E}^n$ but not to a subspace of lower dimension. The constant rank of the maximal osculating spaces of the orbit is called the *rank of the orbit*. Obviously, this rank also is a property of the corresponding 1-parameter subgroup. We may find all the orbits of 1-parameter subgroups of $\mathfrak{E}(n)$ step by step finding at every step in $\boldsymbol{E}^k$ the orbits of rank k, these are the generally curved curves of constant curvatures with highest curvature $k_{k-1} \neq 0$. Clearly, for $k = 1$ the line E^1 itself is such an orbit. For the plane $\boldsymbol{E}^2$ besides of the lines also the circles of radius $r > 0$ are curves of constant curvature $k = 1/r$ as easily can be proved establishing the Frenet formulas for the circles. As we shall show in the example below, in the 3-space $\boldsymbol{E}^3$ the helices are the orbits of rank 3.

The Fundamental Theorem yields a classification of all generally curved curves by their curvatures. Using Proposition 10 we may find a classification of the 1 -parameter subgroups of the motion group with respect to conjugation: two elements or subgroups of a group are *conjugated*, $g \cong \gamma$, if there exists an element $a \in G$ such that $\gamma = aga^{-1}$. Now let $\boldsymbol{x}(t) = g(t)\boldsymbol{x}_0 \in \boldsymbol{E}^n$ be an orbit of a 1-parameter subgroup of the motion group with rank n. Then its curvatures are constant fulfilling $k_{n-1} \neq 0$. Let $\boldsymbol{y}(t)$ be any curve with the same constant curvatures. By Proposition 10 there exists a uniquely defined 1-parameter subgroup $\gamma(t)$ of $\mathfrak{E}(n)$ such that $\boldsymbol{y}(t) = \gamma(t)\boldsymbol{y}_0$. Let z_0, w_0 be the Frenet frames of the curves at the points $\boldsymbol{x}_0, \boldsymbol{y}_0$ respectively. Denote by a the uniquely defined motion with $w_0 = az_0$. Then the curves $a\boldsymbol{x}(t), \boldsymbol{y}(t)$ are solution of the Frenet formulas with the same constant curvatures and the same starting conditions. By the uniqueness of the solutions they coincide. We conclude for the corresponding moving frames:

$$az(t) = ag(t)a^{-1}w_0 = \gamma(t)w_0.$$

It follows $\gamma(t) = ag(t)a^{-1}$, the subgroups are conjugated. Summarizing we formulate

Proposition 11. *Let* $\mathfrak{C} = (\mathfrak{C}(k_1, \ldots, k_{n-1}))$ *be a representing family of curves of constant curvatures* $k_i > 0$ *for* $i = 1, \ldots, n-2$, $k_{n-1} \neq 0$, *of rank* n. *Denote by* $\mathfrak{H}(k_1, \ldots, k_{n-1})$ *the* 1*-parameter subgroup corresponding to* $\mathfrak{C}(k_1, \ldots, k_{n-1})$. *Then any* 1*-parameter subgroup of rank* n *is conjugated to a subgroup of the family* $\mathfrak{H}$. $\square$

We emphasize that as a rule the family $\mathfrak{H}$ contains conjugated subgroups with distinct curvature parameters. In the plane we have only two classes of conjugated 1-parameter subgroups: the parallel transformations generating the lines, and the rotations, generating the circles. In the next example we discuss the 1-parameter subgroups of rank 3 in the Euclidean group $\mathfrak{E}(3)$.

Example. Let $(\boldsymbol{o}; \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ be a fixed orthonormal frame in the Euclidean space $\boldsymbol{E}^3$. We consider the group of *rotations* in the x_1, x_2-plane:

$$\gamma(t)\boldsymbol{x} := \boldsymbol{a}_1(x_1 \cos(t) - x_2 \sin(t)) + \boldsymbol{a}_2(x_1 \sin(t) + x_2 \cos(t)) + \boldsymbol{a}_3 x_3, \quad (29)$$

where x_i denote the coordinates of $\boldsymbol{x}$ with respect to the fixed frame. Its fixed points are the points of the x_3-axis, and its 1-dimensional orbits are the circles with centers on the x_3-axis in planes parallel to the x_1, x_2-plane. Since, as remarked above, the oriented circles are curves having arbitrary constant curvatures $k = 1/r \neq 0$, the rotation group $\gamma(t)$ is the only 1-parameter Euclidean subgroup of rank 2 up to conjugation.

