Math. Nachr. 96, 165–183 (1980)

Submanifolds of the Möbius space

Dedicated to Professor Hans Reichardt

By Christina Schiemangk and Rolf Sulanke in Berlin

(Eingegangen am 16. 8. 1979)

1. Introduction

The differential geometry of the Möbius space has seldom been treated in contemporary geometric literature. Here we only mention the papers of O. Kowalski [8] and M. A. Akивис [2], [3] where bibliographies of earlier works can be found. Both authors proved general and complicated fundamental theorems, but, as far as we know, the details of differential geometry of the n-dimensional Möbius space remained an almost undeveloped field. Very detailed investigations only exist for $n \leqq 3$, see W. Blaschke and G. Thomsen [4], T. Takasu [13], but from the modern conceptual point of view they are written in a manner partly unsatisfactory.

In this paper we will investigate immersions $f\colon Y^m \to S^n$ into the Möbius space with the help of E. Cartan's method of moving frames. A geometric interpretation and an exact version of this method was presented in R. Sulanke and A. Švec [11]; the terminology and results of this paper will be used in the following. The necessary preliminaries about the Möbius space are assembled in section 2. Section 3 contains some remarks on the theory of curves in the Möbius space S^n. Section 4 is devoted to the general theory of immersions $f\colon Y^m \to S^n$; we give a characterization of the spheres $S^m \subset S^n$, and for non-spherical immersions we describe a natural reduction of the frame bundles allowing us to define their first and second fundamental forms in a Möbius-invariant manner. In the rest of the paper we consider surfaces $f\colon Y^2 \to S^4$ in the 4-dimensional sphere. Of course one could prove a fundamental theorem for hypersurfaces of S^n too, see M. A. Akивис [1]. In section 5 we construct a canonical moving frame for generic immersions and prove a fundamental theorem for them. In particular we characterize the immersions lying in an $S^3 \subset S^4$. Section 6 is devoted to the non-generic case, the so-called M-isoclinic surfaces. We obtain canonical frames and a fundamental theorem for most of them, but there exists a subclass of surfaces not admitting a canonical framing. We prove that all immersions of this subclass are locally Möbius-equivalent to the Veronese surface, and that all immersions of the sphere belonging to this subclass are Möbius-equivalent to this surface. Partly the paper contains results given in the unpublished thesis of Ch. Schiemangk [10], which are presented here in a revised form.

2. Möbius geometry

Let V^{n+2} be an $(n+2)$-dimensional vector space over the real field $\mathbf{R}$, and $\langle , \rangle$ a non-degenerate symmetric bilinear form of index 1 on V. Then we have the corresponding projective space P^{n+1} and the canonical map

$$\pi : \mathfrak{x} \in V - \{\mathfrak{o}\} \mapsto x = \pi(\mathfrak{x}) := \{\mathfrak{x}\lambda\}_{\lambda \in \mathbf{R}^*} \in P^{n+1}, \tag{2.1}$$

where $\mathbf{R}^*$ denotes the multiplicative group of $\mathbf{R}$. The isotropic cone $J \subset V$ is defined by

$$J := \{\mathfrak{x} \in V \mid \langle \mathfrak{x}, \mathfrak{x} \rangle = 0, \text{ and } \mathfrak{x} \neq \mathfrak{o}\}; \tag{2.2}$$

it is projected by π on the sphere $S^n = \pi(J)$, considered as a hyperquadric in P^{n+1}. By $\mathbf{O}(n+1, 1)$ we denote the pseudo-orthogonal group of V^{n+2} defined as the set of all those $g \in \mathbf{GL}(V)$ leaving the scalar product invariant. Obviously, $\mathbf{O}(n+1, 1)$ acts transitively, but not effectively on S^n, the action being defined by $g\pi(\mathfrak{x}) = \pi(g\mathfrak{x})$, $\mathfrak{x} \in J$, $g \in \mathbf{O}(n+1, 1)$. The kernel of this action is $\{e, -e\}$, $e = id_V$. It is well known that $\mathbf{O}(n+1, 1)$ has four connected components, see e.g. P. K. Raschewski [9], § 50. Let us distinguish one of the connected components J^+ of the isotropic cone J. Then the Möbius *group* $\mathfrak{G}_n := \mathbf{O}(n+1, 1)/\{e, -e\}$ can be identified with the isotropy group of J^+, consisting of two components of $\mathbf{O}(n+1, 1)$, namely the component of e, $\mathbf{S}\mathfrak{G}_n := \mathbf{O}(n+1, 1)_0$ and its complement $\mathfrak{G}_n - \mathbf{S}\mathfrak{G}_n$; an element $g \in \mathfrak{G}_n$ preserves the orientation of S^n if and only if $g \in \mathbf{S}\mathfrak{G}_n$. In the following we will represent the points $x \in S^n$ by $\mathfrak{x} \in J^+$ only, $\pi(\mathfrak{x}) = x$, and we will assume $\mathbf{S}\mathfrak{G}_n \subset \mathfrak{G}_n \subset \mathbf{O}(n+1, 1)$ in the manner described above. Then both groups $\mathbf{S}\mathfrak{G}_n$ and $\mathfrak{G}_n$ act transitively and effectively on S^n.

The transformation group $[\mathfrak{G}_n, S^n]$ is called the Möbius *space*, and the Möbius *geometry* is concerned with the properties of figures and objects of S^n invariant under $\mathfrak{G}_n$. It is well known that for $n > 1$ the group $\mathfrak{G}_n$ can be defined as the group of all those diffeomorphisms of S^n transforming hyperspheres $S^{n-1} \subset S^n$ in hyperspheres, or equivalently as the group of conformal transformations of the sphere S^n equipped with a Riemannian structure of constant positive sectional curvature.

To obtain a matrix representation of $\mathfrak{G}_n$ adapted to the purposes of Möbius geometry one uses *i.-o. bases* (read *isotropic-orthogonal bases*) in V^{n+2}. A base $(\mathfrak{a}_i)$, $i = 0, 1, \ldots, n+1$, of V^{n+2} is called an i.-o. base, if the scalar product with respect to it has the form

$$\langle \mathfrak{x}, \mathfrak{y} \rangle = \sum_{k=1}^{n} x^k y^k - x^0 y^{n+1} - x^{n+1} y^0, \tag{2.3}$$

or, equivalently, the vectors $\mathfrak{a}_i$ fulfil the conditions

$$\langle \mathfrak{a}_k, \mathfrak{a}_l \rangle = \delta_{k,l} \tag{2.4}$$

$$\langle \mathfrak{a}_0, \mathfrak{a}_0 \rangle = \langle \mathfrak{a}_{n+1}, \mathfrak{a}_{n+1} \rangle = \langle \mathfrak{a}_0, \mathfrak{a}_k \rangle = \langle \mathfrak{a}_l, \mathfrak{a}_{n+1} \rangle = 0 \tag{2.5}$$

for $k, l = 1, \ldots, n$,

$$\langle \mathfrak{a}_0, \mathfrak{a}_{n+1} \rangle = -1. \tag{2.6}$$

The equations

$$\mathfrak{a}_0 = \frac{(\mathfrak{e}_0 - \mathfrak{e}_{n+1})}{\sqrt{2}}, \qquad \mathfrak{a}_{n+1} = \frac{(\mathfrak{e}_0 + \mathfrak{e}_{n+1})}{\sqrt{2}} \tag{2.7}$$

$$\mathfrak{a}_k = \mathfrak{e}_k, \qquad k = 1, \ldots, n,$$

$$\mathfrak{e}_0 = \frac{(\mathfrak{a}_0 + \mathfrak{a}_{n+1})}{\sqrt{2}}, \qquad \mathfrak{e}_{n+1} = \frac{(\mathfrak{a}_{n+1} - \mathfrak{a}_0)}{\sqrt{2}}$$

establish a 1-1-correspondence between i.-o. bases and p.-o. bases (*pseudo-orthogonal bases*) $(\mathfrak{e}_i)$ with

(2.8) $\langle \mathfrak{e}_0, \mathfrak{e}_0 \rangle = -1; \quad \langle \mathfrak{e}_i, \mathfrak{e}_j \rangle = \delta_{ij} \quad \text{for} \quad (i, j) \neq (0, 0),$

$i, j = 0, \ldots, n+1$. Let us now fix an i.-o. base $(\mathfrak{a}_i)$. Then corresponds to any endomorphism X of V a uniquely defined matrix $(\vartheta^i_j) \in M_{n+2}(\mathbf{R})$ defined by

(2.9) $X\mathfrak{a}_j = \mathfrak{a}_i \vartheta^i_j, \quad i, j = 0, \ldots, n+1;$

here and in the following we use the sum convention. In particular we identify the element $g \in \mathfrak{G}_n$ with the matrix (g^i_j) corresponding to the endomorphism g by (2.9), which gives us the desired matrix representation of $\mathfrak{G}_n$. An endomorphism X belongs to the LIE algebra $\mathfrak{g}_n = \mathfrak{o}(n+1, 1)$ if and only if it fulfils the condition

(2.10) $\langle X\mathfrak{x}, \mathfrak{y} \rangle + \langle \mathfrak{x}, X\mathfrak{y} \rangle = 0, \quad \mathfrak{x}, \mathfrak{y} \in V^{n+2}.$

