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Wir erbitten umgehende Rücksendung eines korrigierten Fahnenabzugs

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I: Nachkorrekturen!

Submanifolds of the MÖBIUS space, II
FRENET Formulas and Curves of Constant Curvatures

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Introduction

This article is an immediate continuation of the paper by CH. SCHIEMANGK, R. SULANKE [6] cited in the following with the number I. The first section is an elaboration of the results already announced in § I.3. It contains the fundamental theorem for the generic curves in the MÖBIUS space S^n proved with E. CARTAN's method of moving frames. We applied the precised version of E. CARTAN's theory given in R. SULANKE, A. ŠVEC I [11], compare also G. JENSEN [3] and especially for the theory of curves M. L. GREEN [2]. The natural conformal parameter of the curve and the $n-1$ curvatures characterizing it are the same as those obtained with different proofs by Л. Л. ВЕРБИЦКИЙ I [14] in 1959 and, in the case $n=3$, by E. VESSIOT I [15] already in 1925. The conformal parameter was found by H. LIEBMANN [4] yet earlier, in 1923, and for $n=2$ already in 1914 by G. PICK [5]. But we proved our results by other methods and indepently of these earlier papers, which we only found after having finished our calculations. The coincidence of the results again confirmes the adequateness of E. CARTAN's group theoretical method of moving frames.

In section 2 we will calculate all the curves of constant conformal curvatures on the sphere S^3. Besides the loxodromes, which lie in an $S^2 \subset S^3$, there are 3 types of curves corresponding to the curves of constant curvatures in hyperbolic, elliptic and euclidean geometry respectively, which are considered as subgeometries of the MÖBIUS geometry in the sense of F. KLEIN's Erlanger programme: its isometry groups are subgroups of the MÖBIUS group in a natural manner. Two of the types have already been described by E. VESSIOT I [15], but he did not prove that the curves he calculated realize all possible values of the curvatures and so he overlooked the third, the euclidean type. All these curves can be described as isogonal trajectories of the families of generating circles of certain cyclids of DUPIN.

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1. The Frenet formulas

Let $f\colon t \in Y \mapsto f(t) \in S^n$ be an immersion of a 1-dimensional connected manifold Y into S^n, and $E_0(Y, \omega_0)$ its $\boldsymbol{g}_n$, $\boldsymbol{H}_0$-structure. Analogous to §I.4, $p_0\colon E_0 \to Y$ is the principal fibre bundle with base Y, induced by f from the principal bundle $p\colon \boldsymbol{G}_n \to S^n = \boldsymbol{G}_n/\boldsymbol{H}_0$, and $\omega_0 = \bar{f}^*\omega$ the inverse image of the structure form of $\boldsymbol{G}_n$ defined by the corresponding bundle homomorphism $\bar{f}\colon E_0 \to \boldsymbol{G}_n$. Using the identification I (2.17) we can describe $E_0(Y, \omega_0)$ in the following more elementary manner. E_0 is the set of moving frames along f:

$$E_0 := \{(t, (\mathfrak{b}_i)) \mid t \in Y,\ (\mathfrak{b}_i)\ \textit{an i.-o. frame with}\ \pi(\mathfrak{b}_0) = f(t)\}\,,$$

$p_0\colon E_0 \to Y$ is the projection of the frame $z = (t, (\mathfrak{b}_i)) \in E_0$ to its origin $p_0(z) = t \in Y$, $\bar{f}\colon E_0 \to \boldsymbol{G}_n$ is the immersion $\bar{f}(z) = g = (g^i_j) \in \boldsymbol{G}_n$, where (g^i_j) is defined by I (2.17), and the form ω_0 on E_0 with values in the Lie algebra $\boldsymbol{g}_n$ of $\boldsymbol{G}_n$ has the matrix representation

$$\omega_0 = (\omega^i_j) = (g^i_l)^{-1}\,(dg^l_j)\,.$$

The Pfaffian forms ω^i_j on E_0 appear as coefficients of the differentials of the vector functions $\mathfrak{b}_j\colon E_0 \to V^{n+2}$ in the derivation equations of the moving frame

$$(1.1) \qquad d\mathfrak{b}_j = \sum_{i=0}^{n+1} \mathfrak{b}_i \omega^i_j\,.$$

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Corresponding to I (2.18) we used the following notations here:

$$\omega^i_j = \bar{f}^*\vartheta^i_j, \quad \omega^k = \bar{f}^*\vartheta^k.$$

Formulas I (2.19) yield the following symmetry relations and notations:

$$(1.2) \qquad \begin{aligned} &\omega^0 := \omega^0_0 = -\omega^{n+1}_{n+1}, \quad && \omega^0_k = \omega^k_{n+1}\,, \\ &\omega^k := \omega^k_0 = \omega^{n+1}_k, && \omega^k_l + \omega^l_k = 0, \quad k, l = 1, \ldots, n\,. \end{aligned}$$

The canonical form on E_0 is

$$1.3) \qquad \Theta_0 := p_0\omega_0 = (\omega^k)\colon TE_0 \to \boldsymbol{g}_n/\boldsymbol{h}_0 = T_0S^n.$$

$p \circ \omega_0$

Since rank $\Theta_0(z) = 1$, and the isotropy representation I (2.22) acts transitively on the Grassmann manifold $G_{n,1}$ of $\boldsymbol{g}_n/\boldsymbol{h}_0$, we get the *first reduction* by

$$(1.4) \qquad z \in E_1 :\leftrightarrow \omega^k(z) = 0 \quad \text{for} \quad k = 2, \ldots, n, \quad z \in E_0\,.$$

Geometrically, $E_{1t} = p_1^{-1}(t)$, $p_1 := p_0 \mid E_1$, consists of all frames $z = (b_i)$ tangential to the curve at t:

$$(1.5) \qquad \pi(\mathfrak{b}_0) = f(t), \quad d\pi(\mathfrak{b}_1) \in df_t(T_tY)\,.$$

The structure group $\boldsymbol{H}_1 \subset \boldsymbol{H}_0$ is the subgroup of all matrices (see I (2.15))

$$(1.6) \qquad h(A, \mathfrak{c}, \mu) \in \boldsymbol{H}_0 \quad \text{with} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & h^r_s \end{pmatrix} \in \boldsymbol{SO}(n), \quad r, s = 2, \ldots, n\,.$$

The Lie algebra $\boldsymbol{h}_1$ of $\boldsymbol{H}_1$ is characterized as a subalgebra of $\boldsymbol{h}_0$ by

$$(1.7) \qquad \vartheta^r_1 = 0, \quad r = 2, \ldots, n\,.$$

Here we suppose to consider *oriented immersions*, Y oriented, t an oriented parameter, *into the oriented sphere* $S^n = \boldsymbol{SG}_n/\boldsymbol{SH}_0$, and require

$$(1.8) \qquad d\pi(\mathfrak{b}_1(z)) = \dot{x}(t)\, b(t) \quad \text{with} \quad b(t) > 0, \quad p_1(z) = t, \quad \dot{x} = df/dt\,.$$

So we must put $h^1_1 = 1$ in (1.6) instead of $h^1_1 = \pm 1$.

