Submanifolds of the Möbius Space III
The Analogue of O. Bonnet's Theorem for Hypersurfaces

Dedicated to Professor Akitsugu Kawaguchi
on the occasion of his 80th birthday

by
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Introduction

The goal of this paper is to prove a fundamental theorem for hypersurfaces of the sphere S^n, considered as the Möbius space $S^n = G_n/H_o$, where G_n denotes the conformal group of the standard sphere and H_o the isotropy group of a fixed point, the origin $o \in S^n$. It turns out that for each immersion f: $Y^{n-1} \longrightarrow S^n$, $n \geq 4$, without M-flat points two G_n-invariant tensors can be defined, which yield a complete system of invariants of the immersion, and that a corresponding fundamental theorem can be proved. For n = 3 one has to add a G_3-invariant on Y^2 to obtain the corresponding result. The method applied in this paper is a precise and global version of E. Cartan's method of moving frames, described in R. Sulanke, A. Švec [5], see also [4]. In a paper of K. Yano and Y. Mutô [8], which has not been available to the author, a similar fundamental theorem seems to be proved by methods of Riemannian geometry, cf. MR 7, p. 332. Two years later, 1944, there appeared a paper of A. Fialkow [2], in which, among other more general results, a fundamental theorem for hypersurfaces of S^n was obtained by Riemannian methods; he needed 3 fundamental tensors to characterize the hypersurface. Finally, in 1952 M.A. Akivis [1] proved a fundamental theorem by the method of G.F. Laptev; he used 3 tensors of valence 2 and a covector field as a complete invariant system. Therefore it may be justified to give a new and global treatment of this old subject considered already by E. Vessiot [7] for the case $n = 3$; see also the book of T. Takasu [6], where hints to earlier literature can be found.

In the following we use the notations and results of sections 2,4 of Ch. Schiemangk, R. Sulanke [3], where we introduced the so-called Möbius structure of an immersion $f: Y^m \longrightarrow S^n$, $2 \leq m \leq n-1$, without M-flat points. In the euclidean picture a point $y \in Y^m$ is an M-flat point of f iff it is flat or umbilical. An immersion, all the points of which are M-flat, is an open submanifold of an m-sphere $S^m \subset S^n$. Since already the first G_n-invariant fundamental form remains undefined at M-flat points, these points have to be excluded. In section 1 we define a reduced Möbius structure of the immersion f and discuss G_n-invariant objects and constructions associated with it. In section 2 we restrict ourselves to the case $m=n-1$ and prove the fundamental theorem. For simplicity we assume that all the manifolds and maps under consideration are of class C^∞.

1. The reduced Möbius structure

Let $\bar{E}(Y^m, \bar{\omega}_1)$ be the Möbius structure of the immersion $f: Y^m \longrightarrow S^n$, $2 \leq m \leq n-1$, without M-flat points ([3], prop. 4.2). The structure group $\bar{H}_1$ of the principal bundle $\bar{E}_1$ is the set of all matrices

$$\begin{pmatrix} 1 & {}^t c & 0 & b \\ 0 & A_1 & 0 & A_1 c \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c \in \mathbb{R}^m,\ A_1 \in O(m),\ A_2 \in O(n-m), \quad b = \langle c,c\rangle/2. \tag{1.1}$$

The structure form $\bar{\omega}_1$ is a 1-form on $\bar{E}_1$ with values in the Lie algebra $\underline{g_n}$ of the Möbius group G_n. With respect to the matrix representation $\bar{\omega}_1(z) = (\omega^J{}_I(z))$, $z \in \bar{E}_1$, I, J = 0,..., n+1, the properties of the Möbius structure can be summed up as follows:

$$\omega^k(z) := \omega^k{}_0(z) = 0, \quad k = m+1,\dots,n; \tag{1.2}$$

putting $\omega^\alpha(z) := \omega^\alpha{}_0(z)$, $\alpha = 1,\dots,m$, we have

$$\omega^1 \wedge \dots \wedge \omega^m(z) \neq 0; \tag{1.3}$$

the ω^α, $\alpha = 1,\dots,m$, are the basis forms of the immersion. With respect to a local section $y \in U \longrightarrow z(y) \in \bar{E}_1$ of $\bar{E}_1$, a 'moving frame' of f, the G_n-invariant first fundamental form of the immersion is given by