The only 1-parameter subgroups of rank 1 are the *translation groups*, for example

$$g_b(t)\boldsymbol{x} := \boldsymbol{x} + \boldsymbol{a}_3 bt, \; b \in \boldsymbol{R}, b \neq 0. \quad (30)$$

Its orbits are the parallels to the x_3-axis. Clearly the groups γ, g_b commute:

$$\gamma(t)g_b(s) = g_b(s)\gamma(t), \; s, t \in \boldsymbol{R}.$$

Therefore the composition of both is a 1-parameter group again, the group of *screw motions*

$$h_b(t) := g_b(t)\gamma(t). \quad (31)$$

The orbits of $h_b(t)$ of a point of the x_3-axis is the x_3-axis itself, and the orbits of all other points are helices with the x_3-axis as axis, for example the orbits of points $\boldsymbol{x} = \boldsymbol{o} + \boldsymbol{a}_1 a, a > 0$, of the positive x_2-axis:

$$h_b(t)(\boldsymbol{a}_1 a) = \boldsymbol{a}_1 a \cos(t) + \boldsymbol{a}_2 a \sin(t) + \boldsymbol{a}_3 bt. \quad (32)$$

This is a family of orbits of rank 3 depending on two parameters $a > 0, b \neq 0$. The corresponding family of 1-parameter subgroups depends on one parameter b only. The curvature and the torsion of the helix (32) can easily be calculated as

$$k_1 = \frac{a}{a^2 + b^2},\ k_2 = \frac{b}{a^2 + b^2}, \tag{33}$$

see e. g. A. Gray [2],section 7.5, or my notebook [8], section 3.3.2. The system (33) has the uniquely defined inversion

$$a = \frac{k_1}{k_1^2 + k_2^2},\ b = \frac{k_2}{k_1^2 + k_2^2}, \tag{34}$$

which shows that any pair of curvatures (k_1, k_2) with $k_1 > 0, k_2 \neq 0$ appears as the curvatures of one of the helices (32). Applying Proposition 11 we conclude:*Any* 1*-parameter Euclidean motion group of rank* 3 *is conjugated to a group* $h_b(t)$ *of screw motions of the family (31).* □

In my notebook [8] one also finds the calculation and description of the curves of constant curvatures of rank 4 in the 4-dimensional Euclidean space. We summarize here the results; later we describe a representing family for the curves of constant curvatures of rank n in the n-dimensional Euclidean space. As a special case of Proposition 13 below we have: Any 1-parameter subgroup of the Euclidean group $\mathfrak{E}(4)$ of rank 4 is conjugated to one of the family

$$g_a(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(at) & -\sin(at) \\ 0 & 0 & \sin(at) & \cos(at) \end{pmatrix}, t \in \boldsymbol{R}, a \neq 0. \tag{35}$$

The orbits of the points $(r, 0, R, 0)$ generate a representing family of the curves of constant curvatures of rank 4:

$$\begin{gathered} cc(a, r, R)(t) = (r\cos(t), r\sin(t), R\cos(at), R\sin(at)), \\ r, R > 0, a \neq 0, 1, -1. \end{gathered} \tag{36}$$

Obviously, the curve $cc(a, r, R)$ lies on the torus

$$\text{torus}(r, R)(u, v) = (r\cos(u), r\sin(u), R\cos(v), R\sin(v)) \tag{37}$$

being an orbit of a 2-dimensional subgroup of the motion group containing $g_a(t)$. These tori lie in hyperspheres of radii $\sqrt{r^2 + R^2}$ the stereographic projection of which transforms the tori into tori of the Euclidean 3-space and the curves $cc(a,r,R)$ into isogonal trajectories of the generator s of the tori (the coordinate lines $u = u_0$ resp. $v = v_0$). Indeed, we have

Lemma 12. *The curve of constant curvatures $cc(a,r,R)$ intersects the generating circles $v = v_0$ of the* torus(r,R) *under a constant angle α with*

$$\cos(\alpha) = \frac{r}{\sqrt{r^2 + a^2 R^2}}.$$

□

For the proof it suffices to remark that the motions $g_a(t)$ preserve the generating circle families and the angles. The formula for $\cos(\alpha)$ follows by at́simple calulation.