In order to carry out the second reduction, we have to consider the transformation rule for the coefficients c^r_1 of the first order,

$$(1.9) \qquad \omega^r_1 = c^r_1 \omega^1, \quad r = 2, \ldots, n\,,$$

under the action of $\boldsymbol{H}_1$. Using again the sum convention we put

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$$(1.10) \qquad \bar{z} = (\bar{\mathfrak{b}}_i) = h^{-1} \times z = (\mathfrak{b}_j h^j_i), \quad z = (\mathfrak{b}_j) \in E_1, \quad h = (h^i_j) \in \boldsymbol{H}_1\,,$$

and calculate

$$(1.11) \qquad \bar{\omega}^1 = \langle d\bar{\mathfrak{b}}_0, \bar{\mathfrak{b}}_1 \rangle = \langle d(\mu^{-1}\mathfrak{b}_0), \mathfrak{b}_0 h^0_1 + \mathfrak{b}_1 \rangle = \mu^{-1}\omega^1,$$

⊥ (1.12) ⊥

$$\bar{\omega}^s_1 = \langle d\bar{\mathfrak{b}}_1, \bar{\mathfrak{b}}_s \rangle = \Big(\sum_r c^r_1 h^r_s - h^0_s\Big)\, \mu \bar{\omega}^1 = \bar{c}^s_1 \bar{\omega}^1.$$

Giving the components h^0_s of h suitable values we always can reach $\bar{c}^s_1 = 0$. So we define

$$(1.13) \qquad z \in E_2 :\leftrightarrow z \in E_1 \quad \text{and} \quad \omega^r_1(z) = 0, \quad r = 2, \ldots, n\,.$$

With that the *second reduction* step is finished, for all coefficients of the first order have the value $c^r_1(z) = 0$ on E_2; there are no invariants of second order; we also do not have any invariant 1-form $\omega^\alpha \neq 0$, compare (1.11), and yet we can not define a natural parameter. From (1.12) we see, that $\boldsymbol{H}_2 \subset \boldsymbol{H}_1$ consists of all matrices

$$(1.14) \qquad h = h(A, \mathfrak{c}, \mu) \in \boldsymbol{H}_1 \quad \text{with} \quad {}^t\mathfrak{c} = (h^0_1, 0, \ldots, 0), \quad h^0_1 \in \boldsymbol{R}\,,$$

leaving the condition (1.13) invariant. The Lie algebra $\boldsymbol{h}_2$ of $\boldsymbol{H}_2$ is the subalgebra of $\boldsymbol{h}_1$ defined by

$$(1.15) \qquad \vartheta^0_r = 0, \quad r = 2, \ldots, n\,.$$

The structure form $\omega_2 := \omega_0 \mid TE_2$ of E_2 is

$$(1.16) \qquad \omega_2 = (\omega^i_j) = \begin{pmatrix} \omega^0 & \omega^0_1 & \omega^0_2 & \ldots & \omega^0_n & 0 \\ \omega^1 & 0 & 0 & \ldots & 0 & \omega^0_1 \\ 0 & 0 & & (\omega^r_s) & & \omega^0_2 \\ \vdots & \vdots & r, s = 2, \ldots, n & & & \vdots \\ 0 & 0 & & & & \omega^0_n \\ 0 & \omega^1 & 0 & \ldots & 0 & -\omega^0 \end{pmatrix}.$$

For the *third reduction* we calculate the transformation rule of the forms

U h^s_r μ

$$(1.17) \qquad \bar{\omega}^0_r = -\langle d\bar{\mathfrak{b}}_r, \bar{\mathfrak{b}}_{n+1} \rangle = \omega^0_s h^s_r \mu = \bar{c}^0_r \bar{\omega}^1$$

analogously to the preceding reduction step. This gives us for the coefficients of second order

$$(1.18) \qquad \bar{c}^0_r = c^0_s h^s_r \mu^2, \quad (h^r_s) \in \boldsymbol{SO}\,(n-1)\,.$$

Here we meet for the first time the necessity to distinct two types of immersions: *Type* A_2^0: for the structure form ω_2 of the corresponding $\boldsymbol{g}_n$, $\boldsymbol{H}_2$-structure for all $z \in E_2$

$$(1.19) \qquad \omega_2^0(z) = \ldots = \omega_n^0(z) = 0$$

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holds true; *type* A_0: for all $z \in E_0$ there is at least one r, $2 \leqq r \leqq n$, with $\Theta_r^0(z) \neq 0$.

Obviously for the type A_2^0 the $\boldsymbol{g}_n$, $\boldsymbol{H}_2$-structure E_2 fulfils the suppositions of theorem 4.4 of I[11]. Thus we have $\boldsymbol{H}_3 = \boldsymbol{H}_2$, the reduction procedure stops, and locally the immersion must be an orbit. We prove

U L S^n

Proposition 1.1. *Let* $f\colon Y^1 \to S_n$ *be an immersion,* Y *connected, and* $E_2(Y, \omega_2)$ *its* $\boldsymbol{g}_n$, $\boldsymbol{H}_2$*-structure. Then there exists a circle* $S^1 \subset S_n$ *with* $f(Y) \subset S^1$ *if and only if* f *is of type* A_2^0.

