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SUBMANIFOLDS OF THE MÖBIUS SPACE IV: CONFORMAL INVARIANTS OF IMMERSIONS INTO SPACES OF CONSTANT CURVATURE

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Let $E^n(c)$ denote the simply connected space of constant curvature c, considered as an open submanifold of the Möbius space S^n, see (3) below. Then each immersion $f: Y^m \longrightarrow E^n(c)$ is at the same time an immersion into the Möbius space S^n. Since the isometry group $O_c(n)$ of $E^n(c)$ is a subgroup of the Möbius group G_n, the G_n-invariants of the immersion f, described in [6], can be expressed in terms of the Riemannian invariants of f, see proposition 3. Proposition 2 characterizes the totally umbilical submanifolds of $E^n(c)$, and proposition 4 shows that the M-isoclinic immersions $f: Y^2 \longrightarrow S^4$ defined in [6] coincide with the twistor-holomorphic immersions introduced by Th. Friedrich [2]. Most of the results are known at least in special cases; but the simple proofs given here seem to be new.

Let (e_I) denote a fixed p.-o. frame and (a_I) the corresponding i.-o. frame of the pseudo-euclidean vector space V^{n+2} of index one, $I = 0,\dots, n+1$, see [6]. We consider the Möbius space S^n as the set of the generators [b] of the positive light cone $J^+ = \{b \in V \mid \langle b, b\rangle = 0, \langle b, e_0\rangle < 0\}$. In V we have the following standard models of the simply connected, complete spaces $E^n(c)$:

$$x \in E^n(c) \;:\leftrightarrow$$

$$(1)\quad \begin{cases} x = e_i x^i, \quad i = 1,\dots, n, & \text{if } c = 0;\\ x = e_0 x^0 + e_i x^i, \quad \langle x, x\rangle = -r^2, & \text{if } c = -r^{-2};\\ x = e_i x^i + e_{n+1} x^{n+1}, \quad \langle x, x\rangle = r^2, & \text{if } c = r^{-2}; \end{cases}$$

here is $r > 0$, and we use the sum convention. Put

$$(2)\quad \begin{cases} F_c(x) = a_0 + x + a_{n+1}\langle x, x\rangle/2, & \text{if } c = 0,\\ F_c(x) = x - e_{n+1} r, \quad r > 0, & \text{if } c < 0,\\ F_c(x) = e_0 r + x, & \text{if } c > 0. \end{cases}$$

Then we have $F_c(x) \in J^+$, and the map

$$(3)\qquad \iota_c: x \in E^n(c) \longmapsto [F_c(x)] \in S^n$$

defines the usual conformal embedding of $E^n(c)$ into the Möbius space. The sets $F_c(E^n(c))$ are the intersections of J^+ with the hyperplanes $\langle n_c, b\rangle = 1$, where $n_o = -a_{n+1}$, $n_c = -e_{n+1}/r$, if $c<0$, and $n_c = -e_o/r$, if $c>0$. One easily proves

(4) $\quad \langle dF_c(t), dF_c(s)\rangle = \langle t, s\rangle, \quad t, s \in T_xE^n(c),$

what means that F_c is an isometric embedding of $E^n(c)$ into the isotropic cone J^+ with the induced metric.

In each of the $E^n(c)$ we take the point $(x^i) = o$ as origin ($x^o = r$ if $c<0$ and $x^{n+1} = -r$ if $c>0$), and the $e_i \in T_oE^n(c)$ as a fixed orthonormal frame at o. Since the isometry groups act simply transitively on the bundles $O(E^n(c))$ of orthonormal frames over $E^n(c)$, we identify $O_c(n) = O(E^n(c))$. The following proposition describes the natural embedding of the isometry group $O_c(n)$ into the Möbius group G_n, corresponding to the conformal model, in terms of the frame bundles.

<u>Proposition 1.</u> The map $z = (x, v_i) \in O(E^n(c)) \mapsto \overline{F}_c(z) \in G_n$ with

(5) $\overline{F}_c(x, v_i) := (F_c(x), (dF_c)_x(v_i), b_{n+1}(x)),$

where $b_{n+1}(x)$ is defined by

$$(6)\quad b_{n+1} = \begin{cases} a_{n+1}, & \text{if } c = 0, \\ (x + e_{n+1}r)/2r^2, & \text{if } c < 0, \\ (e_or - x)/2r^2, & \text{if } c > 0, \end{cases}$$

is an embedding homomorphism of principal fibre bundles with the corresponding embedding of the structure group

$$A \in O(n) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H_o. \ \square$$

For the proof one verifies easily that $\overline{F}_c(z)$ is an i.-o. frame, thus identified with an element of G_n, see the remark in [6], p. 168, and that $\overline{F}_c$ commutes with the right action of $h \in O(n) \subset H_o$.