Since stereographic projections are conformal maps transforming k-spheres, in particular circles, and tori not containing the projection center into k-spheres respectively tori of the Euclidean space we may give conformal immages of the curves $cc(a,r,R)$ on the torus(r,R). In the notebook [8] we use the graphics tools of Mathematica to show pictures of the curves obtained this way, see Abbildung 1. The orbits are closed if and only if a is a rational number. The closed orbit 1 has the parameters

$$a = 5/3,\ r = 1/2,\ R = \sqrt{3}/2.$$

The curvatures of the curve $cc(a,r,R)$ are calculated in [8]:

$$k_1 = \frac{\sqrt{r^2 + a^4 R^2}}{r^2 + a^2 R^2},\ k_2 = \frac{arR|a^2 - 1|}{(r^2 + a^2 R^2)\sqrt{r^2 + a^4 R^2}},\ k_3 = \frac{a}{\sqrt{r^2 + a^4 R^2}}. \quad (38)$$

In case of $a^2 = 1$ the orbit is of rank 2, i.e. a circle, and these cases as well as $a = 0$ have to be excluded in formula (36).

For treating the n-dimensional case we need the elements of Lie theory and a bit more linear algebra. An elementary approach to Lie theory one finds in many textbooks, e.g. in our book R. Sulanke, P. Wintgen [7]. To the Euclidean group $\mathfrak{E}(n)$ and to the orthogonal group $\boldsymbol{O}(n)$ belong the corresponding Lie algebras $\mathfrak{e}(n)$ and $\mathfrak{o}(n)$. With respect to n-dimensional orthonormal frames the orthogonal Lie algebra consists of all skew symmetric matrices of order n. The 1-parameter subgroups of a Lie group G correspond bijectively to the 1-dimenional linear subspaces of the associated Lie algebra $\mathfrak{g}$; there exists a map, generalizing the exponential map

$$\exp : A \in \mathfrak{g} \longmapsto \exp A \in G$$

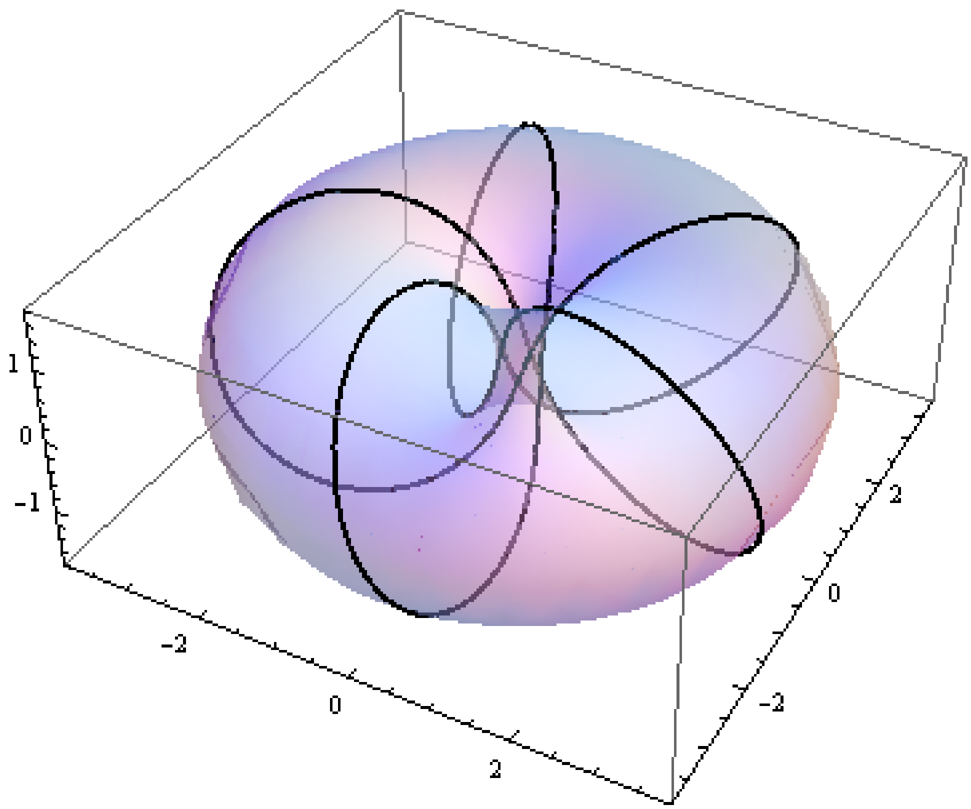

Figure 1: Conformal image of a curve with constant curvatures in $\boldsymbol{E}^4$.

such that any 1-parameter subgroup can be represented in the form

$$g(t) = \exp(tA) \text{ for certain } A \in \mathfrak{g}. \tag{39}$$

For matrix representations of Lie algebras the function exp can be written as the matrix series

$$\exp(tA) = \sum_{\nu=0}^{\infty} \frac{t^\nu A^\nu}{\nu!},$$

uniformly and absolutly converging in all arguments t, A. Therefore one obtains the generating element A of the 1 parameter subgroup $g(t)$ as the derivative

$$A = \frac{dg(t)}{dt}|_{t=0}.$$

For example, the generating element of the group (35) is

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \\ 0 & 0 & a & 0 \end{pmatrix} \tag{40}$$