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Proof. If (1.19) is fulfilled we obtain from (1.16)

$$(1.20) \qquad d\mathfrak{b}_0 = \mathfrak{b}_0 \omega^0 + \mathfrak{b}_1 \omega^1, \quad d\mathfrak{b}_1 = \mathfrak{b}_0 \omega_1^0 + \mathfrak{b}_{n+1} \omega^1,$$

$$(1.21) \qquad d\mathfrak{b}_{n+1} = \mathfrak{b}_1 \omega_1^0 - \mathfrak{b}_{n+1} \omega^0.$$

From this $d(\mathfrak{b}_0 \wedge \mathfrak{b}_1 \wedge \mathfrak{b}_{n+1}) = 0$ follows. Thus the trivector $\mathfrak{b}_0 \wedge \mathfrak{b}_1 \wedge \mathfrak{b}_{n+1}$ is constant, and therefore the linear hull $V_2 := \mathfrak{L}(\mathfrak{b}_0, \mathfrak{b}_1, \mathfrak{b}_{n+1})$ is a constant, 3-dimensional, pseudo-euclidean subspace of V^{n+2}, since Y is connected. Its intersection with J^{n+1} is the isotropic cone $J^2 \subset V_2$ containing $\mathfrak{b}_0 = \mathfrak{b}_0(t)$. This cone is projected on $S^1 \subset S^n$ with $f(t) = \pi(\mathfrak{b}_0(t)) \in S^1$. Conversely, if $f(Y) \subset S^1$, we have a constant pseudo-euclidean subspace $W^3 \subset V^{n+2}$ with $S^1 = \pi(J^{n+1} \cap W^3)$ and $\mathfrak{b}_0(t) \in W^3$ for all $t \in Y$. Thus all derivatives $d^k \mathfrak{b}_0 / dt^k$ belong to W^3 for all $t \in Y$. Since (1.20) holds true for the $\boldsymbol{g}_n$, $\boldsymbol{H}_2$-structure of any curve, we see that $\mathfrak{b}_{n+1} \in \mathfrak{L}(\mathfrak{b}_0, \dot{\mathfrak{b}}_0, \ddot{\mathfrak{b}}_0) \subseteqq W^3$. Therefore we also have $d\mathfrak{b}_{n+1} \in W^3$, and all ω_r^0, $r = 2, \ldots, n$, must vanish. □

Now let f *be an immersion of type* A_2. We obtain from (1.18): It is always possible to choose $(h_r^s) \in \boldsymbol{SO}(n-1)$ and $\mu \in \boldsymbol{R}^+$ such that $\bar{c}_2^0 = \varepsilon = \pm 1$; $\bar{c}_r^0 = 0$ for $r = 3, \ldots, n$, where $\varepsilon = +1$ if $n > 2$; in the case $n = 2$ the sign of ε indicates whether the curve turns right ($\varepsilon = 1$) or left ($\varepsilon = -1$). Therefore *the next reduction* can be defined by

$$(1.22) \qquad z \in E_3 :\leftrightarrow z \in E_2, \quad \omega_2^0(z) = \varepsilon \omega^1(z) \quad \text{and} \quad \omega_r^0(z) = 0 \quad \text{for} \quad r = 3, \ldots, n.$$

The structure group $\boldsymbol{H}_3$ contains all the matrices (1.14) with the additional property

$$(1.23) \qquad \mu = 1, \quad h_r^2 = h_2^r = \delta_r^2, \quad r = 2, \ldots, n,$$

see also (1.6). The Lie algebra $\boldsymbol{h}_3$ of $\boldsymbol{H}_3$ is defined as subalgebra of $\boldsymbol{h}_2$ by

$$(1.24) \qquad \vartheta^0 = 0, \quad \vartheta_r^2 = 0, \quad r = 3, \ldots, n.$$

The structure form ω_3 has the matrix (1.16) with condition (1.22). From (1.11) and (1.23) it follows that ω^1 is a $\boldsymbol{G}_n$-invariant of the third order. It can be used to define a *natural parameter* s for curves of type A_2 up to an integration constant requiring

$$ds = z^*\omega^1 \tag{1.25}$$

for any moving frame $t \in U \mapsto z(t) \in E_3$, $U \subseteq Y$, of third order.

We will calculate ds in terms of the derivatives of a parameter representation. Let $f(t) = \pi(\mathfrak{b}(t))$, $t \in Y$. The function $t \in Y \mapsto \mathfrak{b}(t) \in J^+$ is called a *normed representation* of the immersion f, if, for all $t \in Y$, we have $\langle \dot{\mathfrak{b}}, \dot{\mathfrak{b}} \rangle = 1$, the point $\cdot$ denoting derivation with respect to t. Obviously there exist many normed representations of a given immersion. Particularly, if $(\mathfrak{b}_i(s))$ is a section of E_3 and s a natural parameter, we conclude from (1.16) and (1.22)

$$d\mathfrak{b}_0/ds = \mathfrak{b}_0 k^0 + \mathfrak{b}_1, \quad \langle d\mathfrak{b}_0/ds, d\mathfrak{b}_0/ds \rangle = 1;$$

thus $\mathfrak{b} = \mathfrak{b}_0(s)$ is a normed representation. We prove

Proposition 1.2. *If $t \in Y^1 \mapsto \mathfrak{b}(t) \in J^+$ is a normed representation of the immersion $f: Y^1 \to S^n$ of type A_2, the formula of* H. Liebmann [4], *p.* 81,

$$ds = (\langle d^3\mathfrak{b}/dt^3, d^3\mathfrak{b}/dt^3 \rangle - \langle d^2\mathfrak{b}/dt^2, d^2\mathfrak{b}d/t^2 \rangle^2)^{1/4}\, dt \tag{1.26}$$

defines the differential of the natural parameter s.

Proof. Let $'$ denote the derivation with respect to s. Since s and t are normed parameters, we obtain derivating $\mathfrak{b}_0(s) = \mathfrak{b}(t(s))\, \mu^{-1}(t(s))$

$$\mathfrak{b}_0' = \dot{\mathfrak{b}} \cdot (dt/ds)\, \mu^{-1} + \mathfrak{b} \cdot (d\mu^{-1}/ds).$$

Forming the scalar square we get

$$ds/dt = \mu^{-1}. \tag{1.27}$$

Thus it suffices to calculate μ^{-1}. Now let $z(t) = (\mathfrak{b}_i(t))$ be a section of E_2 with $\mathfrak{b} = \mathfrak{b}_0(t)$ a normed representation and $\omega^1 = dt$. From (1.18) we see:

$$\mu^{-4} = \sum_{r=2}^{n} (c_r^0)^2 = \langle \dot{\mathfrak{b}}_{n+1}, \dot{\mathfrak{b}}_{n+1} \rangle - \langle \dot{\mathfrak{b}}_{n+1}, \mathfrak{b}_1 \rangle^2 \tag{1.28}$$

is necessary and sufficient for

$$\sum_{r=2}^{n} (\bar{c}_r^0)^2 = 1,$$

and this condition defines μ and by (1.27) ds. To construct such a section of E_2 we prove

Lemma 1.1. *For any normed representation $\mathfrak{b}(t)$ of an immersion $f: Y^1 \to S^n$ of type A_2 the vectors*

$$\mathfrak{b}_0 = \mathfrak{b}, \quad \mathfrak{b}_1 = \dot{\mathfrak{b}}, \quad \mathfrak{b}_{n+1} = \ddot{\mathfrak{b}} + \mathfrak{b}\langle \ddot{\mathfrak{b}}, \ddot{\mathfrak{b}} \rangle/2, \tag{1.29}$$