$$\varphi_y(s,t) = \sum_{\alpha=1}^{m} \omega^\alpha(y,s)\, \omega^\alpha(y,t), \quad s,t \in T_yY^m \tag{1.4}$$

here and in the following we write by abuse of notation

$$\omega^I{}_J(y,dy) = (z^* \omega^I{}_J)(y,dy). \tag{1.5}$$

The coefficients $c^\kappa_{\alpha\beta}$ are defined by

$$\omega^\kappa{}_\alpha(z) = c^\kappa_{\alpha\beta}(z)\, \omega^\beta(z), \quad \kappa = m+1,\dots,n, \quad \alpha,\beta = 1,\dots,m; \tag{1.6}$$

here and in the following we apply the sum convention. For $z \in \bar{E}_1$ we have the symmetry

$$c^\kappa_{\alpha\beta} = c^\kappa_{\beta\alpha}, \tag{1.7}$$

the trace condition

$$\sum_{\alpha=1}^{m} c^\kappa_{\alpha\alpha}(z) = 0, \tag{1.8}$$

and the norming

$$\sum_{\kappa=m+1}^{n} \sum_{\alpha,\beta=1}^{m} (c^{\kappa}_{\alpha\beta}(z))^2 = 1 , \qquad (1.9)$$

As in euclidean geometry the $c^{\kappa}_{\alpha\beta}(y) := (z^{*} c^{\kappa}_{\alpha\beta})(y)$ define a G_n-invariant symmetric bilinear map $\alpha_y : T_y \times T_y \longrightarrow N_y$, the vectorial 2nd fundamental form of the immersion:

$$\alpha_y(s,t) = c_{\kappa}(y) c^{\kappa}_{\alpha\beta}(y) \omega^{\alpha}(y,s) \omega^{\beta}(y,t), \qquad (1.10)$$

the trace of which is zero; here $c_i = d\pi_{b_o}(b_i) \in T_{f(y)}S^n$, $i = 1,\ldots,n$, and the $c_{\kappa}(y) \in N_y := (df_y T_y)^{\perp}$, $\kappa = m+1,\ldots,n$, are an orthogonal basis of the normal space at y. Of course, the form φ_y is positive definite; it defines a G_n-invariant Riemannian metric on Y^m.

Following literally E. Cartan's method one would now have to find normal forms for the action of $\overline{H}_1$ on the coefficients $c^{\kappa}_{\alpha\beta}(z)$. This action has $\{h \in \overline{H}_1 | A = 1_m, A_2 = 1_{n-m}, c \in \mathbb{R}^m\}$ (see (1.1)) as its kernel of non-effectivity; thus it coincides with the action of $O(m) \times O(n-m)$ on the $c^{\kappa}_{\alpha\beta}$ known from euclidean geometry. To get normal forms for this action one would have to assume further restrictions to the class of immersion under consideration, e.g. for $m = n-1$ that all the eigenvalues of the 2nd fundamental form are different. To avoid this we now deviate form E. Cartan's method and proceed as follows. We recall that the Lie subalgebras $\overline{\underline{h}}_1 \subset \underline{h}_1 \subset \underline{h}_o \subset \underline{\underline{g}}_n$ of the structure groups of the successive reductions are successively defined by the following equations in terms of the Maurer-Cartan forms ϑ^J_I of G_n (cf. [3],(2.16),(4.6),(4.20)):

$\underline{h}_0 \subset \underline{g}_n$: $\vartheta^i = 0$, $i = 1,\dots,n$; (1.11)

$\underline{h}_1 \subset \underline{h}_0$: $\vartheta^\varkappa_\alpha = 0$, $\alpha = 1,\dots,m$, $\varkappa = m+1,\dots,n$; (1.12)

$\underline{\overline{H}}_1 \subset \underline{h}_1$: $\vartheta^o_o = 0$, $\vartheta^o_\varkappa = 0$. (1.13)