Analogously to (18),(19) it follows that the skew symmetric matrices form the Lie algebra of the orthogonal group. The Frenet formulas yield a uniquely defined map of the curve into the Euclidean Lie algebra $\mathfrak{e}(n)$ consisting of all square $(n+1)$-matrices with the block structure

$$X(\boldsymbol{a}, A) = \begin{pmatrix} 0 & 0 \\ \boldsymbol{a} & A \end{pmatrix} \text{ with } \boldsymbol{a} \in \boldsymbol{R}^n, A \in \mathfrak{o}(n), \tag{41}$$

i. e. A skew symmetric. Representing the points of $\boldsymbol{E}^n$ as column $(n+1)$-vectors with first element 1 and the vectors as column $(n+1)$-vectors with first element 0:

$$\boldsymbol{x} \hat{=} \begin{pmatrix} 1 \\ x_i \end{pmatrix}, \ \boldsymbol{a} \hat{=} \begin{pmatrix} 0 \\ a_i \end{pmatrix},$$

where x_i respectively a_i, $i = 1, \ldots, n$, are the Cartesian coordinates of the point $\boldsymbol{x} \in \boldsymbol{E}^n$ and the vector $\boldsymbol{a} \in \boldsymbol{V}^n$ with respect to a fixed orthonormal frame, the Euclidean group is lineareły represented by the block matrices

$$g \hat{=} \begin{pmatrix} 1 & 0 \\ b & B \end{pmatrix}, \ g \in \mathfrak{E}(n), \ b \in \boldsymbol{R}^n, \ B \in \boldsymbol{O}(n). \tag{42}$$

Now we remember that a linear endomorphism A of the Euclidean vector space $\boldsymbol{V}^n$ is named *skew symmetric* if

$$\langle A\boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, A\boldsymbol{y} \rangle = 0 \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}^n$$

is satisfied and mention the following result of linear algebra, the proof of which may be found e.g. in the book A. I. Maltzev [4], section V.6, p.186:

Proposition 13. *For every skew symmetric operator B of the Euclidean vector soace $\boldsymbol{V}^n$ there exists a decomposition of $\boldsymbol{V}^n$ in a direct sum of pairwise orthogonal under B invariant subspaces*

$$\boldsymbol{V}^n = \boldsymbol{W}(a_1) \oplus \boldsymbol{W}(a_2) \oplus \ldots \oplus \boldsymbol{W}(a_k) \oplus \boldsymbol{W}_0 \tag{43}$$

where $\boldsymbol{W}(a_i)$ *are* 2*-dimensional, and in an appropriately adapted orthonormal basis the restrictions of* B *have the matrix*

$$B|_{\boldsymbol{W}(a_i)} \hat{=} \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}, a_i \in \boldsymbol{R}, a_i \neq 0, \tag{44}$$

and $B|_{\boldsymbol{W}_0}$ *is the null operator. Moreover, the orthonormal positively oriented basis may be enumerated in such a way that*

$$a_1 \geq a_2 \geq \ldots \geq a_{k-1} \geq |a_k| \tag{45}$$

is satisfied. □

Now we are going to classify the 1-parameter subgroups of the Euclidean group. We first show

Proposition 14. *Let* $g(t) = \exp(tX) \in \mathfrak{E}(n)$*,* n *even, be a* 1*-parameter subgroup of the Euclidean group for which the generating element* $X = X(\boldsymbol{b}, B)$ *has a non degenerated orthogonal part* B*, i. e.* $n = 2k$ *in the decomposition* (43) *and* $\boldsymbol{W}_0$ *the null space. Then* $g(t)$ *has a fixed point.*

P r o o f. The method for finding fixed points of a 1-parameter transformation group $\exp(tX)$ is to calculate the kernel of the generator X. The solution of

$$\begin{pmatrix} 0 & 0 \\ \boldsymbol{b} & B \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{o} \end{pmatrix}$$

is the uniquely defined point $(1, -B^{-1}\boldsymbol{b})$, since B has rank n. Inserting the solution into the exponential formula for $g(t)$ proves the statement. □