$\mathfrak{b}_r$, $r = 2, \ldots, n$, any orthonormal base of $\mathfrak{L}(\mathfrak{b}, \dot{\mathfrak{b}}, \ddot{\mathfrak{b}})^\perp$, define a moving frame $z = z(t)$ of the $\boldsymbol{g}$[illegible], $\boldsymbol{H}_2$-structure E_2 of f with

$$\omega^1 = dt, \quad \omega^0 = 0, \quad \omega_1^0 = -\langle \ddot{\mathfrak{b}}, \ddot{\mathfrak{b}} \rangle\, dt/2. \tag{1.30}$$

Proof. From the assumption we get deriving three times

$$\langle\mathfrak{b},\mathfrak{b}\rangle=0,\quad \langle\dot{\mathfrak{b}},\dot{\mathfrak{b}}\rangle=1,\quad \langle\mathfrak{b},\dot{\mathfrak{b}}\rangle=0,\quad \langle\dot{\mathfrak{b}},\ddot{\mathfrak{b}}\rangle=0\,,$$
$$\langle\mathfrak{b},\ddot{\mathfrak{b}}\rangle=-1,\quad \langle\mathfrak{b},d^3\mathfrak{b}/dt^3\rangle=-\langle\dot{\mathfrak{b}},\ddot{\mathfrak{b}}\rangle=0\,,$$
$$\langle\dot{\mathfrak{b}},d^3\mathfrak{b}/dt^3\rangle=-\langle\ddot{\mathfrak{b}},\ddot{\mathfrak{b}}\rangle\,. \tag{1.31}$$

These relations imply that $(\mathfrak{b}_i)$ is an i.-o. base, if it is defined as formulated in the lemma. From this definition we obtain

$$d\mathfrak{b}_0=\mathfrak{b}_1dt,\quad d\mathfrak{b}_1=-\mathfrak{b}_0\langle\ddot{\mathfrak{b}},\ddot{\mathfrak{b}}\rangle\,dt/2+\mathfrak{b}_{n+1}dt\,,$$

what proves the assertion. □

Now the proof of proposition 1.2 can easily be finished. (1.29) gives us

$$d\mathfrak{b}_{n+1}/dt=\mathfrak{b}^{(3)}+\dot{\mathfrak{b}}\cdot a^2/2+\mathfrak{b}a\dot{a},\quad \text{with}\quad a:=\langle\ddot{\mathfrak{b}},\ddot{\mathfrak{b}}\rangle^{1/2}.$$

Applying (1.31) we obtain

$$\langle d\mathfrak{b}_{n+1}/dt,d\mathfrak{b}_{n+1}/dt\rangle=\langle\mathfrak{b}^{(3)},\mathfrak{b}^{(3)}\rangle-3a^4/4$$
$$\langle d\mathfrak{b}_{n+1},\mathfrak{b}_1\rangle=a^2/2\,.$$

From (1.28), (1.27) one gets the stated formula (1.26). □

Proposition 1.3. *Let $f\colon t\in Y^1\mapsto x(t)\in S^n\subseteq E^{n+1}$ be an immersion of Y^1 into the sphere S^n, considered as hypersphere of radius r in the euclidean vector space $E^{n+1}\subseteq\subseteq V^{n+2}$, t its euclidean arc length, $k(t)$, $\varkappa(t)$ its first and second curvatures in the sense of euclidean geometry. Further let $\mathfrak{e}_0$ denote a vector orthogonal to E^{n+1} with $\langle\mathfrak{e}_0,\mathfrak{e}_0\rangle=-1$. Then $\mathfrak{b}(t)=\mathfrak{e}_0r+\mathfrak{x}(t)$ is a normed representation of the immersion f, and its natural conformal parameter s is defined by* (compare T. Takasu I [13], (520), p. 123)

$$ds=(k^2+\varkappa^2k^2)^{1/4}\,dt\,. \tag{1.32}$$

Proof. The first assertion is trivial. Thus we can apply (1.26). Remarking $\mathfrak{b}^{(q)}=\mathfrak{x}^{(q)}$, $q=1,2,\ldots$ and using the Frenet frame $(\mathfrak{c}_j(t))$ of $\mathfrak{x}(t)$ in the euclidean space E^{n+1} we get

$$\ddot{\mathfrak{b}}=\mathfrak{c}_2k,\quad \mathfrak{b}^{(3)}=-\mathfrak{c}_1k^2+\mathfrak{c}_2k+\mathfrak{c}_3k\varkappa\,.$$

Now (1.26) gives us (1.32). □

Now we return to the reduction procedure. Following E. Cartan's method we have to calculate the transformation rule for the coefficients of third order $\omega^0=c^0\omega^1$, $\omega^2_r=c^2_r\omega^1$, $r=3,\ldots,n$, under the action of $\boldsymbol{H}_3$, compare (1.23), (1.24). With notations analogously to (1.11) we get

$$\bar{\omega}^0=\omega^0-\omega^1h^1_{n+1},\quad h^1_{n+1}\in\boldsymbol{R}\,.$$

So we always can find frames $z\in E_3$ with

$$\omega^0(z)=0\,. \tag{1.33}$$

For $n=2$, 3 we have $\boldsymbol{H}_4=\{e\}$, and the reduction is finished. In the case $n=2$ the

Frenet *formulas are* (see (1.4), (1.13), (1.22), (1.33)):

$$\mathfrak{b}_0' = \mathfrak{b}_1, \quad \mathfrak{b}_1' = \mathfrak{b}_0\lambda_1 + \mathfrak{b}_3, \tag{1.34}$$
$$\mathfrak{b}_2' = \mathfrak{b}_0\varepsilon, \quad \mathfrak{b}_3' = \mathfrak{b}_1\lambda_1 + \mathfrak{b}_2, \quad f(s) = \pi(\mathfrak{b}_0(s)), \quad \varepsilon = \pm 1 .$$

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Here s denotes the natural parameter and $' = d/ds$. The Frenet formulas for $n=3$ are easily seen to be a special case of (1.45).