Therefore we calculate now the transformation rule for $\omega^o := \omega^o_o$ under the action of $\overline{H}_1$. Using (1.1) and putting $z = (y,(b_I))$, $\hat{z} = h^{-1} \times z = (y,(\hat{b}_I))$, $\omega^I_J = \omega^I_J(z)$, $\hat{\omega}^I_J = \omega^I_J(\hat{z})$ we obtain:

$$\hat{\omega}^o = -\langle d\hat{b}_o, \hat{b}_{n+1}\rangle = \omega^o - \sum_{\alpha,\beta=1}^{m} \omega^\alpha h^\alpha_\beta c^\beta \qquad (1.14)$$

with $c = (c^\beta) = (h^o_\beta) \in \mathbb{R}^m$, and it follows

Proposition 1.1. Let $f: Y^m \longrightarrow S^n$, $2 \leq m \leq n-1$, be an immersion without M-flat points. Then

$$\tilde{E} := \{z \in \overline{E}_1 \mid \omega^o(z) = 0\}, \quad \tilde{\omega} := \tilde{\omega}_1 | T\tilde{E}, \qquad (1.15)$$

defines a natural reduction of the Möbius structure $\overline{E}_1$ of f to the subgroup $O(m) \times O(n-m) \subset \overline{H}_1$ characterized by

$$h^o_\beta = 0, \quad \beta = 1,\dots,m. \qquad (1.16)$$

The proof is standard: Since $\overline{E}_1$ is a G_n, $\overline{H}_1$-structure, condition (3.4) of [5] and (1.13) imply $\omega^o(z,t) = 0$ for all vertical tangential vectors t of $\overline{E}_1$ at z. Thus we have $\omega^o(z) = a_\alpha(z)\,\omega^\alpha(z)$, and $\tilde{E}$ is defined by the equations $a_\alpha(z) = 0$, $\alpha = 1,\dots,m$. From (1.14) we see that each $z \in \overline{E}_{1y}$ can be transformed into a $\tilde{z} \in \tilde{E}_y$ by putting $h^\alpha_\beta = \delta^\alpha_\beta$, $c^\beta = a_\beta(z)$. Applying this to a family of smooth local sections of $\overline{E}_1$ we obtain another family of smooth local sections of $\overline{E}_1$, and from (1.14) with $\omega^o = \hat{\omega}^o = 0$ we see that the transition functions of the transformed sections take values in the subgroup of $\overline{H}_1$ defined by (1.16). Thus the constructed family defines the reduction. Clearly, isomorphic G_n, $\overline{E}_1$-structures lead to isomorphic reductions, and therefore the reduction is natural. □

Looking at the transformation rules for the moving frames of the Möbius structure $\overline{E}_1$ or its reduction $\tilde{E}$ we obtain the following G_n-invariant objects for the immersion f: 1. A distinguished tangential m-sphere at y, the so-called central m-sphere S^m_y, defined by

$$\langle b_\kappa(z),x\rangle = 0, \quad \kappa = m+1,\dots,n, \quad z\in\overline{E}_{1\,y}. \qquad (1.17)$$

2. A uniquely defined representing vector $b_o(y) = b_o(z)$, $z\in\overline{E}_{1y}$; it defines an immersion $y\in Y^m \longrightarrow b_o(y)\in J^+$ into the isotropic upper half cone J^+ of the pseudo-euclidean vector space V^{n+2} such that

$$db_o = b_o\omega^o + b_\alpha\,\omega^\alpha \qquad (1.18)$$

induces the Riemannian metric φ defined by (1.4).

3. A uniquely defined normal (n-m)-sphere Σ^{n-m}_y characterized by

$$\langle b_\alpha(z),x\rangle = 0,\ \alpha = 1,\dots,m,\ z\in\tilde{E}_y. \qquad (1.19)$$

4. A uniquely defined associated isotropic vector

$$b(y) := b_{n+1}(z), \qquad z\in\tilde{E}_y, \qquad (1.20)$$

and the corresponding associated map $y\in Y^m \longrightarrow b(y) := \pi(b(y))\in S^n$. Obviously we have $f(y) \neq b(y)$ and

$$S^m_y \cap \Sigma^{n-m}_y = \{f(y), b(y)\}. \qquad (1.21)$$

5. The invariantly defined vectors b_o, b yield the so-called 3rd fundamental form

$$\xi(s,t) := \langle db_{oy}(s), db_y(t)\rangle. \qquad (1.22)$$

With respect to a local section of $\tilde{E}$ we obtain

$$\xi = \omega^\alpha \overset{o}{\omega}_\alpha = a_{\alpha\beta}\,\omega^\alpha \omega^\beta \quad \text{with} \tag{1.23}$$

$$\overset{o}{\omega}_\alpha(y) = a_{\alpha\beta}(y)\,\omega^\beta(y). \tag{1.24}$$

