Surfaces
in the Euclidean 3-Space

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Preface

In this notebook we develop *Mathematica* tools for applications to Euclidean differential geometry of surfaces. We construct modules for the calculation of all Euclidean invariants like fundamental forms and curvatures for surfaces in the 3-space, and some also for immersions into the n-dimensional Euclidean spaces. This notebook may be considered as a continuation of the notebook EuCurvesv4.nb devoted to Euclidean curve theory contained in the zipfile eudiffgeov4.zip or may be downloaded from my homepage. It may be useful to compare and to complete this notebook studying the pioneering work of Alfred Gray on applications of *Mathematica* to differential geometry, see the book [G06], or the German translation of the first edition [G94]. The third edition [G06] is accompanied by a series of 27 *Mathematica* notebooks corresponding to the chapters of the book, which can be downloaded from the website [G06nb]. Alfred Gray presented Euclidean differential geometry with many applications of *Mathematica*. Writing his book Alfred Gray collected about 200 definitions of concrete curves and as many of concrete surfaces with short descriptions (formulated as usages) and implementations into *Mathematica*. These catalogs are contained in his *Mathematica* packages CURVES.m and SURFS.m available also from my homepage. They open many opportunities to apply the functions and mini programs developed here, to deepen the

insight into the differential geometry of curves and surfaces comparing the calculated invariants, e.g. the curvatures, with the shape of the geometric objects using the graphical tools of *Mathematica*. I am much obliged to Alfred Gray for providing this material and for many discussions introducing me into *Mathematica*. Preparing the fourth version I completely reworked the notebook, introduced many improvements and some corrections intending to make it usable as an interactive introduction to Euclidean differential geometry.

Chapter 1 of the notebook is devoted to E. Cartan's method of moving frames using exterior differential forms. It gives the structure forms and the Frenet formulas including the integrability conditions for surfaces in the Euclidean space. Here I followed the presentation of W. Blascke and H. Reichardt using the differential forms ad hoc as elementary as possible. Also the first and the second fundamental forms of surface theory are treated as fields of bilinear forms not using concepts of tensor analysis. Spherical image and Gauss curvature are the subject of Chapter 2 containing a proof of Gauss famous Theorema Egregium. Chapter 3 is devoted to the second fundamental form of the surface, studying the geometry of curves on the surface, in particular their normal curvature. In particular, ruled surfaces and surfaces of revolution are presented very detailed. Chapter 4 contains the absolute differentiation on the surface, expressed by the Christoffel sym-

bols, as a part of the inner geometry of the surface, describing the properties of the surfaces which are invariant under isometries. It is applied to describe the geodesic curvature of a curve on the surface and to define the geodesics as auto-parallel curves. The basic concepts are formulated avoiding the tensor concept, but admitting a straightforward extension to higher dimensional Riemannian and more general geometries, as shown in the *Mathematica* notebook [RG] about pseudo-Riemannian geometry.

I hope that the notebook will help students to understand the Euclidean differential geometry of surfaces. By its interactive character it may serve as a starting point for further excursions into this interesting field, which joins algebraic and analytic methods to explore the heart of mathematics: the geometry. Studying older or modern textbooks or monographs of classical differential geometry it may be an exciting challenge to express the objects and concepts in *Mathematica* terms aiming to explore or illustrate their properties, to verify well known facts, and, with some luck, to gain new insight.

I shall be very thankful for comments and corrections, please send them to sulanke@mathematik.hu-berlin.de.

Important Hint.

Before starting working with the notebook read the Copyrigth and make the necessary adaption of the Section Initialization to your computer!

Keywords

absolute differential, absolute parallel, arc length, area, asymptotic curves, Christoffel symbols, cone, conical surface, cylindrical surface, developable surface, diffeomorphism, differential, envelope, Euler's formula, evolute, exterior differential, exterior product, flat points, Frenet formulas, fundamental forms, Gauss curvature, Gauss map, geodesic, geodesic curvature, graph, helix, hypersurface, immersion, integrability conditions, integral curvature, involute, isogonal trajectory, Jacobian matrix, line congruences, loxodrome, mean curvature, Meusnier's proposition, minimal surface, moving frame, normal congruence, normal curvature, normal vector, orientation, osculating sphere, parallel displacement, parameter, Plücker's conoid, principal curvatures, principal curves, principal directions, regular, Riemannian curvature tensor, Riemannian manifold, Rückkehrkante, ruled surface, schiebfläche, singular, spherical image, spiral, structure forms, submani-

fold, surface, surface of revolution, surface with constant Gauss curvature, tangent plane, tangent surface, tangent vector, Theorema egregium, torus, total differential, umbilic point, vector field, volume element.