For $n>3$ we remark that relation (1.33) remains invariant only for those $h \in \boldsymbol{H}_3$ with $h^1_{n+1} = h^0_1 = 0$. So we get the transformation rule

$$\bar{\omega}^2_r = \omega^2_s h^s_r, \quad r, s = 3, \ldots, n .$$

Since $\omega^1 = \bar{\omega}^1$ we obtain

$$\bar{c}^2_r = c^2_s h^s_r, \quad (h^s_r) \in \boldsymbol{SO}\,(n-2) . \tag{1.35}$$

We define

$$z \in E_4 :\leftrightarrow z \in E_3, \quad \omega^0(z) = 0, \quad \omega^3_2(z) = \lambda_2\omega^1 \quad \textit{with} \quad \lambda_2 \geqq 0 , \tag{1.36}$$
$$\textit{and} \quad \omega^r_2(z) = 0 \quad \textit{for} \quad r = 4, \ldots, n .$$

Now all coefficients of the third order have well defined values, and *the fourth reduction is finished*. Supposing $\lambda_2 > 0$, we get a uniquely defined vector $\mathfrak{b}_3$, and the structure group $\boldsymbol{H}_4 \subseteqq \boldsymbol{H}_3$ is given by $h^0_1 = 0$, $h^r_3 = h^3_r = \delta^r_3$, $(h^r_s) \in \boldsymbol{SO}\,(n-3)$ with $r, s = 4, \ldots, n$; the defining equations of its Lie algebra $\boldsymbol{h}_4 \subseteqq \boldsymbol{h}_3$ are

$$\vartheta^0_1 = 0, \quad \vartheta^r_3 = 0, \quad r = 4, \ldots, n . \tag{1.37}$$

By definition $\lambda_2 = \lambda_2(s)$ is constant on the fibres of E_4, and therefore it defines an $\boldsymbol{SG}_n$*-invariant of order* 4 of the immersion f. Furthermore the vectors $\mathfrak{b}_\varrho(z)$, $\varrho = 0, 1, 2, 3, n+1$, are constant on the fibres of E_4; so we obtain *two* $\boldsymbol{SG}_n$*-invariants of order* 5: 2

$$\lambda_1 = -\langle \mathfrak{b}_1', \mathfrak{b}_1' \rangle / 2 = -\langle \mathfrak{b}_1', \mathfrak{b}_{n+1} \rangle, \quad \omega^0_1 = \lambda_1\omega^1, \tag{1.38}$$

$$(\lambda_3)^2 = \langle \mathfrak{b}_3', \mathfrak{b}_3' \rangle - (\lambda_2)^2 \geqq 0 . \tag{1.39}$$

For $n=4$ *the reduction procedure is finished:* $\boldsymbol{H}_4 = \{e\}$. For $n \geqq 5$ let us suppose $\lambda_3^2 > 0$. Then there is a uniquely defined vector $\mathfrak{b}_4 = \mathfrak{b}_4(s)$ such that $\lambda_3 > 0$, and

$$z \in E_5 :\leftrightarrow z \in E_4, \quad \omega^4_3(z) = \lambda_3\omega^1, \quad \omega^a_3(z) = 0 \quad \textit{for} \quad a = 5, \ldots, n , \tag{1.40}$$

defines the $\boldsymbol{g}_n$, $\boldsymbol{H}_5$*-structure* E_5 *with the structure group* $\boldsymbol{H}_5 = \boldsymbol{SO}\,(n-4)$; $\boldsymbol{H}_5$ transfroms the vectors $(b_5, \ldots, b_n)$ of the moving frames.

For $k \geqq 5$ the reduction procedure is the same as in the euclidean geometry. Let us define the osculating space $V_k(t)$ of the k-th order of an immersion f at $t \in Y^1$ as the linear hull

$$V_k(t) := \mathfrak{L}(\mathfrak{b}(t), \dot{\mathfrak{b}}(t), \ldots, \mathfrak{b}^{(k)}(t)) . \tag{1.41}$$

Then $V_k(t)$ depends neither on the choice of the parameter t nor on the chosen representation $\mathfrak{b}(t) \in J^+$ with $f(t) = \pi(\mathfrak{b}(t))$. One easily proves

Lemma 1.2. *If* $\dim V_k(t)=k+1$ *for all* $t\in Y^1$, *and* $z=(\mathfrak{b}_i(t))$ *denotes a moving frame in* E_k, $k\geqq 2$, *then we have*

$$V_0=\mathfrak{L}(\mathfrak{b}_0),\quad V_1=\mathfrak{L}(\mathfrak{b}_0,\mathfrak{b}_1)\,, \tag{1.42}$$
$$V_k=\mathfrak{L}(\mathfrak{b}_0,\mathfrak{b}_1,\ldots,\mathfrak{b}_{k-1},\mathfrak{b}_{n+1}),\quad 2\leqq k\leqq n+1\,.$$

The space V_k; $k=2,\ldots,n+1$, *is a pseudo-euclidean vector space of index* 1; *it defines the osculating sphere of order* k *of the immersion at the point* $t\in Y^1$:

$$S^{k-1}(t)=\pi\,(J^+\cap V_k(t)),\quad k>1\,.\quad \square \tag{1.43}$$

For $k\leqq 5$ the proof is obvious, and for $k>5$ the assertion will become clear from the general reduction step described below:

An immersion $f\colon Y^1\to S^n$ is called *generic*, if $\dim V_n(t)=n+1$ for all $t\in Y^1$. Suppose the $\boldsymbol{g}_n$, $\boldsymbol{H}_k$-structure E_k, $\boldsymbol{H}_k=\boldsymbol{SO}\,(n-k+1)$, $k\geqq 5$, of the generic immersion f is already defined; $\boldsymbol{H}_k$ transforms the vectors $(\mathfrak{b}_k,\ldots,\mathfrak{b}_n)$ of the moving frames; its Lie algebra $\boldsymbol{h}_k\subset\boldsymbol{h}_{k-1}$ is characterized by $\vartheta^a_{k-1}=0$, $a=k,\ldots,n$. The vectors $\mathfrak{b}_0,\ldots,\mathfrak{b}_{k-1},\mathfrak{b}_{n+1}$ are constant on the fibres of E_k, and we have the $\boldsymbol{SG}_n$-invariants $\lambda_1,\ldots,\lambda_{k-2}$ of order $\leqq k$. Then *the reduction* E_{k+1} *is defined by*

$$z\in E_{k+1}:\leftrightarrow z\in E_k,\quad \omega^k_{k-1}=\lambda_{k-1}\omega^1\quad \textit{with}\quad \lambda_{k-1}>0, \tag{1.44}$$
$$\omega^b_{k-1}=0\quad \textit{for}\quad b=k+1,\ldots,n\,.$$

The inequality $\lambda_{k-1}>0$ is fulfilled for $3\leqq k\leqq n-1$ since f is generic. The reduction procedure breaks off with E_n, $\boldsymbol{H}_n=\boldsymbol{SO}(1)=\{e\}$. Choosing $\mathfrak{b}_n$ in correspondence to the orientation of S^n we get the canonical frame which delivers us the last $\boldsymbol{SG}_n$-invariant $\lambda_{n-1}=\langle\mathfrak{b}'_{n-1},\mathfrak{b}_n\rangle$. Thus we proved