Corollary 15. *Any* 1*-parameter transformation group* $g(t) = \exp(tX(\boldsymbol{b}, B))$ *with* $\operatorname{rank} B = n$ *of the Euclidean space* $\boldsymbol{E}^n$*,* $n = 2k$*, is conjugated to a subgroup with a matrix representation*

$$g(t) \hat{=} \begin{pmatrix} D(a_1 t) & 0 & \ldots & 0 \\ 0 & D(a_2 t) & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & D(a_k t) \end{pmatrix}, \tag{46}$$

where the diagonal blocks denote the rotation matrix

$$D(a_j t) = \begin{pmatrix} \cos(a_j t) & -\sin(a_j t) \\ \sin(a_j t) & \cos(a_j t) \end{pmatrix}, t \in \boldsymbol{R}, \tag{47}$$

and the a_j *satisfy* (45) *with* $2k = n$*.*

P r o o f. We take the uniquely defined fixed point as the origion. Since then $g(t)\boldsymbol{o} = \boldsymbol{o}$, we omit the first row and the first column of the matrix and write down only the transformation for the vector coordinates coinciding with that of the point coordinates. Applying Proposition 14 we get in any of the components appearing in (43) (with $\boldsymbol{W}_0 0\boldsymbol{o}$) the corresponding rotation group, and (47) follows.□

Now generalizing the 4-dimensional case one easily proves

Corollary 16. *Any curve of constant curvatures of rank $n = 2k$ in the Euclidean space $\boldsymbol{E}^n$ is congruent to a curve with the parameter representation*

$$\boldsymbol{x}(t) = \begin{pmatrix} D(a_1 t) & 0 & \dots & 0 \\ 0 & D(a_2 t) & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & D(a_k t) \end{pmatrix} \begin{pmatrix} r_1 \\ 0 \\ r_2 \\ 0 \\ \vdots \\ r_k \\ 0 \end{pmatrix}, t \in \boldsymbol{R}, \tag{48}$$

with $a_1 = 1$ and the a_j satisfying (43). □

Since the homomorphism parameter is defined up to a constant parameter only we may assume $a_1 = 1$. Since any orbit of a subgroup with rank $B < n$ belongs to a hyperplane it may not be a curve with rank n. Therefore we may apply Corollary 15 for the generating group of the curve of constant curvatures of maximal rank. We remark that not all the curves (48) have maximal rank; if $a_i = a_{i+1}$ for at least one i the rank diminishes.

For the case of odd dimensions one obtaines

Corollary 17. *The curves with constant curvatures of maximal rank $n = 2k + 1$ in the odd-dimensional Eucldean space $\boldsymbol{E}^n$ are orbits of* 1*-parameter subgroups having a matrix representation of the form*

$$h(t) \hat{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & g(t) & 0 \\ bt & 0 & 1 \end{pmatrix}, \ b \neq 0, \tag{49}$$

where $g(t)$ denotes a block with shape (46) *and the a_i satisfa the conditions mentioned in Corollary 16. As starting points $\boldsymbol{x}_0$ of the orbits one may take the points with coordinates $(r_1, 0, r_2, =, \dots, r_k, 0, 0)$, $r_j > 0$ for $j = 1 \dots, k$.* □

For the prove we remark that the vector part always leaves a 1-dimensional subspace invariant. We span it by the origin and the vector $\boldsymbol{e}_n$ of the orthonormal frame. Since the curve has maximal rank we have $b \neq 0$ and the statement follows. We leave the details to the reader. Clearly, the helices result in the case $n = 3$.

References

[1] E. Cartan. *La théorie des groupes fini et la géométrie différentielle, traitées par la méthode du repère mobile.* Gauthier-Villars, Paris, 1937.

[2] Alfred Gray. *Differentialgeometrie. Klassische Theorie in moderner Darstellung.* Spektrum Akademischer Verlag, Heidelberg, Berlin, Oxford, 1994.

[3] E. Kamke. *Differentialgleichungen reeller Funktionen.* Akademische Verlagsgesellschaft Geest und Portig K.-G., Leipzig, 1952.

[4] A. I. Maltzev. *Grundlagen der linearen Algebra (russ.).* Gosisdat Technisch-Theoretischer Literatur, Moskau, 1956.

[5] A. L. Onishchik and R. Sulanke. *Algebra und Geometrie I.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1986. 2. Auflage.

[6] R. Sulanke. On e. cartan's method of moving frames. *Colloquia Mathem. Soc. J. Bolyai. Differential Geometry, Budapest 1979*, 31:681–704, 1979.

[7] R. Sulanke and P. Wintgen. *Differentialgeometrie und Faserbündel.* VEB Deutscher Verlag der Wissenschaften, Birkhäuser, Berlin, Basel, 1972.

My homepage:
http://www-irm.mathematik.hu-berlin.de/∼sulanke/