So this LIE algebra can be identified with the matrix LIE algebra $\mathfrak{g}_n \subset \mathfrak{gl}(n+2, \mathbf{R})$ consisting of all matrices of the form (see S. KOBAYASHI [6], p. 134)

$$(2.11) \qquad (\vartheta^i_j) = \begin{pmatrix} \vartheta^0 & {}^t\tau^0 & 0 \\ \vartheta & D & \tau^0 \\ 0 & {}^t\vartheta & -\vartheta^0 \end{pmatrix}, \quad \begin{array}{l} \vartheta^0 \in \mathbf{R} \\ \vartheta = (\vartheta^k), \quad \tau^0 = (\vartheta^0_k) \in \mathbf{R}^n, \\ D = (\vartheta^k_l) \in \mathfrak{o}(n), \end{array}$$

here (ϑ^k) denotes a column-n-vector, and ${}^tB$ the matrix transposed to B. Considering the ϑ^i_j as 1-forms on $\mathfrak{G}_n$, the left invariant structure form of $\mathfrak{G}_n$ can be written as $\omega = g^{-1}\, dg = (\vartheta^i_j)$. The MAURER-CARTAN *structure equations of* $\mathfrak{G}_n$ are given by

$$(2.12) \qquad \begin{aligned} d\vartheta^0 &= -\vartheta^0_k \wedge \vartheta^k, \\ d\vartheta^l &= -\vartheta^l \wedge \vartheta^0 - \vartheta^l_k \wedge \vartheta^k, \\ d\vartheta^k_l &= -\vartheta^k \wedge \vartheta^0_l - \vartheta^k_h \wedge \vartheta^h_l - \vartheta^0_k \wedge \vartheta^l, \\ d\vartheta^0_l &= -\vartheta^0 \wedge \vartheta^0_l - \vartheta^0_k \wedge \vartheta^k_l, \qquad h, k, l = 1, \ldots, n. \end{aligned}$$

This can easily be proved remembering that the structure form of $\mathfrak{G}_n$ is induced by that of $\mathbf{GL}(n+2, \mathbf{R})$:

$$\omega_{\mathfrak{G}_n} = \iota^* \omega_{\mathbf{GL}(n+2, \mathbf{R})},$$

where $\iota\colon \mathfrak{G}_n \to \mathbf{GL}(n+2, \mathbf{R})$ denotes the inclusion; then the structure equation of $\mathbf{GL}(n+2, \mathbf{R})$

(2.13) $d\vartheta^i_j = -\vartheta^i_p \wedge \vartheta^p_j, \quad i, j, p = 0, \ldots, n+1,$

immediately leads to (2.12).

Now we calculate the isotropy group $\mathbf{H}_0 \subset \mathfrak{G}_n$ of the point $o = \pi(\mathfrak{a}_0) \in S^n$, $\mathfrak{a}_0 \in J^+$. Obviously we have $g \in \mathbf{H}_0$ if and only if $g\mathfrak{a}_0 = \mathfrak{a}_0 \mu^{-1}$, $\mu > 0$ because $\mathfrak{G}_n J^+ \subset J^+$. Noting that

(2.14) $g \in \mathbf{O}(n+1, 1) \leftrightarrow g \in \mathbf{GL}(n+2, \mathbf{R}) \quad \text{and} \quad \langle g\mathfrak{a}_i, g\mathfrak{a}_j \rangle = \langle \mathfrak{a}_i, \mathfrak{a}_j \rangle,$

we obtain $\mathbf{H}_0$ as the matrix group

$$(2.15) \qquad \mathbf{H}_0 = \left\{ \begin{pmatrix} \mu^{-1} & {}^t\mathfrak{c} & b \\ o & A & A\mathfrak{c}\mu \\ 0 & o & \mu \end{pmatrix} \middle| \begin{array}{l} A \in \mathbf{O}(n), \\ \mu \in \mathbf{R}^+, \qquad \mathfrak{c} \in \mathbf{R}^n, \\ b = \frac{1}{2} \langle \mathfrak{c}, \mathfrak{c} \rangle \mu \end{array} \right\};$$

$\langle , \rangle$ denotes here the standard scalar product in the space $\mathbf{R}^n$ of column-n-vectors, and $\mathbf{R}^+$ the multiplicative group of the positive real numbers. The Lie algebra $\mathbf{h}_0$ of $\mathbf{H}_0$ is defined by the system of linear equations

$$(2.16) \qquad \vartheta^k = 0, \qquad k = 1, \ldots, n;$$

at the same time (2.16) can be considered as the involutive system of Pfaffian equations having the fibres of the principal fibre bundle $p: \mathfrak{G}_n \to S^n = \mathfrak{G}_n/\mathbf{H}_0$ as (non connected) integral manifolds.

Remark. In elementary differential geometry one identifies $\mathfrak{G}_n$ with a set of i.-o. frames ($(\mathfrak{a}_i)$ fixed) via the map

$$(2.17) \qquad g \mapsto (\mathfrak{b}_i) = (\mathfrak{a}_j g_i^j), \qquad g = (g_i^j) \in \mathfrak{G}_n.$$

The structure form $\omega = (\vartheta_j^i)$ appears as the coefficient matrix in the derivation equations

$$(2.18) \qquad d\mathfrak{b}_j = \mathfrak{b}_i \vartheta_j^i,$$

where $d\langle \mathfrak{b}_i, \mathfrak{b}_j \rangle = 0$ yields (2.11), i.e. the relations

$$(2.19) \qquad \vartheta^0 = \vartheta_0^0 = -\vartheta_{n+1}^{n+1}, \qquad \vartheta_k^0 = \vartheta_{n+1}^k, \qquad \vartheta^k = \vartheta_0^k = \vartheta_k^{n+1},$$
$$\vartheta_l^k + \vartheta_k^l = 0, \qquad k, l = 1, \ldots, n,$$

characterizing $\mathfrak{g}_n$ as Lie subalgebra of $\mathbf{gl}(n + 2, \mathbf{R})$. One gets the structure equations (2.12) (or (2.13)) by exterior derivation $d(d\mathfrak{b}_j) = \mathfrak{o}$ of (2.18) as 'integrability conditions'. The principal fibre bundle $p: \mathfrak{G}_n \to S^n$ can be interpreted as the manifold of "Möbius *frames*" of S^n; at each point $x \in S^n$ the fibre at x

$$p^{-1}(x) = \{(\mathfrak{b}_i) \mid (\mathfrak{b}_i) \in \mathfrak{G}_n \text{ with } \pi(\mathfrak{b}_0) = x\}$$

consists of the set of i.-o. frames $(\mathfrak{b}_i)$ whose first vector $\mathfrak{b}_0$ represents x.

For simplicity let us write $p = (dp)_e: \mathfrak{g}_n \to \mathfrak{g}_n/\mathbf{h}_0$. The fibre bundle and Lie group notations not explained in this paper can be found in R. Sulanke, P. Wintgen [12].

Using the *canonical form θ of the homogeneous space*

$$(2.20) \qquad \theta := p \circ \omega = (\vartheta^k) \in \mathfrak{g}_n/\mathbf{h}_0, \qquad k = 1, \ldots, n,$$

the identification $T_0S^n = \mathfrak{g}_n/\mathbf{h}_0$ and the equivalence $dl_h \approx \dot{A}d(h)$, where $\dot{A}d$ denotes the representation of $\mathbf{H}_0$ on $\mathfrak{g}_n/\mathbf{h}_0$ defined by

$$(2.21) \qquad \dot{A}d(h) \circ p = p \circ Ad(h), \qquad h \in \mathbf{H}_0,$$

one obtaines the *linear isotropy representation*

(2.22) $$h = h(A, \mathfrak{c}, \mu) \in \mathbf{H}_0, \qquad \theta \in \mathfrak{g}_n/\mathfrak{h}_0 \to \dot{A}d(h)\,\theta = (A\mu)\,\theta;$$

here $h(A, \mathfrak{c}, \mu)$ means the matrix in (2.15). Therefore the kernel of $\dot{A}d$ is the group

(2.23) $$\ker \dot{A}d = \left\{ \begin{pmatrix} 1 & {}^t\mathfrak{c} & \langle \mathfrak{c}, \mathfrak{c} \rangle \frac{1}{2} \\ \mathfrak{o} & I_n & \mathfrak{c} \\ 0 & \mathfrak{o} & 1 \end{pmatrix} \middle| \ \mathfrak{c} \in \mathbf{R}^n \right\} \approx \mathbf{R}^n .$$

Let $U \subset S^n$ be an open set and

(2.24) $$z : x \in U \mapsto z(x) = (\mathfrak{b}_i(x)) \in \mathfrak{G}_n$$

a local section of $p : \mathfrak{G}_n \to S^n$ over U, i.e. a locally defined moving frame. Then $(\sigma^k) := (z^*\vartheta^k)$, $k = 1, \ldots, n$, defines a coframe on U, and the bilinear from

(2.25) $$\varphi_x(\mathfrak{s}, \mathfrak{t}) := \sum_k \sigma_x^k(\mathfrak{s})\, \sigma_x^k(\mathfrak{t}), \qquad x \in U, \qquad \mathfrak{s}, \mathfrak{t}, \in T_xS^n,$$

changes under a transformation of the moving frame

$$\tilde{z}(x) = h(x) \times z(x), \qquad \text{i.e.} \qquad \tilde{\mathfrak{b}}_i(x) = \mathfrak{b}_j(x)\, \overset{-1}{h}{}^j_i(x), \qquad x \in U \cap \tilde{U},$$

according to

(2.26) $$\tilde{\varphi}_x(\mathfrak{s}, \mathfrak{t}) = \mu^2(h(x))\, \varphi_x(\mathfrak{s}, \mathfrak{t}),$$

where μ is the character of $\mathbf{H}_0$ defined by

(2.27) $$h = h(A, \mathfrak{c}, \mu) \in \mathbf{H}_0 \mapsto \mu(h) = \mu \in \mathbf{R}^+ .$$

Therefore φ yields a *conformal structure on* S^n, which can be used to define angles. It can easily be proved that this conformal structure coincides with the structure defined by a RIEMANNian metric of constant sectional curvature on S^n.