Theorem 1.4. *Let* $f\colon Y^1\to S^n$ *be a generic immersion into the oriented Möbius space* S^n, Y^1 *connected and oriented. Then the* $\boldsymbol{g}_n$, $\boldsymbol{H}_N$*-structure* E_N, $N=\max\,(4,n)$, *defines a canonical frame* $z(s)=(\mathfrak{b}_i(s))$, $i=0,\ldots,n+1$, s *the natural conformal parameter of the immersion, which is a moving isotropic-orthogonal frame satisfying the following Frenet formulas* ($n\geqq 3$; *for* $n=2$ *see* (1.34)):

$$\begin{cases} f(s)=\pi(\mathfrak{b}_0(s))\,, \\ \mathfrak{b}_0'=\mathfrak{b}_1,\quad \mathfrak{b}_1'=\mathfrak{b}_0\lambda_1+\mathfrak{b}_{n+1},\quad \mathfrak{b}_2'=\mathfrak{b}_0+\mathfrak{b}_3\lambda_2\,, \\ \mathfrak{b}_k'=-\mathfrak{b}_{k-1}\lambda_{k-1}+\mathfrak{b}_{k+1}\lambda_k,\quad k=3,\ldots,n-1\,, \\ \mathfrak{b}_n'=-\mathfrak{b}_{n-1}\lambda_{n-1},\quad \mathfrak{b}_{n+1}'=\mathfrak{b}_1\lambda_1+\mathfrak{b}_2\,, \end{cases} \tag{1.45}$$

$$\lambda_\varrho>0\quad \textit{for}\quad \varrho=2,\ldots,n-2\,. \tag{1.46}$$

The $\lambda_k(s)$, $k=1,\ldots,n-1$, *and* $\omega^1=ds$ *make up a complete system of* $\boldsymbol{SG}_n$*-invariants for the class of the generic immersions.* $\square$

Remarks. 1. The last assertion of theorem 1.4 means the following: If on $Y^1=]a,b[\subset\boldsymbol{R}$ a 1-form ω^1 and $n-1$ functions λ_k with (1.46) are given, then there exists one and up to $\boldsymbol{SG}_n$-congruence only one generic immersion $f\colon Y^1\to S^n$ having a natural conformal parameter s with $\omega^1=ds$ and the given conformal curvatures λ_k. If $Y^1=S^1$, it may happen that we only obtain an immersion of a certain covering $\hat{Y}^1$ of S^1. The statement is an immediate consequence of E. Cartan's theory, compare R. Sulanke, A. Švec I [11], theorems 3.2, 4.2, 4.3. — 2. The invariants λ_k, $k=2,\ldots,n-1$, are differential invariants of order $k+2$, λ_1 is of order 5. — 3. The vector $\mathfrak{b}_k \in V_{k+1}$, $2 \leqq k \leqq n-1$, is orthogonal to $V_k \subseteqq V_{k+1}$; it is defined uniquely by the condition $\lambda_{k-1} > 0$. If $\lambda_k = 0$ for all $s \in Y^1$, $k \geqq 2$ fixed, one can prove as in proposition 1.1 that the immersion lies in a certain subsphere $S^k \supseteqq f(Y^1)$. Thus λ_k, $k \geqq 2$, may be considered as a measure for the variation of the osculating sphere S^k. The SG_n-invariant λ_1 defines the parameter of the *loxodrome* (curve of constant curvature) on $S^2(s)$ osculating the immersion. — 4. Particularly, we have $\lambda_{n-1} = 0$ if and only if the generic immersion lies on a hypersphere $S^{n-1} \subseteqq S^n$; if $\lambda_{n-1} \neq 0$ we can distinguish whether the curve turns right or left.

2. Curves of constant conformal curvatures in the Möbius space S^3

In order to determine all the curves of constant curvatures λ_1, λ_2 we first prove (compare E. Vessiot I [15], (86), p. 90)

Proposition 2.1. *Let $\mathfrak{b}_0 = \mathfrak{b}_0(s)$ be the canonical representing vector of a curve of constant curvatures λ_1, λ_2 of the* Möbius *space S^3, and s its natural conformal parameter. Then $\mathfrak{b}_0(s)$ fulfils the differential equation*

(2.1) $$\mathfrak{b}_0^{(4)} = \mathfrak{b}_0^{(2)} A + \mathfrak{b}_0 B + \mathfrak{c}, \quad \mathfrak{c} = \text{const.},$$

with

(2.2) $$A = 2\lambda_1 - \lambda_2^2, \quad B = 1 + 2\lambda_1 \lambda_2^2 .$$

Proof. From the Frenet formulas (1.45) for $n=3$ we obtain taking into account $\lambda_1, \lambda_2 = \text{const.}$:

(2.3) $$\mathfrak{b}_0' = \mathfrak{b}_1, \quad \mathfrak{b}_0'' = \mathfrak{b}_0 \lambda_1 + \mathfrak{b}_4, \quad \mathfrak{b}_0^{(3)} = 2\mathfrak{b}_1 \lambda_1 + \mathfrak{b}_2 ,$$

(2.4) $$\mathfrak{b}_0^{(4)} = \mathfrak{b}_0 \, (1 + 2\lambda_1^2) + \mathfrak{b}_3 \lambda_2 + 2\mathfrak{b}_4 \lambda_1 ,$$

(2.5) $$\mathfrak{b}_0^{(5)} = \mathfrak{b}_1 \, (1 + 4\lambda_1^2) + \mathfrak{b}_2 \, (2\lambda_1 - \lambda_2^2) .$$

By a simple calculation we get

$$\mathfrak{b}_0^{(5)} - \mathfrak{b}_0^{(3)} A - \mathfrak{b}_0' = 0 ,$$

from which (2.1) follows by an integration. □

Of course there exist solutions of (2.1) not representing curves of constant curvatures. To single out the solutions interesting for our problem we prove

Lemma 2.1. *Let $s \in \mathbf{R} \mapsto \mathfrak{x}(s) \in V^5$ be a solution of (2.1) with values in the pseudo-euclidean vector space V^5 of index 1. Then $\mathfrak{b}_0(s) = \mathfrak{x}(s)$ is the canonical representing vector of a curve of constant curvatures λ_1, λ_2 with the natural conformal parameter s if and only if the scalar products of the derivatives $\mathfrak{x}^{(i)}(s)$, $i=0,\ldots,4$, have the* *following values:*

$$\langle \mathfrak{x}^{(i)}, \mathfrak{x}^{(j)} \rangle = (-1)^{i+1} \quad \text{for} \quad i+j=2\,,$$
$$= (-1)^{i+1} 2\lambda_1 \quad \text{for} \quad i+j=4\,,$$
$$= (-1)^{i+1}(1+4\lambda_1^2) \quad \text{for} \quad i+j=6\,,$$
$$= \lambda_2^2 - 8\lambda_1^3 - 4\lambda_1 \quad \text{for} \quad i=j=4$$
$$= 0 \quad \text{for all other} \quad i, j = 0, \ldots, 4\,. \tag{2.6}$$