Applying Cartan's lemma we get from $d\overset{o}{\omega} = 0$ the symmetry

$$a_{\alpha\beta}(y) = a_{\beta\alpha}(y). \tag{1.25}$$

6. Suppose the sphere S^n to be oriented and restrict to the orientation preserving subgroup SG_n. In the case $m = n-1$ of an oriented hypersurface the normal vector $b_n(y)$ of f at y is uniquely defined, and the same is true for the 2nd fundamental form

$$\psi_y(s,t) = -\langle db_{ny}(s), db_{oy}(t)\rangle = c_{\alpha\beta}(y)\,\omega^\alpha(s)\,\omega^\beta(t) \tag{1.26}$$

being the unique component of α_y in (1.10). Changing the orientation implies a change of the sign of ψ. Furthermore we obtain the invariantly defined covector field on Y^{n-1}

$$\overset{o}{\omega}_n(y) = \langle db_y, b_n(y)\rangle = b_\alpha(y)\,\omega^\alpha(y) \tag{1.27}$$

associated to $\tilde{E}$; it also changes the sign under a changing of the orientation.

2. The fundamental theorem for hypersurfaces

In this section we only consider orientation preserving $g \in SG_n$, i.e. positively oriented frames (b_I) such that $(c_i) = (d\pi_b(b_i))$, $i = 1,\dots,n$, belongs to the fixed orientation of S^n. If $f\colon Y^{n-1} \longrightarrow S^n$ is a hypersurface without M-flat points, the fundamental forms φ, ψ are defined. It is now our task to characterize the reduced Möbius structure $\tilde{E}$ of f by means of φ and ψ. First we remark

Lemma 2.1. The principal bundle $\tilde{E}$ of $f\colon Y^{n-1} \to S^n$ coincides with the bundle of orthonormal frames of the Riemannian manifold $[Y^{n-1}, \varphi]$.

Proof. Each $z \in \tilde{E}_y$ is represented as an i.-o. frame $(y,(b_o(y),b_\alpha,b_n, b(y)))$, where b_o and b are constant on $\tilde{E}_y$ and (b_α,b_n), $\alpha=1,\ldots,n-1$, is positively oriented and belongs to the orthogonal complement $\mathcal{L}(b_o(y),b(y))^\perp$ in V^{n+2}. Given (b_α), the vector b_n is uniquely defined by orthogonality and orientation. By (1.4) and (1.18) the (b_α) represent an orthonormal frame of the Riemannian metric φ. □

The next step is to express the structure form $\check{\omega}$ of $\check{E}$ by the fundamental forms φ,ψ. Since $\check{E}$ is the principal bundle of orthonormal frames of the Riemannian manifold, the components $\omega^\alpha(z)$ of the canonical form and the components $\omega^\alpha_\beta(z)$ of the connection form of the Levi-Civita connection are well defined by φ; indeed, according to [3], (2.11) and (2.12), we have

$$\omega^\alpha_\beta + \omega^\beta_\alpha = 0, \quad d\omega^\alpha = -\omega^\alpha_\beta \wedge \omega^\beta, \ \alpha, \ \beta = 1,\ldots,n-1. \qquad (2.\ )$$

Here we used that for $z \in \tilde{E}$

$$\omega^o(z) = 0, \quad \omega^n(z) = 0 \qquad (2.2)$$

are fulfilled. The 2nd fundamental form ψ provides the forms

$$\omega^n_\alpha(z) = c_{\alpha\beta}(z)\omega^\beta(z). \qquad (2.3)$$

It remains to determine the forms $\omega^o_\alpha = \omega^\alpha_{n+1}$ and $\omega^o_n = \omega^n_{n+1}$ of $\tilde{\omega}$, see [3], (2.11). For this purpose we shall evaluate the integrability conditions corresponding to [3], (2.12). In particular, we have

$$d\omega^n{}_\alpha + \omega^n{}_\beta \wedge \omega^\beta{}_\alpha = -\omega^o{}_n \wedge \omega^\alpha \quad . \tag{2.4}$$