3. Remarks on the theory of curves

Applying E. CARTAN's method of moving frames we proved FRENET formulas for generic curves $f\colon Y^1 \to S^n$. The results are essentially the same as those obtained by Л. Л. ВЕРБИЦКИЙ [14] using tensorial methods. Especially, our natural conformal parameter coincides with that parameter used in [14] and discovered by H. LIEBMANN, Münchener Berichte 1923. We only mention the following formula expressing the differential ds of the conformal parameter s by EUCLIDian invariants:

(3.1) $$ds = (\dot{k}^2 + \varkappa^2 k^2)^{\frac{1}{4}}\, dt$$

where t is the EUCLIDian arc length, k, $\varkappa$ are the first and second curvature of the curve

(3.2) $$f_1 := c \circ f\colon Y^1 \underset{f}{\to} S^n \underset{c}{\to} E^{n+1}$$

and

$$\dot{k} := dk/dt.$$

For the dimension $n = 3$ formula (3.1) was found by T. Takasu in 1928, compare [13], (520), p. 123. Obviously, the conformal parameter s is well defined for curves with $\dot{k}^2 + \varkappa^2 k^2 \neq 0$ only. The geometrical meaning of this condition is that the osculating circles osculate the curve exactly of order two. The "generic" curves in S^n are characterized up to $\mathfrak{G}_n$-congruence by the form ds and $n - 1$ curvatures coinciding with those given by Л. Л. Верблицкий [14], and in the case $n = 3$ already found by E. Vessiot [15] in 1925.

4. The Möbius structure of an immersion

In this section we consider immersions $f\colon Y^m \to S^n$ of connected m-dimensional manifolds Y^m, $m \geqq 2$, into the Möbius space S^n. According to E. Cartan's method of moving frames we would have to start with the bundle E_0:

$$\begin{array}{ccc} E_0 & \xrightarrow{\ \bar{f}\ } & \mathfrak{G}_n \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\ Y^m & \xrightarrow{\ f\ } & S^n = \mathfrak{G}_n/\mathbf{H}_0 \end{array}$$

induced by f from the bundle of Möbius frame over S^n, i.e. with the $\mathfrak{g}_n$, $\mathbf{H}_0$-structure $E_0(Y^m, \omega_0)$, $\omega_0 = \bar{f}^*\omega$, of the immersion f, compare R. Sulanke, A. Švec [11]. Yet, since $\mathbf{SG}_n$ acts transitively on the Grassmann bundle $G_m(S^n)$ of m-dimensional subspaces of the tangential spaces of S^n, we can immediately begin with the bundle $E_1 \subset E_0$ of the tangential frames of f, defining the $\mathfrak{g}_n$, $\mathbf{H}_1$-structure $E_1(Y^m, \omega_1) \xrightarrow{\bar{f}_1} \mathfrak{G}_n$ of f, where $\mathbf{H}_1$ denotes the isotropy group of the space $\mathscr{L}(d\pi(\mathfrak{a}_1), \ldots, d\pi(\mathfrak{a}_m)) \subset T_0(S^n)$, and $\omega_1 = \bar{f}_1^*\omega$ is the first order structure form. Using the identification (2.17) we obtain the bundle $p_1 : E_1 \to Y^m$ as the set of all i.-o. frames $z = (\mathfrak{b}_i)$, $i = 0, \ldots, n + 1$, with

$$z \in p_1^{-1}(y) \leftrightarrow \pi(\mathfrak{b}_0) = f(y), \qquad d\pi(\mathfrak{b}_\alpha) \in df_y(T_y), \qquad \alpha = 1, \ldots, m. \tag{4.1}$$

Defining

$$\mathfrak{c}_k = (d\pi)_{\mathfrak{b}_0}(\mathfrak{b}_k), \qquad k = 1, \ldots, n, \tag{4.2}$$

we get an orthogonal base $(\mathfrak{c}_\alpha)$, $\alpha = 1, \ldots, m$, of the *tangential space at* $f(y) = \pi(\mathfrak{b}_0)$, and an orthogonal base $(\mathfrak{c}_\varkappa)$, $\varkappa = m + 1, \ldots, n$, of the *normal space*

$$N_y := \big(df_y(T_y)\big)^\perp \subset T_{f(y)}(S^n). \tag{4.3}$$

In the following we identify $T_y = df_y(T_y)$ via the map df_y. The structure form $\omega_1 = \bar{f}_1^*\omega$ is a $\mathfrak{g}_n$-valued 1-form on E_1

$$\omega_1 = (\omega^i_j) \text{ with } \omega^\varkappa = 0 \quad \text{for} \quad \varkappa = m + 1, \ldots, n; \tag{4.4}$$

here we use the notations corresponding to (2.11), (2.19) with ϑ replaced by ω. The forms ω^i_j fulfil the corresponding integrability conditions (2.12). The structure group $\mathbf{H}_1$ of E_1 consists of all elements $h \in \mathbf{H}_0$, see (2.15), of the shape $h = h(A, \mathfrak{c}, \mu)$ with

$$A = A_1 \oplus A_2, \qquad A_1 = (h^\alpha_\beta) \in 0(m), \qquad A_2 = (h^\varkappa_\lambda) \in 0(n - m). \tag{4.5}$$

The bilinear form (2.25) induces scalar products over T_y, N_y which can be used to define angles, but no length. The Lie algebra $\mathfrak{h}_1 \subset \mathfrak{h}_0$ is defined as subalgebra of $\mathfrak{h}_0$ by the equations

(4.6) $$\vartheta^\varkappa_\alpha = 0, \qquad \alpha = 1, \ldots, m; \qquad \varkappa = m+1, \ldots, n.$$

So the *coefficients of the first order* $c^\varkappa_{\alpha\beta}(z)$, $z \in E_1$, are given by

(4.7) $$\omega^\varkappa_\alpha = c^\varkappa_{\alpha\beta}\omega^\beta.$$

From the second integrability condition (2.12) and (4.4) we get

(4.8) $$0 = d\omega^\varkappa = -\omega^\varkappa_\alpha \wedge \omega^\alpha.$$

By E. Cartan's Lemma we obtain the symmetry

(4.9) $$c^\varkappa_{\alpha\beta} = c^\varkappa_{\beta\alpha}.$$

The transformation rule for the coefficients and forms under the action of the structure group $\mathbf{H}_1$ of E_1 can be calculated as follows: We put $(\tilde{\mathfrak{b}}_i) = h^{-1} \times (\mathfrak{b}_i)$, i.e.

(4.10) $$\tilde{\mathfrak{b}}_0 = \mathfrak{b}_0 \mu^{-1}, \qquad \tilde{\mathfrak{b}}_\alpha = \mathfrak{b}_\beta h^\beta_\alpha + \mathfrak{b}_0 h^0_\alpha, \qquad \tilde{\mathfrak{b}}_\varkappa = \mathfrak{b}_\lambda h^\lambda_\varkappa + \mathfrak{b}_0 h^0_\varkappa,$$

differentiate these vectors and use the relations (2.3)—(2.5) for $(\mathfrak{b}_i)$, $(\tilde{\mathfrak{b}}_i)$. So it results

(4.11) $$\bar{\omega}^\alpha := \omega^\alpha(h^{-1} \times z) = \mu^{-1} \overset{-1}{h}{}^\alpha_\beta \omega^\beta(z),$$

(4.12) $$\bar{c}^\varkappa_{\alpha\gamma} := c^\varkappa_{\alpha\gamma}(h^{-1} \times z) = \mu\left(\overset{-1}{h}{}^\varkappa c^\lambda_{\beta\delta}(z)\, h^\beta_\alpha h^\delta_\gamma - h^0_\varkappa \delta_{\alpha\gamma}\right).$$

In (4.12) we take the trace and obtain for $\varkappa = m+1, \ldots, n$:

(4.13) $$\sum_{\alpha=1}^{m} \bar{c}^\varkappa_{\alpha\alpha} = \mu\left(\overset{-1}{h}{}^\varkappa \sum_{\alpha=1}^{m} c^\lambda_{\alpha\alpha} - m h^0_\varkappa\right).$$

So we see: it is always possible to find frames $z \in p_1^{-1}(y)$ with

(4.14) $$\sum_{\alpha=1}^{m} c^\varkappa_{\alpha\alpha}(z) = 0, \qquad \varkappa = m+1, \ldots, n;$$

(4.14) is invariant under the subgroup $\tilde{\mathbf{H}}_1 \subset \mathbf{H}_1$ of all $h \in \mathbf{H}_1$ with