Proof. The necessity is easily seen using (2.3), (2.4). Conversely, let $\mathfrak{x}=\mathfrak{x}(s)$ be a solution of (2.1) the derivatives of which fulfil (2.6) with the constants λ_1, λ_2 defining A, B by (2.2). Assuming for simplicity $\lambda_2 \neq 0$ we define

$$\mathfrak{b}_0(s)=\mathfrak{x}(s), \quad \mathfrak{b}_1(s)=\mathfrak{x}'(s), \quad \mathfrak{b}_2(s)=\mathfrak{x}^{(3)}(s)-2\mathfrak{x}'(s)\,,$$
$$\mathfrak{b}_3(s)=[\mathfrak{x}^{(4)}-2\mathfrak{x}''\lambda_1-\mathfrak{x}]\,(s)/\lambda_2\,, \tag{2.7}$$
$$\mathfrak{b}_4(s)=\mathfrak{x}''(s)-\mathfrak{x}(s)\,\lambda_1\,.$$

As a consequence of (2.1), (2.6) we prove by a straightforward calculation that $(\mathfrak{b}_i(s))$ is a moving i.-o. frame fulfilling the Frenet formulas. Therefore it is the canonical frame of the curve $f(s)=\pi(\mathfrak{b}_0(s))$ of constant curvatures λ_1, λ_2 (see remark 1. after theorem 1.4), what implies the statement. □

Remark. The assumption $\lambda_2 \neq 0$ could be avoided defining $\mathfrak{b}_3$ as the oriented orthonormal complementary vector $\mathfrak{b}_3 = -\mathfrak{b}_0 \times \mathfrak{b}_1 \times \mathfrak{b}_2 \times \mathfrak{b}_4$.

Now we resolve the linear differential equation (2.1) by the usual methods. First we remark:

Lemma 2.2. *The equations* (2.2) *imply* $A^2+B^2 \neq 0$. *In the case* $B \neq 0$ *the constant*

$$\mathfrak{b}_0 = -\mathfrak{c}B^{-1} \tag{2.8}$$

is a solution of (2.1). *For* $B=0$ *the function*

$$\mathfrak{b}_0(s) = -\mathfrak{c}s^2/2A \tag{2.9}$$

solves (2.1). □

The characteristic equation of the homogeneous part of (2.1) is

$$x^4 - Ax^2 - B = 0\,. \tag{2.10}$$

It has the solutions $\pm\sqrt{\nu_+}$, $\pm\sqrt{\nu_-}$ with

$$\nu_\pm = (A \pm \sqrt{4B+A^2})/2\,, \tag{2.11}$$

here ν_+, ν_- are distinct real numbers since we have

$$A^2+4B = 4+(2\lambda_1+\lambda_2^2)^2 \geqq 4\,. \tag{2.12}$$

Now we distinguish the following three cases:

I. $B>0$. Then $\nu_+>0$, $\nu_- \neq 0$, and we obtain the four distinct roots

$$\pm\alpha, \quad \pm i\beta \quad \text{with} \quad \alpha=\sqrt{|\nu_+|}, \quad \beta=\sqrt{|\nu_-|}\,. \tag{2.13}$$

II. $B<0$. This implies $A<0$, and we have the four distinct roots

$$\pm i\alpha, \quad \pm i\beta\,. \tag{2.14}$$

III. $B=0$. Then we have the root $\nu_+=\alpha=0$ of multiplicity 2 and the two distinct roots

(2.15) $\qquad \pm i\beta \quad \text{with} \quad \beta^2=-A\,.$

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Since for each pair (λ_1, λ_2) there exists up to Möbius equivalence one and only one immersion of $\boldsymbol{R}$ of constant curvatures λ_1, λ_2, it suffices to find such a curve in each of the three cases. This can be done choosing the constants of the general solution in such a manner that (2.6) is satisfied. We choose a fixed pseudo-orthogonal frame $(\mathfrak{e}_j)$, $j=0, \ldots, 4$, and identify the representing vector $\mathfrak{b}_0(s)$ of the curve with the column vector of its coordinates $(x_j(s))$,

$$\mathfrak{b}_0(s)=\sum_{j=0}^{4} \mathfrak{e}_j x_j(s)\,.$$

Carrying out the necessary calculations one gets

Theorem 2.2. *To each pair $(\lambda_1, \lambda_2)\in \boldsymbol{R}^2$ exists up to $\boldsymbol{SG}_3$-equivalence one and only one immersion $f\colon \boldsymbol{R}\to S^3$ of $\boldsymbol{R}$ into the oriented Möbius space S^3 with constant curvatures λ_1, λ_2. These immersions are represented as orbits of the corresponding 1-parameter subgroups of $\boldsymbol{SG}_3$ in the following way:*

I. $B>0$.

$$(2.16) \qquad \begin{pmatrix} \cosh \alpha s & \sinh \alpha s & 0 & & 0 \\ \sinh \alpha s & \cosh \alpha s & & & 0 \\ & & \cos \beta s & -\sin \beta s & 0 \\ 0 & & \sin \beta s & \cos \beta s & 0 \\ & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varrho \\ 0 \\ r \\ 0 \\ c \end{pmatrix} = \mathfrak{b}_0(s)$$

with $\varrho>0$, $r>0$, $c=(\varrho^2-r^2)^{1/2}$, *and*

(2.17) $\qquad \varrho^2=(\beta^2+2\lambda_1)/\alpha^2\,(\alpha^2+\beta^2), \quad r^2=(\alpha^2-2\lambda_1)/\beta^2\,(\alpha^2+\beta^2);$

II. $B<0$.

$$(2.18) \qquad \begin{pmatrix} 1 & 0 & 0 & & \\ 0 & \cos \alpha s & -\sin \alpha s & 0 & \\ 0 & \sin \alpha s & \cos \alpha s & & \\ 0 & 0 & & \cos \beta s & -\sin \beta s \\ 0 & & & \sin \beta s & \cos \beta s \end{pmatrix} \begin{pmatrix} C \\ R \\ 0 \\ Q \\ 0 \end{pmatrix} = \mathfrak{b}_0(s)$$

with $R>0$, $Q>0$, $C=(R^2+Q^2)^{1/2}$, *and*

(2.19) $\qquad Q^2=(\alpha^2+2\lambda_1)/\beta^2\,(\alpha^2-\beta^2), \quad R^2=(\beta^2+2\lambda_1)/\alpha^2\,(\beta^2-\alpha^2);$