We substitute (1.27) and compare the basis decompostitions of both sides of (2.4):

$$d\omega^n{}_\alpha + \omega^n{}_\beta \wedge \omega^\beta{}_\alpha = \frac{1}{2} A_{\alpha,\gamma\delta}\, \omega^\gamma \wedge \omega^\delta$$

$$= \frac{1}{2}(\delta_{\alpha\gamma}\; b_\delta - \delta_{\alpha\delta}\; b_\gamma) \tag{2.5}$$

The $A_{\alpha,\gamma\delta}$ with

$$A_{\alpha,\gamma\delta} + A_{\alpha,\delta\gamma} = 0 \tag{2.6}$$

can be expressed by φ and ψ; thus they are known. In

$$A_{\alpha,\gamma\delta} = \delta_{\alpha\gamma}\; b_\delta - \delta_{\alpha\delta}\, b_\gamma \tag{2.7}$$

we take $\gamma = \alpha \neq \delta$ and obtain

$$b_\delta = A_{\alpha,\alpha\delta} \;, \quad \alpha \neq \delta, \quad 1 \leq \alpha,\ \delta \leq n-1 \,. \tag{2.8}$$

Now remember the Riemannian curvature tensor $R_{\alpha\beta\gamma\delta}$ defined by

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta = \frac{1}{2} R_{\alpha\beta\gamma\delta}\, \omega^\gamma \wedge \omega^\delta \,. \tag{2.9}$$

We introduce the tensor

$$\Pi^\alpha{}_\beta := -\omega^n{}_\alpha \wedge \omega^n{}_\beta = -\frac{1}{2}(c_{\alpha\gamma}c_{\beta\delta} - c_{\alpha\delta}c_{\beta\gamma})\,\omega^\gamma \wedge \omega^\delta \tag{2.10}$$

and define

$$\overset{o}{\Omega}{}^\alpha{}_\beta := \Omega^\alpha{}_\beta + \Pi^\alpha{}_\beta = \frac{1}{2} \overset{o}{R}_{\alpha\beta\gamma\delta}\, \omega^\gamma \wedge \omega^\delta \qquad \text{with} \tag{2.11}$$

$\overset{\circ}{R}_{\lambda\beta\gamma\delta} + \overset{\circ}{R}_{\lambda\beta\delta\gamma} = 0;$

clearly, the $\overset{\circ}{R}_{\alpha\beta\gamma\delta}$ are uniquely defined by φ and ψ. Substituting them on the left and (1.24) on the right-hand side of

$$d\omega^{\alpha}_{\beta} + \omega^{\alpha}_{\gamma} \wedge \omega^{\gamma}_{\beta} + \omega^{\alpha}_{n} \wedge \omega^{n}_{\beta}$$
$$= -\omega^{o}_{\alpha} \wedge \omega^{\beta} - \omega^{\alpha} \wedge \omega^{o}_{\beta} \qquad (2.12)$$

we obtain

$$\overset{\circ}{R}_{\alpha\beta\gamma\delta} = -(a_{\alpha\gamma}\delta_{\beta\delta} - a_{\alpha\delta}\delta_{\beta\gamma} - a_{\beta\gamma}\delta_{\alpha\delta} + a_{\beta\delta}\delta_{\alpha\gamma}), \qquad (2.13)$$

as a system of equations equivalent to (2.12). In particular, (2.13) implies

$$\overset{\circ}{R}_{\alpha\beta\alpha\beta} = -a_{\alpha\alpha} - a_{\beta\beta}, \quad \alpha \neq \beta, \quad 1 \leq \alpha, \beta \leq n-1. \qquad (2.14)$$

If $n \geq 4$, we get for each triple of different indices α, β, γ:

$$-2a_{\alpha\alpha} = \overset{\circ}{R}_{\alpha\beta\alpha\beta} + \overset{\circ}{R}_{\alpha\gamma\alpha\gamma} - \overset{\circ}{R}_{\beta\gamma\beta\gamma}, \quad \alpha \neq \beta, \gamma; \quad \beta \neq \gamma. \qquad (2.15)$$

Again for $n \geq 4$, we get from (2.13):

$$-a_{\alpha\gamma} = \overset{\circ}{R}_{\alpha\beta\gamma\beta}, \quad \alpha \neq \gamma, \beta, \quad \beta \neq \gamma. \qquad (2.16)$$

The obtained formulas yield the following global uniqueness theorem:

Theorem 2.1. Let $f_\iota : Y^{n-1}_\iota \longrightarrow S^n$ be two oriented immersions into the oriented Möbius space S^n, $n \geq 4$, Y_ι connected, $\iota = 1,2$.