(4.15) $$h^0_\varkappa = 0, \qquad \varkappa = m+1, \ldots, n.$$

For $h \in \tilde{\mathbf{H}}_1$ we obtain from (4.12), (4.15) and the orthogonality of A_1, A_2: If we define

(4.16) $$S^2(z) := \sum_{\varkappa,\alpha,\gamma} \left(c^\varkappa_{\alpha\gamma}(z)\right)^2,$$

we have the following transformation rule for S^2:

(4.17) $$S^2(h^{-1} \times z) = \mu^2 S^2(z), \qquad h \in \tilde{\mathbf{H}}_1.$$

An immersion $f: Y^m \to S^n$ is called *M-flat at a point y* if there exists a frame $z \in E_1$ with $p_1(z) = Y$ and $S^2(z) = 0$; it is called an *M-flat immersion* if it is M-flat at each point $y \in Y^m$. Obviously M-flatness is a $\mathfrak{G}_n$-invariant property. We prove

Proposition 4.1. *Let $f: Y^m \to S^n$ be an immersion, $m \geqq 2$, Y^m connected. Then there exists a sphere $S^m \subset S^n$ with $f(Y^m) \subseteqq S^m$ if and only if f is M-flat.*

Proof. If f is M-flat, we have $\omega_\alpha^\varkappa(z) = 0$ for all $z \in \tilde{E}_1$, and, by (4.4) and (2.19), $\omega^\varkappa = \omega_\varkappa^{n+1} = 0$. From the third integrability condition (2.12) we conclude

$$0 = \omega_\varkappa^0 \wedge \omega^\alpha = c_{\varkappa\beta}^0(y)\, \omega^\beta \wedge \omega^\alpha$$

for any moving frame $y \in U \mapsto z(y) \in \tilde{E}_1$, $U \subset Y^m$ open; here and sometimes in the following we put $z^*\omega_j^i = \omega_j^i$ by abuse of notation. Since $m \geqq 2$ we get $\omega_\varkappa^0 = 0$, and this implies $d\mathfrak{b}_\varkappa = \mathfrak{b}_\lambda \omega_\varkappa^\lambda$; λ, $\varkappa = m+1, \ldots, n$. Therefore the $(n-m)$-vector $\mathfrak{b}_{m+1} \wedge \cdots \wedge \mathfrak{b}_n$ is constant:

$$d(\mathfrak{b}_{m+1} \wedge \cdots \wedge \mathfrak{b}_n) = \mathfrak{b}_{m+1} \wedge \cdots \wedge \mathfrak{b}_n \sum_{\lambda=m+1}^{n} \omega_\lambda^\lambda = 0,$$

since Y is connected. Thus the vector space $\mathfrak{L}(\mathfrak{b}_{m+1}, \ldots, \mathfrak{b}_n)$ and also its orthogonal complement $\mathfrak{L}(\mathfrak{b}_0, \ldots, \mathfrak{b}_m, \mathfrak{b}_{n+1})$ are constant on Y. The intersection of the latter with J^+ determines the sphere S^m with $f(y) = \pi(\mathfrak{b}_0(y)) \in S^m$ for all $y \in Y$. The converse statement follows from the next example.

Example. Let $(\mathfrak{a}_i)$, $i = 0, \ldots, n+1$, be a fixed i.o. base of V^{n+2}. Put $\mathfrak{x} = \mathfrak{a}_\alpha x^\alpha$, $(x^\alpha) \in \mathbf{R}^m$. Then

$$(x^\alpha) \in \mathbf{R}^m \mapsto \mathfrak{b}_0(x^\alpha) := \mathfrak{a}_0 + \mathfrak{x} + \mathfrak{a}_{n+1} \frac{1}{2} \langle \mathfrak{x}, \mathfrak{x} \rangle,$$

$$f(x^\alpha) := \pi(\mathfrak{b}_0(x^\alpha)) \tag{4.18}$$

defines an injective immersion $f: \mathbf{R}^m \to S^n$ with $f(\mathbf{R}^m) = S^m - \{x_\infty\}$, $x_\infty = \pi(\mathfrak{a}_{n+1})$. Obviously any m-sphere can be locally represented in such a manner. We define

$$\mathfrak{b}_\alpha(x) = \mathfrak{b}_{n+1} x^\alpha + \mathfrak{a}_\alpha, \qquad \mathfrak{b}_\varkappa = \mathfrak{a}_\varkappa, \qquad \mathfrak{b}_{n+1} = \mathfrak{a}_{n+1}.$$

Then $(\mathfrak{b}_i(x))$ is easily proved to be a moving i.o. frame of f with respect to E_1. Since $d\mathfrak{b}_\varkappa = d\mathfrak{a}_\varkappa = \mathfrak{o}$ the immersion f is M-flat. □

Corollary 4.1. *Let Y^m be a compact, connected manifold with $m \geqq 2$ and $f: Y^m \to S^n$ an M-flat immersion. Then $f(Y^m) = S^m$ is an m-sphere in S^n, and Y^m is diffeomorphic to S^m.*

Proof. By proposition 4.1 there exists an m-sphere S^m with $f(Y^m) \subset S^m$. Since $f(Y^m)$ is open and closed in S^m we have $f(Y^m) = S^m$. Thus $f: Y^m \to S^m$ is a covering map, which must be a diffeomorphism. □

Now let us suppose $f: Y^m \to S^n$ to be an immersion without M-flat points, $m \geqq 2$. We choose a constant $c > 0$ and define

$$z \in \bar{E}_1 \Leftrightarrow z \in E_1, \quad z \text{ fulfils (4.14) and } S^2(z) = c. \tag{4.19}$$

Obviously, $\bar{E}_1$ is a reduction of E_1 to the structure group

$$\bar{\mathbf{H}}_1 = \{h \in \mathbf{H}_1 \mid \mu = 1,\ h_\varkappa^0 = 0 \text{ for } \varkappa = m+1, \ldots, n\}. \tag{4.20}$$

The structure form is $\overline{\omega}_1 := \omega_1 \mid T\overline{E}_1$, and $\overline{E}_1(Y^m, \overline{\omega}_1)$ is a $\mathfrak{g}_n$, $\overline{\mathbf{H}}_1$-structure on Y^m, called the MÖBIUS *structure of the immersion f*. From the left invariance of the structure form ω of $\mathfrak{G}_n$ we see immediately that the MÖBIUS structures of $\mathfrak{G}_n$-equivalent immersions are isomorphic, i.e. $\overline{E}_1$ is a natural reduction of E_1. We summarize the properties of $\overline{E}_1$ in the following

Proposition 4.2. *Let $\overline{E}_1(Y^m, \overline{\omega}_1)$ be the Möbius structure of an immersion $f: Y^m \to S^n$, $m \geqq 2$, without M-flat points. Then the induced tangential bundle splits*

$$TS^n \mid Y^m = TY \oplus NY \tag{4.21}$$

into the tangential and the normal bundle of the immersion. Let $U \subset Y$ be open, $z: U \to \overline{E}_1$ a local section of $\overline{E}_1$ and put with abuse of notation

$$\omega_j^i(y) := (z^*\omega_j^i)(y), \qquad y \in U. \tag{4.22}$$

Then: 1. *The field of bilinear forms*

$$\varphi_y(\mathfrak{t}, \mathfrak{s}) := \sum_{\alpha=1}^{m} \omega^\alpha(y, \mathfrak{t})\, \omega^\alpha(y, \mathfrak{s}), \qquad \mathfrak{s}, \mathfrak{t} \in T_y,$$

defines a $\mathfrak{G}_n$-invariant Riemannian metric on Y, called the first Möbius fundamental form of the immersion f.

2. *Let $\mathfrak{u}, \mathfrak{v} \in N_y$ have the coordinate representations $\mathfrak{u} = \mathfrak{c}_\varkappa u^\varkappa$, $\mathfrak{v} = \mathfrak{c}_\varkappa v^\varkappa$, $\varkappa = m + 1, \ldots, n$, with the notation (4.2). Then*

$$\varphi_y^\perp(\mathfrak{u}, \mathfrak{v}) := \sum_{\varkappa=m+1}^{n} u^\varkappa v^\varkappa$$

defines a positive definite scalar product on N_y not depending on the choice of the moving frame in $\overline{E}_1$.

3. *The density $dF_y := \omega^1 \wedge \cdots \wedge \omega^m(y)$ does not depend on the chosen moving frame in $\overline{E}_1$ up to sign; it defines a $\mathfrak{G}_n$-invariant oriented, resp. absolute, volume measure on the oriented, resp. non-oriented, manifold Y.*

4. *The field of symmetric bilinear maps $y \mapsto \alpha_y$ with*

$$\alpha_y: \mathfrak{s}, \mathfrak{t} \in T_y \mapsto \alpha_y(\mathfrak{s}, \mathfrak{t}) := \mathfrak{c}_\varkappa(y)\, \omega_\alpha^\varkappa(\mathfrak{s})\, \omega^\alpha(\mathfrak{t})$$

does also not depend on the chosen moving frame in E_1; it is called the second Möbius fundamental form of the immersion f.