Copyright

Initialization

Preparation

Start Initialization

The Symbols and their Usages

Some Hints to Usages and Help Pages

tensalgv3.m

euvecv2.m

eudiffgeov4

SURFS.m

CURVES.m

Global

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1. The Local Structure Forms of a Surface

1.1. Parameter Representation of Surfaces.Differential

1.1.1. Surfaces. Tangent Planes. Regular and Singular Points

1.1.2. Differential. Linearity. Product Rule.Chain Rule

1.1.3. Translation Surfaces

1.2. An Adapted Orthonormal Moving 3-Frame of the Surface

1.2.1. The Definition

1.2.2. First Fundamental Form. Inner Geometry

1.2.3. Isometries

1.2.4. Transformations. Tangent Bundle

1.3. Differential Forms

In the following we will use E. Cartan's method of moving frames. We apply the calculus of differential forms on the parameter manifold U in an elementary way in dimension 2, as it is presented in the Introduction of W. Blaschke and H. Reichardt [B-R], avoiding tensor algebra. The general calculus of differential forms on manifolds is presented in the notebook

Pseudo – Riemannian Geometry and Tensor Analysis.

2. Spherical Image and Gaussian Curvature

3. The Second Fundamental Form

3.5. Normal and Principal Curvatures

3.6. Graphs of Functions

3.7. Ruled Surfaces: Criterion of Developability

3.8. The Normal Congruences

3.8.1. Normal Congruence of a Regular Curve

3.8.2. Normal Congruence of the Helix

3.8.3. Normal Surfaces of a Surface

3.9. Surfaces of Revolution

Any body produced on a lathe or a potter's wheel is bordered by a surface of revolution. Such surfaces are not only easy to produce; they may serve also as easy to handle examples in surface

geometry. The reason is that the geometric properties of such a surface are defined by the properties of its generating curve, the profile curve or meridian of the surface. Thus two-dimensional problems for surfaces can be reduced to one-dimensional problems for curves, e.g. partial differential equations for surfaces to ordinary differential equations for the meridians, which are much easier to solve. In this section we consider some of these applications.

3.9.1. Definition and Fundamental Invariants

3.9.2. Minimal Surfaces of Revolution

3.9.3. Surfaces of Revolution of Constant Gaussian Curvature

3.9.4. Surfaces of Revolution of Constant Positive Gaussian Curvature

3.9.5. Surfaces of Revolution of Constant Negative Gaussian Curvature

3.9.6. Alfred Gray's Exposition

3.9.6.a. Surfaces of Revolution of Constant Positive Gaussian Curvature

3.9.6.b. Surfaces of Revolution of Constant Negative Gaussian Curvature

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4. Absolute Differential and Christoffel Symbols

In general, the differential and the derivatives of a tangent vector field on a surface are not tangent to the surface. The tangential component of the differential of a vector field is called its absolute differential; we will calculate it in this section. The calculation leads to the Christoffel symbols which depend on the metric (the first fundamental form) of the surface only and so belong to the inner geometry. These concepts play a fundamental role in the n-dimensional differential geometry of Riemannian manifolds and in Einstein's general relativity theory. We begin this section considering m-dimensional submanifolds as immersions of m-dimensional parameter manifolds into the n-dimensional Euclidean space.

4.1. The First Fundamental Form of an Immersion

4.2. Absolute Differential of Tangent Vector Fields

We consider a general immersion smf[f[u...] into the n-dimensional Euclidean space, n = dim, e.g. a hypersurface

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? smf
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In[◦]:= `dim = 4; smf[f][u, v, w]`

4.2.1. The Total Differential of a Tangent Vector Field

4.2.2. The Absolute Differential of a Tangent Vector Field

4.2.3. Absolute Parallel Vector Fields

4.2.4. Parallel Displacement Along Curves

4.2.5. Geodesics as Auto-Parallel Curves