Assume: 1. The sets

$\hat{Y}_\iota := \{y \in Y_\iota \mid f_\iota$ is not M-flat at $y\}$, $\iota = 1,2$,

are connected and dense in Y_ι. - 2. There exists a diffeomorphism $\Phi: Y_1 \longrightarrow Y_2$ with the following properties: a) Φ preserves the orientation; b) $\Phi(\hat{Y}_1) = \hat{Y}_2$; for $y \in \hat{Y}_1$ we have $\Phi^*\varphi_2(y) = \varphi_1(y)$, $\Phi^*\psi_2(y) = \psi_1(y)$, where φ_ι, ψ_ι denote the 1st and 2nd fundamental forms of f_ι. Then there exists a $g \in SG_n$ such that $f_2 \circ \Phi = g \circ f_1$, i.e. f_1 and f_2 are SG_n-equivalent.

Proof. We identify Y_1 and Y_2 via Φ. Then we have two immersion f_ι of Y_1 inducing the same 1st and 2nd fundamental forms on $\hat{Y}_1$. From $\varphi_1 = \varphi_2$ on the open and dense submanifold $\hat{Y}_1 \subset Y_1$ we conclude $\tilde{E}_1 = \tilde{E}_2$, for the principal bundles $\tilde{E}$ of the immersions $f_\iota \mid \hat{Y}_1$ (Lemma 2.1). From (2.8), (2.15) and (2.16) we see that the structure forms $\tilde{\omega}_\iota$ of the reduced Möbius structures of the immersion $f_\iota \mid \hat{Y}_1$ coincide. Enlarging the structure group $\tilde{H} \subset H_0$ to H_0 and extending the common structure form $\tilde{\omega}$ to ω_0 we see that the SG_n, H_0-structure $E_0(\hat{Y}_1, \omega_0)$ of both immersion $f_\iota \mid \hat{Y}_1$ coincide. Since $\hat{Y}_1$ is connected, Lemma 3.2 of [5] shows the existence of a $g \in SG_n$ such that $f_2 \mid \hat{Y}_1 = g \circ f_1 \mid \hat{Y}_1$. Since $\hat{Y}_1$ is dense in Y_1, we obtain $f_2 = g \circ f_1$ by continuity. □

Now we shall prove a corresponding existence theorem. Let us assume:

I. Y^{n-1} is a connected and simply connected manifold, φ and ψ are two symmetric covariant tensor fields of valence 2 on Y^{n-1}, φ positive definite, and ψ has the properties $\mathrm{Tr}\,\psi(y) = 0$, $\|\psi(y)\| = 1$ for all $y \in Y^{n-1}$.

The latter properties are special cases of (1.8), (1.9), therefore they are necessary. To prove the existence we shall apply Folgerung 3.2 of [5]. From the uniquess theorem we know how to construct a reduced Möbius structure $\tilde{E}(Y^{n-1},\tilde{\omega})$ from the given φ, ψ : $\tilde{E}$ is the bundle of orthonormal frames of the Riemannian manifold $[Y, \varphi]$; $\tilde{\omega} = (\omega^I{}_J)$ is defined as follows: $\omega^\alpha = \omega^\alpha{}_o$, $\omega^\alpha{}_\beta$ are the components of the canonical and the connection form of $[Y,\varphi]$; $\omega^o = \omega^o{}_o = 0$, $\omega^n = \omega^n{}_o = 0$ by (2.2), $\omega^n{}_\alpha$ by (2.3). Now we calculate the coefficients $A_{\alpha,\gamma\delta}$, $\mathring{R}_{\alpha\beta\gamma\delta}$ by the left equation of (2.5), and (2.11) respectively. If the following assumptions are satisfied:

II.a. The equations (2.7) for the b_α are compatible and have the uniquely defined solution (2.8). -b. The equations (2.13) for the $a_{\alpha\beta}$ are compatible and have the uniquely defined solution (2.15), (2.16), -

and if $n \geq 4$, we define $\omega^o{}_\alpha$ and $\omega^o{}_n$ by (1.25) and (1.27). Since these forms are tensorial, i.e. vanish for vertical tangential vectors of $\tilde{E}$, we can construct $\tilde{\omega}$ globally on $\tilde{E}$ or by a family $(\tilde{\omega}_\iota)$ of local structure forms with respect to a covering of Y by local sections (z_ι) of $\tilde{E}$ as well. Completing $\tilde{\omega} = (\omega^I{}_J)$ by the symmetry properties expressed in [3], (2.11). we get a 1-form on $\tilde{E}$ with values in the Lie algebra $\underline{g}_n$. By construction the conditions 1., 2. and 4. of Definition 3.1 in [5] (for $H = \tilde{H}$) are fullfilled. It remains to check the structure equation, condition 3., which appears here as the system of integrability conditions

$$d\omega^I{}_J = -\omega^I{}_K \wedge \omega^K{}_J, \quad I,J,K=0,\dots,n+1, \qquad (2.17)$$

which can be simplified by using the symmetry properties of $\tilde{\omega}$, see (2.12) in [3]. The symmetry

$$\mathring{R}_{\alpha\beta\gamma\delta} = \mathring{R}_{\gamma\delta\alpha\beta} \tag{2.18}$$

implies (1.25) and the condition $d\,\omega^o = 0$; analogously the symmetry of ψ implies the condition $d\omega^n = 0$. Since (2.1) is fulfilled for the Riemannian connection, we obtain the conditions for $d\,\omega^\alpha$. By assumption II the equations (2.7), (2.13) are satisfied, and they imply the integrability conditions (2.4) for $d\omega^n_{\ \alpha}$ and (2.12) for $d\omega^\alpha_{\ \beta}$. The integrability conditions for the remaining forms $\omega^o_{\ i}$, $i = 1,\dots,n$, must be required:

III. The forms $\omega^o_{\ \alpha} = a_{\alpha\beta}\,\omega^\beta$, $\omega^o_{\ n} = b_\alpha\,\omega^\alpha$ defined by II satisfy the following integrability conditions:

$$d\,\omega^o_{\ \alpha} = -\,\omega^o_{\ \beta} \wedge \omega^\beta_{\ \alpha} - \omega^o_{\ n} \wedge \omega^n_{\ \alpha} \tag{2.19}$$

$$d\,\omega^o_{\ n} = -\,\omega^o_{\ \beta} \wedge \omega^\beta_{\ n}. \tag{2.20}$$

If I, II, III are fulfilled, we obtain a $SG_n, \check{H}$-structure $\tilde{E}(Y^{n-1},\tilde{\omega})$ on Y^{n-1}. By Folgerung 3.2 in [5] it defines an immersion $\tilde{\check{f}}: Y^{n-1} \longrightarrow SG_n/\tilde{\check{H}}$, realizing the constructed structure: $\tilde{\check{E}}(Y,\tilde{\omega}) = \tilde{E}(\check{f})$, up to SG_n-congruence. The SG_n-map β : $SG_n/\tilde{\check{H}} \longrightarrow S^n = G_n/H_o$ induced by $\tilde{\check{H}} \subset H_o$ defines an immersion $f = \beta \circ \tilde{\check{f}}$ of Y^{n-1} into S^n up to SG_n-congruence, the reduced Möbius structure $\tilde{E}(f)$ of which coincides with the constructed $\tilde{E}(Y,\tilde{\omega})$; therefore the 1st and 2nd fundamental forms of f are the given ones φ and ψ. Thus we proved:

Theorem 2.2. If for the triple $[Y^{n-1},\varphi,\psi]$, $n \geq 4$, the which can be simplified by using the symmetry properties of $\tilde{\omega}$, see (2.12) in [3]. The symmetry

$$\mathring{R}_{\alpha\beta\gamma\delta} = \mathring{R}_{\gamma\delta\alpha\beta} \qquad (2.18)$$

implies (1.25) and the condition $d\,\omega^o = 0$; analogously the symmetry of ψ implies the condition $d\omega^n = 0$. Since (2.1) is fulfilled for the Riemannian connection, we obtain the conditions for $d\,\omega^\alpha$. By assumption II the equations (2.7), (2.13) are satisfied, and they imply the integrability conditions (2.4) for $d\omega^n_\alpha$ and (2.12) for $d\omega^\alpha_\beta$. The integrability conditions for the remaining forms ω^o_i, $i= 1,\dots,n$, must be required:

III. The forms $\omega^o_\alpha = a_{\alpha\beta}\,\omega^\beta$, $\omega^o_n = b_\alpha\,\omega^\alpha$ defined by II satisfy the following integrability conditions:

$$d\,\omega^o_\alpha = -\,\omega^o_\beta \wedge \omega^\beta_\alpha - \omega^o_n \wedge \omega^n_\alpha \qquad (2.19)$$

$$d\,\omega^o_n = -\,\omega^o_\beta \wedge \omega^\beta{}_n . \qquad (2.20)$$

If I, II, III are fulfilled, we obtain a $SG_n, \check{H}$-structure $\tilde{E}(Y^{n-1},\tilde{\omega})$ on Y^{n-1}. By Folgerung 3.2 in [5] it defines an immersion $\tilde{f}$: $Y^{n-1} \longrightarrow SG_n/\tilde{H}$, realizing the constructed structure: $\tilde{E}(Y,\tilde{\omega}) = \tilde{E}(\tilde{f})$, up to SG_n-congruence. The SG_n-map β : $SG_n/\tilde{H} \longrightarrow S^n = G_n/H_o$ induced by $\tilde{H} \subset H_o$ defines an immersion $f = \beta \circ \tilde{f}$ of Y^{n-1} into S^n up to SG_n-congruence, the reduced Möbius structure $\tilde{E}(f)$ of which coincides with the constructed $\tilde{E}(Y,\tilde{\omega})$; therefore the 1st and 2nd fundamental forms of f are the given ones φ and ψ. Thus we proved:

Theorem 2.2. If for the triple $[Y^{n-1},\varphi,\psi]$, $n \geq 4$, the assumptions I, II, III are satisfied, there exists an oriented immersion f: $Y^{n-1} \longrightarrow S^n$ without M-flat points for which φ and ψ are the 1st and 2nd SG_n-invariant fundamental forms. □

Finally we consider the case n = 3. From (1.8), (1.9) it follows that the eigenvalues of ψ with respect to φ are $\mu_1 = \frac{1}{\sqrt{2}} = -\mu_2$.*) Assuming again II.a we can define $\omega^0{}_3$ by (2.8).

Assumption II.b can omitted since (2.13) consists of one equation only:

$$R^0{}_{1212} = -a_{11} - a_{22}. \tag{2.21}$$

Let us assume now that ω^1, ω^2 correspond to the eigendirections of ψ:

$$\omega^3{}_1 = \frac{\omega^1}{\sqrt{2}}, \qquad \omega^3_2 = -\frac{\omega^2}{\sqrt{2}}. \tag{2.22}$$

From (2.20) we obtain the equation

$$d\omega^0{}_3 = -\sqrt{2}\, a_{12}\omega^1 \wedge \omega^2, \tag{2.23}$$

which can be used to determine $a_{12} = a_{21}$. Since we could not define a_{11}, a_{22} from the given φ, ψ, we must suppose another invariant to be given. Let

$$D(y) := a_{11}(y) - a_{22}(y) \tag{2.24}$$

be defined with respect to a basis satisfying (2.22). In the same way as theorems 2.1, 2.2 one can prove:

Theorem 2.3. For n=3 let assumptions I., II.a be valid, and the G_3-invariant function D be given; assume (2.19) for

*) "Krümmungslinienparameter" in B.-Th. III, p. 313, (86) (?)

the $\omega^o{}_i$, i= 1,2,3, defined by (1.27), (2.8), resp. (2.21), (2.23) and (2.24). Then there exists one and up to G_n-equivalence only one immersion $f:Y^2 \longrightarrow S^3$ with the given φ, ψ, D as the induced G_n-invariants. □

This theorem has already been proved in [3], theorem 5.4, for the special class of immersion $f:Y^2 \longrightarrow S^4$, the image of which is contained in an $S^3 \subset S^4$.

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