Now we consider an immersion $f: Y^m \to E^n(c)$. The bundle $E_1(f)_c$ of first order frames $(f(y), v_i)$ with $v_\alpha \in \operatorname{Im} df_y$ for $\alpha = 1, \ldots, m$ is the first reduction of the induced bundle $f'O(E^n(c))$. Its structure form $\sigma_1 : TE_1(f)_c \to \mathfrak{o}_c(n)$ can be calculated from the Frenet formulas

$$
(7)\qquad
\begin{aligned}
dv_o &= v_\alpha \sigma^\alpha, & x &= v_o r, & \sigma^\kappa &= 0,\ \alpha = 1,\dots, m,\\
dv_\alpha &= - v_o \sigma^\alpha \varepsilon + v_i \sigma^i{}_\alpha, & & & i &= 1,\dots, n,\\
dv_\kappa &= v_i \sigma^i{}_\kappa, & & & \kappa &= m+1,\dots, n,
\end{aligned}
$$

with $r = 1$ and $\varepsilon = 0$ for $c = 0$, and $\varepsilon = \operatorname{sgn} c$, $c = \varepsilon/r^2$. It is well known that all the metric invariants of second order of f can be expressed in terms of the σ^α, $\sigma^i{}_j$. Because of $O_c(n) \subset G_n$ the same must be true for the Möbius invariants, too. Indeed, the map $\overline{F}_c$ defines an embedding of $E_1(f)_c$ into the bundle $E_1(f)$ of first order Möbius frames over Y^m, yielding a reduction of $E_1(f)$ to $O(m) \times O(n-m)$, which is not Möbius invariant, of course. But we can calculate the $\mathfrak{g}_n$-valued structure form ω_1 of this reduction from the Frenet formulas with respect to the i.-o. Möbius frames $(b_I) = \overline{F}_c(f(y), v_i)$:

$$
db_I = b_J \omega^J{}_I, \qquad I, J = 0, 1,\dots, n+1,
$$

and express $\omega_1 = (\omega^J{}_I)$ by $\sigma_1 = (\sigma^\alpha, \sigma^j{}_i)$. It results

$$
(8)\qquad \omega_1 = \begin{pmatrix} 0 & -\varepsilon\sigma^\beta/2r & 0 & 0 \\ r\sigma^\alpha & \sigma^\alpha{}_\beta & \sigma^\alpha{}_\lambda & -\varepsilon\sigma^\alpha/2r \\ 0 & \sigma^\kappa{}_\beta & \sigma^\kappa{}_\lambda & 0 \\ 0 & r\sigma^\beta & 0 & 0 \end{pmatrix}
$$

We consider the Möbius invariants of f associated to the so-called Möbius structure of the immersion, see [6], proposition 4.2. To express them in terms of the metric invariants we proceed in the following manner: we transform $\overline{F}_c(E_1(f)_c) \subset E_1(f)$ into the Möbius structure $\overline{E}_1(f)$ and calculate its structure form $\overline{\omega}_1$ by means of ω_1. To this goal we simply have to carry out the reduction steps described in [6], § 4. If in [6], (4.13) we take

$$
h^\kappa{}_\lambda = \delta^\kappa{}_\lambda, \qquad h^o{}_\kappa = m^{-1} \sum_{\alpha=1}^{m} c^\kappa{}_{\alpha\alpha}
$$

where the $c^\kappa{}_{\alpha\beta}$ are defined by (8) and

$$
(9)\qquad \sigma^\kappa{}_\alpha = c^\kappa{}_{\alpha\beta} r \sigma^\beta,
$$

condition [6], (4.14) is fulfilled; from [6], (4.12) we see that in Riemannian terms the new coefficients $c^\kappa{}_{\alpha\beta}$ are the coordinates of the symmetric bilinear map

(10) $\widetilde{II}$: $s, t \in T_yY^m \mapsto (II - \bar{h}\cdot I)(s, t) \in N_y$,

where I, II denote the first and the second fundamental forms, N_y the normal space at y, and $\bar{h}$ the mean curvature vector of f: $Y^m \to E^n(c)$. The condition that f is a totally umbilical im= mersion can easily be proved to be equivalent to $\widetilde{II} = 0$. By defi= nition [6], (4.16) we obtain