Proof. The transformation rules (4.11), (4.12) applied to $h \in \mathbf{H}_1$ show that (ω^α) and $(c_{\alpha\gamma}^\varkappa)$ change under orthogonal transformations. Thus 1. and 3. hold true. Equations (4.2), (4.10) and the remark $d\pi(\mathfrak{b}_0) = \mathfrak{o}$ yield the same property for the bases $(\mathfrak{c}_\alpha)$ resp. $(\mathfrak{c}_\varkappa)$. From this the statements 2. and 4. follow. □

We notice that the value of the constant c in (4.19) is not essential; one can choose it in an appropriate manner for each class of immersions.

5. Generic surfaces in S^4

In the remaining two sections of this paper we will only consider immersions $f\colon Y^2 \to S^4$ without M-flat points. Firstly we have to finish the reduction of the $\mathfrak{g}_4$, H_1-structure E_1, that means, we must reduce the Möbius structure $\bar{E}_1$ investigating the transformation rule

$$\bar{c}^{\varkappa}_{\alpha\gamma} = \overset{-1}{h^{\varkappa}_{\lambda}} c^{\lambda}_{\beta\delta} h^{\beta}_{\alpha} h^{\delta}_{\gamma}, \tag{5.1}$$

where $(h^{\alpha}_{\beta}) \in 0(2)$, $(h^{\varkappa}_{\lambda}) \in 0(2)$ are orthogonal transformations of the tangential and normal base. From the symmetry (4.9) and from (4.14) we see that there are only four coefficients, namely

$$c^3_{11} = -c^3_{22}, \qquad c^3_{12} = c^3_{21}, \qquad c^4_{11} = -c^4_{22}, \qquad c^4_{12} = c^4_{21}. \tag{5.2}$$

Choosing $c = 2$ in (4.19) we get

$$(c^3_{11})^2 + (c^3_{12})^2 + (c^4_{11})^2 + (c^4_{12})^2 = 1. \tag{5.3}$$

Let $z(y) = (\mathfrak{b}_i(y))$ be a local moving frame in E_1. By proposition 4.2 we have scalar products in TY, NY, and thus it makes sense to consider vectors of length one in TY: $\mathfrak{s} = \mathfrak{c}_1 \cos\beta + \mathfrak{c}_2 \sin\beta$, and to calculate the length square of their images under the second fundamental form

$$\alpha(\mathfrak{s}, \mathfrak{s}) = \mathfrak{c}_{\varkappa}(c^{\varkappa}_{11} \cos 2\beta + c^{\varkappa}_{12} \sin 2\beta), \tag{5.4}$$

$$|\alpha(\mathfrak{s}, \mathfrak{s})|^2 = \big((c^3_{11})^2 + (c^4_{11})^2\big) \cos^2 2\beta + \big((c^3_{12})^2 + (c^4_{12})^2\big) \sin^2 2\beta + 2(c^3_{11}c^3_{12} + c^4_{11}c^4_{12}) \cos 2\beta \sin 2\beta; \tag{5.5}$$

here we applied (5.2). Now we define: An immersion $f\colon Y^2 \to S^4$ without M-flat points is called *M-isoclinic at a point* $y \in Y^2$, if the function $F(\beta) = |\alpha(\mathfrak{s}, \mathfrak{s})|^2$ defined by (5.5) is constant; f is called *M-isoclinic* if it is M-isoclinic at each $y \in Y^2$. The name M-isoclinic is chosen by a purely formal analogy to the isoclinic minimal immersions $f\colon Y^2 \to E^4$. Obviously, f is M-isoclinic at y if and only if the following conditions are fulfilled:

$$(c^3_{11})^2 + (c^4_{11})^2 = (c^3_{12})^2 + (c^4_{12})^2 = \frac{1}{2} \tag{5.6}$$

$$c^3_{11}c^3_{12} + c^4_{11}c^4_{12} = 0. \tag{5.7}$$

An immersion $f\colon Y^2 \to S^4$ is said to be *generic* if it has neither M-flat nor M-isoclinic points. Considering the function $F(\beta) = Q(s_1, s_2)$, $s_1 = \cos 2\beta$, $s_2 = \sin 2\beta$, as a quadratic form in s_1, s_2 we see that f is generic if and only if at each point y the form $Q(s_1, s_2)$ has two distinct eigenvalues. We define the bundle E_2 to be the set of all frames $z = (\mathfrak{b}_i) \in \bar{E}_1$ having the following properties: 1. With $\mathfrak{c}_i = d\pi(\mathfrak{b}_i)$ we have

$$|\alpha(\mathfrak{c}_1, \mathfrak{c}_1)|_y = \max\big(|\alpha(\mathfrak{s}, \mathfrak{s})| \mid \mathfrak{s} \in T_yY,\ |\mathfrak{s}| = 1\big); \tag{5.8}$$

2. $\mathfrak{c}_3$, $\mathfrak{c}_4$ are chosen in such a manner that

(5.9) $$\alpha(\mathfrak{c}_1, \mathfrak{c}_1) = \mathfrak{c}_3 \cos\vartheta, \qquad \alpha(\mathfrak{c}_1, \mathfrak{c}_2) = \mathfrak{c}_4 \sin\vartheta, \qquad |\vartheta| < \frac{\pi}{4};$$

3. the frame $(\mathfrak{c}_k)$, $k = 1, \ldots, 4$, corresponds to the chosen orientation of S^4.

To prove the existence of such $(\mathfrak{b}_i)$ we first remark that turning $(\mathfrak{c}_1, \mathfrak{c}_2)$, i.e. $(\mathfrak{b}_1, \mathfrak{b}_2)$, we can reach 1. In this frame $\beta = 0$ gives the maximum, and thus $F'(0) = 0$ implies

(5.10) $$\langle\alpha(\mathfrak{c}_1, \mathfrak{c}_1), \alpha(\mathfrak{c}_1, \mathfrak{c}_2)\rangle = c^3_{11}c^3_{12} + c^4_{11}c^3_{12} = 0.$$

Since $\alpha(\mathfrak{c}_1, \mathfrak{c}_1) \neq 0$, we may choose $\mathfrak{c}_3 = d\pi(\mathfrak{b}_3)$ by a rotation in the normal plane such that $\alpha(\mathfrak{c}_1, \mathfrak{c}_1) = \mathfrak{c}_3 a$, $a > 0$, which is equivalent to

(5.11) $$c^3_{11} > 0, \qquad c^4_{11} = 0.$$

Then (5.10) implies

(5.12) $$c^3_{12} = 0,$$

(5.13) $$\alpha(\mathfrak{c}_1, \mathfrak{c}_2) = \mathfrak{c}_4 c^4_{12},$$

where $(\mathfrak{c}_k)$ corresponds to the orientation of S^4. (5.2) leads to

(5.14) $$\alpha(\mathfrak{c}_2, \mathfrak{c}_2) = -\mathfrak{c}_3 c^3_{11}.$$

From (5.5) and (5.8) we get $|c^3_{11}| > |c^4_{12}|$ and (5.3) implies the existence of a unique ϑ with

(5.15) $$c^3_{11} = \cos\vartheta > 0, \qquad c^4_{12} = \sin\vartheta, \qquad |\vartheta| < \frac{\pi}{4}.$$

Thus the existence of moving frames in $\bar{E}_1$ with the required properties is proved. Since now all coefficients of the first order, $c^\varkappa_{\alpha\beta}$, have well defined values (for $z \in E_2$), the bundle E_2 is the $\mathfrak{g}_4$, $\mathbf{H}_2$-structure of the generic immersion f.

Proposition 5.1. *Let E_2 be the bundle of frames of second order of the generic immersion $f\colon Y^2 \to S^4$. Then the structure group $\mathbf{H}_2 \subset \mathbf{H}_1$ is the subgroup of all $h \in \mathbf{H}_1$ of the form*

(5.16) $$h = \begin{pmatrix} 1 & h^1 & h^2 & 0 & 0 & b \\ 0 & \multicolumn{2}{c}{} & & & k^1 \\ 0 & \multicolumn{2}{c}{(\varepsilon^\alpha_\beta)} & \multicolumn{2}{c}{0} & k^2 \\ 0 & & & & & 0 \\ 0 & \multicolumn{2}{c}{0} & \multicolumn{2}{c}{(\varepsilon^\varkappa_\lambda)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \begin{cases} ((\varepsilon^\alpha_\beta), (\varepsilon^\varkappa_\lambda)) \in \Gamma, \\ \Gamma \subset \mathbf{O}(2) \times \mathbf{O}(2) \text{ certain discrete subgroup; } h^1, h^2 \in \mathbf{R} \\ k^\alpha = \varepsilon^\alpha_\beta h^\beta; \\ b = \frac{1}{2}\left((h^1)^2 + (h^2)^2\right). \end{cases}$$

Its Lie *algebra $\mathfrak{h}_2 \subset \mathfrak{h}_1$ is defined as subalgebra of $\mathfrak{h}_1$ by the equations*

(5.17) $$\vartheta^0 = 0, \qquad \vartheta^0_\varkappa = 0, \qquad \vartheta^1_2 = 0, \qquad \vartheta^3_4 = 0.$$

The structure form $\omega_2 = (\omega^i_j)$ *is a matrix the elements of which fulfil relations corresponding to* (2.19) *and*

$$\omega^3 = \omega^4 = 0, \qquad \omega^3_1 = \cos\vartheta\omega^1, \qquad \omega^3_2 = -\cos\vartheta\omega^2,$$
$$\omega^4_1 = \sin\vartheta\omega^2, \qquad \omega^4_2 = \sin\vartheta\omega^1, \quad \text{with} \quad |\vartheta| < \frac{\pi}{4}, \tag{5.18}$$

where the angle $\vartheta = \vartheta(y)$ *is an* $\mathbf{SG}_4$*-invariant of the immersion.*