III. $B=0$.

$$(2.20) \qquad \begin{pmatrix} 1+\dfrac{a^2s^2}{2} & 0 & 0 & as & -\dfrac{a^2s^2}{2} \\ 0 & \cos \alpha s & -\sin \alpha s & 0 & 0 \\ 0 & \sin \alpha s & \cos \alpha s & 0 & 0 \\ as & 0 & 0 & 1 & -as \\ \dfrac{a^2s^2}{2} & 0 & 0 & as & 1-\dfrac{a^2s^2}{2} \end{pmatrix} \begin{pmatrix} r \\ r \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathfrak{b}_0(s)$$

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with $r > 0$, $a > 0$, and

(2.21) $\qquad r^2 = \lambda_2^2/(1+\lambda_2^4)^2, \quad a^2 = \lambda_2^2\,(1+\lambda_2^4)$. □

The orbit representations (2.16), (2.18), (2.20) are well appropriated for a geometric discussion of the curves. The invariant directions of the corresponding screwings in V^5 and the fixed points in S^3 can be determined as follows: Let

(2.22) $\qquad X := \lim_{s\to 0}\,(g_s - e)/s$

be the infinitesimal generator of the 1-parameter subgroup

$$g_s = \exp\,(sX)\,.$$

The invariant directions $\mathfrak{m}$ of $\{g_s\}$ are determined by the eigenvectors of X; if μ is the corresponding eigenvalue, we get

(2.23) $\qquad g_s(\mathfrak{m}) = \mathfrak{m}\cdot\exp\,(\mu s), \quad \text{if} \quad X\mathfrak{m} = \mathfrak{m}\mu\,.$

Since we consider only real μ and g_s preserves the scalar product, we see that for $\mu = 0$ the vector $\mathfrak{m}$ itself is fixed by g_s, and for $\mu \neq 0$ the vector $\mathfrak{m}$ must be isotropic, thus defining a point of S^3. Performing the easy calculations we obtain

Proposition 2.3. *The curves of constant curvatures λ_1, λ_2 in S^3 are orbits of the 1-parameter subgroups of $\boldsymbol{SG}_3$ described by the matrices in* (2.16), (2.18), (2.20). *The decompositions of V^5 into invariant subspaces are*

case I: $\quad V^5 = \mathfrak{L}(\mathfrak{e}_0, \mathfrak{e}_1) \oplus \mathfrak{L}(\mathfrak{e}_2, \mathfrak{e}_3) \oplus \mathfrak{L}(\mathfrak{e}_4)$,

case II: $\quad V^5 = \mathfrak{L}(\mathfrak{e}_0) \oplus \mathfrak{L}(\mathfrak{e}_1, \mathfrak{e}_2) \oplus \mathfrak{L}(\mathfrak{e}_3, \mathfrak{e}_4)$,

case III: $\quad V^5 = \mathfrak{L}(\mathfrak{e}_0, \mathfrak{e}_3, \mathfrak{e}_4) \oplus \mathfrak{L}(\mathfrak{e}_1, \mathfrak{e}_2)$.

Here $\mathfrak{L}(\mathfrak{e}_4)$, $\mathfrak{L}(\mathfrak{e}_0)$, and $\mathfrak{L}\,(\mathfrak{e}_0 + \mathfrak{e}_4)$ are the eigenspaces for the eigenvalue $\mu = 0$ of X in case I, II, *and* III *respectively. In case* I *we have two fixed points $\pi\,(\mathfrak{e}_0 \pm \mathfrak{e}_1) \in S^3$, in case* III *only one fixed point $\pi\,(\mathfrak{e}_0 + \mathfrak{e}_4) \in S^3$, and in case* II *there are no fixed points in S^3.* □

It is well-known, compare W. Blaschke, G. Thomsen I [4], § 19, in the case $n = 2$, that the isometry groups of the simply connected spaces $E^n(c)$ of constant curvatures c are the subgroups of the Möbius group $\boldsymbol{G}_n$ fixing a vector $\mathfrak{k} \in V^{n+2}$, $\mathfrak{k} \neq 0$, $\langle \mathfrak{k}, \mathfrak{k}\rangle = 1$ in the case $c < 0$, $\langle \mathfrak{k}, \mathfrak{k}\rangle = -1$ in the case $c > 0$, and $\langle \mathfrak{k}, \mathfrak{k}\rangle = 0$ in the euclidean case, see I(2.15). Since clearly each conformal image of a curve of constant curvatures in one of the space forms $E^n(c)$ must be a curve of constant conformal curvatures in S^3, we get the following corollary of theorem 2.2 and proposition 2.3:

Corollary 2.1. *The generic curves of constant conformal curvatures λ_1, λ_2 in the* Möbius *space S^3 coincide with the loxodromes and the conformal images of the curves of constant curvatures in the simply connected space forms. The correspondence is*

$B = 1, \quad \lambda_2 = 0$: *loxodromes in S^2, of curvature λ_1* ,

$B > 0, \quad \lambda_2 \neq 0$: $E^3(-c^2)$, $\quad c \neq 0$,

$B < 0$: $\qquad E^3(c^2)$, $\quad c \neq 0$,

$B = 0$: $\qquad E^3(0)$, $\quad$ *euclidean space* :

Such a curve is closed if and only if $B < 0$ and α/β rational.

Proof. It only remains to prove the statement in the case $B=1$, $\lambda_2=0$. It follows that $\varrho^2=r^2$. By (2.16) the curve lies in the sphere S^2 defined by $\langle \mathfrak{e}_4, \mathfrak{x}\rangle=0$. It is a loxodrome turning around the 'south pole' $\pi\,(\mathfrak{e}_0-\mathfrak{e}_1)\in S^2$ for $s\to-\infty$, and around the 'north pole' $\pi\,(\mathfrak{e}_0+\mathfrak{e}_1)$ for $s\to+\infty$, see W. Blaschke, G. Thomsen I [4], p. 110. $\square$

Euclidean models for the curves under consideration can be obtained by stereographical projection from one of the fixed points in the cases I or III. It results: case I. Isogonal trajectories of a circular cone, especially the loxodromes in the plane for $\lambda_2=0$; case II. Isogonal trajectories of the generators of a flat torus $T^2\subseteqq S^3$; case III. Circular helices, i.e. the isogonal trajectories of a cylinder in $E^3(0)$. All the surfaces we mentioned are *cyclids of* Dupin, compare W. Blaschke, G. Thomsen I [4], and T. E. Cecil [1].

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