(11) $S^2(z) = |\widetilde{II}|^2 = \sum_{\varkappa,\alpha,\beta} (\tilde{c}^{\varkappa}{}_{\alpha\beta}(z))^2 = |II|^2 - mH^2 \geq 0$

with $H = |\bar{h}|$ the mean curvature. Thus, proposition [6], 4.1 yields the following simple generalization of a result proved by B.-Y. Chen [1]:

<u>Proposition 2.</u> An immersion f: $Y^m \to E^n(c)$ is totally umbili= cal if and only if its Möbius picture $\iota_c f(Y^m)$ is an open subset of an m-sphere, i.e. it is totally geodesic or m-spherical in the case $c \geq 0$, and it is totally geodesic, m-spherical, m-equidistant or m-horocyclic in the case $c < 0$. □

To obtain the Möbius structure $E_1(Y^m, \bar{\omega}_1)$ of the immersion f without umbilical points we put c = 1 in [6], (4.19) and have to consider the transformation rules with

$$\mu = S^{-1}(z),\quad h^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta},\ h^{\varkappa}{}_{\lambda} = \delta^{\varkappa}{}_{\lambda},\ h^{o}{}_{i} = 0.$$

Then the G_n-invariants of the immersion, described in [6], propo= sition 4.2, can be expressed by the Riemannian invariants in the following way:

<u>Proposition 3.</u> Let f: $Y^m \to E^n(c)$ be an immersion without umbilical points:

(12) $\gamma(y) := (|II|^2 - mH^2)^{1/2} > 0$,

and choose c = 1 in [6], (4.19). Then the G_n-invariant Riemann= ian metric [6], (4.22), is

(13) $\varphi_y(s, t) = \gamma^2(y)\cdot I_y(s, t)$.

The corresponding volume element is

(14) $dY = \gamma^m(y)\cdot dy$,

dy the Riemannian volume element.
The G_n-invariant second vectorial fundamental form α_y is given by

(15) $\alpha_y(s, t) = \gamma(y)(II_y - \bar{H}\cdot I_y)(s, t), \quad s, t \in T_yY^m.$ □

We remark that the volume (14) up to a constant factor coincides with the volume calculated by Ch.-Ch. Hsiung and G. R. Mugridge [3].

Furthermore we obtain as a corollary the well-known result (see M. A. Matsumoto [4]) that the principal curvature directions of a hypersurface $F^{n-1} \subset E^n(c)$ are G_n-invariant; when λ_α denote the Riemannian principal curvatures of F^{n-1}, we have the corresponding G_n-invariant principal curvatures

(16) $\hat{\lambda}_\alpha = \gamma^{-1}(\lambda_\alpha - H);$

for n = 3 this yields $\hat{\lambda}_1 = -\hat{\lambda}_2 = 2^{-1/2}$.

Since the 'normal vectors' $b_\varkappa$ remain unchanged under the reduction procedure we obtain with respect to the Möbius frames $z \in \bar{E}_1(f)$:

(17) $\bar{\omega}^\varkappa{}_\lambda = \sigma^\varkappa{}_\lambda, \quad \varkappa, \lambda = m+1, \ldots, n,$

which yields a simple proof for the known result (see J. D. Moore and J. H. White [5]) that the Riemannian connection of the normal bundle of f remains invariant under the action of G_n.

Now we consider surfaces $f(Y^2) \subset E^4(c)$. In [6] we introduced the so-called M-isoclinic surfaces in S^4 by the condition

(18) $\alpha(s, s) = \text{const.}$ for all $s \in T_yY^2$ with $\varphi(s, s) = 1$.

If one evaluates this condition one gets from [6], (5.6), (5.7) (with 1/2 replaced by 1/4, since we normed by c = 1 now):

Proposition 4. An immersion $f: Y^2 \to E^4(c)$ without umbilical points has M-isoclinic Möbius-picture if and only if under an appropriate orientation of $E^4(c)$ the coefficients (9) of the Riemannian second fundamental form II with respect to each tangential moving frame fulfil the conditions

(19) $c^3{}_{11} - c^3{}_{22} = 2c^4{}_{12}, \quad c^4{}_{11} - c^4{}_{22} = -2c^3{}_{12}.$ □

Obviously, these equations show that the class of M-isoclinic immersions coincides with the class of twistor-holomorphic immersions introduced by Th. Friedrich [2], compare proposition 2.

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