Proof. The elements of Γ permute the possible values of $(\mathfrak{c}_k)$. Let us denote

$$\delta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If an orientation of T_y should be preserved, we have the subgroup Γ_1 of $\mathbf{SO}(2)$ of order 4 generated by δ as the tangential component $\{(\varepsilon^\alpha_\beta)\}$ of Γ. If $T_y(Y^2)$ is not oriented, $\{(\varepsilon^\alpha_\beta)\}$ will be the subgroup of $\mathbf{O}(2)$ of order 8 generated by δ and σ. At any rate, if $\mathfrak{c}_1$, $\mathfrak{c}_2$ are chosen, the normal base $(\mathfrak{c}_3, \mathfrak{c}_4)$ is uniquely defined: $\mathfrak{c}_3$ by (5.11) and $\mathfrak{c}_4 = \mathfrak{c}_1 \times \mathfrak{c}_2 \times \mathfrak{c}_3$ as orthogonal complement corresponding to the orientation of S^4. By (5.1) the variation of (h^1, h^2) does not influence on the coefficients $c^\varkappa_{\alpha\beta}$ of the first order; therefore $(h^1, h^2) \in \mathbf{R}^2$ remains variable. (5.17) is easily obtained from the restriction defining $\mathbf{H}_2 \subset \mathbf{H}_1$. We notice that the sign of c^4_{12} does not depend on the chosen frame; by elementary calculations one shows that it remains unchanged under the transformations of Γ. Therefore ϑ is uniquely defined by (5.18) (or (5.15)) and yields an $\mathbf{SG}_4$-invariant (of order 2) of the immersion. □

We define the generic immersion f to be *right-winding* (resp. *left-winding*) *at a point* $y \in Y$, if $\vartheta > 0$ (resp. $\vartheta < 0$). Of course, this concept makes sense in an oriented sphere S^4 only.

Proposition 5.2. *Let* $f\colon Y^2 \to S^4$ *be as in proposition* 5.1. *Then the* $\mathfrak{g}_4$, $\mathbf{H}_3$*-structure* E_3 *of* f *is defined to be the set of all frames* $z \in E_2$ *for which* $\omega^0(z) = 0$. *The structure group* $\mathbf{H}_3$ *is the discrete group* Γ *described in the proof of proposition* 5.1. *The forms* (*on* Y^2)

$$\omega^1, \omega^2, \omega^0_k, k = 1, \ldots, 4, \omega^1_2, \omega^3_4 \bmod \Gamma \tag{5.19}$$

together with the angle ϑ *are a complete system of* $\mathbf{SG}_4$*-invariants of the immersion.*

Proof. The form ω^0 changes under the action of $\mathbf{H}_2$ as follows:

$$\tilde\omega^0 = -\langle d\tilde{\mathfrak{b}}_0, \tilde{\mathfrak{b}}_5\rangle = \omega^0 - \omega^1 h^1 - \omega^2 h^2. \tag{5.20}$$

Thus it is possible to choose h^1, h^2 in such a manner that $\tilde\omega^0 = 0$. This equation remains invariant if an only if $h^1 = h^2 = 0$. Therefore $\mathbf{H}_3 = \Gamma$ is discrete and E_3 is the bundle of canonical frames. To prove the last assertion we remark that the structure form ω_3 of E_3 is well defined if the forms (5.19) and ϑ are given, compare (5.18) and use $\omega^0 = 0$. Thus the assertion follows from the theorems 3.2, 4.2 of R. Sulanke, A. Švec [11]. □

Proposition 5.2 is a "fundamental theorem", but the immersion is characterized by too many $\mathbf{SG}_4$-invariants which are not mutually independent. We shall now evaluate the integrability conditions (2.12) with ϑ^i_j replaced by ω^i_j to reduce the number of invariants.

Let $f: Y^2 \to S^4$ be a generic immersion and $(\mathfrak{b}_i(y))$ a locally defined canonical fram for f. For the corresponding forms on Y^2 we put

(5.21) $$\omega_j^i = c_{j\alpha}^i(y)\, \omega^\alpha, \qquad \alpha = 1, 2.$$

The first condition (2.12) and $\omega^0 = \omega^3 = \omega^4 = 0$ give us

(5.22) $$c_{12}^0 = c_{21}^0.$$

As in Euclidean geometry the second condition (2.12) for $l = 1, 2$ implies

(5.23) $$\omega_1^2 = \frac{d\omega^1}{\omega^1 \wedge \omega^2}\, \omega^1 + \frac{d\omega^2}{\omega^1 \wedge \omega^2}\, \omega^2;$$

for $l = 3, 4$ we get the symmetry conditions (4.9) already applied.

Now let us return to the Möbius structure $\overline{E}_1$ of the immersion f. One easily proves

Lemma 5.1. *Let $f: Y^2 \to S^4$ be an immersion without M-flat points and $\overline{E}_1$ its Möbius structure with $c = 2$, S^4 oriented. Then*

(5.24) $$K^\perp := \frac{d\omega_4^3}{\omega^1 \wedge \omega^2}$$

does not depend on the moving frame in $\overline{E}_1$, and we have

(5.25) $$|K^\perp| \leqq 1.$$

Proof. From (4.20) we get the following transformation rule for the normal vectors:

$$\tilde{\mathfrak{b}}_3 = \mathfrak{b}_3 \cos \psi + \mathfrak{b}_4 \sin \psi, \qquad \tilde{\mathfrak{b}}_4 = \pm(-\mathfrak{b}_3 \sin \psi + \mathfrak{b}_4 \cos \psi).$$

It follows

(5.26) $$\tilde{\omega}_4^3 = \pm(\omega_4^3 - d\psi),$$

and so we get $d\tilde{\omega}_4^3 = \pm d\omega_4^3$, where the sign $-$ stands if the orientation of the normal plane changes. But then the orientation of the tangent plane has to change too, because we consider the oriented sphere S^4. Thus $\omega^1 \wedge \omega^2$ changes the sign, and the exterior curvature $K^\perp$ as defined by (5.24) does not depend on the orientation of T_y. From the third integrability condition (2.12) and (5.2) we obtain

(5.27) $$K^\perp = 2(c_{11}^3 c_{12}^4 - c_{12}^3 c_{11}^4),$$

and this together with (5.3) implies (5.25). □

Corollary 5.1. *Let $f: Y^2 \to S^4$ be a generic immersion. Then*

(5.28) $$K^\perp = \sin 2\vartheta, \qquad |K^\perp| < 1.$$

This follows immediately from (5.18) and (5.27). □

Now we return to the canonical moving frame of a generic immersion. The third condition (2.12) yields

(5.29) $$d\omega_1^2 = (1 + c_{11}^0 + c_{22}^0)\, \omega^1 \wedge \omega^2.$$

If we put

(5.30) $dk = d_1k\omega^1 + d_2k\omega^2$

for any function $k \in C_1(Y^2)$ we obtain evaluating $d\omega_1^3$, $d\omega_2^3$, $d\omega_1^4$, $d\omega_2^4$ from (2.12) and (5.18):

(5.31) $c_{32}^0 = -d_2 \cos\vartheta + 2\cos\vartheta c_{11}^2 + \sin\vartheta c_{41}^3$,

(5.32) $c_{31}^0 = d_1 \cos\vartheta + 2\cos\vartheta c_{12}^2 + \sin\vartheta c_{42}^3$,

(5.33) $c_{42}^0 = d_1 \sin\vartheta + 2\sin\vartheta c_{12}^2 + \cos\vartheta c_{42}^3$,

(5.34) $c_{41}^0 = d_2 \sin\vartheta - 2\sin\vartheta c_{11}^2 - \cos\vartheta c_{41}^3$.

Finally we have to consider the last condition (2.12). Applying (5.22) we get

(5.35) $d\omega_3^0 + \omega_4^0 \wedge \omega_3^4 = 2\cos\vartheta c_{12}^0\omega^1 \wedge \omega^2$,

(5.36) $d\omega_4^0 + \omega_3^0 \wedge \omega_4^3 = \sin\vartheta(c_{11}^0 - c_{22}^0)\,\omega^1 \wedge \omega^2$.

There remain two conditions

(5.37) $d\omega_1^0 + \omega_2^0 \wedge \omega_1^2 = -\omega_3^0 \wedge \omega_1^3 - \omega_4^0 \wedge \omega_1^4$,

(5.38) $d\omega_2^0 + \omega_1^0 \wedge \omega_2^1 = -\omega_3^0 \wedge \omega_2^3 - \omega_4^0 \wedge \omega_2^4$.

Theorem 5.3. *For the class of generic immersions $f\colon Y^2 \to S^4$ with $K^\perp \neq 0$ the forms $(\omega^1, \omega^2, \omega_4^3) \bmod \Gamma$ of a canonical frame are a complete system of $\mathfrak{G}_4$-invariants. The necessary and sufficient conditions that locally exists a generic immersion with prescribed $\omega^1, \omega^2, \omega_4^3$ with $K^\perp \neq 0$ are 1. $\omega^1 \wedge \omega^2 \neq 0$, 2. the inequality (5.28), 3. $d\omega_4^3 \neq 0$, and 4. the integrability conditions (5.37), (5.38) for the forms ω_j^i, calculated from the given invariants.*

Proof. We have to show, that all the forms ω_j^i entering in ω_3 can be calculated by ω^1, ω^2 and ω_4^3. We have $\omega^0 = \omega^3 = \omega^4 = 0$. The form ω_1^2 is defined by (5.23). Formulas (5.24) and (5.28) give the angle $\vartheta = \frac{1}{2}\operatorname{arc\,sin} K^\perp$, $|\vartheta| < \frac{\pi}{4}$. Now (5.18) defines the forms ω_α^x. (5.31)—(5.34) yield the forms ω_3^0, ω_4^0. Since $K^\perp \neq 0$ implies $\vartheta \neq 0$ we finally obtain ω_1^0, ω_2^0 from (5.22), (5.29), (5.35) and (5.36). Obviously, the conditions 1.—4. are fulfilled for any generic immersion with $K^\perp \neq 0$. Conversely, if the conditions 1.—3. are fulfilled, all the calculations mentioned above can be carried out. The structure form ω_3 obtained in this manner fulfils the integrability condition of the $\mathfrak{g}_4, \mathbf{H}_3$-structure E_3 if and only if the equations (5.37), (5.38) hold true. Thus the general existence and uniqueness theorem 3.2 of [11] implies the assertion. □

Theorem 5.4. *The image of a generic immersion $f\colon Y^2 \to S^4$, Y^2 connected, is contained in certain $S^3 \subset S^4$ if and only if we have $\omega_4^3 = 0$ in the canonical frame. In this case the forms ω^1, ω^2 $\bmod \Gamma$ and the function*

(5.39) $D(y) = c_{11}^0(y) - c_{22}^0(y)$

are a complete system of $\mathfrak{G}_4$-invariants.

Proof. Since $\omega_4^3 = 0$, it follows that $K^\perp = 0$ and therefore $\vartheta = 0$. (5.33) and (5.34) imply $\omega_4^0 = 0$, and (5.18) gives $\omega_4^1 = \omega_4^2 = 0$. Thus $d\mathfrak{b}_4 = \mathfrak{o}$, and $\mathfrak{b}_4 =$ constant. Then $\mathfrak{b}_0(y)$ lies in the constant hyperplane $\langle \mathfrak{b}_4, \mathfrak{b}_0 \rangle = 0$, the intersection of which with J^+ defines the S^3 containing $f(Y^2)$. Conversely, if $f(Y^2) \subset S^3$, there exists a constant vector $\mathfrak{c}$, $\langle \mathfrak{c}, \mathfrak{c} \rangle = 1$, with $\langle \mathfrak{c}, \mathfrak{b}_0(y) \rangle = 0$ for all $y \in Y$. Let $(\mathfrak{b}_i)$ be the canonical moving frame. We put $\mathfrak{c} = \mathfrak{b}_i c^i(y)$. From $\langle \mathfrak{c}, \mathfrak{b}_0 \rangle = 0$ we get $c^5 = 0$, and $0 = \langle \mathfrak{c}, d\mathfrak{b}_0 \rangle$ gives $c^1(y) = c^2(y) = 0$. Therefore we obtain from (5.18)

$$0 = \langle \mathfrak{c}, d\mathfrak{b}_1 \rangle = (c^3 \cos \vartheta - c^0)\, \omega^1 + c^4 \sin \vartheta \omega^2,$$

$$0 = \langle \mathfrak{c}, d\mathfrak{b}_2 \rangle = -(c^3 \cos \vartheta + c^0)\, \omega^2 + c^4 \sin \vartheta \omega^1.$$

It follows that $c^0 = 0$, $c^3 \cos \vartheta = 0$, $c^4 \sin \vartheta = 0$. Since $|\vartheta| \leqq \frac{\pi}{4}$, we have $\cos \vartheta > 0$ and $c^3 = 0$. This implies $\mathfrak{c} = \mathfrak{b}_4 =$ constant, $\vartheta = 0$ and $\omega_4^3 = 0$. If we only consider hyperspherical generic immersions f, $f(Y^2) \subset S^3 \subset S^4$, we can calculate all the forms ω_j^i, except ω_1^0, ω_2^0, as in the proof of theorem 5.3. Since (5.36) is trivial in this case we must suppose the function D to be given. □

6. M-isoclinic surfaces

Let now $f: Y^2 \to S^4$ be an M-isoclinic immersion, and $\overline{E}_1$ its Möbius structure. Then (5.6) and (5.7) are valid for any moving frame $z \in \overline{E}_1$. Therefore (5.10) holds true for any orthogonal tangential base $\mathfrak{c}_1, \mathfrak{c}_2$. We agree to choose $\mathfrak{b}_3, \mathfrak{b}_4$ in such a manner, that

(6.1) $$\alpha(\mathfrak{c}_1, \mathfrak{c}_1) = \mathfrak{c}_3 c_{11}^3, \qquad \alpha(\mathfrak{c}_1, \mathfrak{c}_2) = \mathfrak{c}_4 c_{12}^4$$

are valid. This implies

(6.2) $$c_{12}^3 = c_{11}^4 = 0.$$

Orienting S^4 in an appropriate way we get from (5.6)

(6.3) $$\omega_1^3 = \frac{\omega^1}{\sqrt{2}}, \qquad \omega_2^3 = -\frac{\omega^2}{\sqrt{2}},$$

$$\omega_1^4 = \frac{\omega^2}{\sqrt{2}}, \qquad \omega_2^4 = \frac{\omega^1}{\sqrt{2}}, \qquad \vartheta = \frac{\pi}{4}.$$

Now all the coefficients of the first order have well defined values, and the reduction of E_1 is finished: $E_2 \subset \overline{E}_1$ is the set of all those frames $z = (\mathfrak{b}_i) \in \overline{E}_1$ for which the equations (6.3) are fulfilled. To characterize the structure group we proceed as follows. Since the rotation acts on $(\mathfrak{b}_k)$ and on $(\mathfrak{c}_k)$, $k = 1, \ldots, 4$, in the same way, it suffices to consider the $(\mathfrak{c}_k)$. Transforming

(6.4) $$A(\beta): \tilde{\mathfrak{c}}_1 = \mathfrak{c}_1 \cos \beta + \mathfrak{c}_2 \sin \beta, \qquad \tilde{\mathfrak{c}}_2 = \pm(-\mathfrak{c}_1 \sin \beta + \mathfrak{c}_2 \cos \beta)$$

we obtain from (6.1), (6.3)

(6.5) $$\tilde{\mathfrak{c}}_3 = \sqrt{2}\, \alpha(\tilde{\mathfrak{c}}_1, \tilde{\mathfrak{c}}_1), \qquad \tilde{\mathfrak{c}}_4 = \sqrt{2}\, \alpha(\tilde{\mathfrak{c}}_1, \tilde{\mathfrak{c}}_2),$$

and this implies

$$A(2\beta)^{\perp}: \tilde{c}_3 = c_3 \cos 2\beta + c_4 \sin 2\beta, \qquad (6.6)$$

$$\tilde{c}_4 = \pm(-c_3 \sin 2\beta + c_4 \cos 2\beta).$$

Thus the structure group $\mathbf{H}_2$ is the group of matrices of the form (5.16), where now

$$(\varepsilon^{\alpha}_{\beta}) = A(\beta), \quad (\varepsilon^{\varkappa}_{\lambda}) = A(2\beta)^{\perp}. \qquad (6.7)$$

The Lie algebra $\mathfrak{h}_2 \subset \mathfrak{h}_1$ is characterized by the equations

$$\vartheta^0 = 0, \quad \vartheta^0_3 = 0, \quad \vartheta^0_4 = 0, \quad \vartheta^3_4 = 2\vartheta^1_2. \qquad (6.8)$$

Now we begin with the third reduction. As in the generic case we can reach $\omega^0 = 0$, see (5.20). Next we remark

Lemma 6.1. *Let E_2 be the $\mathfrak{g}_4$, $\mathbf{H}_2$-structure of an M-isoclinic immersion. Then the form $\omega^3_4 - 2\omega^1_2$ is invariant under all transformations $h \in \mathbf{H}_2$ with $h^1 = h^2 = 0$.*

Indeed, we have $\tilde{\omega}^3_4 = \omega^3_4 \mp 2d\beta$, $\tilde{\omega}^1_2 = \omega^1_2 \mp d\beta$, and this yields the assertion. We define E_3 to be the set of all those frames $z = (\mathfrak{b}_i) \in E_2$ having the properties

$$\omega^0(z) = 0, \quad \omega^3_4(z) - 2\omega^1_2(z) = k\omega^1, \qquad (6.9)$$

and state

Proposition 6.1. *Let $f: Y^2 \to S^4$ be an M-isoclinic immersion. Then $E_3 \subset E_2$ as defined by (6.9) is the third reduction of the $\mathfrak{g}_4$, $\mathbf{H}_0$-structure induced by f. The function $k = k(z)$ only depends on $p(z)$, up to a sign; therefore it defines a $\mathfrak{G}_4$-invariant of f. In case $k \neq 0$ on Y^2 the structure group $\mathbf{H}_3$ of E_3 is discrete. For $k = 0$ on Y^2 the structure group $\mathbf{H}_3$ of E_3 is the 1-parameter subgroup $\mathbf{H}_3 \subset \mathbf{H}_2$ characterized by $h^1 = h^2 = 0$.*

Proof. First we remark that the structure form $\omega_3 := \omega_2 \mid TE_3$ has the same shape as in the generic case; it is only more special since we have $\vartheta = \frac{\pi}{4}$. Therefore, if we put (5.21) again, all the integrability formulas calculated in § 5 remain valid. In particular, (5.31)–(5.34) yield now

$$c^0_{32} = \frac{(c^3_{41} - 2c^1_{21})}{\sqrt{2}} = \frac{k}{\sqrt{2}} = -c^0_{41}, \qquad (6.10)$$

$$c^0_{31} = \frac{(c^3_{42} - 2c^1_{22})}{\sqrt{2}} = 0 = c^0_{42}. \qquad (6.11)$$

This implies

$$\omega^0_3 = \frac{k\omega^2}{\sqrt{2}}, \quad \omega^0_4 = \frac{-k\omega^1}{\sqrt{2}}, \qquad (6.12)$$

and these formulas show that the reduction is finished: all coefficients of the third order have well defined values. If in the case of $k \neq 0$ we require $k > 0$, condition (6.9) remains valid for $\beta = 0$, $h^1 = h^2 = 0$ only, and $\mathbf{H}_3$ is discrete. If $k = 0$ we have $\omega_4^3 - 2\omega_2^1 = 0$ on E_3, and β remains variable. In each case k depends on $p(z)$ only. □

Now let us consider the integrability conditions. (5.22), (5.23) remain unchanged; together with the symmetry relation (4.9) they are equivalent to the first and second equations (2.12). The third equation of (2.12) yields

$$K^\perp = \frac{d\omega_4^3}{\omega^1 \wedge \omega^2} = 1; \tag{6.13}$$

this also follows from $\vartheta = \frac{\pi}{4}$. Formula (5.29) for $d\omega_2^1$ remains valid; the equations for $d\omega_\alpha^\varkappa$, $\alpha = 1, 2$; $\varkappa = 3, 4$, reduce themselves to (6.10), (6.11). Since ω_2^1 is known from (5.23), we get ω_4^3 from (6.9) if k is given. The last equations (2.12) can be used to calculate ω_1^0, ω_2^0. (5.35), (6.12) and $\vartheta = \frac{\pi}{4}$ yield

$$d_1 k + k(c_{12}^2 + c_{42}^3) = 2c_{12}^0. \tag{6.14}$$

The second equation (6.9) gives

$$c_{41}^3 - 2c_{21}^1 = k, \qquad c_{42}^3 = 2c_{22}^1. \tag{6.15}$$

Using (5.22) we obtain

$$c_{12}^0 = c_{21}^0 = \frac{1}{2}\,(d_1 k + k c_{22}^1). \tag{6.16}$$

Analogously we get from (5.36) and (6.12):

$$c_{11}^0 - c_{22}^0 = d_2 k - k^2 - k c_{21}^1. \tag{6.17}$$

From (5.29) we have

$$c_{11}^0 + c_{22}^0 = -K - 1, \tag{6.18}$$

where the *M-curvature* K of the immersion is defined by

$$d\omega_2^1 = K\omega^1 \wedge \omega^2. \tag{6.19}$$

Thus all the components ω_j^i of ω_3 are uniquely defined if ω^1, ω^2 and k are given; this also holds true in the case $k = 0$. If we calculate these forms by the derived formulas, all the integrability conditions with exception of (5.37), (5.38) are fulfilled automatically. The evaluation of these remaining conditions leads to

$$d\omega_1^0 + \omega_2^0 \wedge \omega_1^2 = k\omega^1 \wedge \omega^2, \tag{6.20}$$

$$d\omega_2^0 + \omega_1^0 \wedge \omega_2^1 = 0. \tag{6.21}$$

Thus we proved

Theorem 6.2 *The forms ω^1, ω^2 and the function k are a complete system of $\mathfrak{G}_4$-invariants for the M-isoclinic immersions $f: Y^2 \to S^4$ with the additional property $k \neq 0$. If one calculates all the components ω_j^i of the structure form ω_3 of third order starting with given*

ω^1, ω^2, $k \neq 0$, *the conditions* (6.20), (6.21) *and* $\omega^1 \wedge \omega^2 \neq 0$ *are necessary and sufficient for the local existence of an M-isoclinic immersion* $f\colon Y^2 \to S^4$ *with these prescribed* $\mathfrak{G}_4$*-invariants.* □

Now we consider an M-isoclinic immersion with $k = 0$. Since $\omega_4^3 = 2\omega_2^1$ we obtain for the M-curvature K from (6.13) and (6.19)

$$K = \frac{1}{2}. \tag{6.22}$$

The equations (6.16)—(6.18) imply

$$\omega_1^0 = \frac{-3\omega^1}{4}, \qquad \omega = \frac{-3\omega^2}{4}. \tag{6.23}$$

From (6.12) follows

$$\omega_3^0 = \omega_4^0 = 0. \tag{6.24}$$

Thus all the coefficients of order $\leqq 3$ are constant, and theorem 4.4 of [11] can be applied. One can prove

Theorem 6.3. *There exist M-isoclinic immersions* $f\colon Y^2 \to S^4$ *with* $k = 0$; *the Veronese surface is such an immersion. Any two M-isoclinic immersions of* S^2 *in* S^4 *with* $k = 0$ *are* $\mathfrak{G}_4$*-equivalent, and each M-isoclinic immersion with* $k = 0$ *is locally* $\mathfrak{G}_4$*-equivalent to the Veronese surface.*

Proof. The existence of such immersions can easily be proved in the following way: One starts with a sphere S^2 considered as a Riemannian space of constant curvature $K = \frac{1}{2}$. If (ω^1, ω^2) is an orthonormal local base field on S^2 one constructs the local structure form ω_3 by the given formulas. The principal bundle E_3 is the bundle of orthonormal frames of the Riemannian space S^2, and the local expressions for ω_3 define a global form on E_3 with values in $\mathfrak{g}_4$. The integrability conditions (6.20), (6.21) immediately follow from (5.23) and (6.23). Since S^2 is simply connected, the existence and uniqueness is a consequence of theorem 3.2 of [11]. In the same way one proves that all the M-isoclinic immersions with $k = 0$ are locally $\mathfrak{G}_4$-equivalent. That the Veronese surface is an immersion of the desired type can be proved by an explicite calculation of its $\mathfrak{g}_4, \mathbf{H}_3$-structure $E_3(S^2, \omega_3)$; one starts with a parameter representation of the Veronese surface, compare B.-Y. Chen [5], p. 88, and carries out the reduction procedure. Doing this one obtaines, of course, another more direct proof of the existence of the immersions under consideration. The calculations are straightforward, but a little tiresome, so we will not reproduce them here. □

Bibliography

[1] М. А. Акивис, Инвариантное построение геометрии гиперповерхности конформного пространства, Матем. сб. **31**, № 1, 43—75 (1952).

[2] —, К конформно-дифференциальной геометрии многомерных поверхностей Матем. сб. **53**, № 1, 53—72 (1961).

[3] —, Конформно-дифференциальная геометрия, Итоги науки, Геометрия 1963, 108—137, Москва 1965.
[4] W. Blaschke & G. Thomsen, Vorlesungen über Differentialgeometrie III, Differentialgeometrie der Kreise und Kugeln, Grundlehren der math. Wiss. Bd. **24**, Berlin 1929.
[5] B.-Y. Chen, Geometry of submanifolds, New York 1973.
[6] S. Kobayashi, Transformation groups in differential geometry, Ergebn. der Math. u. ihrer Grenzgeb. **70**, Berlin—Heidelberg—New York 1972.
[7] S. Kobayashi & K. Nomizu, Foundations of differential geometry, Vol. II, New York—London—Sydney 1969.
[8] O. Kowalski, Partial curvature structures and conformal geometry of submanifolds, J. Differential Geometry **8**, 53—70 (1973).
[9] P. K. Raschewski, Riemannsche Geomterie und Tensoranalysis, Berlin 1959.
[10] Ch. Schiemangk, Zur konformen Kurventheorie und zur konformen Differentialgeometrie der 2-Flächen im vierdimensionalen konformen Raum, Dissertation, Humboldt-Univ., Berlin 1977.
[11] R. Sulanke & A. Švec, Zur Differentialgeometrie der Untermannigfaltigkeiten eines Kleinschen Raumes, to appear in Beiträge zur Algebra und Geometrie, Halle.
[12] R. Sulanke & P. Wintgen, Differentialgeometrie und Faserbündel, Berlin 1972.
[13] T. Takasu, Differentialgeometrie in den Kugelräumen, I, Konforme Differentialgeometrie von Liouville und Möbius, Tokyo 1938.
[14] Л. Л. Вербицкий, Основы теории кривых конформного пространства и измерений, Изв. высш. уч. завед., математика, № **6**, 26—37 (1959).
[15] E. Vessiot, Enveloppes de sphéres et courbes gauches, J. École Polytechnique, 2 ser., **25**, 43—91 (1925).

Humboldt-Universität
Sektion Mathematik
DDR - 1080 Berlin
Unter den Linden 6